

Superstrata

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We give a survey of the present status of the microstate geometries called superstrata. Superstrata are smooth, horizonless solutions of six-dimensional supergravity that represent some of the microstates of the D1-D5-P black hole in string theory. They are the most general microstate geometries of that sort whose CFT dual states are identified. After reviewing relevant features of the dual CFT, we discuss the construction of superstratum solutions in supergravity, based on the linear structure of the BPS equations. We also review some of recent work on generalizations of superstrata and physical properties of superstrata. Although the number of superstrata constructed so far is not enough to account for the black-hole entropy, they give us valuable insights into the microscopic physics of black holes.

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1 Introduction

The microstate geometry program aims to explicitly construct as many microstates of black holes as possible, as “microstate geometries”, *i.e.* smooth, horizonless solutions of classical supergravity. In this program, the so-called D1-D5 system has played an important role. This system is obtained by compactifying type IIB string theory on $S^1 \times \mathcal{M}$ with $\mathcal{M} = T^4$ or K3 and wrapping N_1 D1-branes¹ on S^1 and N_5 D5-branes on $S^1 \times \mathcal{M}$. The size of \mathcal{M} is taken to be of the string scale while the radius R_y of S^1 remains macroscopic. If we add a third charge, N_P units of Kaluza-Klein momentum (P) charge along S^1 , we have a 1/8-BPS, 3-charge black hole in five dimensions (or black string in six dimensions, if we include S^1) with a finite entropy which was reproduced by Strominger and Vafa by counting microstates in the brane worldvolume theory [2]. More generally, we can also add left-moving angular momentum J and the area entropy of the resulting 1/8-BPS black hole (the BMPV black hole [3]) is given by²

$$S_{\text{BMPV}} = 2\pi\sqrt{N_1 N_5 N_P - J^2}. \quad (1.1)$$

A central question in the microstate geometry program is how much of this entropy can be accounted for by supergravity solutions.

The precursor of the microstate geometry program was the study of the 2-charge states of the D1-D5 system, namely the ones with $N_P = 0$. In this case, the microstates can be realized as microstate geometries called Lunin-Mathur geometries [4–7], which are parametrized by functions of one variable. The growth of the microscopic entropy, $S_{2\text{-chg}} \sim \sqrt{N_1 N_5}$, can be reproduced by counting Lunin-Mathur geometries [8, 9], although the 2-charge ensemble has vanishing area entropy at the classical level.

This success led to the microstate geometry program to construct microstate geometries for the 3-charge system, namely for the case with $N_P > 0$, for which the area entropy is non-vanishing at the classical level. Many families of 3-charge microstates have been constructed based on five-dimensional multi-center solutions [10–13] (see [14, 15] about smooth multi-center solutions) and other methods, such as solution-generating technique, the matching technique, and BPS equations [16–26] (see also [27, 28]). More recently, a new class of microstate geometries called *superstrata* was constructed [29]. Superstrata are solutions of six-dimensional supergravity parametrized by functions of three variables, and represent the most general microstate geometries known thus far for the D1-D5-P black hole, with understood CFT dual.³

¹For $\mathcal{M} = \text{K3}$, N_1 includes the D1-brane charge induced on the worldvolume of the D5-branes by a curvature coupling [1]. Namely $N_1 = N_1^{\text{explicit}} + N_1^{\text{induced}}$, $N_1^{\text{induced}} = -N_5$.

² $J = J_0^3 \in \mathbb{Z}/2$ where J_0^3 is a generator of $SU(2)_L \in SO(4)$ coming from the rotational symmetry in the directions transverse to the D-branes.

³The CFT dual states of superstrata based on the five-dimensional multi-center solutions with two centers

The existence of superstrata was conjectured based on the idea of double supertube transition [30]. The 2-charge microstates (Lunin-Mathur geometries) have a dipole charge which does not exist in the original configuration of the D1- and D5-branes but is produced by the supertube transition [31]. In [32,33], it was argued that, via a multistage supertube transition, black-hole microstates involve various dipole charges that do not exist in the original configuration.⁴ Based on this idea, in [30], it was conjectured that the double supertube transition in the D1-D5-P system will lead to smooth microstate geometries, *i.e.* superstrata, that depend on the coordinate of the S^1 , v , and are parametrized by functions of (at least) two variables. This v -dependence means that superstrata will live in six-dimensional supergravity.

Although it was conjectured that superstrata exist in six-dimensional supergravity, there were some more steps needed for their actual construction. The first was the realization that the BPS equations of six-dimensional supergravity [34,35] have a linear structure [36]; namely, the BPS equations can be organized so that they are linear if solved in a certain order. Leveraging this structure, one can start with infinitesimal (linear) perturbation around a simple background and then non-linearly complete it to obtain a fully backreacted solution. Another step is so-called “coiffuring” [29]. This means that, when one goes from a linear (infinitesimal) solution to a full non-linear solution, one must turn on fields that were not turned on in the linear solution, in order for the geometry to be regular (plus, in order for the BPS equations to simplify and be solvable). From an AdS/CFT viewpoint, this is due to the fact that [37] single-trace operators and double-trace operators with the same quantum numbers mix and therefore turning on one at linear order implies turning on the other at higher order. The fields particularly relevant for coiffuring were identified in [38–40].

Based on these developments, the first examples of superstrata were explicitly constructed in [29], providing a proof of their existence. After that, superstratum solutions have been generalized in many ways [41–47], and a number of checks of their proposed AdS/CFT dictionary were carried out [37,48]. By now it is fair to say that we have a pretty good picture of possible superstrata solutions, although explicit expressions have been found only for a limited class of solutions. They come in multiple species and are parametrized by functions of three variables, with rich physical content that one can explore. Various physical aspects of the solutions and their implications for black-hole microphysics are being actively investigated; instead of listing the relevant literature here, we will review some of the developments later in this article. It is expected that superstrata will continue to give us useful insights into the microstructure of black holes in string theory.

are known, while duals of multi-center solutions with more than two centers or those of their superstratum generalizations are not known.

⁴The string-theory configurations resulting from such multistage supertube transition should be called (general) superstrata, which in general contain dipole charges that do not allow a description in terms of smooth geometry. Superstrata that do allow a geometric description, like the ones constructed in [29], should properly be called *geometric* superstrata, although they are normally simply called superstrata.

In the remainder of the article, we will review the construction of superstrata with some explicit examples and then attempt a short survey of further developments in the literature. In section 2, we review the CFT picture of the states of the D1-D5 system, focusing on the map between chiral primary states in CFT and 1/4-BPS supergraviton states in the bulk, and between left-descendant states in CFT and 1/8-BPS supergraviton states in the bulk. Superstrata are nothing but coherent superpositions of 1/8-BPS supergravitons. In section 3, we review the supergravity setup and the three layers of equations to be satisfied by supersymmetric solutions. We discuss some 1/4-BPS microstate geometries as examples. In section 4, we review the construction of superstrata based on the formulation of section 3. We emphasize the importance of focusing on solutions for which the base space is fixed and, taking flat base, construct explicit solutions. Superstrata can have arbitrary sets of modes but, for simplicity, we focus on the single-mode superstrata in which only one mode is turned on. In section 5, we review some of the recent developments in generalizing superstratum solutions and the studies of their properties. We end with a conclusion in section 6.

2 CFT

After the decoupling limit, the geometry of the D1-D5 system becomes asymptotically $\text{AdS}_3 \times S^3 \times \mathcal{M}$. This means that this system can be equivalently described by a holographic CFT, which is known as the D1-D5 CFT.⁵ This theory is a $d = 2, \mathcal{N} = (4, 4)$ CFT with a symmetry group $SU(1, 1|2)_L \times SU(1, 1|2)_R$, which is generated by the affine generators $L_n, G_n^{\alpha A}, J_n^i$ and their right-moving versions $\tilde{L}_n, \tilde{G}_n^{\dot{\alpha} A}, \tilde{J}_n^{\dot{i}}$. Here, $\alpha = \pm$ is a doublet index and $i = 1, 2, 3$ is a triplet index for $SU(2)_L \subset SU(1, 1|2)_L$, while $\dot{\alpha}, \dot{i}$ are their right-moving counterparts. The index $A = 1, 2$ is the doublet index for an additional $SU(2)_B$ symmetry group which acts as an outer automorphism on the superalgebra. In its moduli space, the D1-D5 CFT is believed to have an orbifold point where the theory is described by a supersymmetric sigma model with the target space being the symmetric orbifold, $\text{Sym}^N \mathcal{M}$, where [51, 52]⁶

$$N \equiv \begin{cases} N_1 N_5 & (\mathcal{M} = T^4), \\ N_1 N_5 + 1 & (\mathcal{M} = \text{K3}). \end{cases} \quad (2.1)$$

We will be working at the orbifold point henceforth.

⁵For reviews of the D1-D5 CFT, see *e.g.* [49, 50].

⁶For $\mathcal{M} = T^4$, the low-energy dynamics of the D-brane bound state can be described by a supersymmetric sigma model with target space $\mathbb{R}^4 \times T^4 \times \text{Sym}^{N_1 N_5}(T^4)$, where the \mathbb{R}^4 part describes the center-of-mass motion of the D-branes in the noncompact \mathbb{R}^4 , the T^4 part describes worldvolume Wilson lines along the internal T^4 , and the $\text{Sym}^{N_1 N_5}(T^4)$ part describes the moduli space of D1-branes as instantons inside the D5 worldvolume [51, 53]. Here we are focusing on the last part. For $\mathcal{M} = \text{K3}$, the target space is $\mathbb{R}^4 \times \text{Sym}^{N_1 N_5 + 1}(\text{K3})$ [2, 51] where the \mathbb{R}^4 part describes the center-of-mass motion in the noncompact \mathbb{R}^4 and the $\text{Sym}^{N_1 N_5 + 1}(\text{K3})$ part describes the instanton moduli space which we are focusing on.

If we want to preserve supersymmetry in the D1-D5 system, we must impose the periodic boundary condition for the worldvolume fermions along the S^1 on which the D-branes are wrapped. This means that we are naturally in the RR (Ramond-Ramond) sector of the SCFT.

In this section, we review the structure of the BPS states in the D1-D5 CFT, whose holographic dual we are after. We start with the NS (Neveu-Schwarz) sector of the theory in which the spectrum of states is somewhat more transparent, and then discuss the R sector.

2.1 NS sector

2.1.1 1/4-BPS supergraviton states

In the NS-NS sector, the theory has the unique vacuum with $L_0 = \tilde{L}_0 = 0$ which preserves all the 8+8 supercharges of the theory. The bulk dual of the NS-NS vacuum is empty $\text{AdS}_3 \times S^3$.

As excited states above the NS-NS vacuum, the theory has *single-particle* chiral primary states which are in one-to-one correspondence with the Dolbeault cohomology of \mathcal{M} [54, 55]. For $\mathcal{M} = T^4$, we have 16 species of states

$$T^4 : \begin{aligned} & |\alpha\dot{\alpha}\rangle_k, \quad h = j = \frac{k+\alpha}{2}, \quad \tilde{h} = \tilde{j} = \frac{k+\dot{\alpha}}{2}, \quad \text{bosonic}, \\ & |\alpha\dot{A}\rangle_k, \quad h = j = \frac{k+\alpha}{2}, \quad \tilde{h} = \tilde{j} = \frac{k}{2}, \quad \text{fermionic}, \\ & |\dot{A}\dot{\alpha}\rangle_k, \quad h = j = \frac{k}{2}, \quad \tilde{h} = \tilde{j} = \frac{k+\dot{\alpha}}{2}, \quad \text{fermionic}, \\ & |\dot{A}\dot{B}\rangle_k, \quad h = j = \frac{k}{2}, \quad \tilde{h} = \tilde{j} = \frac{k}{2}, \quad \text{bosonic}, \end{aligned} \quad (2.2)$$

where $k = 1, \dots, N$. $\dot{A}, \dot{B} = 1, 2$ are doublet indices for an $SU(2)_C$ that is not part of the symmetry group of the theory. h, j are the values of L_0, J_0^3 , while \tilde{h}, \tilde{j} are those of $\tilde{L}_0, \tilde{J}_0^3$. At the orbifold point, these states correspond to twist operators of order k ; namely, they intertwine k copies of \mathcal{M} (out of N copies). We refer to these k copies, thus intertwined together, as a strand of length k . Because spin is $j - \tilde{j}$, the states $|\alpha\dot{\alpha}\rangle, |\dot{A}\dot{B}\rangle$ are bosonic while $|\alpha\dot{A}\rangle, |\dot{A}\dot{\alpha}\rangle$ are fermionic. The $SU(2)_C$ -invariant linear combination $\frac{1}{\sqrt{2}}\epsilon_{\dot{A}\dot{B}}|\dot{A}\dot{B}\rangle_k$ is denoted by $|00\rangle_k$, which corresponds to the Kähler form of T^4 . For $\mathcal{M} = K3$, there are 24 species of single-particle chiral primary states and they are all bosonic:

$$K3 : \begin{aligned} & |\alpha\dot{\alpha}\rangle_k, \quad j = \frac{k+\alpha}{2}, \quad \tilde{j} = \frac{k+\dot{\alpha}}{2} \\ & |I\rangle_k, \quad j = \frac{k}{2}, \quad \tilde{j} = \frac{k}{2}, \quad I = 1, \dots, 20. \end{aligned} \quad (2.3)$$

Among $|I\rangle_k$, the one that corresponds to the Kähler form of K3 is denoted by $|00\rangle_k$.

All these states (2.2), (2.3) preserve 8 supercharges, 4 from the left and another 4 from the right.⁷ Conventionally, they are said to be 1/4-BPS, relative to the amount of supersymmetry (32 supercharges) of type IIB superstring in ten dimensions (although they preserve half of the supersymmetry of the D1-D5 CFT).

⁷Except for the case with $\alpha = -$ ($\dot{\alpha} = -$) and $k = 1$ for which 8 left-moving (right-moving) supercharges are preserved.

Among the states in (2.2), (2.3), the state $|--\rangle_1 = |\alpha = -, \dot{\alpha} = -\rangle_1$ is special because it has $h = j = \tilde{h} = \tilde{j} = 0$ and actually represents the vacuum (of a single copy of \mathcal{M}). All other states can be thought of as excitations and, via AdS/CFT, correspond to the possible excitations in linearized supergravity around empty $\text{AdS}_3 \times S^3$, called “supergravitons”. In other words, each of the chiral primary states (2.2), (2.3) (except $|--\rangle_1$) is in one-to-one correspondence with a particular single-particle, 1/4-BPS state of the supergraviton propagating in the bulk $\text{AdS}_3 \times S^3$ background [54, 56–58].

If we multiply together single-particle chiral primary states, we obtain *multi-particle* chiral primary states, which are the most general 1/4-BPS states. Explicitly, they can be written as

$$\prod_{\psi} \prod_{k=1}^N [|\psi\rangle_k]^{N_k^{\psi}}, \quad (2.4)$$

where $|\psi\rangle$ runs over different species in (2.2) or (2.3). The general chiral primary state is specified by the set of numbers $\{N_k^{\psi}\}$, which correspond to the number of strands of species $|\psi\rangle$ and length k . The values that N_k^{ψ} can take are $0, 1, 2, \dots$ if $|\psi\rangle$ is bosonic and $0, 1$ if $|\psi\rangle$ is fermionic. The strand numbers $\{N_k^{\psi}\}$ must satisfy the constraint that the total strand length is equal to N :

$$\sum_{\psi} \sum_k k N_k^{\psi} = N. \quad (2.5)$$

The non-trivial part of the multi-particle chiral primary state (2.4) is made of single-particle chiral primary states that are not the trivial state $|--\rangle_1$. The trivial part of the state is made of N_1^{--} copies of the trivial state $|--\rangle_1$, so that the total strand length is N .

In the bulk, the states (2.4) correspond to multi-particle, 1/4-BPS states of supergravitons (“supergraviton gas”). Namely, the states (2.4) span the Fock space of 1/4-BPS supergravitons, modulo the constraint (2.5). When $N_k^{\psi} = \mathcal{O}(N)$ (where $|\psi\rangle_k \neq |--\rangle_1$), the bulk picture of supergravitons propagating in undeformed $\text{AdS}_3 \times S^3$ is no longer valid but the geometry becomes deformed by backreaction.

In order to correspond to the supergravity point, the boundary CFT must be perturbed away from the orbifold point where the chiral primary states have the above simple description. Even if we go away from the orbifold point, the number of chiral primary states remains the same, although individual states can mix into each other [59].⁸

⁸For $\mathcal{M} = \text{K3}$, supersymmetry implies that the number of chiral primary states do not change [60]. For $\mathcal{M} = T^4$, such supersymmetry argument is not enough for showing that the number stays constant, although we expect that it does, on physical grounds (single-particle supergravitons and their gas must exist everywhere in the moduli space).

2.1.2 1/8-BPS supergraviton states

The single-particle chiral primary states in (2.2) and (2.3) are the highest-weight states with respect to the rigid $SU(1, 1|2)_L \times SU(1, 1|2)_R$ symmetry and more general, descendant states in the $SU(1, 1|2)_L \times SU(1, 1|2)_R$ multiplet can be obtained by the action of the rigid generators $\{L_{-1}, G_{-1/2}^{-,A}, J_0^-\}$ and $\{\tilde{L}_{-1}, \tilde{G}_{-1/2}^{-,A}, \tilde{J}_0^-\}$. To preserve supersymmetry, we will only consider descendants obtained by the action of the left-moving generators $\{L_{-1}, G_{-1/2}^{-,A}, J_0^-\}$. If we start with a chiral primary states with $h = j$, which we denote by $|j, j\rangle\rangle$, we generate the following states:

$$\begin{aligned}
& |j+n, j\rangle\rangle \xrightarrow{J_0^-} |j+n, j-1\rangle\rangle \xrightarrow{J_0^-} \cdots \xrightarrow{J_0^-} |j+n, -j\rangle\rangle \\
& \quad \downarrow G_{-1/2}^{-,A} \\
& |j+\frac{1}{2}+n, j-\frac{1}{2}\rangle\rangle \xrightarrow{J_0^-} |j+\frac{1}{2}+n, j-\frac{3}{2}\rangle\rangle \xrightarrow{J_0^-} \cdots \xrightarrow{J_0^-} |j+\frac{1}{2}+n, -(j-\frac{1}{2})\rangle\rangle \\
& \quad \downarrow G_{-1/2}^{-,B} \\
& |j+1+n, j-1\rangle\rangle \xrightarrow{J_0^-} |j+1+n, j-2\rangle\rangle \xrightarrow{J_0^-} \cdots \xrightarrow{J_0^-} |j+1+n, -(j-1)\rangle\rangle
\end{aligned} \tag{2.6}$$

Here, $|h, j\rangle\rangle$ means a state with $(L_0, J_0^3) = (h, j)$. The states in the second line are doubly degenerate, because we can use $G_{-1/2}^{-,A}$ with either $A = 1$ or $A = 2$ to descend from the first line to the second. The third line has no such degeneracy because we can only descend from the first line with $G_{-1/2}^{-,1}G_{-1/2}^{-,2}$. More precisely, to get a genuinely new state, we must act instead with $G_{-1/2}^{-,1}G_{-1/2}^{-,2} + \frac{1}{2h}L_{-1}J_0^-$ where h is the value of L_0 for the chiral primary state [46, 50]. Moreover, the number $n = 0, 1, \dots$ corresponds to the number of times we act on the state with L_{-1} . We denote the states thus obtained building on $|\psi\rangle_k$ by⁹

$$|\psi; k, m, n\rangle = \frac{1}{m!n!} (J_0^-)^m (L_{-1})^n |\psi\rangle_k, \tag{2.7a}$$

$$|\psi; k, m, n, A\rangle = \frac{1}{(m-1/2)!(n-1/2)!} (J_0^-)^{m-1/2} (L_{-1})^{n-1/2} G_{-1/2}^{-,A} |\psi\rangle_k, \tag{2.7b}$$

$$|\psi; k, m, n, 12\rangle = \frac{1}{(m-1)!(n-1)!} (J_0^-)^{m-1} (L_{-1})^{n-1} \left(G_{-1/2}^{-,1}G_{-1/2}^{-,2} + \frac{1}{2h}L_{-1}J_0^- \right) |\psi\rangle_k. \tag{2.7c}$$

The range of m, n is: $m = 0, 1, \dots, 2h$, $n = 0, 1, 2, \dots$ for (2.7a); $m = \frac{1}{2}, \frac{3}{2}, \dots, 2h - \frac{1}{2}$, $n = \frac{1}{2}, \frac{3}{2}, \dots$ for (2.7b); and $m = 1, 2, \dots, 2h - 1$, $n = 1, 2, 3, \dots$ for (2.7c). The numbers m and n give the increase in $-J_0^3$ and L_0 relative to the chiral primary state $|\psi\rangle_k$. If $h = 0$ the states (2.7b) and (2.7c) do not exist, and if $h = 1/2$ the state (2.7c) does not exist. If the chiral primary state $|\psi\rangle_k$ is bosonic (fermionic), the states (2.7a) and (2.7c) are bosonic (fermionic) while the state (2.7b) is fermionic (bosonic). These states break all left-moving supersymmetry but preserve 4 right-moving supercharges. In the bulk, they correspond to single-particle,

⁹These states are not normalized.

1/8-BPS supergraviton states obtained by the bulk action of the rigid $SU(1, 1|2)_L$ generators. Although we only considered descendants obtained by left rigid generators here, if we also included descendants obtained by right rigid generators, we could reproduce the complete spectrum of linearized supergravity around $\text{AdS}_3 \times S^3$ [54, 56–58].

Just as in the 1/4-BPS case, we can multiply together single-particle 1/8-BPS states to construct a more general, multi-particle 1/8-BPS state:

$$\prod_{\psi, k, m, n, f} [|\psi; k, m, n, f\rangle]^{N_{k, m, n, f}^\psi}, \quad \sum_{\psi, k, m, n, f} k N_{k, m, n, f}^\psi = N, \quad (2.8)$$

where $f = \text{null}, A, 12$ so that it covers all the three kinds in (2.7). If the state $|\psi; k, m, n, f\rangle$ is bosonic (fermionic), $N_{k, m, n, f}^\psi = 0, 1, 2, \dots$ ($N_{k, m, n, f}^\psi = 0, 1$). The state (2.8) corresponds in the bulk to a 1/8-BPS state of the supergraviton gas. Namely, (2.8) spans the Fock space of 1/8-BPS supergravitons, modulo the constraint on $N_{k, m, n, f}^\psi$.

2.2 R sector

By spectral flow transformation, we can map all the above statements into the R sector, which more directly corresponds to the bulk states of the D1-D5 system. By spectral transformation, the charges (h, j) of a state on a strand of length k are transformed as follows:

$$h' = h + 2\eta j + k\eta^2, \quad j' = j + k\eta. \quad (2.9)$$

If we take the flow parameter $\eta = -1/2$, NS states get mapped into R states. However, to match the convention of charges to that in the literature [29, 42, 44], we further flip the sign of the $SU(2)_L$ charge, as $j \rightarrow -j$. So, the map from NS to R that we will be using is

$$h^{\text{R}} = h^{\text{NS}} - j^{\text{NS}} + \frac{k}{4}, \quad j^{\text{R}} = \frac{k}{2} - j^{\text{NS}}. \quad (2.10)$$

The same transformation in the right-moving sector is understood.

The map (2.10) transforms single-particle chiral primary states into R ground states on a single strand of length k . For example,

$$\begin{aligned} |--\rangle_k^{\text{NS}}, \quad h^{\text{NS}} = j^{\text{NS}} = \frac{k-1}{2} &\rightarrow |++\rangle_k^{\text{R}}, \quad h^{\text{R}} = \frac{k}{4}, j^{\text{R}} = \frac{1}{2}, \\ |00\rangle_k^{\text{NS}}, \quad h^{\text{NS}} = j^{\text{NS}} = \frac{k}{2} &\rightarrow |00\rangle_k^{\text{R}}, \quad h^{\text{R}} = \frac{k}{4}, j^{\text{R}} = 0. \end{aligned} \quad (2.11)$$

The NS vacuum (empty $\text{AdS}_3 \times S^3$ in the bulk) goes to the following R ground state:

$$[|--\rangle_1^{\text{NS}}]^N \rightarrow [++\rangle_1^{\text{R}}]^N \quad (2.12)$$

The general R ground states, which are general 1/4-BPS states, are

$$\prod_{\psi} \prod_{k=1}^N [|\psi\rangle_k^{\text{R}}]^{N_k^\psi}, \quad \sum_{\psi} \sum_k k N_k^\psi = N. \quad (2.13)$$

where $|\psi\rangle$ runs over the species in (2.2) or (2.3), now understood as R ground states on a strand of length k . Coherent superpositions [7, 61, 62] of these supergraviton states are dual to smooth 1/4-BPS geometries called Lunin-Mathur geometries [4–7], as mentioned in the introduction.

The 1/8-BPS states in the NS sector, (2.8), map into the R states of the following form:

$$\Psi(\{N_{k,m,n,f}^\psi\}) \equiv \prod_{\psi,k,m,n,f} \left[|\psi; k, m, n, f\rangle^{\text{R}} \right]^{N_{k,m,n,f}^\psi}, \quad (2.14a)$$

$$\sum_{\psi,k,m,n,f} k N_{k,m,n,f}^\psi = N, \quad (2.14b)$$

where now the single-particle supergraviton states are given by

$$|\psi; k, m, n\rangle^{\text{R}} = \frac{1}{m!n!} (J_{-1}^+)^m (L_{-1} - J_{-1}^3)^n |\psi\rangle_k^{\text{R}}, \quad (2.15a)$$

$$|\psi; k, m, n, A\rangle^{\text{R}} = \frac{1}{(m-1/2)!(n-1/2)!} (J_{-1}^+)^{m-1/2} (L_{-1} - J_{-1}^3)^{n-1/2} G_{-1}^{+,A} |\psi\rangle_k^{\text{R}}, \quad (2.15b)$$

$$|\psi; k, m, n, 12\rangle^{\text{R}} = \frac{1}{(m-1)!(n-1)!} (J_{-1}^+)^{m-1} (L_{-1} - J_{-1}^3)^{n-1} (G_{-1}^{+,1} G_{-1}^{+,2} + \frac{1}{2h^{\text{NS}}} (L_{-1} - J_{-1}^3) J_{-1}^+) |\psi\rangle_k^{\text{R}}. \quad (2.15c)$$

The range of m and n is the same as for (2.7). The operators acting on the R ground states $|\psi\rangle_k^{\text{R}}$ have charges that have been shifted and sign-flipped due to spectral flow. Coherent superpositions of the states (2.14) are realized as superstrata in the bulk, as we will expand below.

2.3 General 1/8-BPS states

The states we discussed above represent a large class of 1/8-BPS states that are nicely in correspondence with the multi-particle supergraviton states around the bulk $\text{AdS}_3 \times S^3$ background. However, they are *not* the most general 1/8-BPS states. This is because we used only the rigid generators, $L_{-1}, G_{-1/2}^{\alpha A}, J_0^i$, to excite the left-moving sector of the theory. In the Cardy regime, $N_P \gg N$, this class of 1/8-BPS states has entropy $S_{\text{supergravitons}}^{1/8\text{-BPS}} \sim N^{1/2} N_P^{1/4}$ [63].

The most general 1/8-BPS states are obtained by exciting the left-moving sector by general modes of the fields of the theory. In the case of $\mathcal{M} = T^4$, the fundamental fields of the CFT are free bosonic and fermionic fields, which we collectively denote by $X, \Psi, \tilde{\Psi}$.¹⁰ On a strand of length k , these fields have left-moving modes $\alpha_{-\frac{n}{k}}, \Psi_{-\frac{n+1/2}{k}}$ (in the NS sector), and the most general states can be obtained by exciting them in an arbitrary way on all strands, except that the symmetric orbifold symmetry requires $L_0 - \tilde{L}_0$ on each strand to be an integer. For $\mathcal{M} = \text{K3}$, we can work at the orbifold point of the K3 moduli space [64], where K3 is an orbifold of T^4 , and project out states that are not invariant under the orbifold action as

¹⁰These fields have symmetry indices as $X^{A\dot{A}}(z, \bar{z}), \Psi^{\alpha\dot{A}}(z), \tilde{\Psi}^{\dot{\alpha}\dot{A}}(\bar{z})$ and each has four components on each strand [49, 50].

well as include the twisted sector. For both $\mathcal{M} = T^4$ and K3, this will give central charge $c = 6N$ worth of 1/8-BPS states. Alternatively, we can think of the action of the Affine generators $L_{-\frac{n}{k}}, G_{-\frac{n+1/2}{k}}^{\alpha A}, J_{-\frac{n}{k}}^i$. On each copy of \mathcal{M} , these generate an $SU(1,1|2)_L$ current algebra of level 1, leading to central charge $c_{SU(1,1|2)_L} = \frac{3}{2}N$ worth of 1/8-BPS states [65]. So, In the Cardy regime, the most general 1/8-BPS states have entropy $S_{\text{general}}^{1/8\text{-BPS}} \sim \sqrt{cN_P} \sim N^{1/2}N_P^{1/2} \gg S_{\text{supergravitons}}^{1/8\text{-BPS}}$.

The above description of general 1/8-BPS states is valid at the orbifold point of the D1-D5 CFT. If we perturb the CFT away from the orbifold point, some of those 1/8-BPS states will lift. Supergraviton states (namely, superstrata) are expected to remain supersymmetric on physical grounds, but more general 1/8-BPS states can lift. Elliptic genus and its generalization [53, 66, 67] give partial information about the number of states that lift, but precisely which states lift is a highly non-trivial problem and a satisfactory understanding has not emerged yet.¹¹

2.4 Phase diagram

The ‘‘phase diagram’’ of the states of D1-D5(-P) system on the J - N_P plane in the R sector is shown in Figure 1. Here, $N_P = L_0 - N/4$. This is only for the left-moving sector; for supersymmetry, right-moving sector must be in one of the R ground states.

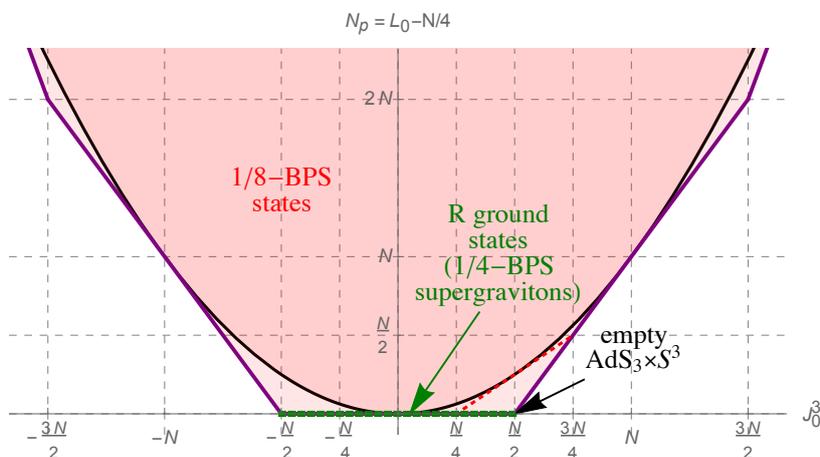


Figure 1: The J_0^3 - N_P plane of the D1-D5(-P) system in the R sector.

States exist only in the region bounded below by the unitarity bound (the purple polygon in Figure 1). The empty $\text{AdS}_3 \times S^3$ corresponds to the point $(J, N_P) = (N/2, 0)$. We can think of other states as excitation of this state. The 1/4-BPS states are on the interval $J \in [-N/2, N/2]$, $N_P = 0$ (the green dashed line in Figure 1). The 1/8-BPS states (both supergraviton states and more general states) have $N_P > 0$. The single-center, 3-charge

¹¹For recent progress, see [68, 69].

BMPV black hole exists only above the parabola $N_P = J^2/N$, which is finitely away from the empty $\text{AdS}_3 \times S^3$ point.

3 Supergravity setup

3.1 The 10-dimensional solution ansatz

The D1-D5-P black hole is a configuration in type IIB string theory and is 1/8 BPS, meaning that it preserves 8 supercharges out of the 32 supercharges in 10 dimensions. Every microstate of the D1-D5-P black hole must preserve the same supersymmetry. The most general solutions of type IIB supergravity that preserve the same 1/8 of supersymmetry and preserve the symmetry of the internal manifold \mathcal{M} were studied in [25, 40]. By preserving the symmetry of \mathcal{M} we mean that all fields are independent of the coordinates of \mathcal{M} and all form fields have legs of the form $dx^\mu \wedge dx^\nu \wedge \dots$ or $\text{vol}(\mathcal{M}) \wedge dx^\mu \wedge dx^\nu \wedge \dots$, where $\text{vol}(\mathcal{M})$ is the volume 4-form of \mathcal{M} and μ, ν, \dots are not along \mathcal{M} .¹² So, we can forget about the internal manifold \mathcal{M} , except for its overall volume, and consider the remaining six directions.

Preserving the same supersymmetry as the D1-D5-P black hole implies that the solution must have a null Killing vector,¹³ which is chosen to be the direction of a coordinate u , and all fields must be independent of u . The null Killing vector introduce a 2 + 4 split of the six directions and it is natural to introduce a second retarded time coordinate v and a four-dimensional spatial base \mathcal{B} with coordinates x^m , $m = 1, 2, 3, 4$. All fields, including the metric of the base, are independent of u but can depend on v and x^m .

The ten-dimensional fields are given by [25, Appendix E]:

$$ds_{10}^2 = \sqrt{\frac{Z_1 Z_2}{\mathcal{P}}} ds_6^2 + \sqrt{\frac{Z_1}{Z_2}} ds^2(\mathcal{M}), \quad (3.1a)$$

$$ds_6^2 = -\frac{2}{\sqrt{\mathcal{P}}}(dv + \beta) \left[du + \omega + \frac{\mathcal{F}}{2}(dv + \beta) \right] + \sqrt{\mathcal{P}} ds^2(\mathcal{B}), \quad (3.1b)$$

$$e^{2\Phi} = \frac{Z_1^2}{\mathcal{P}}, \quad B_2 = -\frac{Z_4}{\mathcal{P}}(du + \omega) \wedge (dv + \beta) + a_4 \wedge (dv + \beta) + \delta_2, \quad (3.1c)$$

$$C_0 = \frac{Z_4}{Z_1}, \quad C_2 = -\frac{Z_2}{\mathcal{P}}(du + \omega) \wedge (dv + \beta) + a_1 \wedge (dv + \beta) + \gamma_2, \quad (3.1d)$$

$$C_4 = \frac{Z_4}{Z_2} \text{vol}(\mathcal{M}) - \frac{Z_4}{\mathcal{P}} \gamma_2 \wedge (du + \omega) \wedge (dv + \beta) + x_3 \wedge (dv + \beta), \quad (3.1e)$$

$$C_6 = \text{vol}(\mathcal{M}) \wedge \left[-\frac{Z_1}{\mathcal{P}}(du + \omega) \wedge (dv + \beta) + a_2 \wedge (dv + \beta) + \gamma_1 \right] \quad (3.1f)$$

where

$$\mathcal{P} \equiv Z_1 Z_2 - Z_4^2. \quad (3.2)$$

¹²For supersymmetric solutions that do not preserve this symmetry, see [45].

¹³There are also supersymmetric solutions with a timelike Killing vector, but they are not relevant for the microstates of the D1-D5-P black hole whose Killing spinor squares to a null Killing vector [70].

Here, ds_{10}^2 is the string-frame metric of the ten-dimensional spacetime, $ds^2(\mathcal{M})$ is the metric of the internal manifold, and ds_6^2 is the Einstein-frame metric of the six-dimensional spacetime which involves u, v and the 4-dimensional manifold \mathcal{B} whose (possibly ambi-polar) metric is

$$ds^2(\mathcal{B}) = h_{mn}(x, v)dx^m dx^n. \quad (3.3)$$

The solution ansatz (3.1) contains various quantities: $Z_{1,2,4}, \mathcal{F}$ are scalars, $\beta, \omega, a_{1,2,4}$ are 1-forms, $\gamma_{1,2}, \delta_2$ are 2-forms, and x_3 is a 3-form, all on \mathcal{B} , and can in general depend on v but not u . The RR potentials C_p can have extra terms proportional to a four-form \mathcal{C} on \mathcal{B} , but it has been set to zero by using an appropriate gauge [40].

The diffeomorphism that preserves the form of the solution ansatz (3.1) is

$$v \rightarrow v + V(x), \quad u \rightarrow u + U(x, v), \quad (3.4)$$

which induces the following gauge transformation:

$$\beta \rightarrow \beta - \tilde{d}V, \quad \mathcal{F} \rightarrow \mathcal{F} - 2\dot{U}, \quad \omega \rightarrow \omega - \tilde{d}U + \dot{U}\beta. \quad (3.5)$$

Here $\dot{} \equiv \partial_v$ and we introduced the exterior derivative restricted to \mathcal{B} ,

$$\tilde{d} \equiv dx^m \partial_m. \quad (3.6)$$

It will be useful to introduce a differential operator \mathcal{D} defined by

$$\mathcal{D} \equiv \tilde{d} - \beta \wedge \partial_v, \quad (3.7)$$

which is invariant under the gauge transformation (3.4) and (3.5) provided that everything is u -independent. The full exterior derivative¹⁴ can be written as

$$d = \mathcal{D} + (dv + \beta) \wedge \partial_v. \quad (3.8)$$

The u, v coordinates are related to the time coordinate t and the coordinate y parametrizing the S^1 with periodicity $2\pi R_y$, on which D1- and D5-branes are wrapped. In view of the gauge symmetry (3.5), the identification is not unique but, in the current article, we take it to be¹⁵

$$u = \frac{1}{\sqrt{2}}(t - y), \quad v = \frac{1}{\sqrt{2}}(t + y). \quad (3.9)$$

Ignoring the u direction on which nothing depend, we can regard v as the coordinate of the compact S^1 direction.

¹⁴The six-dimensional exterior derivative acting on u, v, x^m , although nothing depends on u .

¹⁵For example, when one relates 6D and 5D solutions, other choices are more convenient; see [43].

The quantities $a_{1,2,4}, \gamma_{1,2}, \delta_2, x_3$ that appear in the NSNS and RR potentials in (3.1) are not invariant under the gauge symmetry of these potentials. Gauge-invariant combinations are [25, 29]

$$\Theta_1 \equiv \mathcal{D}a_1 + \dot{\gamma}_2 - \dot{\beta} \wedge a_1, \quad \Theta_2 \equiv \mathcal{D}a_2 + \dot{\gamma}_1 - \dot{\beta} \wedge a_2, \quad \Theta_4 \equiv \mathcal{D}a_4 + \dot{\delta}_2 - \dot{\beta} \wedge a_4, \quad (3.10a)$$

$$\begin{aligned} \Sigma_1 \equiv \mathcal{D}\gamma_2 - a_1 \wedge \mathcal{D}\beta, \quad \Sigma_2 \equiv \mathcal{D}\gamma_1 - a_2 \wedge \mathcal{D}\beta, \quad \Sigma_4 \equiv \mathcal{D}\delta_2 - a_4 \wedge \mathcal{D}\beta, \\ \Xi_4 \equiv \mathcal{D}x_3 - \dot{\beta} \wedge x_3 - \Theta_4 \wedge \gamma_2 + a_1 \wedge \Sigma_4, \end{aligned} \quad (3.10b)$$

where Θ_I are 2-forms and Ξ_4 is a 4-form, and the field strengths can be written in terms of these quantities (see section 3.5 for the explicit expressions). From this definition (3.10) and the relations $\mathcal{D}^2 = -(\mathcal{D}\beta) \wedge \partial_v, \dot{\mathcal{D}} = -\dot{\beta} \wedge \partial_v$, we can show that the following relations hold between Θ_I and Σ_I :

$$\mathcal{D}\Sigma_I = -\Theta_I \wedge \mathcal{D}\beta, \quad \partial_v(\Sigma_I + \beta \wedge \Theta_I) = \tilde{d}\Theta_I. \quad (3.11)$$

The scalars Z_1, Z_2 , and Z_4 can be regarded as the electrostatic potentials sourced by D1(v), D5($v\mathcal{M}$), and F1(v), respectively, where D1(v) means D1-branes extending along v . The 2-forms Θ_2, Θ_1 , and Θ_4 can be regarded as the magnetic fields sourced by D1(\mathcal{C}), D5($\mathcal{C}\mathcal{M}$), and F1(\mathcal{C}), respectively, where \mathcal{C} is a curve in the base \mathcal{B} .

3.2 The zeroth layer

The BPS equations satisfied by the ansatz quantities can be organized in three layers. The zeroth layer is about the base space \mathcal{B} and the 1-form β on it. The base \mathcal{B} must be an almost hyper-Kähler space with three anti-self-dual 2-forms

$$J^{(A)} \equiv \frac{1}{2} J_{mn}^{(A)} dx^m \wedge dx^n, \quad *_4 J^{(A)} = -J^{(A)} \quad (3.12)$$

where $A = 1, 2, 3$ and $*_4$ is the Hodge star with respect to the metric $ds^2(\mathcal{B})$. The 2-forms satisfy the quaternionic relation

$$J^{(A)m}{}_p J^{(B)p}{}_n = \epsilon^{ABC} J^{(C)m}{}_n - \delta^{AB} \delta_n^m \quad (3.13)$$

where the indices are raised and lowered using h_{mn} and its inverse h^{mn} . Unlike in hyper-Kähler spaces, these 2-forms are not closed; instead, they are required to satisfy

$$\tilde{d}J^{(A)} = \partial_v(\beta \wedge J^{(A)}). \quad (3.14)$$

Furthermore, β must satisfy

$$\mathcal{D}\beta = *_4 \mathcal{D}\beta. \quad (3.15)$$

The integrability condition for (3.14), obtained by acting on it with \tilde{d} , is $\partial_v(\mathcal{D}\beta \wedge J^{(A)}) = 0$, which is guaranteed to hold because $\mathcal{D}\beta$ is self-dual and $J^{(A)}$ is anti-self-dual.

One quantity that is defined by the data of the zeroth layer is the anti-self-dual 2-form

$$\psi \equiv \frac{1}{8} \epsilon^{ABC} J^{(A)mn} j^{(B)}_{mn} J^{(C)}, \quad (3.16)$$

which will show up in higher layers.

In the zeroth layer, we must find almost complex structures $J^{(A)}$ and a 1-form β which in general depend on v and satisfy the non-linear conditions (3.13)–(3.15). If we solve the zeroth layer, the remaining two layers can be written as linear differential equations on \mathcal{B} . In practice, in most of the explicit superstratum solutions in the literature, it is assumed that the \mathcal{B} is flat \mathbb{R}^4 or a Gibbons-Hawking space [43, 71, 72], and that β is independent of v .

3.3 The first layer

The first-layer equations determine the scalars Z_I and the flux forms Θ_I, Σ_I . They must satisfy [25] the following linear differential equations

$$*_4(\mathcal{D}Z_1 + \dot{\beta}Z_1) = \Sigma_2, \quad *_4(\mathcal{D}Z_2 + \dot{\beta}Z_2) = \Sigma_1, \quad *_4(\mathcal{D}Z_4 + \dot{\beta}Z_4) = \Sigma_4, \quad (3.17)$$

and duality relations

$$(1 - *_4)\Theta_2 = 2Z_1\psi, \quad (1 - *_4)\Theta_1 = 2Z_2\psi, \quad (1 - *_4)\Theta_4 = 2Z_4\psi. \quad (3.18)$$

Another condition is

$$\Xi_4 = Z_2^2 \partial_v \left(\frac{Z_4}{Z_2} \right) *_4 1. \quad (3.19)$$

By acting with ∂_v and \mathcal{D} on (3.17) and using the identities (3.11), we can derive equations involving only Z_I, Θ_I :

$$\partial_v[*_4(\mathcal{D}Z_1 + \dot{\beta}Z_1) + \beta \wedge \Theta_2] = \tilde{d}\Theta_2, \quad (3.20a)$$

$$\partial_v[*_4(\mathcal{D}Z_2 + \dot{\beta}Z_2) + \beta \wedge \Theta_1] = \tilde{d}\Theta_1, \quad (3.20b)$$

$$\partial_v[*_4(\mathcal{D}Z_4 + \dot{\beta}Z_4) + \beta \wedge \Theta_4] = \tilde{d}\Theta_4 \quad (3.20c)$$

and

$$\mathcal{D} *_4(\mathcal{D}Z_1 + \dot{\beta}Z_1) = -\Theta_2 \wedge \mathcal{D}\beta, \quad (3.21a)$$

$$\mathcal{D} *_4(\mathcal{D}Z_2 + \dot{\beta}Z_2) = -\Theta_1 \wedge \mathcal{D}\beta, \quad (3.21b)$$

$$\mathcal{D} *_4(\mathcal{D}Z_4 + \dot{\beta}Z_4) = -\Theta_4 \wedge \mathcal{D}\beta. \quad (3.21c)$$

Eqs. (3.21) can be regarded as the integrability condition for (3.17) or (3.20).

Given the solution of the zeroth layer, Eqs. (3.18) and (3.20) give a system of linear equations defined on \mathcal{B} which can be solved to determine Z_I, Θ_I . Each line of (3.20) contains four equations, which can be used to find four independent components of Z_I, Θ_I in view of the duality relation (3.18).

3.4 The second layer

Given the solution to the first-layer equations, the second-layer equations give a system of linear equations for ω, \mathcal{F} with sources quadratic in the first-layer fields [25]:

$$(1 + *_4)\mathcal{D}\omega + \mathcal{F}\mathcal{D}\beta = Z_1\Theta_1 + Z_2\Theta_2 - 2Z_4\Theta_4 - 2(Z_1Z_2 - Z_4^2)\psi, \quad (3.22a)$$

$$\begin{aligned} & *_4\mathcal{D}*_4L + 2\dot{\beta}_m L^m - *_4(\psi \wedge \mathcal{D}\omega) \\ &= -\frac{1}{4}(Z_1Z_2 - Z_4^2)\dot{h}^{mn}\dot{h}_{mn} + \frac{1}{2}\partial_v[(Z_1Z_2 - Z_4^2)h^{mn}\dot{h}_{mn}] \\ &+ (\dot{Z}_1\dot{Z}_2 - \dot{Z}_4^2) + (Z_1\ddot{Z}_2 + Z_2\ddot{Z}_1 - 2Z_4\ddot{Z}_4) \\ &- \frac{1}{2}*_4\left[(\Theta_1 - Z_2\psi) \wedge (\Theta_2 - Z_1\psi) - (\Theta_4 - Z_4\psi) \wedge (\Theta_4 - Z_4\psi) \right. \\ &\quad \left. + (Z_1Z_2 - Z_4^2)\psi \wedge \psi\right], \end{aligned} \quad (3.22b)$$

where

$$L \equiv \dot{\omega} + \frac{\mathcal{F}}{2}\dot{\beta} - \frac{1}{2}\mathcal{D}\mathcal{F}. \quad (3.23)$$

3.5 Field strengths

Using the relations above, the NSNS and RR field strengths can be written solely in terms of $\beta, Z_I, \Theta_I, \omega$. The explicit expression for the NSNS field strength is [40]

$$H_3 = dB_2 = -d\left[\frac{Z_4}{\mathcal{P}}(du + \omega) \wedge (dv + \beta)\right] + (dv + \beta) \wedge \Theta_4 + *_4(\mathcal{D}Z_4 + \dot{\beta}Z_4). \quad (3.24)$$

The RR field strengths are defined by $G_{p+1} = dC_p - H_3 \wedge C_{p-2}$. In the present case, their explicit form can be conveniently written in terms of F_p, \tilde{F}_p defined by

$$G = \sum_{p=1,3,4,5,7,9} G_p =: F_1 + F_3 + F_5 + (\tilde{F}_1 + \tilde{F}_3 + \tilde{F}_5) \wedge \text{vol}(\mathcal{M}). \quad (3.25)$$

The explicit expressions for F_p are

$$F_1 = \mathcal{D}\left(\frac{Z_4}{Z_1}\right) + (dv + \beta)\partial_v\left(\frac{Z_4}{Z_1}\right), \quad (3.26)$$

$$\begin{aligned} F_3 = & -(du + \omega) \wedge (dv + \beta) \wedge \left[\mathcal{D}\left(\frac{1}{Z_1}\right) - \frac{1}{Z_1}\dot{\beta} + \frac{Z_4}{Z_1}\mathcal{D}\left(\frac{Z_4}{\mathcal{P}}\right)\right] \\ & + (dv + \beta) \wedge \left(\Theta_1 - \frac{Z_4}{Z_1}\Theta_4 - \frac{1}{Z_1}\mathcal{D}\omega\right) + \frac{1}{Z_1}(du + \omega) \wedge \mathcal{D}\beta \\ & + *_4\left[(\mathcal{D}Z_2 + \dot{\beta}Z_2) - \frac{Z_4}{Z_1}(\mathcal{D}Z_4 + \dot{\beta}Z_4)\right], \end{aligned} \quad (3.27)$$

$$\begin{aligned} F_5 = & \frac{1}{\mathcal{P}}(du + \omega) \wedge (dv + \beta) \wedge *_4\left[Z_2(\mathcal{D}Z_4 + \dot{\beta}Z_4) - Z_4(\mathcal{D}Z_2 + \dot{\beta}Z_2)\right] \\ & + (dv + \beta) \wedge Z_2^2\partial_v\left(\frac{Z_4}{Z_2}\right) *_4 1. \end{aligned} \quad (3.28)$$

\tilde{F}_p can be obtained from F_p by setting $Z_1 \leftrightarrow Z_2, \Theta_1 \leftrightarrow \Theta_2$.

3.6 A covariant form of BPS equations

It is possible to write the above BPS equations in a more concise form [43, Appendix A]. Define the matrix

$$C^{IJ} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad C_{IJ} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad I, J = 1, 2, 4, \quad (3.29)$$

which has origin in an M-theory frame as intersection numbers of 2-cycles [29, 73]. The matrices C_{IJ} and C^{IJ} are inverse matrices of each other. Furthermore, define $\mathbf{a}^I, \gamma^I, Z_I, \Theta^I, \Sigma^I$ by

$$\begin{aligned} \mathbf{a}^1 &\equiv a_1, & \mathbf{a}^2 &\equiv a_2, & \mathbf{a}^4 &\equiv 2a_4, & \gamma^1 &\equiv \gamma_2, & \gamma^2 &\equiv \gamma_1, & \gamma^4 &\equiv 2\delta_2, \\ Z_1 &\equiv Z_1, & Z_2 &\equiv Z_2, & Z_4 &\equiv -Z_4, & \Theta^1 &\equiv \Theta_1, & \Theta^2 &\equiv \Theta_2, & \Theta^4 &\equiv 2\Theta_4, \\ \Sigma^1 &\equiv \Sigma_1, & \Sigma^2 &\equiv \Sigma_2, & \Sigma^4 &\equiv 2\Sigma_4. \end{aligned} \quad (3.30)$$

We raise and lower indices using C^{IJ} and C_{IJ} ; for example, $Z^I = C^{IJ}Z_J$ and $\Theta_I = C_{IJ}\Theta^J$. We define the inner product by

$$(\mathbf{A}, \mathbf{B}) \equiv A_I B^I. \quad (3.31)$$

Then we can rewrite the BPS equations as follows.

Definitions of Θ_I, Σ_I , (3.10):

$$\Theta = \mathcal{D}\mathbf{a} + \dot{\gamma} - \dot{\beta} \wedge \mathbf{a}, \quad \Sigma = \mathcal{D}\gamma - \mathbf{a} \wedge \mathcal{D}\beta, \quad (3.32)$$

The first-layer equations (3.17)–(3.21):

$$*_4(\mathcal{D}\mathbf{Z} + \dot{\beta}\mathbf{Z}) = \Sigma, \quad (1 - *_4)\Theta = 2\mathbf{Z}\psi \quad (3.33)$$

$$\partial_v[*_4(\mathcal{D}\mathbf{Z} + \dot{\beta}\mathbf{Z}) + \beta \wedge \Theta] = \tilde{d}\Theta, \quad \mathcal{D} *_4(\mathcal{D}\mathbf{Z} + \dot{\beta}\mathbf{Z}) = -\Theta \wedge \mathcal{D}\beta \quad (3.34)$$

The second-layer equations (3.22):

$$(1 + *_4)\mathcal{D}\omega + \mathcal{F}\mathcal{D}\beta = (\mathbf{Z}, \Theta) - (\mathbf{Z}, \mathbf{Z})\psi \quad (3.35a)$$

$$\begin{aligned} *_4\mathcal{D} *_4 L + 2\dot{\beta}_m L^m - *_4(\psi \wedge \mathcal{D}\omega) &= -\frac{1}{8}(\mathbf{Z}, \mathbf{Z})\dot{h}^{mn}\dot{h}_{mn} + \frac{1}{4}\partial_v[(\mathbf{Z}, \mathbf{Z})h^{mn}\dot{h}_{mn}] \\ &+ \frac{1}{2}(\dot{\mathbf{Z}}, \dot{\mathbf{Z}}) + (\mathbf{Z}, \ddot{\mathbf{Z}}) \\ &- \frac{1}{4} *_4 \left[(\Theta - \mathbf{Z}\psi \wedge \Theta - \mathbf{Z}\psi) + (\mathbf{Z}, \mathbf{Z})\psi \wedge \psi \right] \end{aligned} \quad (3.35b)$$

The 10-dimensional solution (3.1), having no dependence in the internal manifold \mathcal{M} , can also be studied within 6-dimensional supergravity. In $d = 6$, $\mathcal{N} = (1, 0)$ supergravity [74, 75], a graviton multiplet consists of a graviton $g_{\mu\nu}$, a left-handed symplectic Majorana-Weyl

gravitino ψ_μ , and a tensor gauge field $B_{\mu\nu}^+$ with self-dual dressed field-strength. A tensor multiplet consists of a two-form $B_{\mu\nu}^-$ with anti-self-dual dressed field-strength, a right-handed symplectic Majorana-Weyl fermion χ , and a scalar field φ . Classification of supersymmetric solutions in minimal $d = 6, \mathcal{N} = (1, 0)$ supergravity was done in [34] and later extended to include other multiplets in [35, 76, 77]. Classification of supersymmetric solutions in $d = 6, \mathcal{N} = (2, 0)$ supergravity was carried out in [70]. The theory without (Z_4, Θ_4) corresponds to minimal $\mathcal{N} = (1, 0)$ supergravity plus a tensor multiplet, and including (Z_4, Θ_4) means to add another tensor multiplet [7]. The index I above corresponds to the label for the (self-dual and anti-self-dual) tensor gauge fields $B_{\mu\nu}^\pm$.

3.7 v -independent case

In the above, we wrote down the BPS equations in the general case where the base space metric $ds^2(\mathcal{B})$ and the 1-form β are v -dependent. To consider general microstate geometries, such general base space is unavoidable. However, because of technical limitation, it is normally assumed that $ds^2(\mathcal{B})$ and β are independent of v . This certainly restricts the class of microstate geometries, but there are superstrata with such a base whose entropy scales the same as the general superstrata ($S \sim N^{1/2} N_P^{1/4}$); see section 4.2.

Here, we assume that the zeroth-layer ansatz quantities, namely the base space metric $ds^2(\mathcal{B})$, the 1-form β , and 2-forms $J^{(A)}$, do not depend on v and write down the form of the BPS equations we introduced above. In the zeroth layer, the complex structures $J^{(A)}$ are closed and therefore the base space \mathcal{B} becomes hyper-Kähler. The condition on β is that it is self-dual,

$$d\beta = *_4 d\beta. \quad (3.36)$$

Also, the anti-self-dual 2-form ψ defined in (3.16) vanishes.

Under the above assumptions, the first-layer equations (3.18)–(3.21) become

$$*_4 \mathcal{D} \dot{Z}_1 = \mathcal{D} \Theta_2, \quad \mathcal{D} *_4 \mathcal{D} Z_1 = -\Theta_2 \wedge d\beta, \quad \Theta_2 = *_4 \Theta_2, \quad (3.37a)$$

$$*_4 \mathcal{D} \dot{Z}_2 = \mathcal{D} \Theta_1, \quad \mathcal{D} *_4 \mathcal{D} Z_2 = -\Theta_1 \wedge d\beta, \quad \Theta_1 = *_4 \Theta_1, \quad (3.37b)$$

$$*_4 \mathcal{D} \dot{Z}_4 = \mathcal{D} \Theta_4, \quad \mathcal{D} *_4 \mathcal{D} Z_4 = -\Theta_4 \wedge d\beta, \quad \Theta_4 = *_4 \Theta_4. \quad (3.37c)$$

The second-layer equations (3.22) simplify to

$$(1 + *_4) \mathcal{D} \omega + \mathcal{F} d\beta = Z_1 \Theta_1 + Z_2 \Theta_2 - 2Z_4 \Theta_4, \quad (3.38a)$$

$$*_4 \mathcal{D} *_4 \left(\dot{\omega} - \frac{1}{2} \mathcal{D} \mathcal{F} \right) = (\dot{Z}_1 \dot{Z}_2 - \dot{Z}_4^2) + (Z_1 \ddot{Z}_2 + Z_2 \ddot{Z}_1 - 2Z_4 \ddot{Z}_4) - \frac{1}{2} *_4 (\Theta_1 \wedge \Theta_2 - \Theta_4 \wedge \Theta_4), \quad (3.38b)$$

where

$$L = \dot{\omega} - \frac{1}{2} \mathcal{D} \mathcal{F}. \quad (3.39)$$

3.8 2-charge microstate geometries

Let us see how the 2-charge microstate geometries (Lunin-Mathur geometries), which are dual to the 1/4-BPS supergraviton states (2.13), are described in the supergravity setup above. The Lunin-Mathur geometries that respect the symmetry of \mathcal{M} are parametrized by profile functions $g_A(\lambda)$ with $A = 1, 2, 3, 4 =: i$ and $A = 5$.¹⁶ Given such a profile, the ansatz data are given by [4, 5, 7, 29]

$$Z_1 = 1 + \frac{Q_5}{L} \int_0^L d\lambda \frac{|\partial_\lambda g_i(\lambda)|^2 + |\partial_\lambda g_5(\lambda)|^2}{|x_i - g_i(\lambda)|^2}, \quad Z_4 = -\frac{Q_5}{L} \int_0^L d\lambda \frac{\partial_\lambda g_5(\lambda)}{|x_i - g_i(\lambda)|^2}, \quad (3.40a)$$

$$Z_2 = 1 + \frac{Q_5}{L} \int_0^L \frac{d\lambda}{|x_i - g_i(\lambda)|^2}, \quad d\gamma_2 = *_4 dZ_2, \quad d\delta_2 = *_4 dZ_4, \quad (3.40b)$$

$$A = -\frac{Q_5}{L} dx^j \int_0^L d\lambda \frac{\partial_\lambda g_j(\lambda)}{|x_i - g_i(\lambda)|^2}, \quad dB = -*_4 dA, \quad ds^2(\mathcal{B}) = dx^i dx^i, \quad (3.40c)$$

$$\beta = \frac{-A + B}{\sqrt{2}}, \quad \omega = \frac{-A - B}{\sqrt{2}}, \quad \Theta_I = \mathcal{F} = a_{1,4} = x_3 = 0. \quad (3.40d)$$

The base \mathcal{B} is always flat \mathbb{R}^4 with coordinates x^i , and $*_4$ is the Hodge dual with respect to its flat metric $ds_4^2 = dx^i dx^i$. The functions $g_A(\lambda)$ are periodic with period $L = 2\pi Q_5/R_y$. The D1 charge is given by

$$Q_1 = \frac{Q_5}{L} \int_0^L d\lambda (|\partial_\lambda g_i(\lambda)|^2 + |\partial_\lambda g_5(\lambda)|^2). \quad (3.41)$$

The quantities Q_1, Q_5 are related to the quantized D1 and D5 numbers N_1, N_5 by

$$Q_1 = \frac{N_1 g_s \alpha'^3}{v_4}, \quad Q_5 = N_5 g_s \alpha', \quad (3.42)$$

where $(2\pi)^4 v_4$ is the coordinate volume of \mathcal{M} .

If we drop “1” from $Z_{1,2}$ in (3.40), the geometry becomes asymptotically AdS_3 . The AdS/CFT dictionary between the profile $g_A(\lambda)$ and the 1/4-BPS supergraviton states (2.13) is that the Fourier coefficient with mode k in g_A is related to the excitation number N_k^ψ as follows.¹⁷

$$g_1 \pm ig_2 \leftrightarrow N_k^{\pm\pm}, \quad g_3 \pm ig_4 \leftrightarrow N_k^{\pm\mp}, \quad g_5 \leftrightarrow N_k^{00}. \quad (3.43)$$

N_k^ψ for other species $|\psi\rangle$ are not turned on, because they would break the symmetry of \mathcal{M} .

3.8.1 Empty $\text{AdS}_3 \times \mathcal{S}^3$

The simplest example is the circular profile in the 1-2 plane:

$$g_1 + ig_2 = ae^{2\pi i\lambda/L}, \quad g_3 + ig_4 = g_5 = 0, \quad (3.44)$$

¹⁶The $A \geq 6$ components [5, 7] break the symmetry of \mathcal{M} .

¹⁷The absolute value square of the Fourier coefficient is proportional to N_k^ψ with a non-trivial coefficient. For the precise map, see, e.g., [37].

where $a > 0$ is a constant. According to the dictionary (3.43), this case is dual to the following RR ground state:

$$[[++\rangle_1]^N, \quad J_L = J_R = \frac{N}{2}, \quad N_P = 0. \quad (3.45)$$

To write down the ansatz data, it is convenient to write the flat metric for the base \mathbb{R}^4 in the following form:

$$ds^2(\mathcal{B}) = \Sigma \left(\frac{dr^2}{r^2 + a^2} + d\theta^2 \right) + (r^2 + a^2) \sin^2 \theta d\phi^2 + r^2 \cos^2 \theta d\psi^2, \quad (3.46)$$

$$\Sigma \equiv r^2 + a^2 \cos^2 \theta. \quad (3.47)$$

The relation to the Cartesian coordinates x^i is

$$x^1 + ix^2 = \sqrt{r^2 + a^2} \sin \theta e^{i\phi}, \quad x^3 + ix^4 = r \cos \theta e^{i\psi}. \quad (3.48)$$

In this coordinate system, some of the ansatz data are

$$Z_1 = \frac{R_y^2 a^2}{Q_5 \Sigma}, \quad Z_2 = \frac{Q_5}{\Sigma}, \quad Z_4 = 0, \quad \Theta_I = 0, \quad (3.49a)$$

$$\beta = \frac{R_y a^2}{\sqrt{2} \Sigma} (\sin^2 \theta d\phi - \cos^2 \theta d\psi) \equiv \beta_0, \quad (3.49b)$$

$$\omega = \frac{R_y a^2}{\sqrt{2} \Sigma} (\sin^2 \theta d\phi + \cos^2 \theta d\psi) \equiv \omega_0, \quad \mathcal{F} = 0. \quad (3.49c)$$

In (3.49a), we have dropped “1” in (3.40), so the geometry is asymptotically AdS₃. Eq. (3.41) relates the parameters a, R_y, Q_1, Q_5 as

$$a^2 = \frac{Q_1 Q_5}{R_y^2}. \quad (3.50)$$

This geometry is empty global AdS₃ × S³, as we can see by doing the coordinate transformation

$$\tilde{\phi} = \phi - \frac{t}{R_y}, \quad \tilde{\psi} = \psi - \frac{y}{R_y} \quad (3.51)$$

after which the 6D metric becomes

$$ds_6^2 = \sqrt{Q_1 Q_5} \left(-\frac{r^2 + a^2}{a^2 R_y^2} dt^2 + \frac{r^2}{a^2 R_y^2} dy^2 + \frac{dr^2}{r^2 + a^2} + d\theta^2 + \sin^2 \theta d\tilde{\phi}^2 + \cos^2 \theta d\tilde{\psi}^2 \right). \quad (3.52)$$

This is indeed global AdS₃ × S³ with radius $\mathcal{R} = (Q_1 Q_5)^{1/4}$. This is consistent with the fact that the R ground state (3.45) is mapped via (2.11) into the unique NS vacuum $[|--\rangle_1^{\text{NS}}]^N$. In the dual CFT, the coordinate transformation (3.51) corresponds to the spectral flow transformation between the R and NS sectors.

3.8.2 Z_4 excitation

A slightly more non-trivial example is given by the profile

$$g_1 + ig_2 = ae^{2\pi i\lambda/L}, \quad g_3 + ig_4 = 0, \quad g_5 = -\frac{b}{k} \sin \frac{2\pi k\lambda}{L} \quad (3.53)$$

which is dual to the state

$$[|++\rangle_1]^{N_0} [|00\rangle_k]^{N_k^{00}}, \quad N_0 + kN_k^{00} = N. \quad (3.54)$$

The ansatz data are, in the coordinate system (3.46),

$$\begin{aligned} Z_1 &= \frac{R^2}{Q_5} \left[\frac{a^2 + b^2/2}{\Sigma} + b^2 a^{2k} \frac{\sin^{2k} \theta \cos(2k\phi)}{2(r^2 + a^2)^k \Sigma} \right], & Z_2 &= \frac{Q_5}{\Sigma}, \\ Z_4 &= Rba^k \frac{\sin^k \theta \cos(k\phi)}{(r^2 + a^2)^{k/2} \Sigma}, & \beta &= \beta_0, \quad \omega = \omega_0, \quad \mathcal{F} = \Theta_I = 0. \end{aligned} \quad (3.55)$$

The relation (3.50) is modified to

$$a^2 + \frac{b^2}{2} = \frac{Q_1 Q_5}{R_y^2}. \quad (3.56)$$

Because $a^2 \propto N_0$, $b^2 \propto N_k^{00}$, this relation is the bulk dual of the strand budget constraint given by the second equation of (3.54).

4 Superstrata

In section 2, we discussed states that correspond to populating $\text{AdS}_3 \times S^3$ with “supergravitons”. For the 1/4-BPS supergraviton gas state (2.13), when the number of excited supergravitons is macroscopic, namely, when $N_k^\psi = \mathcal{O}(N)$ (excluding N_1^{++} , which corresponds to the $\text{AdS}_3 \times S^3$ background), the bulk spacetime gets backreacted and is given by the Lunin-Mathur geometry (3.40). Similarly, for the 1/8-BPS supergraviton gas state (2.14), when the excitation number $N_{k,m,n,f}^\psi$ is $\mathcal{O}(N)$, the bulk spacetime must become a backreacted geometry, which can be described within the supergravity setup of section 3. In the original incarnation, this is nothing but the superstratum. Here let us review the construction of the superstratum.

4.1 General remarks

4.1.1 Coherent states

The states of the 1/4-BPS supergraviton gas, (2.13), is holographically described by the Lunin-Mathur microstate geometries (3.40). However, this is not a precise statement. The state (2.13) with a fixed distribution $\{N_k^\psi\}$ generally corresponds to a highly quantum situation in the bulk and cannot be described by a well-defined classical geometry. Instead,

the state that corresponds to a smooth microstate geometry is a coherent superposition of the state (2.13) with different distributions $\{N_k^\psi\}$ peaked around an average distribution. This has been well established for 1/4-BPS supergraviton states and Lunin-Mathur geometries [7, 61, 62]. Similarly, for the 1/8-BPS case, the state that corresponds to a well-defined classical geometry is not (2.14) but a coherent state obtained by taking a superposition of (2.14) with different distributions $\{N_{k,m,n,f}^\psi\}$ [44, 48].

That we must use a coherent superposition for a classical configuration is clear from the coherent state for the harmonic oscillator of quantum mechanics, but an argument in our context is as follows. A state like (2.14) is an eigenstate of charges such as L_0, J_L and therefore the vev of operators that have non-vanishing charges will vanish for the state. However, the vev is holographically related to the falloff of bulk fields near the boundary, which generally does not vanish in microstate geometries. To resolve this contradiction, one should take a coherent superposition of states with different values of L_0, J_L . Another, related way to argue that a superposition is necessary is as follows [25]. The CFT state (2.14) is an eigenstate of the momentum operator $N_P = L_0 - \tilde{L}_0$ and is therefore invariant under translation up to phase. On the other hand, in the bulk, the geometry carrying $N_P > 0$ involves traveling waves along v and thus is v -dependent; namely, it is not invariant under translation. This problem is resolved if the precise CFT dual of a bulk geometry is not (2.14) but a coherent superposition of states with different values of N_P .

Specifically, the supergraviton state (2.14) must more properly be replaced by the following coherent superposition state characterized by complex parameters $\{A_{k,m,n,f}^\psi\}$:

$$\begin{aligned} \Psi(\{A_{k,m,n,f}^\psi\}) &= \sum'_{\{N_{k,m,n,f}^\psi\}} \prod_{\psi,k,m,n,f} (A_{k,m,n,f}^\psi)^{N_{k,m,n,f}^\psi} \Psi(\{N_{k,m,n,f}^\psi\}) \\ &= \sum'_{\{N_{k,m,n,f}^\psi\}} \prod_{\psi,k,m,n,f} \left[A_{k,m,n,f}^\psi |\psi; k, m, n, f\rangle \right]^{N_{k,m,n,f}^\psi}, \end{aligned} \quad (4.1)$$

where the sum is restricted to distributions $\{N_{k,m,n,f}^\psi\}$ that satisfy (2.14b). This state is not normalized. We restrict to the case where $N_{k,m,n,f}^\psi \neq 0$ only for bosonic $|\psi; k, m, n, f\rangle$. In the large N limit, the sum is dominated by a particular distribution $\{\bar{N}_{k,m,n,f}^\psi\}$, which can be obtained by computing the norm $|\Psi(\{A_{k,m,n,f}^\psi\})|^2$ and taking its variation with respect to $\{N_{k,m,n,f}^\psi\}$. Generally, we have $\bar{N}_{k,m,n,f}^\psi \propto |A_{k,m,n,f}^\psi|^2$, although the detail depends on the normalization of the state $\Psi(\{N_{k,m,n,f}^\psi\})$. In the current article, we will not explicitly use coherent states and loosely talk about the geometry corresponding to the supergraviton state (2.14) specified by the distribution $\{N_{k,m,n,f}^\psi\}$. However, strictly speaking, we should instead use the coherent state (4.1) whose average distribution is equal to $\{N_{k,m,n,f}^\psi\}$. For more detail of the coherent superposition for 1/8-BPS states, see [44, 48].

4.1.2 Fixing the base

On physical grounds, the bulk geometry dual to (2.14) for any $\{N_{k,m,n,f}^\psi\}$, or more precisely its coherent state version, is expected to exist and, being BPS, must be obtainable by solving the BPS equations of section 3. They are superstrata. However, for constructing such general solutions, we must confront the non-linear problem in the zeroth layer of finding the almost hyper-Kähler base and the associated 1-form, $(ds^2(\mathcal{B}), \beta)$, both v -dependent, appropriate for the state. At the time of writing, this is an unsolved technical problem, because little is known about the relevant almost hyper-Kähler space, and because we do not know in general what almost hyper-Kähler base to take for a given CFT state. Instead, let us assume that there is a set of modes $(\psi, k, m, n, f) \in \mathcal{K}$ that correspond to excitations with the *same* particular base. Namely, no matter what $\{N_{k,m,n,f}^\psi\}$ is for $(\psi, k, m, n, f) \in \mathcal{K}$, the bulk geometries have the same zeroth-layer data $(ds^2(\mathcal{B}), \beta)$. The relevant CFT state is assumed to take the form

$$\Psi_{\text{bg}} \times \prod_{(\psi,k,m,n,f) \in \mathcal{K}} [|\psi; k, m, n, f\rangle]^{N_{k,m,n,f}^\psi}, \quad N_{\text{bg}} + \sum_{(\psi,k,m,n,f) \in \mathcal{K}} k N_{k,m,n,f}^\psi = N. \quad (4.2)$$

Here, Ψ_{bg} is a “background” part that is made of strands of total length N_{bg} and corresponds to the fixed base (\mathcal{B}, β) . Depending on the number $N_{k,m,n,f}^\psi$ of supergravitons added to the system, the total strand length N_{bg} must change to accommodate them, if N is to be fixed. Alternatively, we can fix Ψ_{bg} and change the system size N to accommodate the change in $N_{k,m,n,f}^\psi$.

If this assumption holds, the problem reduces to that of solving the BPS equations in the first and second layers, defined on a *fixed base* \mathcal{B} . In the first layer, because the equations are linear and homogeneous (sourceless), it must be possible to write the solution as a sum of modes with arbitrary coefficients. For example, assume that we have solved the equations for the pair (Z_4, Θ_4) . Then they can be written as

$$Z_4(x, v) = \sum_{\mathbf{k}} b_4^{\mathbf{k}} z_{\mathbf{k}}(x, v), \quad \Theta_4(x, v) = \sum_{\mathbf{k}} b_4^{\mathbf{k}} \vartheta_{\mathbf{k}}(x, v), \quad (4.3a)$$

where the modes $(z_{\mathbf{k}}, \vartheta_{\mathbf{k}})$ span a basis of solutions, with \mathbf{k} labeling different modes. Different solutions are parametrized by the expansion coefficients $b_4^{\mathbf{k}}$. Because the equations for the pairs (Z_1, Θ_2) and (Z_2, Θ_1) have the same form as the one for (Z_4, Θ_4) , they must be expandable in the same modes:

$$Z_1 = \sum_{\mathbf{k}} b_1^{\mathbf{k}} z_{\mathbf{k}}, \quad \Theta_2 = \sum_{\mathbf{k}} b_1^{\mathbf{k}} \vartheta_{\mathbf{k}}, \quad (4.3b)$$

$$Z_2 = \sum_{\mathbf{k}} b_2^{\mathbf{k}} z_{\mathbf{k}}, \quad \Theta_1 = \sum_{\mathbf{k}} b_2^{\mathbf{k}} \vartheta_{\mathbf{k}}. \quad (4.3c)$$

When the number of supergravitons in (4.2) is small, namely if $N_{k,m,n,f}^\psi \ll N$, we are in a linear regime and $b_i^{\mathbf{k}}$ are also small; they are linearly related to $(N_{k,m,n,f}^\psi)^{1/2}$, or more precisely

to the parameters $A_{k,m,n,f}^\psi$ in the coherent state (4.1).¹⁸ In the linear regime, the source terms on the right-hand side of the second-layer equations (3.22) (or (3.38)) vanish, because they are quadratic in the first-layer fields. This means that the second-layer fields are the ones that correspond to the base Ψ_{bg} .

When the number of supergravitons in (4.2) is not small, namely if $N_{k,m,n,f}^\psi = \mathcal{O}(N)$, the parameters b_I^k are finite. Because the first-layer equations are linear, Eqs. (4.3) are a valid solution even for finite b_I^k . In this case, the second-layer equations have non-vanishing source and the solution becomes non-trivial. Also, the linear relation between b_I^k and $A_{k,m,n,f}^\psi$ gets non-linear correction. This correction must be in a very specific form so that the full geometry is regular. This is a powerful constraint which in some cases can be used to determine the form of the non-linear correction, without input from CFT. We will see how this mechanism (“coiffuring”) works in explicit examples below.

Because the source terms in the second-layer equations are quadratic in the first-order fields, the expansion (4.3) means that the source has the schematic form

$$\sum_{\mathbf{k}, \mathbf{k}'} b^{\mathbf{k}} b^{\mathbf{k}'} (\text{some function}), \quad (4.4)$$

where we ignored the structure related to the indices I, I' . Therefore, we only have to solve the second-layer equations for each pair of modes $(\mathbf{k}, \mathbf{k}')$; let us call the resulting second-layer fields $\mathcal{F}_{\mathbf{k}, \mathbf{k}'}, \omega_{\mathbf{k}, \mathbf{k}'}$. Once we have them, we can construct the solution for the general first-layer fields (4.3) by superposing solutions for different pairs of modes as $\sum_{\mathbf{k}, \mathbf{k}'} b^{\mathbf{k}} b^{\mathbf{k}'} \mathcal{F}_{\mathbf{k}, \mathbf{k}'}$. The working assumption here is that, if we can make the geometry regular for each pair of modes, the general solution obtained by superposition is also regular. This does work for known solutions [29].

For the arguments above, the assumption that the base is unchanged, no matter what b_I^k are, is crucial. If that is not the case, we will have to change the mode functions $z_{\mathbf{k}}, \vartheta_{\mathbf{k}}$ as we change b_I^k , and the expansions (4.3) lose their meaning as linear superposition. In this article, we will restrict ourselves to the cases where this assumption holds. However, we emphasize that this is a technical assumption that does not hold true for the completely general superstrata dual to the CFT state (2.14). For constructing such general solutions, we would have to face the problem of changing the base depending on the state.

4.2 A class of superstrata with a flat base

4.2.1 Linear spectrum

As discussed above, it is interesting to focus on 1/8-BPS excitations that do not change the base. To find the candidate states for which that is true, let us look at the spectrum of

¹⁸Generally, the large N scaling is $b_I^k \sim A_{k,m,n,f}^\psi N^{-1/2} \sim (N_{k,m,n,f}^\psi/N)^{1/2}$ [7, 61, 62].

linearized supergravity around $\text{AdS}_3 \times S^3$. This means that we are taking the base to be the flat base (3.46) equipped with $\beta = \beta_0$ of (3.49b) and the CFT background Ψ_{bg} to be (3.45).¹⁹

The spectrum of linearized supergravity around $\text{AdS}_3 \times S^3$ has been long known [54, 56–58]. As mentioned before, chiral primary states in CFT are in one-to-one correspondence with 1/4-BPS supergraviton states in linearized supergravity. Likewise, $SU(1, 1|2)_L \times SU(1, 1|2)_R$ descendants of chiral primary states are in one-to-one correspondence with 1/8-BPS supergraviton states.

Let us see how these states are expressed in terms of the ansatz data of section 3. The spectrum of 1/8-BPS supergravitons and the non-trivial fields that they involve were worked out in [56] in $d = 6$ supergravity and reinterpreted in [46] in the formulation of sections 2 and 3. From [46, Appendix C], we see that the following fields in the zeroth and first layers get excited:

$$|--'\rangle, |\mp\pm\rangle, GG|--'\rangle, GG|+-\rangle : ds^2(\mathcal{B}), \beta, Z_I, \Theta_I \quad (4.5a)$$

$$|00\rangle, |++'\rangle, GG|00\rangle, GG|++'\rangle : Z_I, \Theta_I \quad (4.5b)$$

$$GG|+-\rangle : ds^2(\mathcal{B}) \quad (4.5c)$$

Here, $|\psi\rangle$ represents states of the form $(L_{-1} - J_{-1}^3)^n (J_{-1}^+)^m |\psi\rangle_k$ and $GG|\psi\rangle$ represents states of the form $(L_{-1} - J_{-1}^3)^n (J_{-1}^+)^m G_{-1}^{+1} G_{-1}^{+2} |\psi\rangle_k$, in the R sector. Also, $|++'\rangle$ is a particular superposition of $|++\rangle$ and $|--\rangle$ which corresponds to the “density mode” of the 2-charge solution (3.40) that changes the λ -parametrization of the profile but not its shape [41, 78]. On the other hand, $|--'\rangle$ is a superposition that is linearly independent of $|++'\rangle$, namely, the “transverse mode” which changes the shape of the profile. We see that, at the linear level, the states listed in (4.5b) do not change the base. If we consider non-linear correction, there is no guarantee that the base stays undeformed. However, we will see that they in fact lead to superstrata with a fixed base.

4.2.2 A class of superstrata with a flat base

Based on the linear spectrum (4.5b), let us consider the class of superstrata that corresponds to the following set of states:

$$[|++\rangle_1]^{N_0} \prod_{k,m,n} \left\{ \left[|00; k, m, n\rangle \right]^{N_{k,m,n}} \left[|00; k, m, n, 12\rangle \right]^{\widehat{N}_{k,m,n}} \right\}, \quad (4.6a)$$

$$N_0 + \sum_{k,m,n} (kN_{k,m,n} + k\widehat{N}_{k,m,n}) = N. \quad (4.6b)$$

¹⁹If one wants to consider some other background state Ψ_{bg} , then one needs to study the spectrum of linearized supergravity around the background geometry dual to Ψ_{bg} , in order to carry out the procedure of this section.

Here we wrote $N_{k,m,n}^{00} =: N_{k,m,n}$, $N_{k,m,n,12}^{00} =: \widehat{N}_{k,m,n}$. The first factor $[|++\rangle_1]^{N_0}$ is the background part Ψ_{bg} in (4.2), and we wrote its total strand length as $N_0 := N_{\text{bg}}$. This corresponds to empty $\text{AdS}_3 \times S^3$, as in (3.45). In this case, although there is no general construction or proof yet, experience shows [29, 41–44, 46, 47, 79] that we can take the base \mathcal{B} to be flat \mathbb{R}^4 with metric (3.46) and the 1-form β to be β_0 given in (3.49b). In this subsection, we will discuss the superstratum solutions dual to the class of state (4.6). Although this is a subclass of all possible superstrata, their entropy growth rate for large charges is expected to be the same as that for more general superstrata ensemble, $S \sim N^{1/2} N_P^{1/4}$ [63].

In (4.5b), we also have states based on $|++'\rangle, GG|++'\rangle$. They will be discussed in section 4.3.8.

4.2.3 Linear solutions

If the excitation numbers $N_{k,m,n}, \widehat{N}_{k,m,n}$ in the state (4.6) are much smaller than N , it describes small fluctuations around $\text{AdS}_3 \times S^3$; namely, solutions of linearized supergravity in the $\text{AdS}_3 \times S^3$ background. The explicit form of such linear solutions in $d = 6$ supergravity is known [56, 57] and we can read off the ansatz data from them, although that requires knowledge of how our ansatz is embedded in $d = 6$ supergravity.

Another way to find the ansatz data for linear solutions is the so-called *solution-generating technique* [16], which was used in constructing explicit superstrata [29, 44, 46]. The AdS/CFT dictionary for 1/4-BPS supergravitons is given in (3.43), so we know the linear solution dual to $|\psi\rangle_k$. For example, the dual of $|00\rangle_k$ is obtained from (3.55) by taking infinitesimal b . By the bulk spectral flow (3.51), we can transform the background to $\text{AdS}_3 \times S^3$ with the super-isometry group $SU(1, 1|2)_L \times SU(1, 1|2)_R$, which is dual to the symmetry group of the boundary CFT. Its bosonic generators, including L_{-1}, J_0^- ,²⁰ can be realized in the bulk as Killing vectors and, by acting with the corresponding diffeomorphism on the linear solution dual to $|00\rangle_k$, we can obtain the linear solution dual to $(L_{-1})^n (J_0^-)^m |00\rangle_k = |00; k, m, n\rangle$. If we reorganize the resulting linear solution in the form of the ansatz (3.1), we can read off the ansatz data [29, 44, 78]. Likewise, the fermionic generators $G_{-1/2}^{-,A}$ are realized as the supersymmetry transformations with Killing spinors preserved by the $\text{AdS}_3 \times S^3$ background, and its action allows us to construct the linear solution dual to $(L_{-1})^n (J_0^-)^m G_{-1/2}^{-,1} G_{-1/2}^{-,2} |00\rangle_k = |00; k, m, n, 12\rangle$ and read off the corresponding ansatz data [46].

This procedure leads to the following ansatz data:

$$|00; k, m, n\rangle \quad \leftrightarrow \quad Z_4 = b_4 z_{k,m,n}, \quad \Theta_4 = b_4 \vartheta_{k,m,n}, \quad (4.7a)$$

$$|00; k, m, n, 12\rangle \quad \leftrightarrow \quad Z_4 = 0, \quad \Theta_4 = \widehat{b}_4 \widehat{\vartheta}_{k,m,n}, \quad (4.7b)$$

where b_4, \widehat{b}_4 are small constants which are proportional to $(N_{k,m,n})^{1/2}, (\widehat{N}_{k,m,n})^{1/2}$, or more

²⁰Here we using the NS language, appropriate for the $\text{AdS}_3 \times S^3$ background.

precisely the parameters $A_{k,m,n} \equiv A_{k,m,n}^{00}$, $\widehat{A}_{k,m,n} \equiv A_{k,m,n,12}^{00}$ that appear in the coherent superposition in (4.1) (see footnote 18). All other fields, $ds^2(\mathcal{B})$, β , $Z_{1,2}$, $\Theta_{1,2}$, ω , \mathcal{F} are still given by the empty $\text{AdS}_3 \times S^3$ ones, (3.49). In particular, the base is undeformed. The explicit form of the mode function $z_{k,m,n}$ and 2-forms $\vartheta_{k,m,n}$, $\widehat{\vartheta}_{k,m,n}$ are given by

$$z_{k,m,n} \equiv R_y \frac{\Delta_{k,m,n}}{\Sigma} \cos v_{k,m,n}, \quad (4.8a)$$

$$\vartheta_{k,m,n} \equiv -\sqrt{2} \Delta_{k,m,n} \left[\left((m+n) r \sin \theta + n \left(\frac{m}{k} - 1 \right) \frac{\Sigma}{r \sin \theta} \right) \Omega^{(1)} \sin v_{k,m,n} + \left(m \left(\frac{n}{k} + 1 \right) \Omega^{(2)} + \left(\frac{m}{k} - 1 \right) n \Omega^{(3)} \right) \cos v_{k,m,n} \right], \quad (4.8b)$$

$$\widehat{\vartheta}_{k,m,n} \equiv \sqrt{2} \Delta_{k,m,n} \left[\frac{\Sigma}{r \sin \theta} \Omega^{(1)} \sin \widehat{v}_{k,m,n} + (\Omega^{(2)} + \Omega^{(3)}) \cos \widehat{v}_{k,m,n} \right]. \quad (4.8c)$$

where

$$\Delta_{k,m,n} \equiv \left(\frac{a}{\sqrt{r^2 + a^2}} \right)^k \left(\frac{r}{\sqrt{r^2 + a^2}} \right)^n \cos^m \theta \sin^{k-m} \theta, \quad (4.9a)$$

$$v_{k,m,n} \equiv (m+n) \frac{\sqrt{2} v}{R_y} + (k-m) \phi - m \psi, \quad (4.9b)$$

and the $\Omega^{(i)}$ are a (unnormalized) basis of self-dual 2-forms:

$$\begin{aligned} \Omega^{(1)} &\equiv \frac{dr \wedge d\theta}{(r^2 + a^2) \cos \theta} + \frac{r \sin \theta}{\Sigma} d\phi \wedge d\psi, \\ \Omega^{(2)} &\equiv \frac{r}{r^2 + a^2} dr \wedge d\psi + \tan \theta d\theta \wedge d\phi, \\ \Omega^{(3)} &\equiv \frac{dr \wedge d\phi}{r} - \cot \theta d\theta \wedge d\psi. \end{aligned} \quad (4.10)$$

One can check that (4.7) satisfy the first-layer equations (3.37c) for the flat base (3.46). More generally, one could include constant phase in (4.8) by setting $v_{k,m,n} \rightarrow v_{k,m,n} + \alpha_{k,m,n}$, but we do not consider such generalization here.

Because we are in linearized supergravity, we can freely take a linear superposition of different modes in (4.7), obtaining

$$Z_4 = \sum_{k,m,n} b_4^{k,m,n} z_{k,m,n}, \quad \Theta_4 = \sum_{k,m,n} \left(b_4^{k,m,n} \vartheta_{k,m,n} + \widehat{b}_4^{k,m,n} \widehat{\vartheta}_{k,m,n} \right). \quad (4.11)$$

where $b_4^{k,m,n}$ and $\widehat{b}_4^{k,m,n}$ are infinitesimal and proportional to $A_{k,m,n}$ and $\widehat{A}_{k,m,n}$, respectively. All other fields remain undeformed. This is the solution that corresponds to the state (4.6) with general $N_{k,m,n}$, $\widehat{N}_{k,m,n} \ll N$.

4.2.4 Non-linear solutions and coiffuring

If the excitation numbers $N_{k,m,n}$, $\widehat{N}_{k,m,n}$ are $\mathcal{O}(N)$ (or $A_{k,m,n}$, $\widehat{A}_{k,m,n} = \mathcal{O}(N^{1/2})$), we must go beyond the linear approximation and consider backreaction. Such non-linear solutions

are nothing but superstrata. We must find a solution to all three layers of BPS equations, generalizing the linear solution (4.11). The working assumption in doing so is that, even in such a non-linear regime, the base remains undeformed and is still given by (3.46) and (3.49b). Because the first-layer equations are linear, we can still use (4.11) but now with the coefficient $b_4^{k,m,n}, \widehat{b}_4^{k,m,n}$ finite. This means that the source terms will be non-vanishing in the second-layer equations, which we must solve, imposing regularity.

At linear order, $b_4^{k,m,n}, \widehat{b}_4^{k,m,n}$ were proportional to $A^{k,m,n}, \widehat{A}^{k,m,n}$. However, in the non-linear regime, there can be non-linear corrections to the relation. Moreover, there can be non-linear correction to other first-layer fields, $Z_{1,2}, \Theta_{1,2}$. Because $(Z_1, \Theta_2), (Z_2, \Theta_1)$ satisfy the same equation satisfied by (Z_4, Θ_4) , we must be able to expand Z_I, Θ_I as

$$Z_1 = \frac{R_y^2 a^2}{Q_5 \Sigma} + \sum_{k,m,n} b_1^{k,m,n} z_{k,m,n}, \quad \Theta_2 = \sum_{k,m,n} (b_1^{k,m,n} \vartheta_{k,m,n} + \widehat{b}_1^{k,m,n} \widehat{\vartheta}_{k,m,n}), \quad (4.12a)$$

$$Z_2 = \frac{Q_5}{\Sigma} + \sum_{k,m,n} b_2^{k,m,n} z_{k,m,n}, \quad \Theta_1 = \sum_{k,m,n} (b_2^{k,m,n} \vartheta_{k,m,n} + \widehat{b}_2^{k,m,n} \widehat{\vartheta}_{k,m,n}), \quad (4.12b)$$

$$Z_4 = \sum_{k,m,n} b_4^{k,m,n} z_{k,m,n}, \quad \Theta_4 = \sum_{k,m,n} (b_4^{k,m,n} \vartheta_{k,m,n} + \widehat{b}_4^{k,m,n} \widehat{\vartheta}_{k,m,n}), \quad (4.12c)$$

where in $Z_{1,2}$ we included “zero mode” terms from (3.49a). b_I, \widehat{b}_I are finite numbers with $b_4 = \mathcal{O}(A), \widehat{b}_4 = \mathcal{O}(\widehat{A})$ and $b_1, \widehat{b}_1, b_2, \widehat{b}_2 = \mathcal{O}(A^2, \widehat{A}^2, A\widehat{A})$. Or, alternatively, we can write the relation as $b_1, \widehat{b}_1, b_2, \widehat{b}_2 = \mathcal{O}(b_4^2, \widehat{b}_4^2, b_4 \widehat{b}_4)$. These non-linear corrections in the first layer feed into the second layer as source. We must solve the second-layer equations with the source, and determine the correction so that the full solution represent a regular geometry. In principle, the coefficients can receive corrections from all orders:

$$B_1^{\mathbf{k}} \stackrel{?}{=} \sum_{k_1, k_2} c_{k_1, k_2}^{\mathbf{k}} B_4^{k_1} B_4^{k_2} + \sum_{k_1, k_2, k_3} c_{k_1, k_2, k_3}^{\mathbf{k}} B_4^{k_1} B_4^{k_2} B_4^{k_3} + \dots \quad (4.13)$$

where $\mathbf{k} = (k, m, n)$ and $B_I^{\mathbf{k}}$ collectively denotes $b_I^{\mathbf{k}}$ and $\widehat{b}_I^{\mathbf{k}}$. Namely, even if one turns on $B_4^{\mathbf{k}}$ for one or two particular values of \mathbf{k} , it can make $B_1^{\mathbf{k}}, B_2^{\mathbf{k}}$ non-vanishing for infinitely many values of \mathbf{k} . Also, it is expected that the corrections are not unique, due to the possibility to turn on new states at higher order. Determining all the corrections seems to be a formidable task.

However, fortunately, we can gain an idea about how to proceed by using a finite version of the solution-generating technique [25, 29]. Namely, one starts with the Lunin-Mathur geometry with profile (3.53) with finite b , and furthermore acts on it with a finite $SU(2)_L$ rotation. This procedure generates a particular solution of all three layers and gives us an idea about what the general solution must look like. From this, we can extract a rule of thumb, called *coiffuring* [29], which can be stated as follows. Let (Z_4, Θ_4) be given by (4.12c) with general finite coefficients $B_4^{\mathbf{k}} = (b_4^{\mathbf{k}}, \widehat{b}_4^{\mathbf{k}})$. First, we set $B_2^{\mathbf{k}} = (b_2^{\mathbf{k}}, \widehat{b}_2^{\mathbf{k}}) = 0$. Then, we choose

$B_1^{\mathbf{k}} = (b_1^{\mathbf{k}}, \widehat{b}_1^{\mathbf{k}})$ as follows. If (Z_4, Θ_4) have modes $\mathbf{k}_1 = (k_1, m_1, n_1)$ and $\mathbf{k}_2 = (k_2, m_2, n_2)$ turned on, they will produce, when fed into the second-layer equations (3.38), sources with “high-frequency” phase $v_{\mathbf{k}_1+\mathbf{k}_2} \equiv v_+$ and “low-frequency” phase $v_{\mathbf{k}_1-\mathbf{k}_2} \equiv v_-$. This is because the source in (3.38) includes quadratic terms in Z_4, Θ_4 , and because of the product formula for trigonometric functions. Because a high-frequency source leads to a singularity in the 1-form ω , we must set the coefficients in (Z_1, Θ_2) so that the high-frequency terms get canceled in the source. Because Z_2 has a zero-mode term (Q_5/Σ), this can be achieved by setting $B_1^{\mathbf{k}_1+\mathbf{k}_2}$ in (Z_1, Θ_2) to be proportional to $B_4^{\mathbf{k}_1} B_4^{\mathbf{k}_2}$. The actual procedure of coiffuring can be messy and the detail depends on the values of the mode numbers $\mathbf{k}_1, \mathbf{k}_2$. If $\mathbf{k}_1 - \mathbf{k}_2$ is an allowed wave number, we may also have to turn on $B_1^{\mathbf{k}_1-\mathbf{k}_2} \propto B_4^{\mathbf{k}_1} B_4^{\mathbf{k}_2}$ in order to cancel the low-frequency source with phase v_- , to avoid a singular term in ω .²¹ In any case, $B_1^{\mathbf{k}}$ are quadratic in $B_4^{\mathbf{k}}$, and the expansion (4.13) actually terminates at quadratic order. Also, if one turns on $B_4^{\mathbf{k}}$ for the pair $(\mathbf{k}_1, \mathbf{k}_2)$, it makes $B_1^{\mathbf{k}}$ non-vanishing only for finite (actually, up to two) values of \mathbf{k} . Therefore, as mentioned below (4.4), we only have to solve the second-layer equations for each pair of modes $(\mathbf{k}_1, \mathbf{k}_2)$ to find a regular solution for general (Z_4, Θ_4) in (4.12c). If we can find the solution to the second layer, call it $\mathcal{F}_{\mathbf{k}_1, \mathbf{k}_2}, \omega_{\mathbf{k}_1, \mathbf{k}_2}$, for the pair $(\mathbf{k}_1, \mathbf{k}_2)$, the solution \mathcal{F}, ω for the general case (4.12c) can be obtained by summing over all pairs.

At the time of writing, no closed formula for the coiffured Z_1 for a general pair of modes $(\mathbf{k}_1, \mathbf{k}_2)$ and the resulting second-layer fields ω, \mathcal{F} is known. However, for some sets of pairs of modes (some of which are infinite sets), coiffuring has been explicitly carried out and the full solution has been shown to be completely regular. The interested reader are referred to [29, 47] for detail.

Precision holography [37, 48] indicates that, the relation between the mode coefficients $b_4^{k,m,n}, \widehat{b}_4^{k,m,n}$ and the coherent state parameters $A_{k,m,n}, \widehat{A}_{k,m,n}$ is not modified at higher order; they are simply proportional to each other. This suggests that coiffuring is the way preferred by CFT to fix the mode coefficients. This is presumably related to the fact that coiffuring is in some sense the minimal way to achieve regularity by writing $B_1^{\mathbf{k}}$ as a mere quadratic expression in $B_4^{\mathbf{k}}$.

In the early stages of the development, attempts were made to construct smooth superstrata based on states that change the shape of the Lunin-Mathur geometries, $|\alpha\dot{\alpha}\rangle_k$ where $\alpha, \dot{\alpha} = \pm$, but only singular solutions were obtained [80]. In retrospect, having a fixed base and β is technically much easier and, turning on g_5 dual to $|00\rangle_k$ is the most natural way to go. However, even so, it is miraculous that coiffuring allows us to explicitly construct

²¹This coiffuring for low-frequency source is more non-trivial than the high-frequency one. For low-frequency coiffuring, the term in Z_1 to be turned on is proportional to $\Delta_{k_1-k_2, m_1-m_2, n_1-n_2}$, whereas one naively expects terms proportional to $\Delta_{k_1, m_1, n_1} \Delta_{k_2, m_2, n_2} = \Delta_{k_1+k_2, m_1+m_2, n_1+n_2}$, the second-layer source being quadratic in Z_I, Θ_I .

a class of superstrata for which the base is fixed, no matter what modes (k, m, n) we turn on. This could have failed at any stage, because it could be that, once the deformation is finite, regularity requirement inevitably leads to uncontrollable non-linear correction to the first-layer fields, or even to deformation of the base. Currently we lack a deep understanding of why coiffuring works.

4.3 Explicit superstratum solutions

4.3.1 Single-mode superstrata

In section 4.2, we explained how to construct superstrata dual to the CFT states of the form (4.6) for general $N_{k,m,n}, \widehat{N}_{k,m,n}$. By coiffuring, the construction reduces to solving the BPS equations for a general pair of modes (k_1, m_1, n_1) and (k_2, m_2, n_2) in Z_4 . For the solutions of the class (4.6), such multi-mode superstrata have been constructed for some particular pair of modes on a case-by-case basis, but the general solution has not been found yet at the time of writing (see [47] for recent development). So, just as in much of the literature, we will mostly focus on *single-mode* superstrata, for which only one particular mode (k, m, n) is turned on. We will mention the multi-mode case when appropriate.

So, we take the first-layer fields to be

$$Z_4 = b_4 z_{k,m,n}, \quad \Theta_4 = b_4 \vartheta_{k,m,n} + \widehat{b}_4 \widehat{\vartheta}_{k,m,n}, \quad (4.14)$$

where we have suppressed the mode index on the coefficients; more precisely b_4, \widehat{b}_4 must be written as $b_4^{k,m,n}, \widehat{b}_4^{k,m,n}$. Coiffuring is not trivial even in this case, because the mode (k, m, n) quadratically interacts with itself. The CFT state dual to this superstratum is the same as (4.6) but without the product. Namely,

$$[|++\rangle_1]^{N_0} [|00; k, m, n\rangle]^{N_{k,m,n}} [|00; k, m, n, 12\rangle]^{\widehat{N}_{k,m,n}}, \quad (4.15a)$$

$$N_0 + k(N_{k,m,n} + \widehat{N}_{k,m,n}) = N. \quad (4.15b)$$

The “original” superstratum constructed in [29, 42, 44] corresponds to the one with $b_4 \neq 0, \widehat{b}_4 = 0$, while the “supercharged” superstratum constructed in [46] corresponds to the one with $b_4 = 0, \widehat{b}_4 \neq 0$. The case with $b_4, \widehat{b}_4 \neq 0$ is called the “hybrid” superstratum and was constructed in [47]. Our presentation of the solution follows [47].

4.3.2 The first layer

In the presence of a single mode $\mathbf{k} = (k, m, n)$, the second-layer source will have high-frequency terms with phase $v_{2\mathbf{k}} = 2v_{\mathbf{k}}$ and low-frequency terms with constant phase. The constant-phase terms, or the “RMS” terms, do not lead to singularities in ω, \mathcal{F} , while the high-frequency

terms do and must be coiffured away. This means that we must turn on a term with mode numbers $2\mathbf{k}$ in Z_1 . Therefore, we are led to the following ansatz for the first-layer fields:

$$\begin{aligned} Z_1 &= \frac{R_y^2 a^2}{Q_5 \Sigma} + b_1 \frac{R_y}{2Q_5} z_{2k,2m,2n}, & \Theta_2 &= b_1 \frac{R_y}{2Q_5} \vartheta_{2k,2m,2n} + \widehat{b}_1 \frac{R_y}{2Q_5} \widehat{\vartheta}_{2k,2m,2n}, \\ Z_2 &= \frac{Q_5}{\Sigma}, & \Theta_1 &= 0, \\ Z_4 &= b_4 z_{k,m,n}, & \Theta_4 &= b_4 \vartheta_{k,m,n} + \widehat{b}_4 \widehat{\vartheta}_{k,m,n}, \end{aligned} \quad (4.16)$$

where the coefficients b_1, \widehat{b}_1 in Z_4 must more properly be written as $b_1^{2k,2m,2n}, \widehat{b}_1^{2k,2m,2n}$.

4.3.3 The second layer

If we plug in the ansatz (4.16) into the second-layer equations (3.38), we find that the high-frequency terms cancel if we the coefficients satisfy the following coiffuring constraints:

$$b_1 = b_4^2, \quad \widehat{b}_1 = 2b_4 \widehat{b}_4. \quad (4.17)$$

Let us solve the second-layer equations. When the coiffuring relation (4.17) is satisfied, the source in the second-layer equation (3.38) consists only of an RMS term. So, ω and \mathcal{F} can be written as

$$\omega = \omega_0 + \omega_{k,m,n}, \quad \mathcal{F} = \mathcal{F}_{k,m,n} \quad (4.18)$$

where $\omega_{k,m,n}, \mathcal{F}_{k,m,n}$ are RMS modes, independent of $v_{k,m,n}$. The second-layer equations (3.38) are

$$\begin{aligned} (1 + *_4) d\omega_{k,m,n} + \mathcal{F}_{k,m,n} d\beta &= \sqrt{2} R_y \frac{\Delta_{2k,2m,2n}}{\Sigma} \left[\left(\frac{m(k+n)}{k} b_4 - \widehat{b}_4 \right) \Omega^{(2)} \right. \\ &\quad \left. - \left(\frac{n(k-m)}{k} b_4 + \widehat{b}_4 \right) \Omega^{(3)} \right], \end{aligned} \quad (4.19)$$

$$\begin{aligned} \widehat{\mathcal{L}} \mathcal{F}_{k,m,n} &= \frac{4}{(r^2 + a^2) \Sigma \cos^2 \theta} \left[\left(\frac{m(k+n)}{k} b_4 - \widehat{b}_4 \right)^2 \Delta_{2k,2m,2n} \right. \\ &\quad \left. + \left(\frac{n(k-m)}{k} b_4 + \widehat{b}_4 \right)^2 \Delta_{2k,2m+2,2n-2} \right], \end{aligned} \quad (4.20)$$

where $\widehat{\mathcal{L}}$ is the scalar Laplacian on the base $\mathcal{B} = \mathbb{R}^4$:

$$\widehat{\mathcal{L}} F \equiv \frac{1}{r\Sigma} \partial_r (r(r^2 + a^2) \partial_r F) + \frac{1}{\Sigma \sin \theta \cos \theta} \partial_\theta (\sin \theta \cos \theta \partial_\theta F). \quad (4.21)$$

The solution to Eq. (4.20) is given by

$$\mathcal{F}_{k,m,n} = 4 \left[\left(\frac{m(k+n)}{k} b_4 - \widehat{b}_4 \right)^2 F_{2k,2m,2n} + \left(\frac{n(k-m)}{k} b_4 + \widehat{b}_4 \right)^2 F_{2k,2m+2,2n-2} \right], \quad (4.22)$$

where $F_{2k,2m,2n}$ solves the equation

$$\widehat{\mathcal{L}}F_{2k,2m,2n} = \frac{\Delta_{2k,2m,2n}}{(r^2 + a^2) \cos^2 \theta \Sigma} \quad (4.23)$$

and its explicit form is [44]

$$F_{2k,2m,2n} = - \sum_{j_1, j_2, j_3=0}^{j_1+j_2+j_3 \leq k+n-1} \binom{j_1 + j_2 + j_3}{j_1, j_2, j_3} \frac{\binom{k+n-j_1-j_2-j_3-1}{k-m-j_1, m-j_2-1, n-j_3}^2}{\binom{k+n-1}{k-m, m-1, n}^2} \frac{\Delta_{2(k-j_1-j_2-1), 2(m-j_2-1), 2(n-j_3)}}{4(k+n)^2(r^2 + a^2)} \quad (4.24)$$

with the multinomial coefficients defined by

$$\binom{j_1 + j_2 + j_3}{j_1, j_2, j_3} \equiv \frac{(j_1 + j_2 + j_3)!}{j_1! j_2! j_3!}. \quad (4.25)$$

On the other hand, $\omega_{k,m,n}$ can be written as

$$\omega_{k,m,n} = \mu_{k,m,n}(d\psi + d\phi) + \zeta_{k,m,n}(d\psi - d\phi). \quad (4.26)$$

If we define

$$\widehat{\mu}_{k,m,n} \equiv \mu_{k,m,n} + \frac{R_y}{4\sqrt{2}} \frac{r^2 + a^2 \sin^2 \theta}{\Sigma} \mathcal{F}_{k,m,n} + \frac{R_y b_4^2}{4\sqrt{2}} \frac{\Delta_{2k,2m,2n}}{\Sigma}, \quad (4.27)$$

then $\widehat{\mu}_{k,m,n}$ is found to satisfy

$$\widehat{\mathcal{L}}\widehat{\mu}_{k,m,n} = \frac{R_y}{\sqrt{2}} \frac{1}{(r^2 + a^2)\Sigma \cos^2 \theta} \left[\left(\frac{(k-m)(k+n)}{k} b_4 + \widehat{b}_4 \right)^2 \Delta_{2k,2m+2,2n} + \left(\frac{mn}{k} b_4 - \widehat{b}_4 \right)^2 \Delta_{2k,2m,2n-2} \right]. \quad (4.28)$$

Therefore,

$$\mu_{k,m,n} = \frac{R_y}{\sqrt{2}} \left[\left(\frac{(k-m)(k+n)}{k} b_4 + \widehat{b}_4 \right)^2 F_{2k,2m+2,2n} + \left(\frac{mn}{k} b_4 - \widehat{b}_4 \right)^2 F_{2k,2m,2n-2} - b_4^2 \frac{\Delta_{2k,2m,2n}}{4\Sigma} \right] - \frac{R_y}{4\sqrt{2}} \frac{r^2 + a^2 \sin^2 \theta}{\Sigma} \mathcal{F}_{k,m,n} + \frac{R_y X_{k,m,n}}{2\sqrt{2}\Sigma}. \quad (4.29)$$

In the expression for $\mathcal{F}_{k,m,n}$ and $\mu_{k,m,n}$, it should be understood that, when the coefficient of the F function in a term is zero, that term is zero; this rule is necessary because $F_{2k,2m,2n}$ defined in (4.24) can be ill-defined for some values of k, m, n . In $\widehat{\mu}$, the term proportional to $X_{k,m,n}$ is a harmonic piece that can be freely added to the solution of the Poisson equation (4.28).

Finally, once $\mathcal{F}_{k,m,n}$ and $\mu_{k,m,n}$ are known, $\zeta_{k,m,n}$ is determined by the equations

$$\begin{aligned}
\partial_r \zeta_{k,m,n} &= \frac{r^2 c_{2\theta} - a^2 s_\theta^2}{\Lambda} \partial_r \mu_{k,m,n} - \frac{r s_{2\theta}}{\Lambda} \partial_\theta \mu_{k,m,n} \\
&\quad + \frac{\sqrt{2} R_y r}{\Sigma \Lambda} \left[b_4 \left((m s_\theta^2 + n c_\theta^2) b_4 - \left(\frac{mn}{k} b_4 - \widehat{b}_4 \right) c_{2\theta} \right) \Delta_{2k,2m,2n} \right. \\
&\quad \quad \left. - \frac{a^2 (2r^2 + a^2) s_\theta^2 c_\theta^2}{\Sigma} \mathcal{F}_{k,m,n} \right], \\
\partial_\theta \zeta_{k,m,n} &= \frac{r(r^2 + a^2) s_{2\theta}}{\Lambda} \partial_r \mu_{k,m,n} + \frac{r^2 c_{2\theta} - a^2 s_\theta^2}{\Lambda} \partial_\theta \mu_{k,m,n} \\
&\quad + \frac{R_y s_{2\theta}}{\sqrt{2} \Sigma \Lambda} \left[b_4 \left((-mr^2 + n(r^2 + a^2)) b_4 - (2r^2 + a^2) \left(\frac{mn}{k} b_4 - \widehat{b}_4 \right) \right) \Delta_{2k,2m,2n} \right. \\
&\quad \quad \left. + \frac{a^2 r^2 (r^2 + a^2) c_{2\theta}}{\Sigma} \mathcal{F}_{k,m,n} \right],
\end{aligned} \tag{4.30}$$

where $\Lambda \equiv r^2 + a^2 \sin^2 \theta$, $s_\theta \equiv \sin \theta$, $c_\theta \equiv \cos \theta$. For specific values of (k, m, n) , it is easy to solve these to find explicit $\zeta_{k,m,n}$, but the expression for $\zeta_{k,m,n}$ for general (k, m, n) is not known in closed form.

One obstacle to obtaining a general expression for ζ for general (k, m, n) is that the F function is defined only through a sum in (4.24).²² If we set two of the three variables (k, m, n) to specific values, for example as $(k, m, n) = (2, 1, n)$, then the summation for k, m is a finite sum while the summation over n is elementary to carry out; we can plug the result into (4.30) and find ζ . However, for general (k, m, n) , it is hard to evaluate the sum.

4.3.4 Regularity

For the solution to represent a microstate geometry, it must be regular everywhere. Fully establishing regularity, including absence of closed-timelike curves, in full generality is quite challenging and only a case-by-case results are known. Here we discuss some salient features important for the physics of the solutions. Some more detail can be found in, *e.g.*, [44].

First, the point $r = \theta = 0$ is the origin of the flat \mathbb{R}^4 base and all Cartesian components of all forms must be finite there. In particular, for the 1-form ω to be regular there, it is necessary that its component along $d\phi + d\psi$, namely $\mu_{k,m,n}$ vanish there. This fixes the undetermined coefficient $X_{k,m,n}$ to be

$$X_{k,m,n} = B_4^2 + \widehat{B}_4^2, \tag{4.31}$$

where we defined B_4, \widehat{B}_4 by

$$B_4^2 = \frac{k+n}{2k} \binom{k+n}{k-m, m, n}^{-1} b_4^2, \quad \widehat{B}_4^2 = \frac{k}{2(k-m)mn} \binom{k+n}{k-m, m, n}^{-1} \widehat{b}_4^2. \tag{4.32}$$

²²The expression (4.24) can be regarded as a sort of triple hypergeometric function.

The component of ω along $d\phi + d\psi$, namely $\zeta_{k,m,n}$ must also vanish. This can be checked on a case-by-case basis for values of (k, m, n) for which explicit solutions exist and does not lead to a new constraint.

Another dangerous point is $r = 0, \theta = \pi/2$, where the original 2-charge supertube sits. By requiring that the $(d\phi + d\psi)^2$ component of the metric remain finite, one finds that

$$a^2 + B_4^2 + \widehat{B}_4^2 = \frac{Q_1 Q_5}{R_y^2}. \quad (4.33)$$

This can be thought of as the bulk version of the strand-budget relation, (4.15b). The fact that one cannot excite gravity modes by an arbitrary amount is sometimes called the stringy exclusion principle [54]. In linearized supergravity such constraint is not visible, but in fully backreacted geometries it is known that such constraint can arise by requiring the solution to be physical [81].

4.3.5 Examples

Although the explicit expression for single-mode superstrata with general (k, m, n) has not been found (because the expression for ζ is not known), some infinite families of solutions have been explicitly written down in the literature.

For example, for $(k, m, n) = (1, 0, n)$ [42, 44],

$$\mathcal{F}_{1,0,n} = -\frac{b_4^2}{a^2} \left(1 - \frac{r^{2n}}{(r^2 + a^2)^n} \right), \quad \omega_{1,0,n} = \frac{b_4^2 R_y}{\sqrt{2} \Sigma} \left(1 - \frac{r^{2n}}{(r^2 + a^2)^n} \right) \sin^2 \theta d\phi. \quad (4.34)$$

In this case, there is no supercharged mode. This solution is deceptively simple but has a quite non-trivial structure and contains rich physics. For $a \ll b_4$, this geometry is roughly AdS₃ for $r \gtrsim b_4$. For $a \lesssim r \lesssim b_4$, the spacetime is an AdS₂ \times S^1 throat. At $r \sim a$, there is a momentum wave that supports the geometry which smoothly caps off at $r = 0$. For studies of various physical aspects of this solution, see [42, 44, 79, 82–90], some of which are reviewed in section 5.

For more explicit examples of single-mode superstrata – original, supercharged, and hybrid, see [44–47, 72, 86].

4.3.6 Asymptotically flat solution and conserved charges

In order to extend the above asymptotically AdS superstrata to asymptotically flat solutions that represent microstates of the D1-D5-P black hole in flat space, we must add “1” to $Z_{1,2}$ as

$$Z_1 \rightarrow 1 + Z_1, \quad Z_2 \rightarrow 1 + Z_2. \quad (4.35)$$

This does not affect the first-layer equations, but makes the second-layer equations more complicated by introducing new source terms. As a result, unlike asymptotically AdS ones, in

asymptotically flat superstrata, high-frequency sources in the second layer are not completely canceled but are combined so as to remove singularities in the solutions. This means that the coiffuring relations such as (4.17) get modified. The asymptotically flat, single-mode solution with general (k, m, n) was found in [44] for the original superstratum (except for ζ in the RMS part), although the one for the hybrid superstratum has not been written down as of writing. For the original superstratum, $\widehat{b}_4 = 0$, the coiffuring relation (4.17) is just

$$b_1 = b_4^2. \quad (4.36)$$

In the asymptotically flat version, this is modified to [44]

$$b_1 = \frac{b_4^2}{1 + \frac{a^2}{Q_5} \frac{m+n}{k}}. \quad (4.37)$$

In the decoupling limit, $a^2 \ll Q_{1,5}$, this relation falls back to the AdS relation (4.36).

To read off asymptotic charges, we do not have to know the explicit form of the asymptotically flat solution. This is because the extra source terms that appear in the asymptotically flat solution have a non-vanishing wave number in the v direction and thus vanish when integrated over the S^1 . Therefore, we can read off charges from the ansatz quantities of the asymptotically AdS solution [29]. The D1- and D5-brane charges are simply Q_1 and Q_5 , which are related to the quantized numbers N_1 and N_5 as in (3.42). The momentum charge Q_p can be read off from

$$\mathcal{F} \sim -\frac{2Q_p}{r^2} \quad (4.38)$$

and the angular momenta $\mathcal{J}_L, \mathcal{J}_R$ are read off from

$$\beta_\phi + \beta_\psi + \omega_\phi + \omega_\psi \sim \sqrt{2} \frac{\mathcal{J}_L - \mathcal{J}_R \cos 2\theta}{r^2}. \quad (4.39)$$

If we apply these relations to the single-mode hybrid superstratum, we find

$$Q_p = \frac{m+n}{k} (B_4^2 + \widehat{B}_4^2), \quad \mathcal{J}_L = R_y \left[\frac{a^2}{2} + \frac{m}{k} (B_4^2 + \widehat{B}_4^2) \right], \quad \mathcal{J}_R = \frac{R_y}{2} a^2. \quad (4.40)$$

The supergravity quantities Q_p are related to the quantized momentum number N_P by

$$Q_p = \frac{g_s^2 \alpha'^4}{R_y^2 v_4} N_P = \frac{Q_1 Q_5}{R_y^2 N} N_P, \quad (4.41)$$

where in the second equality we used (3.42). On the other hand, $\mathcal{J}_L, \mathcal{J}_R$ are related to the quantized angular momenta J_L, J_R by

$$\mathcal{J}_{L,R} = \frac{g_s^2 \alpha'^4}{R_y v_4} J_{L,R} = \frac{Q_1 Q_5}{R_y N} J_{L,R}. \quad (4.42)$$

As mentioned before, the supergravity amplitudes a, B_4, \widehat{B}_4 are related to the CFT occupation numbers $N_0, N_{k,m,n}, \widehat{N}_{k,m,n}$ characterizing the state (4.15a). Let us identify the supergravity and CFT quantities as

$$a^2 = \frac{Q_1 Q_5}{R_y^2} \frac{N_0}{N}, \quad B_4^2 = \frac{Q_1 Q_5}{R_y^2} \frac{k N_{k,m,n}}{N}, \quad \widehat{B}_4^2 = \frac{Q_1 Q_5}{R_y^2} \frac{k \widehat{N}_{k,m,n}}{N}. \quad (4.43)$$

Then the regularity constraint (4.33) becomes the strand budget equation (4.15b), while (4.40) translate into

$$N_P = (m+n)(N_{k,m,n} + \widehat{N}_{k,m,n}), \quad J_L = \frac{N_0}{2} + m(N_{k,m,n} + \widehat{N}_{k,m,n}), \quad J_R = \frac{N_0}{2}, \quad (4.44)$$

which are exactly equal to the charges of the CFT state (4.15a), giving a strong support for the holographic dictionary.

4.3.7 Multi-mode superstrata

In the above, we focused on the single-mode hybrid superstratum, for which coiffuring can be carried out explicitly. Namely, the high-frequency source can be coiffured away by the choice (4.17), while the low-frequency source is just the RMS mode that does not need coiffuring. All the ansatz quantities can be explicitly found for specific values of $\mathbf{k} = (k, m, n)$.

For the multi-mode superstratum (4.6), for the pair $(\mathbf{k}_1, \mathbf{k}_2)$, there are four possible modes in the second-layer source: the high-frequency mode $\mathbf{k}_1 + \mathbf{k}_2$, the low-frequency mode $\mathbf{k}_1 - \mathbf{k}_2$, and the RMS mode $(\mathbf{k}_1 - \mathbf{k}_1 = \mathbf{k}_2 - \mathbf{k}_2 = \mathbf{0})$. The RMS mode does not need coiffuring. The high-frequency mode can generally be coiffured away [47], as long as the supercharged modes are turned on. On the other hand, how to coiffure the low-frequency mode is known only on a case-by-case basis; see [29, 91] for explicitly worked out examples.

For asymptotic charges, only the RMS modes will contribute and the result will be a simple sum over all modes; for example, (4.40) will be generalized to

$$Q_p = \sum_{k,m,n} \frac{m+n}{k} \left[(B_4^{k,m,n})^2 + (\widehat{B}_4^{k,m,n})^2 \right], \quad (4.45)$$

which is consistent with the CFT side with identifications similar to (4.43). Other charges will similarly be given by a sum over all modes.

4.3.8 Superstrata based on other species

Superstrata based on $|++'\rangle$

In the above, we discussed superstrata based on the state $|00\rangle$. At the linear level, the “density mode” $|++'\rangle$ in (4.5b) is another state that does not change the base \mathbb{R}^4 , and the superstratum based on it is also expected to have flat base.

In [41], a certain family of superstrata based on $|++'\rangle$ were explicitly constructed. If one turns on density fluctuation, generically, infinitely many modes will be turned on in the harmonic functions $Z_{1,2}$. However, if one turns on the Z_4 mode at the same time in a coordinated way, the harmonic functions will involve essentially only one mode. This is called “Style 1” coiffuring in [41] (the superstratum based on $|00\rangle$ is called “Style 2” there). More precisely, in their solution, $(J_{-1}^+)^k|++\rangle_{k+1}$, $(J_{-1}^+)^k|--\rangle_{k-1}$ and $(J_{-1}^+)^k|00\rangle_k$ are turned on.^{23,24}

In [41], only bosonic excitations on top of $|++'\rangle$ were considered but, as we can see from (4.5b), their supercharged version based on $GG|++'\rangle$ are also expected to have flat \mathbb{R}^4 base.

Superstrata based on $|\dot{A}\dot{B}\rangle$

In this article, we are restricting ourselves to superstrata that preserve the symmetry of the internal manifold \mathcal{M} , for which the supergravity fields take the form of (3.1). By relaxing this condition and turning on fields that have legs along \mathcal{M} (but are still independent of the coordinates of \mathcal{M}), one can construct superstrata [45] that are based on the species $|\dot{A}\dot{B}\rangle$ listed in (2.2). Such solutions will include more scalars $Z_{I \geq 5}$ and forms $\Theta_{I \geq 5}$, corresponding to more tensor multiples of $d = 6, \mathcal{N} = (1, 0)$ supergravity.

4.4 Superstrata on the orbifold $(\text{AdS}_3 \times S^3)/\mathbb{Z}_p$

The superstrata reviewed above describe fluctuation around $\text{AdS}_3 \times S^3$ (see section 3.8.1) which corresponds to the circular profile (3.44) and whose CFT dual is $[|++\rangle_1]^N$ in the R sector. Instead, if we consider a p times wound circle,

$$g_1 + ig_2 = ae^{2\pi i p \lambda / L}, \quad g_3 + ig_4 = g_5 = 0, \quad p \geq 1, \quad (4.46)$$

then we obtain the orbifold $(\text{AdS}_3 \times S^3)/\mathbb{Z}_p$ whose CFT dual in the R sector is

$$[|++\rangle_p]^{N/p}. \quad (4.47)$$

In this case, the ansatz data are somewhat changed from the ones given in (3.49) to

$$Z_1 = \frac{(pR_y)^2 a^2}{Q_5 \Sigma}, \quad Z_2 = \frac{Q_5}{\Sigma}, \quad Z_4 = 0, \quad \Theta_I = 0, \quad (4.48a)$$

$$\beta = \frac{pR_y a^2}{\sqrt{2} \Sigma} (\sin^2 \theta d\phi - \cos^2 \theta d\psi) = p\beta_0, \quad (4.48b)$$

$$\omega = \frac{pR_y a^2}{\sqrt{2} \Sigma} (\sin^2 \theta d\phi + \cos^2 \theta d\psi) = p\omega_0, \quad \mathcal{F} = 0, \quad (4.48c)$$

²³Their solutions include generalization to excitations around the orbifold $(\text{AdS}_3 \times S^3)/\mathbb{Z}_p$ with $p \geq 1$, but here we are setting $p = 1$.

²⁴Having a single mode turned on in the bulk means that, on the boundary, infinitely many modes are turned on. Namely, the corresponding CFT state has $(J_{-1}^+)^k|++\rangle_{k+1}$, $(J_{-1}^+)^k|--\rangle_{k-1}$ and $(J_{-1}^+)^k|00\rangle_k$ turned on not just for one value of k but for all integer multiples of k .

and the relation (3.50) is changed to

$$a^2 = \frac{Q_1 Q_5}{(p R_y)^2}. \quad (4.49)$$

We see that we can obtain the result for the p -wound case from the original ($p = 1$) case by the replacement

$$R_y \rightarrow p R_y. \quad (4.50)$$

Because R_y is the radius of identification for the y circle, this replacement means that we have a conical defect where the y circle shrinks. Therefore the geometry is $(\text{AdS}_3 \times S^3)/\mathbb{Z}_p$. If we want to see the structure of the space more explicitly, we can do the following coordinate transformation (cf. (3.51))

$$\tilde{\phi} = \phi - \frac{t}{p R_y}, \quad \tilde{\psi} = \psi - \frac{y}{p R_y}. \quad (4.51)$$

The 6D metric becomes

$$ds_6^2 = \sqrt{Q_1 Q_5} \left(-\frac{r^2 + a^2}{a^2 (p R_y)^2} dt^2 + \frac{r^2}{a^2 (p R_y)^2} dy^2 + \frac{dr^2}{r^2 + a^2} + d\theta^2 + \sin^2 \theta d\tilde{\phi}^2 + \cos^2 \theta d\tilde{\psi}^2 \right). \quad (4.52)$$

Although this is locally $\text{AdS}_3 \times S^3$, because of identification $(y, \tilde{\phi}, \tilde{\psi}) \cong (y + \frac{2\pi R_y}{p}, \tilde{\phi}, \tilde{\psi} - \frac{2\pi}{p})$, there is a \mathbb{Z}_p singularity at $r = 0$. This conical defect singularity is due to p KK monopoles sitting on top each other and is allowed in string theory.

We can say that the orbifolding is done by starting from a “parent” $\text{AdS}_3 \times S^3$ and then quotienting the y -circle by \mathbb{Z}_p , by making the radius p times smaller. This means that, if we consider a fluctuation of the parent $\text{AdS}_3 \times S^3$ (namely, a superstratum) and then divide the y -circle by \mathbb{Z}_p , then we get a superstratum on $(\text{AdS}_3 \times S^3)/\mathbb{Z}_p$, still with a conical defect. For this quotienting to make sense, the parent superstratum must not have general y Fourier mode numbers but only ones that are single-valued after quotienting. If we satisfy this constraint, the resulting superstratum is expected to represent a valid configuration in string theory.

This operation can be stated in the following way. We consider the following CFT state

$$\prod_{\psi, k, m, n} \left[(J_{-1}^+)^m (L_{-1} - J_{-1}^3)^n |\psi\rangle_k \right]^{N_{k, m, n}^\psi} = \prod_{\psi, k, m, n} \left[|\psi; k, m, n\rangle \right]^{N_{k, m, n}^\psi}, \quad (4.53)$$

with $\sum_{\psi, k, m, n} k N_{k, m, n}^\psi = \frac{N}{p}$

which, in the bulk, can be interpreted as a superstratum on the parent $\text{AdS}_3 \times S^3$. Note that the total strand length is $\frac{N}{p}$ and not N . It is assumed that N is divisible by p , namely, $\frac{N}{p} \in \mathbb{Z}$. Also, we require that $N_{k, m, n}^\psi = 0$ unless $\frac{m+n}{p} \in \mathbb{Z}$. For simplicity, we did not include states

excited by G_{-1}^{+A} , but including them is straightforward. Now, given such a state, we define a state

$$\prod_{\psi,k,m,n} \left[(J_{-\frac{1}{p}}^+)^m (L_{-\frac{1}{p}} - J_{-\frac{1}{p}}^3)^n |\psi\rangle_{pk} \right]^{N_{k,m,n}^\psi} \equiv \prod_{\psi,k,m,n} \left[|\psi; k, m, n; p\rangle \right]^{N_{k,m,n}^\psi}, \quad (4.54)$$

with $\sum_{\psi,k,m,n} pk N_{k,m,n}^\psi = N,$

which is interpreted of as a superstratum on $(\text{AdS}_3 \times S^3)/\mathbb{Z}_p$. Although involving fractional modes, this is a physically allowed state of the orbifold CFT because of the restriction $\frac{m+n}{p} \in \mathbb{Z}$. The bulk superstratum depends on y as $e^{i(m+n)y/(pR_y)}$ which is single-valued because of the restriction.

In [41], a special family of single-mode superstrata of this kind was explicitly constructed. The dual CFT state was identified to be the following:²⁵

$$\left[|++\rangle_k \right]^{N_0} \left[(J_{-\frac{1}{p}}^+)^{pk} |00\rangle_{p^2k} \right]^{N_1} = \left[|++\rangle_k \right]^{N_0} \left[|00; pk, pk, 0; p\rangle \right]^{N_1} \quad (4.55)$$

which is indeed of the form of (4.54). The state appearing in the second factor has $\Delta L_0 = \frac{1}{p} \cdot kp = k \in \mathbb{Z}$ and is thus an allowed state of the orbifold CFT. The explicit solution has exactly the same form as the ordinary superstratum in section 4.3 with $(k, m, n) \rightarrow (pk, pk, 0)$ and with $R_y \rightarrow pR_y$. The geometry has a \mathbb{Z}_p orbifold singularity just as the $(\text{AdS}_3 \times S^3)/\mathbb{Z}_p$ space which the solution is excitation of.

This procedure expands the class of CFT states representable by microstate geometries by inclusion of certain fractional modes. These states exist everywhere in the moduli space of the D1-D5 CFT, because $(\text{AdS}_3 \times S^3)/\mathbb{Z}_p$ contains only one non-trivial 3-cycle [70].

5 Further developments

Superstrata have provided a rich paradigm in which to study physics and mathematics of black-hole microstructure. Some of the physical aspects of superstrata that have been explored are: explicit construction of more general class of states, precision holography, relation to other duality frames, scaling limits and asymptotically AdS_2 solutions, scattering off superstrata, counting, and so on. On the mathematics side, some of the aspects that have been investigated include: the structure of the BPS equations, integrability of the geometries, structure of the ambi-polar base, etc. Here, we will give a survey of recent developments concerning superstrata and related subjects.

Generalizations and other duality frames

The MSW black hole [92] in five dimensions obtained by compactifying M-theory on six-dimensional manifold and wrapping M5-branes on 2-cycles in it is another prototypical black

²⁵They also present orbifolded superstrata of “Style 1” mentioned in section 4.3.8.

hole with which to study black-hole microphysics. The near-horizon geometry is $\text{AdS}_3 \times S^2$ and the dual CFT_2 is called the MSW CFT. When the compactification manifold is T^6 , a chain of duality transformations relates the five-dimensional MSW system to the six-dimensional D1-D5-P system. The duality transformations involve T -duality which requires an isometry direction along which the dual is taken. By this chain of duality, D1-D5 superstrata with the required isometry direction can be mapped into superstrata in the MSW system [43, 72], the latter being described in five-dimensional supergravity. This means that there is a map between a subsector of the D1-D5 CFT and a subsector of the MSW CFT. Empty $\text{AdS}_3 \times S^3$ in the D1-D5 system is mapped into a D6- $\overline{\text{D6}}$ configuration in the MSW system. In the latter, one can consider a gas of D0-branes [93, 94]²⁶ which can alternatively be represented by a gas of supergravitons in $\text{AdS}_3 \times S^2$ [81]. The MSW superstrata can be regarded as coherent states of this supergraviton gas. Such microstates in the MSW system are expected to be useful in understanding the MSW CFT which remains mysterious.

As mentioned in 4.3.7, it is not yet known how to construct general multi-mode superstrata including all (k, m, n, f) . In [91], interesting progress was made by considering holomorphic superposition of some family of modes. For example, in section 4.3.5, we discussed $(1, 0, n)$ superstrata. Let us consider superposing different modes in the first-layer fields Z_4, Θ_4 . For example,

$$Z_4 = \sum_n b_4^{1,0,n} z_{1,0,n} = \frac{R_y}{2\Sigma} \left(\chi \sum_n b_4^{1,0,n} \xi^n + \text{c.c.} \right). \quad (5.1)$$

Here, we defined

$$\chi = \frac{a}{\sqrt{r^2 + a^2}} \sin \theta e^{i\phi}, \quad \xi = \frac{r}{\sqrt{r^2 + a^2}} e^{i\frac{\sqrt{2}v}{R_y}}, \quad \eta = \frac{a}{\sqrt{r^2 + a^2}} \cos \theta e^{i(\frac{\sqrt{2}v}{R_y} - \psi)} \quad (5.2)$$

in terms of which $\Delta_{k,m,n} e^{iv_{k,m,n}} = \chi^{k-m} \eta^m \xi^n$. This suggests that we work with the holomorphic function

$$F(\xi) = \sum_n b_4^{1,0,n} \xi^n, \quad (5.3)$$

rather than with the mode coefficients $b_4^{1,0,n}$. It turns out that one can find the explicit expression for all fields in terms of $F(\xi)$. Moreover, regularity and no-CTC analyses can be completed in terms of $F(\xi)$, and conserved charges can be written in terms of $F(\xi)$. In this $(1, 0, n)$ case, there is no coiffuring needed but, in other examples, such as the case (namely, holomorphic superposition of the $(k, 0, 1)$ superstratum with general coefficients $b_4^{k,0,1}$), both

²⁶It was argued that these D0-branes puff out into M2-branes whose Landau level degeneracy accounts for the entropy of the MSW black hole [93, 95]. These M2-branes are supposed to wrap a non-trivial S^2 in the geometry and are sometimes dubbed supereggs. However, it was shown that such M2-branes will violate charge conservation [96] and/or break the supersymmetry [97] preserved by the MSW black hole. Therefore, these superegg M2-branes and their Landau levels cannot be the precise description of the microstates.

high- and low-frequency coiffuring can be done in terms of holomorphic functions. Having arbitrary holomorphic functions allows us to construct physically interesting solutions, such as ones with a momentum wave localized in the y -circle direction, unlike the single-mode superstratum in which the momentum wave is delocalized in the y direction. Turning on all possible (k, m, n) modes corresponds to having holomorphic functions that depend on all three variables, χ, ξ, η . Explicitly constructing the solution would fulfill the promise of general superstrata being parametrized by three variables. Achieving that goal may be quite difficult technically, but having holomorphic functions of one variable is already a promising step forward.

It is interesting to see if the superstratum technology can be generalized to a non-supersymmetric setting. At the linear level, this is straightforward with AdS_3 asymptotics, because all one must do is to use the solution-generating technique to act on a chiral primary state not only with left-moving generators $L_{-1}, G_{-1}^{+A}, J_{-1}^+$ but also with right-moving ones $\tilde{L}_{-1}, \tilde{G}_{-1}^{+A}, \tilde{J}_{-1}^+$. In [98, 99], this was carried out and certain non-supersymmetric microstate geometries were constructed. Furthermore, those geometries were extended to asymptotically-flat solutions by a matching technique. It is desirable to extend such non-supersymmetric superstrata to non-linear solutions; ideas used to construct five-dimensional non-supersymmetric microstates [100] may be useful for such extension. Just as in the supersymmetric fluctuation modes discussed in section 4.2.1, some particular non-supersymmetric modes are expected to be simpler and technically easier to construct than others [46, Appendix C].

By S -duality, the D1-D5 system is related to the F1-NS5 system, which upon T -duality becomes the P-NS5 system. In these duality frames, only NS-NS fields are turned on and an exact worldsheet CFT description of the background is possible. In [101–103], the relevant worldsheet CFT was constructed, and the spectrum of the fundamental string and possible D-branes were analyzed. This duality frame may be useful in studying the bulk realization of the fractional and higher modes that are crucial in understanding the microstates of the D1-D5-P (or the F1-NS5-P) black hole.

Precision holography

Studying the correlation function in a microstate and matching it between bulk geometries and CFT states is sometimes called “precision holography”. Precision holography was developed for 1/4-BPS microstates (Lunin-Mathur geometries) in [7, 62, 104] and extended to 1/8-BPS microstates (superstrata) in [37, 48]. According to the non-renormalization theorem in [105], correlation functions of type $\langle \mathcal{O}_{1/8} \mathcal{O}_{1/4} \mathcal{O}_{1/8} \rangle$ are not renormalized, where $\mathcal{O}_{1/8}$ and $\mathcal{O}_{1/4}$ are 1/8-BPS and 1/4-BPS operators. If $\mathcal{O}_{1/8} = H$ is a backreacted 1/8-BPS geometry (a heavy state with dimension of order N) and $\mathcal{O}_{1/4} = L$ is a light probe (a chiral primary operator with dimension of order one), then $\langle HLH \rangle = \langle H|L|H \rangle$ can be found by computing the one-

point function in the superstratum. In [48], the one-point function of dimension-one chiral primary operators was studied and, in particular, the existence of the term in Z_1 necessary for coiffuring low-frequency source was confirmed from CFT. The existence of this term was shown [48] to be consistent also with holographic entanglement entropy. In [37], the one-point function of dimension-two operators were studied and the existence of the $\mathcal{O}(b^2)$ term in Z_1 was shown to be due to mixing between single-particle operators and two-particle operators in the holographic dictionary. These results are quite intriguing, because the relevant terms in the supergravity ansatz originate from regularity, which requires information about the interior of spacetime, whereas CFT computation only involves operators of small dimension, which are related to deformations of the geometry near the boundary. This demonstrates the power of CFT in predicting non-trivial features of the dual geometry.

Computing the bulk two-point function of a light probe operator in a heavy backreacted geometry gives correlation function of type $\langle H|LL|H\rangle = \langle HLLH\rangle$.²⁷ Two-point functions decay in a black-hole background, because the bulk field gets absorbed into the horizon. It is interesting to see whether and how the correlation function in microstate geometries mimics such behavior. Although the two-point function in a microstate geometry is expected to decay at initial times, it should not go to zero and must show recurrence after a long but finite time. In [106, 107], the correlation function in 1/4-BPS geometries of the type given in section 3.8.2 was studied. The computation was mostly done in the “shallow” limit $b \ll a$, although in [107] an exact expression valid for any b was obtained for $k = 1$. For 1/8-BPS microstates, *i.e.* superstrata, two-point function was studied in [83, 87, 88]. Most of these works focused on the shallow limit but, in [89], the deep throat limit of the $(1, 0, n)$ superstratum was studied using a WKB technique, and it was found that two-point functions decay as in the BTZ black hole for $t \lesssim \sqrt{N_1 N_5} R_y$, while for $t \sim N_1 N_5 R_y$ large echoes are coming back from the cap. Because this is an atypical state, the echo is strong and the correlation function comes back to almost the original value. However, more general superstrata are expected to show less spiky behavior.

Other work on the holography of 1/8-BPS microstates includes [108, 109].

Further aspects of superstrata

Some supersymmetric microstate geometries have been argued to have non-linear instability [110–112], which suggests that they want to evolve into more typical microstates [113]. In [83], based on statistical-mechanical considerations, it was argued that microstate geometries that are classically distinguishable from the black-hole geometry are atypical. Superstrata on $\text{AdS}_3 \times S^3$ are not typical microstates of the three-charge black hole, because their entropy is parametrically smaller than the black-hole entropy [63].

²⁷This correlation function, being really a four-point function, is not protected.

However, these do *not* mean that those superstrata are irrelevant in studying the microscopic physics of black holes.

First, by studying their instability, superstrata must give us information about the nature of more typical microstates that they have tendency to evolve into. For the BTZ black hole, the tidal force felt by an object falling into it can be made small for a large black hole, even at the horizon. However, for a capped BTZ geometry such as superstrata with a deep throat, it was found that, generally, an infalling object experiences a Planckian tidal force [82, 86]. The tidal force is $|\mathcal{A}|_{\text{throat}} \sim a^2 Q_P^2 / (\sqrt{Q_1 Q_5} r^6)$, where a is the scale of the cap determined by the residual angular momentum, which vanishes in the BTZ limit, $a \rightarrow 0$. This large stress force comes from the deviation of the microstate geometry from BTZ geometry, amplified by the relativistic speed of the infalling particle. This means that a particle dropped into a superstratum with a deep throat gets stretched into a string and/or ripped into strings. By following the fate of the string(s), we must be able to get a hint as to the more typical microstates that the superstratum wants to evolve into. If the tidal force tend to transform the particle into a massive string by exciting string oscillator modes on it, that would mean that supergravity is not enough for describing microstates and stringy modes must be included. Instead, if the tidal force turns the particle into many massless strings, that would mean that supergravity is still good but we must consider more general geometries than the known superstrata.

Secondly, even if atypical, they do behave just as a black hole to certain probes; by studying response to such probes we can learn how the non-unitary behavior of black holes emerges from unitary behavior of microstates. For research in this direction see [83, 87–89, 106, 107] mentioned above.

Although superstrata naturally come with AdS_3 asymptotics, one can take a scaling limit of deep superstrata geometries, such as the $(1, 0, n)$ stratum, and find an asymptotically- AdS_2 superstratum [84]. Black-hole microstates with AdS_2 asymptotics are interesting, particularly because of the claim [114, 115] that black-hole microstates must have zero angular momentum in four dimensions ($J_R = 0$ in five dimensions) and fit in an AdS_2 region, and also because of the recent surge of interest in the near- AdS_2/SYK correspondence [116, 117] (for a review of the already large literature see *e.g.* [118]). One cannot have excitation in global AdS_2 [119, 120] because a finite excitation makes the dilaton diverge at the end of the space. However, the existence of a capped AdS_2 does not contradict with the no-go theorem, because the divergent dilaton is interpreted as the collapsing of an internal S_1 which caps off the geometry. In [84], it was found that the non-supersymmetric excitations at the bottom of the capped AdS_2 superstratum are normalizable with spectrum $\Delta E = 4J_R/(NR_y)$,²⁸ which for $J_R = 1/2$ reproduces the CFT expectation. However, it remains to be seen if these excitations preserve the AdS_2

²⁸ The spectrum in the $(1, 0, n)$ geometry was studied in [83] before.

asymptotics, when backreacted. For examples of five-dimensional microstate geometries with AdS_2 asymptotics and their relevance to pure-Higgs branch states [121] of the dual quiver quantum mechanics, see [122].

Further aspects of superstrata and microstate geometries studied in the literature include: scattering off microstate geometries [85, 123]; trailing string and drag force [124] in microstate geometries [90]; integrability of the superstratum backgrounds [72, 79]; ambi-polar hyper-Kähler space, pseudo-harmonic form, and prepotentials of five-dimensional microstate geometries [43, 71, 72].

6 Concluding remarks

In this article, we reviewed aspects of superstrata, a large family of microstate geometries of the D1-D5-P black hole. They can be constructed systematically using the linear structure of BPS equations and represent coherent states of 1/8-BPS supergravitons. They provide an ideal setup in which to study the physics of black holes. Although their holographic dictionary is well understood and has a deceptively simple structure, their bulk physics is surprisingly rich; for example, some superstrata have a long throat with a large redshift and a small gap, which is quite non-trivial from the CFT viewpoint.

As already mentioned, superstrata on $\text{AdS}_3 \times S^3$ involve, in CFT language, only rigid-generator descendants of chiral primary states, and their entropy is not enough to account for the entropy of the D1-D5-P black hole [63]. To reproduce the full entropy, it is crucial to understand the bulk realization of fractional and higher modes mentioned in section 2.3. Some of fractional modes are realized in superstrata on the orbifold $(\text{AdS}_3 \times S^3)/\mathbb{Z}_p$ as reviewed in section 4.4. However, these represent only a special kind of fractional mode on a limited class of chiral primary states; understanding of general fractional modes on general chiral primary states is still missing. Also, the bulk realization of general higher modes is poorly understood (see however [24, 25]). Although multi-center superstrata may correspond to such states [65], it is also possible that intrinsically stringy excitations are essential. In any case, this is one of the most important problems that have to be resolved in order to further our understanding of black-hole microstates.

Superstrata have helped deepen our understanding of black holes by their rich physical and mathematical content. Further investigation is bound to reveal more surprising aspects of superstrata and lead to better understanding of the microscopic working of black holes.

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