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by

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# Public Investment, National Debt, and Economic Growth: The Role of Debt Finance under Dynamic Inefficiency\*

Toshiki Tamai<sup>†</sup> and Akira Kamiguchi<sup>‡</sup>

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## Abstract

Public investment is one of central issues in the dynamic analyses on fiscal policy and economic growth. Debt financing for public investment and its effects has been recently focused because the interest rates has been low and almost remain below economic growth rates. This paper examines the impacts of debt-financed public investment subject to a simple fiscal rule in an overlapping generations model with public capital. This topic includes capital budgeting and the debt/deficit criterion of Maastricht treaty. We show that debt financing for public investment enhances economic growth if the economy is dynamically inefficient and public capital has sufficiently large productivity effect while it reduces economic growth rates in the dynamically efficient economy. Debt and growth could have a monotonic or non-monotonic relationship, depending on steady-state interest rate, growth rate, and productivity effect of public investment. The debt-growth relations match with controversial empirical evidences. Furthermore, existing generations choose perfect debt finance, so-called golden rule of public finance, if dynamic inefficiency exists. In contrast, balanced budget is preferred in a dynamically efficient economy with low productivity effect of public capital. However, an economy with high productivity effect of public capital may choose debt financing. This paper contributes to the understanding of the currently focused issues of public investment.

*Keywords:* National Debt; Public capital; Economic growth

*JEL classification:* H54; H60; O40

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# 1 Introduction

Public investment and its impact on economic performance has been long analyzed as one of core issues in the literatures on fiscal policy and economic growth. Arrow and Kurz (1970) contributed to construct a dynamic framework of analyzing public investment and provided its comprehensive analysis. Futagami et al. (1993) extended Arrow and Kurz’s sophisticated model by developing an endogenous growth model with public capital after Aschauer (1989) found an empirical evidence of the productivity effect of public capital. Empirical studies have supported the positive growth effect of public investment.<sup>1</sup> However, succeeding theoretical studies have shown that debt financing for public investment does not enhance in many cases.

IMF (2014, Ch.3) has recently found that public investment has short- and long-term positive growth effect and debt-financed public investment effectively increases output growth. This finding suggests that “*the time is right for an infrastructure push*”. Blanchard (2019) argued that public debt may have no fiscal costs because safe interest rates are expected to remain below growth rates, and suggested that welfare costs through reducing private capital accumulation may be smaller than typically assumed. The findings above is expressed as low fiscal costs due to low interest rates less than growth rates. Therefore, the controversial outcomes of growth effects of public investment by debt financing mainly arise from the evaluation of debt financing costs, such as interest payments.<sup>2</sup>

Public debt costs essentially concern dynamic efficiency issues. Diamond (1965) showed that private capital can be overaccumulated and public debt can improve efficiency, because debt issuance decreases oversaving. As the counterpart of Diamond’s model, Ihori (1978) and Tirole (1985) demonstrated that public debt or bubbles (non-productive assets) can improve efficiency in a dynamically inefficient economy. The dynamic inefficiency may occur in the overlapping generations models. Public debt is one of important policy instruments if the government has no other ways of solving dynamic inefficiency.

Dynamic inefficiency is characterized that interest rates are lower growth rates. Mishkin (1984) presented the evidence of this phenomenon using the data of seven OECD countries from 1967 to 1979. Von Weizsäcker (2014) recently suggested that real interest rate has become negative in the OECD countries and China. Abel et al. (1989) developed a criterion for dynamic efficiency in lieu of comparing interest rates and growth rates, and found that major OECD countries are dynamically efficient. Moreover, Geerolf (2018) revisited their results, updating data and adding countries. In contrast to Abel et al. (1989), the study clarifies that sufficient conditions for dynamic efficiency are not verified in advanced economies and a criterion for dynamic inefficiency is verified in Japan and South Korea. In sum, dynamic inefficiency is possible in practice.

Debt financing for public investment is related to capital budgeting.<sup>3</sup> Particularly, the budgetary regime is known as the “*golden rule*” of public finance (GRPF). Many countries legally adopted GRPF and its variations.<sup>4</sup> Musgrave (1939) pioneeringly developed the analysis of the budget rule. Debt-financed public investment influences intergenerational welfare through different long-term benefits and costs. Hence, debt financing may have an important role of improving social welfare. Indeed, Bassetto and Sargent (2006) showed that the GRPF can improve efficiency in an overlapping generations economy. Yet there is a necessity to consider a debt financing rule including the GRPF for identifying

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<sup>1</sup>See Bom and Ligthart (2014) for an excellent survey of the empirical analysis on the productivity effect of public capital and Agénor (2013) for a comprehensive analysis and survey of theoretical and empirical literature. Agénor and Neanidis (2015) estimated the output elasticity of public capital to be between 0.2 and 0.4 and concluded that the previous studies may have underestimated its value.

<sup>2</sup>Onori (2018) investigated the role of external debt based on Barro’s (1990) model. The study has shown that financial openness is beneficial in high productivity and low interest rate scenario.

<sup>3</sup>Poterba (1995) provided comparative analyses on the levels and composition of government spending in U.S. states by focusing on budgetary schemes, such as separate budgets for capital and operating expenditures and unified budgets. The empirical findings revealed that states with separate capital budgets spend more on public capital projects than states with unified budgets.

<sup>4</sup>For example, Brazil, Costa Rica, Germany, Japan, Luxembourg, Malaysia, and the United Kingdom. The GRPF incorporates the possibility of borrowing to finance productive public investment that can pay for itself over the long-term with a balanced current budget (IMF, 2014, Ch. 3).

growth effects of debt-financed public investment.<sup>5</sup>

In infrastructure-led growth models without overlapping generations structure, such as Futagami et al. (1993), dynamic inefficiency does not occur and the presence of debt negatively impacts economic performance regardless of the productivity effect of public capital. To address the possibility of dynamic inefficiency and debt financing for public investment, the overlapping generations model of endogenous growth with public capital must be developed. In the overlapping generations models, Yakita (2008) demonstrated the existence of an initial debt threshold for the sustainability of government budget deficits under the GRPF. In addition, Teles and Mussolini (2014) showed that the level of the debt-to-GDP ratio negatively impacts growth effect of fiscal policy. They focus on debt sustainability or the relationship between debt and economic growth.

The goal of this paper is to elucidate why debt financing for public investment is chosen. Thus, we develop an overlapping generations model of endogenous growth with public capital under the generalized version of the GRPF to examine growth effects of debt-financed public investment. Income tax and public bonds are used for financing public investment and interest payment of public bonds. The borrowing rule requires that issuing bonds is allowed for public investment only. Total amount of debt must not exceed public capital stock.

First, we show that public investment under debt finance positively impacts economic growth in two ways. Public investment financed by increased tax enhances economic growth through the productivity effect of public capital. In addition, public investment positively associates with public bond issuance under the borrowing rule. Then, public investment financed by tax increases debt and raises economic growth rate. Public investment financed by bonds may positively affect economic growth if the interest rate is below the growth rate. Increasing debt hampers private capital accumulation and negatively impacts economic growth. However, if the productivity effect of public capital is sufficiently large, an increase in the debt dependency ratio of public investment raises economic growth rate. In either case, excessive public investment diminishes productivity effects of public capital. Therefore, the growth-maximizing level of public investment is verified.

Second, we demonstrate that debt financing for public investment could be politically preferred by existing generations whether the economy is dynamically efficient or inefficient. The existing generations are unwilling to increase public investment for economic growth because far future benefits by economic growth cannot improve their welfare. Thus, they aim to reduce their current burdens. People depend on debt financing to maximize their lifetime income rather than economic growth rate and can ignore crowding-out effect on private capital accumulation. If the economy is dynamically inefficient, then existing generations choose the GRPF due to absence of debt costs. Even if the economy is dynamically efficient, debt financing for public investment is chosen with high productivity effect of public capital because the welfare cost is low. However, people prefer a balanced budget scheme when the productivity effect is not large.

Finally, numerical analyses provide further insights into public investment and debt financing. Under debt financing, public investment generates persistent economic fluctuations. In particular, the economic dynamics exhibits chaos with low total factor productivity. The equilibrium dynamics for high degree of debt dependency is complicated, regardless of total factor productivity. Computed results also show that the equilibrium tax rate and degree of debt financing are close to those in the real economy. Furthermore, an increase in debt-financed public investment enhances short- and long-term economic growth in a dynamically inefficient economy. These results are consistent with certain empirical evidence.

The remainder of this paper is organized as follows. Section 2 discusses the related literature. Section 3 describes the basic setup of our model and characterizes the stationary equilibrium and its transitional dynamics. Section 4 examines the growth effects of debt-financed public investment and considers an endogenous determination of income tax rate and borrowing ratio. Section 5 develops numerical simulations for additional analyses in Section 4. Finally, Section 6 concludes this paper by

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<sup>5</sup>Dombi and Dedák (2019) showed that an increase in debt-to-GDP ratio from 0 to 90 percent decreases the output by 2-3 per cent only on a steady state. They used Blanchard's (1985) model to show this relationship when public capital is not considered and government expenditure does not affect firm production.

noting possible extensions and issues for further research.

## 2 Related literature

Numerous studies have examined the relationship between public investment, public debt, and economic growth. The literature is divided into two categories based on research tasks. First category of the literature develops a dynamic analysis on the growth effects of public investment financed by bonds. Second category of the literature focuses on debt sustainability and fiscal rule. Both categories include theoretical and empirical approaches.

Theoretical analyses in the first category have predicted complicated results. Without any productive government expenditure, economic growth negatively associates with public debt (e.g., Saint-Paul 1992). In contrast, Greiner and Semmler (2000) incorporated public capital into their model and showed that increasing deficit-financed public investment results in a positive growth effect when the government adopts less strict budgetary regimes. They also demonstrated that deficit financing for public investment tends to produce a negative growth effect if interest payments are high. Ghosh and Mourmouras (2004) showed that under the GRPF, less strict budgetary stance lowers optimal level of public investment, interest rate, and growth rate.<sup>6</sup> Using numerical analysis, Minea and Villieu (2009) argued that the GRPF negatively impacts long-run economic growth compared with balanced-budget rules, even though it could improve intertemporal welfare.<sup>7</sup>

Empirical evidence in the first category is controversial to the theoretical predictions. Some empirical studies verified a non-monotonic relationship between economic growth and public debt. For instance, Checherita-Westphal and Rother (2012) found an inverted-U shaped relation between growth and debt-to-GDP ratio from panel data over approximately 40 years in 12 euro area countries. Eberhardt and Presbitero (2015) investigated the long-term relationship between debt and growth using a large panel data of 118 countries and showed that each country has different debt-growth interactions, including non-monotonic relation. They also demonstrated that finding a negative nonlinear debt-growth relationship is extremely difficult and sensitive to modeling choices and data coverage. Debt overhang effect does not arise from common specific debt thresholds.<sup>8</sup>

The second category of related literature includes studies on debt sustainability under various fiscal rules. For instance, the No Ponzi-game condition for government budget has been examined theoretically and empirically (e.g., Hamilton and Flavin 1986; Blanchard et al. 1990; Bohn 1998).<sup>9</sup> Maximum level of government debt has also been clarified in the overlapping generations model based on Diamond (1965). Chalk (2000) showed that current debt stock must not be extremely large for sustainable deficits and debt even if steady state interest rate is less than growth rate. Bräuning (2005) theoretically derived a maximum sustainable debt level in the overlapping generations model with AK production function. These studies considered debt sustainability for non-productive government expenditure under specific fiscal rules, such as a deficit and expenditure rules.

Certain studies in the second category treat productive government expenditure, including public investment. Yakita (2008) considered debt sustainability under three fiscal rules, namely, public investment rule, deficit rule, and allocation rule of tax revenue.<sup>10</sup> Given that the three fiscal rules

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<sup>6</sup>Several studies examined the growth effects of debt-financed public investment and optimal public investment policy in stochastic growth models (e.g., Tamai 2014, 2016). Growth effects depend on the degree of relative risk averse (Tamai 2014). The GRPF replicates optimal equilibrium in a decentralized economy (Tamai 2016).

<sup>7</sup>Greiner (2010) showed that economic growth rate under the balanced-budget rule coincides with that of under the GRPF if the growth rate of public debt is less than that of physical capital. Groneck (2011) demonstrated that the GRPF positively impacts long-run growth and welfare, and mentioned that positive growth effects were observed only if public consumption expenditures were lowered in the long run.

<sup>8</sup>Greiner (2013) and Ueshina (2018) theoretically showed the inverted U-shaped debt-growth relationship under GRPF. However, a non-monotonic relation between debt and growth has not yet been demonstrated under more general class of fiscal rules.

<sup>9</sup>Numerous studies have empirically investigated this issue based on Bohn's (1998) approach (e.g., Greiner and Kauermann 2007; Greiner et al. 2007; Doi et al. 2011).

<sup>10</sup>Arai (2011) examines the same issues of Yakita (2008) using the overlapping generations model with productive

simultaneously bind, tax rate depends on the debt-to-capital ratio as a predetermined variable. It generates multidimensional economic dynamics with private capital, public capital, and public debt. The existence of a stable steady state indicates a sustainable debt level. The effects of debt-financed public investment are also examined in the paper. Growth effect is ambiguous though the model includes the GRPF and productive effect of public investment. In principle, the GRPF is a sustainable fiscal scheme.<sup>11</sup> Hence, other fiscal rules may interfere the stabilizing function and growth effect of public capital.

Teles and Mussolini (2014) extended Bräuning's model by incorporating productive government expenditure as spending flow rather than public capital accumulation. Public expenditure in their model has three different growth effects. An increase in public expenditure positively impacts economic growth through its productivity effect. However, it also raises interest rate and increases tax rate determined endogenously to balance government budget under fiscal rules. Hence, these two indirect effects generate negative growth effect. Teles and Mussolini (2014) found that higher debt-to-GDP ratio decreases positive growth effect of productive public expenditure by empirical estimation based on their theoretical model. Their study implies that the impacts of debt costs on different generations are important to identify the growth effects of fiscal policy.

In our study, we consider the effects of debt-financed public investment on economic growth and equilibrium dynamics in the overlapping generations model. If government issues public debt for public capital investment, then the benefit accrues not only to current individuals but also to future ones. Furthermore, dynamic inefficiency has not been assumed in most studies on this literature. As mentioned in the previous section, the dynamic inefficiency would be the issue we really face. The overlapping generations model with the GRPF and its variations should be used to clearly capture the mechanism of debt financed public investment.

Recent studies in the literatures also examined a relationship between debt financing of public investment and economic growth under the GRPF in the overlapping generations model. Bokan et al. (2016) analyzed the effects of demographic change on economic growth and stability under the GRPF in Diamond's overlapping generations model with fertility choice and death uncertainty. They focused on *pure* the GRPF as the government borrows for public capital investment only. Hence, their study do not cover variations of GRPF as part of capital budgeting and the reason why they are chosen. Kamiguchi and Tamai (2019) examined the growth and welfare effect of fiscal policy under the GRPF in Blanchard's overlapping generations model. They showed that debt-to-GDP ratio and economic growth rate have an inverted-U shaped relationship and that different generations prefer different tax rates depending on their ages.

Heterogeneity of preferred policy is crucial to determining an equilibrium policy through a democratic process. With dynamic inefficiency, debt-financed public investment may have positive growth effect and be politically chosen as a preferable policy. However, the issue of dynamic inefficiency and how fiscal variables relate to fiscal rule determination are not covered in these studies. Our study takes a different tack to consider the growth effect of debt-financed public investment and fiscal rule determination, following the existing literature.

### 3 The model

Consider the overlapping generations model in which each individual lives for two periods. Time is discrete and time period is indexed by the subscript  $t$ . The individuals benefit from their first- and second-period consumptions. The lifetime utility function for an individual born at period  $t$  is

$$U_t = \log c_t^y + \theta \log c_{t+1}^o \quad (1)$$

where  $c_t^y$  is the consumption for the young in period- $t$  generation,  $c_{t+1}^o$  is the consumption for the old in period- $t$  generation, and  $\theta$  is the subjective discount rate ( $0 < \theta \leq 1$ ).

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government expenditure.

<sup>11</sup>See Musgrave (1939). Kamiguchi and Tamai (2019) proved that the GRPF satisfies the No-Ponzi game condition.

In the first period, the individuals are young and supply one unit of labor inelastically. They earn from work and spend their earnings on private consumptions and investment (saving). In the second period, they are old, retiring during the period, expend their savings with interests. The budget equations are

$$c_t^y = w_t - s_t, \quad (2a)$$

$$c_{t+1}^o = (1 + r_{t+1}) s_t, \quad (2b)$$

where  $w_t$  is the post-tax wage rate,  $r_{t+1}$  is the post-tax interest rate and  $s_t$  is the saving.

Each individual chooses consumption levels to maximize the utility function (1) subject to (2a) and (2b) for given  $w_t$  and  $r_{t+1}$ . Solving the optimization problem, we obtain the following saving function:

$$s_t = \beta w_t, \quad (3)$$

where

$$\beta \equiv \frac{\theta}{1 + \theta}.$$

Each firm has an identical production technology. The production function is

$$y_t = A k_t^\alpha g_t^{1-\alpha} l_t^{1-\alpha}, \quad (4)$$

where  $y_t$  is the output,  $A$  is the productivity parameter,  $\alpha$  is the intensity parameter ( $0 < \alpha < 1$ ),  $k_t$  is the private capital,  $g_t$  is the public capital, and  $l_t$  is the labor. We assume that public capital is labor-augmenting and taken as given for each firm. Profit maximization conditions lead to

$$r_t = (1 - \tau) \alpha A k_t^{\alpha-1} g_t^{1-\alpha} l_t^{1-\alpha} - \delta, \quad (5a)$$

$$w_t = (1 - \tau) (1 - \alpha) A k_t^\alpha g_t^{1-\alpha} l_t^{-\alpha}, \quad (5b)$$

where  $\delta$  denotes capital depreciation rate ( $0 \leq \delta \leq 1$ ).

The government issues public bonds and taxes the output to finance government expenditures for interest payments and public investment. The government's budget equation becomes

$$b_{t+1} = (1 + r_t) b_t + [g_{t+1} - (1 - \delta) g_t] - \tau y_t, \quad (6)$$

where  $b_t$  is the government's debt outstanding. Note that the second term in the right-hand side of Equation (6) stands for the investment in public capital.

This economy has two types of assets: private capital and public bonds. The clearing condition of asset market is

$$s_t = b_{t+1} + k_{t+1}. \quad (7)$$

Using Equations (2a), (2b), (5a), (5b), (6), and (7), Walras' law holds.<sup>12</sup>

Suppose that the amount of bond issuance is less than public investment. This borrowing rule has been adopted in numerous countries and regions (e.g., Germany, Japan, the United Kingdom, and some states in the United States). This condition can be formalized as

$$b_{t+1} = \eta g_{t+1}, \quad (8)$$

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<sup>12</sup>Walras' law can be verified as

$$\begin{aligned} y_t &= c_t^y + c_t^o + [k_{t+1} - (1 - \delta) k_t] + [g_{t+1} - (1 - \delta) g_t] \\ &= (1 - \tau) w_t l_t - b_{t+1} - k_{t+1} + (b_t + k_t) (1 + r_t) + k_{t+1} - (1 - \delta) k_t + b_{t+1} - (1 + r_t) b_t + \tau y_t \\ &= w_t l_t + (r_t + \delta) k_t \equiv y_t. \end{aligned}$$

where  $\eta \in [0, 1]$ .  $(\eta, b_0) = (0, 0)$  is referred as a balanced-budget scheme, whereas  $\eta = 1$  is known as the GRPF. Using Equations (4) and (8), the debt rule can be rewritten as

$$\frac{b_t}{y_t} = \frac{\eta x_t^\alpha}{A}, \quad (9a)$$

$$\frac{b_{t+1} - b_t}{y_t} = \left( \frac{b_{t+1} - b_t}{b_t} \right) \frac{b_t}{y_t}, \quad (9b)$$

where  $x_t \equiv g_t/k_t$ . For given private and public capital stocks, Equations (9a) and (9b) lead to the upper bound of the debt-to-GDP ratio and deficit-to-GDP ratios, respectively. Hence, the debt rule corresponds to the Maastricht criteria for the debt outstanding and fiscal deficit. Note that the criterion levels are endogenously determined. Therefore, the variation occurs unless the ratio of public to private capital is constant over time.

For  $0 \leq \eta < 1$ , Equations (3), (6), (7), and (8) lead to

$$g_{t+1} = \frac{[\tau - (1 - \tau) \alpha \eta x_t] y_t + (1 - \delta) (1 - \eta) g_t}{1 - \eta}, \quad (10a)$$

$$k_{t+1} = \beta w_t - b_{t+1} = (1 - \alpha) (1 - \tau) \beta y_t - \eta g_{t+1}, \quad (10b)$$

For  $\eta = 1$ , we have  $(r_t + \delta) b_t = (r_t + \delta) g_t = \tau y_t$ . Then, public-to-private capital ratio under GRPF becomes

$$x_t = \frac{\tau}{(1 - \tau) \alpha} \equiv x^G \text{ for } \eta = 1.$$

Equations (10a) and (10b) yield

$$x_{t+1} = \frac{[\tau - (1 - \tau) \alpha \eta x_t] A + (1 - \delta) (1 - \eta) x_t^\alpha}{(1 - \alpha) (1 - \eta) (1 - \tau) \beta A - \eta \{[\tau - (1 - \tau) \alpha \eta x_t] A + (1 - \delta) (1 - \eta) x_t^\alpha\}} \equiv \psi(x_t), \quad (11)$$

where

$$\psi(0) = \frac{\tau}{(1 - \alpha) (1 - \eta) (1 - \tau) \beta - \eta \tau} \geq 0, \psi(\bar{x}) = 0.$$

The dynamic system of Equation (11) is derived from Equations (2a)–(8). Therefore, the time sequence of  $x_t$  given by Equation (11) characterizes economic dynamics with all equilibrium conditions.

The equilibrium growth path is defined as follows:

**Definition 1.** *Equilibrium growth path is a sequence of  $x_t$ ,  $\{x_t\}_{t=0}^\infty$ , which satisfies (11).*

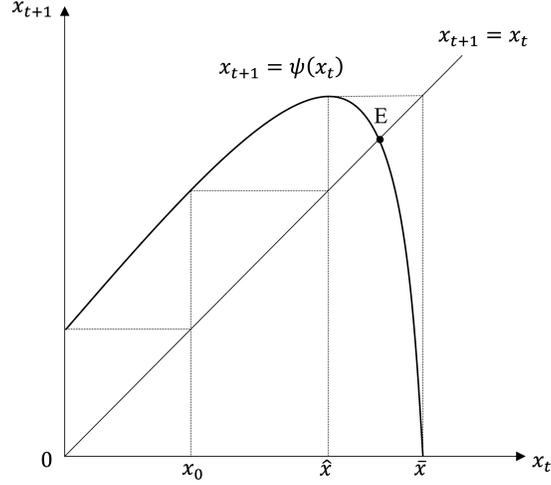
There may exist multiple equilibrium growth paths with different initial values and parameters. In the long-run, the equilibrium growth paths may converge to certain stationary paths with a stationary value of  $x_t$ . Formally, we define the stationary equilibrium as follows:

**Definition 2.** *Stationary equilibrium is a set of  $x_t$ , which satisfies (11) and  $x_{t+1} = x_t$  for  $t \in \mathbb{N}_0$ .*

We now examine the dynamic properties of Equation (11). Differentiation of Equation (11) with respect to  $x_t$  provides

$$\begin{aligned} \psi'(x_t) &= \frac{[(1 - \delta) (1 - \eta) \alpha x_t^{\alpha-1} - (1 - \tau) \alpha \eta A] [1 + \eta \psi(x_t)]}{(1 - \alpha) (1 - \eta) (1 - \tau) \beta A - \eta \{[\tau - (1 - \tau) \alpha \eta x_t] A + (1 - \delta) (1 - \eta) x_t^\alpha\}} \stackrel{>}{<} 0 \\ &\Leftrightarrow x_t \stackrel{>}{<} \hat{x} \equiv \left[ \frac{(1 - \delta) (1 - \eta)}{(1 - \tau) \eta A} \right]^{\frac{1}{1-\alpha}}. \end{aligned}$$

The properties of  $\psi$  function shows that the graph of  $\psi$  in the  $x$ - $\psi$  plane is continuous for  $[0, \bar{x}]$ . Figure 1 illustrates  $\psi$  curve. It displays that there exists at least one stationary equilibrium. Depending on the location of the crossing point between  $\psi$  curve and a 45-degree line, the economy traces the different equilibrium growth paths.



**Figure 1.**  $\psi$  curve and stationary equilibrium

If the stationary equilibrium is located in the left-hand side of  $\hat{x}$ , then  $\psi$  curve has a positive and less than unity slope at the stationary equilibrium. Hence, the economy monotonically converges to the unique stationary equilibrium. For instance, the balanced budget ( $\eta = 0$ ) leads to such a situation. However, deficit financing may cause different scenarios. If the stationary value of public-to-private capital ratio,  $x$ , is larger than  $\hat{x}$ , then  $\psi$  curve has a negative slope, resulting in an oscillation. Oscillating convergence and periodic cycles are conceivable. Figure 1 illustrates  $\psi$  curve with period-3 cycle, which is the source of Li-Yorke chaos. Then, the economy will not converge to stationary point E and persistently fluctuates.

Under a balanced budget scheme, the economy monotonically converges to the unique stationary equilibrium. In contrast, the economy with deficit financing has economic fluctuations. The dynamics may exhibit oscillatory convergence or complicated periodic cycles. Thus, we have the following proposition (See Appendix A for the proof of Proposition 1):

**Proposition 1.** *There exists a unique stationary equilibrium. If  $\eta$  is sufficiently small, then equilibrium growth paths converges to the unique stationary equilibrium. In contrast, when  $\eta$  is insufficiently small to be  $|\psi'(x)| < 1$ , equilibrium growth paths may not reach the unique stationary equilibrium, resulting in economic fluctuations. In particular, equilibrium growth paths exhibit a chaotic behavior if*

$$\eta < \frac{(1-\tau)(1-\alpha)\beta}{(1-\tau)(1-\alpha)\beta + (1-\tau)\alpha + \tau} \text{ and } \left[ \frac{(1-\delta)(1-\eta)}{(1-\tau)\eta A} \right]^{\frac{1}{1-\alpha}} > \frac{\tau}{(1-\alpha)(1-\eta)(1-\tau)\beta - \eta\tau}.$$

If the economy converges to the unique stationary equilibrium, then Equations (10a) and (10b) with the stationary value of  $x_t$  lead to the equilibrium growth factor as follows:

$$\Gamma \equiv \frac{g_{t+1}}{g_t} = \frac{k_{t+1}}{k_t} = \frac{(1-\alpha)(1-\tau)\beta A x^{1-\alpha}}{1 + \eta x}, \quad (12)$$

where  $x$  stands for the value that  $x = \psi(x)$  holds. Growth rate is defined as  $\gamma \equiv \Gamma - 1$ . Focusing on a positive growth rate, we impose  $\Gamma \geq 1$ . Under a balanced budget scheme,  $\gamma^B$  is defined as the equilibrium growth rate with  $x^B \equiv \{x | x = \psi(x), \eta = 0\}$ . For the GRPF,  $\gamma^G$  is defined as the equilibrium growth rate with  $x^G$ . Similarly, all other variables with superscripts  $B$  and  $G$  are used for the economic variables under the balanced budget and GRPF, respectively.

## 4 Fiscal policy and economic growth

This section examines the growth effects of deficit-financed fiscal policy and the long-term relationship between economic growth and government debt. We focus on a unique stable stationary equilibrium to analyze the long-run effect. The government has two policy instruments: output tax rate  $\tau$  and debt-to-public capital ratio (degree of debt financing)  $\eta$ . We will show that these fiscal parameters affect economic growth and debt-to-GDP ratio through different channels.

### 4.1 Growth effects of fiscal policy

We investigate the growth effects of changes in output tax rate and degree of debt financing. Our model has two benchmarks to evaluate the growth effects of the fiscal policy: balanced budget ( $\eta = 0$ ) and the GRPF ( $\eta = 1$ ). The effects of fiscal policy on the accumulation of the two capitals should be considered to analyze growth effects. Using  $x = \psi(x)$  and Equation (11), we obtain the following equations (See Appendix B for the derivation and properties of Equations (13a) and (13b)):

$$\frac{\partial x}{\partial \tau} = \frac{(1 + \alpha\eta x)A + [(1 - \alpha)(1 - \eta)\beta A + (1 + \alpha\eta x)\eta A]x}{[1 - \psi'(x)]\Psi} > 0, \quad (13a)$$

$$\frac{\partial x}{\partial \eta} = \frac{[(1 - \eta)(1 + \gamma)x - (1 + \eta x)(r - \gamma)]x^\alpha}{[1 - \psi'(x)]\Psi} \geq 0, \quad (13b)$$

where  $\Psi \equiv (1 - \alpha)(1 - \eta)(1 - \tau)\beta A - \eta\{\tau - (1 - \tau)\alpha\eta x\}A + (1 - \delta)(1 - \eta)x^\alpha$ .

Equation (13a) shows that a rise in tax rate increases public-to-private capital ratio. Increased tax promotes public capital investment by increased tax revenue and decreased interest payment, whereas it hinders private capital accumulation through decreased saving. Therefore, high tax rate leads to high public-to-private capital ratio. Equation (13b) indicates that debt financing has an ambiguous effect on public-to-private capital ratio. For given stock levels of the two capitals, a rise in  $\eta$  increases interest payment (negatively influences  $x$ ) and public investment (positively affects  $x$ ). The increase in interest payment is the indirect (negative) effect of a rise in  $\eta$  on  $x$ , and the increase in public investment is the direct (positive) effect of a rise in  $\eta$  on  $x$ . Depending on the two opposite effects, debt financing enhances or declines public capital accumulation. On the other hand, debt financing negatively affects private capital accumulation through the crowding out effect. This effect of debt financing works as another indirect (negative) effect of a rise in  $\eta$  on  $x$ . Thus, debt financing has an ambiguous effect on  $x$ .

The positive direct effect of  $\eta$  on  $x$  may dominate over the negative indirect effects depending on the difference between interest and growth rates. If interest rate is less than economic growth rate ( $r < \gamma$ , i.e., the economy is dynamically inefficient), then the government obtains financial resources to increase net public investment per public capital by debt financing because the increase in government revenue passes increase in interest payment. The direct effect of a rise in  $\eta$  on  $x$  overweighs the indirect effects. Therefore, additional debt financing increases public-to-private capital ratio. However, this positive effect scenario may not be realized in a dynamically efficient economy ( $r > \gamma$ ). At least, a rise in  $\eta$  negatively impacts  $x$  for large  $\eta$ .

We consider the growth effects of a change in  $\tau$  for given  $\eta$ . Equations (12) and (13a) yield

$$\frac{\tau}{\Gamma} \frac{\partial \Gamma}{\partial \tau} = -\frac{\tau}{1 - \tau} + (1 - \alpha) \frac{\tau}{x} \frac{\partial x}{\partial \tau} - \frac{\eta x}{1 + \eta x} \frac{\tau}{x} \frac{\partial x}{\partial \tau}. \quad (14)$$

The first and second terms in Equation (14) denote a growth effect of a rise in tax rate through a change in savings. The last term in the equation represents for a negative growth effect of increased tax through the crowding out effect. The crowding out effect occurs because an increase in public investment by increased tax revenue positively associates with public debt under a debt financing scheme of Equation (8). A change in tax rate has direct negative and indirect positive effect on private capital accumulation through a decrease in disposable income and an increase in public capital

accumulation, respectively. Private and public capital accumulation are sources of economic growth. Therefore, the growth-maximizing tax rate exists in the range  $(0, 1)$ .

The analysis on Equation (14) provides the following proposition (See Appendix C for the proof of Proposition 2):

**Proposition 2.** *Suppose that  $\eta$  is fixed. A growth-maximizing tax rate,  $\tau^* \in (0, 1)$ , exists.*

Proposition 2 ensures the existence of a growth-maximizing tax rate. However, it does not display its uniqueness and exact value. Nevertheless, we can verify the values in certain special cases. For instance,  $\tau^* = 1 - \alpha$  holds when  $\eta = 0$ . Thus, the growth-maximizing tax rate under a balanced budget is equal to the output elasticity of public capital (e.g., Futagami et al. 1993).<sup>13</sup> In contrast, the growth-maximizing tax rate may differ from the output elasticity of public capital for  $\eta > 0$ . There are certain channels to be  $\tau^* < 1 - \alpha$ .<sup>14</sup> Issuing bonds ease the needs for tax financing, which has negative growth effect, through raising government revenue. On the other hand, debt financing brings about the crowding out effect. The mixed effects reduce the growth-maximizing tax rate. For example,  $\tau^* < 1 - \alpha$  if  $\delta = 1$ , which reveals that the growth-maximizing tax rate is less than the output elasticity of public capital when capital is completely depreciated in one period.

We move to the analysis of the growth effect of debt financing. Differentiating Equation (12) with respect to  $\eta$  derives

$$\frac{\eta}{\Gamma} \frac{\partial \Gamma}{\partial \eta} = (1 - \alpha) \frac{\eta}{x} \frac{\partial x}{\partial \eta} - \left( 1 + \frac{\eta}{x} \frac{\partial x}{\partial \eta} \right) \frac{\eta x}{1 + \eta x}. \quad (15)$$

The first and second term in Equation (15) correspond to the growth effect of debt financing through saving and crowding out effect, respectively. The effect of a change in  $\eta$  on  $x$ , (13b), is a key determinant of the growth effect of a change in  $\eta$ .

In a dynamically efficient economy ( $r > \gamma$ ), a rise in  $\eta$  may decrease public-to-private capital ratio or have a weakly positive effect on  $x$ . Small  $x$  leads to low wage rate, indicating small saving. Hence, a rise in  $\eta$  engenders totally negative growth effects through saving and crowding out effect. On the other hand, in a dynamically inefficient economy ( $r < \gamma$ ) with sufficiently large productivity effect of public capital, the result supports debt financing to enhance economic growth. The economic intuition behind the result is explained as follows. Public-to-private capital ratio increases with a rise in  $\eta$ . The growth effect through saving positively associates with public-to-private capital ratio. Hence, for small  $\eta$ , the positive growth effect of increasing saving dominates the negative growth effect of crowding out. An analysis of Equation (15) provides the following result (See Appendix D for the proof of Proposition 3):

**Proposition 3.** *Suppose that  $\tau$  is fixed. (i) In a dynamically inefficient equilibrium, there exists a growth-maximizing ratio of debt finance  $\eta^* \in (0, 1)$  if*

$$x^B < \frac{(1 - \alpha)(\gamma^B - r^B)}{(\gamma^B + \delta)\alpha} \text{ and } \frac{(1 - \alpha)(\gamma^G - r^G)}{\gamma^G + \delta - (1 - \alpha)(\gamma^G - r^G)} < x^G.$$

(ii) *If a dynamically efficient equilibrium, further debt financing for public investment decreases the equilibrium growth rate.*

Minea and Villieu (2009) and Kamiguchi and Tamai (2019) implied that the equilibrium growth rate under the GRPF is below to the growth rate under a balanced budget. We generalize the result of these previous studies. In the stationary equilibrium, Equation (6) becomes

$$(\gamma - r)\eta = \gamma - \left( \tau \frac{y}{g} - \delta \right) = \gamma - \zeta \stackrel{\geq}{\leq} 0 \Leftrightarrow \gamma \stackrel{\geq}{\leq} r,$$

<sup>13</sup>Barro (1990) demonstrated that the growth-maximizing tax rate is the output elasticity of productive government expenditure as public input.

<sup>14</sup>In Blanchard's overlapping generations model, Kamiguchi and Tamai (2019) showed that the growth-maximizing tax rate is less than the output elasticity of public capital because of the presence of generation replacement effect.

where  $\zeta$  is the tax revenue minus capital depreciation rate ( $\zeta \equiv \tau Ax^{-\alpha} - \delta$ ). Note that  $\zeta$  is the net public investment and growth rate under the balanced budget for  $\eta = 0$  because  $\zeta = \gamma^B$  holds. The models in the previous studies have no dynamic inefficiency. Similarly, we have  $\gamma < \zeta$ . The equilibrium growth rate under debt financing is less than the growth rate under a balanced budget (Proposition 3 (i)). In contrast,  $\gamma > \zeta$  holds. The equilibrium growth rate under debt-financing may be greater than the growth rate under a balanced budget (Proposition 3 (ii)).

IMF (2014, Ch.3) found evidence from advanced and developing economies suggesting that debt-financed public investment would have short- and long-term positive growth effects. Furthermore, some studies have argued in favor of public debt or dynamic inefficiency because many countries have saving gluts. Von Weizsäcker (2014) implied that the real rate of interest has become negative in the OECD countries and China. Geerolf (2018) found that the criterion for dynamic efficiency is unverified for any advanced countries. These findings support the second result in Proposition 3. However, strong productivity effect is necessary to ensure a positive growth effect of debt-financed public investment even if the growth rate exceeds the interest rate.

Checherita-Westphal and Rother (2012) empirically found an inverted-U shaped relationship between growth and debt-to-GDP ratio. It is worthy to investigate whether such relation exists or not. Under the debt financing rule (8), the debt-to-GDP ratio positively associates with public-to-private capital ratio. Given that Equation (13a) holds, debt-to-GDP ratio increases with output tax rate in the stationary equilibrium. By Equation (9a) and Proposition 2, if  $\eta$  is fixed, then an inverted-U shaped relationship exists between economic growth and debt-to-GDP ratio, in case of increasing the output tax rate. Several studies also showed the inverted-U shaped curve (e.g., Greiner 2010; Ueshina 2018; Kamiguchi and Tamai 2019). However, their results are limited to the GRPF. It corresponds to (8) with  $\eta = 1$ . Our result ensures the robustness of a non-monotonic relationship between economic growth and debt-to-GDP ratio through increasing the output tax rate. We can directly analyze the relationship between economic growth and debt financing.

Using Equation (9a), the response of the debt-to-GDP ratio to a change in  $\eta$  is

$$\eta \frac{\partial}{\partial \eta} \left( \log \frac{b}{y} \right) = 1 + \alpha \frac{\eta}{x} \frac{\partial x}{\partial \eta} > 0. \quad (16)$$

If  $\partial x / \partial \eta > 0$ , then Equation (16) has a positive sign. Equations (13b) and (16) and Proposition 3 yield the following result:

**Proposition 4.** *Suppose  $\tau$  is fixed. (i) In a dynamically inefficient equilibrium, there exists (may not exist) an inverted-U shaped relationship between economic growth and debt-to-GDP ratio in case of increasing the degree of deficit financing if the marginal effect of debt financing on public-to-private capital ratio is (not) sufficiently large. (ii) In a dynamically efficient equilibrium, the economic growth is negatively associated with the debt-to-GDP ratio in accordance with increasing the degree of deficit financing.*

Bokan et al. (2016) derived the growth-maximizing debt-to-GDP ratio under the GRPF in Diamond's overlapping generations model. When  $\eta = 1$ , we have a similar result by controlling the tax rate for maximizing equilibrium growth rate. Furthermore, Proposition 4 implies that the GRPF is not the only contributing factor to the inverted-U shaped curve between debt and growth. This proposition generalizes the results derived by the previous studies in the sense that the degree of debt financing is flexible. Depending on the magnitude relationship between steady state interest rate and growth rate, the debt-to-GDP ratio is differently associated with growth rate. A non-monotonic relationship between debt and growth will be observed if the economy is dynamically inefficient or when the government adopts the GRPF. Reinhart and Rogoff (2010), Checherita-Westphal and Rother (2012), and Baum et al. (2013) found an inverted-U shaped relationship between debt and growth.

On the other hand, certain studies have produced evidence against this relationship. Herndon et al. (2014) refuted Reinhart and Rogoff's (2010) result by emphasizing calculation problems. Égert (2015) demonstrated that finding a negative nonlinear relationship between debt-to-GDP ratio and economic

growth is extremely difficult and is sensitive to modeling choices and data coverage. Furthermore, Eberhardt and Presbitero (2015) found that each country has different debt-growth interactions using a large panel data of 118 countries. The debt overhang effect does not arise from common specific debt thresholds. Proposition 4 implies that the countries have different levels of debt overhang point on debt-to-GDP ratio depending on the tax rate. Whether each country is dynamically efficient or inefficient is important to investigate the public debt-growth nexus. Thus, our results theoretically show the difficulty of identifying the nonlinear relationship suggested by Égert (2015) and also provide a theoretical mechanism behind the empirical evidence found by Eberhardt and Presbitero (2015).

## 4.2 Political determination of fiscal policy

The previous subsection have examined the growth effects of fiscal policy and shown that deficit-financing has a positive growth effect in a dynamically inefficient economy. Future generations, including currently young generation, will benefit from the positive growth effect of debt financing and productivity effect of public capital. On the other hand, the currently old generation does not gain through public investment. This subsection considers whether debt financing is politically chosen or not.

For analytical tractability,  $\delta = 1$  is assumed in this subsection. Then, Equations (5a) and (12) lead to

$$\gamma - r = [(1 - \alpha)\beta - \alpha] (1 - \tau) Ax^{1-\alpha} \gtrless 0 \Leftrightarrow (1 - \alpha)\beta \gtrless \alpha \text{ for } \delta = 1.$$

Therefore, the economy is dynamically efficient ( $\gamma < r$ ) if and only if  $(1 - \alpha)\beta < \alpha$  while it is dynamically inefficient ( $\gamma > r$ ) if and only if  $(1 - \alpha)\beta > \alpha$ .

The government's objective function is based on existing individuals. Specifically, the objective function at period  $t$  is given by

$$\begin{aligned} W_t &= \theta \log c_t^o + \mu [\log c_t^y + \theta \log c_{t+1}^o] \\ &\simeq \theta \log(1 - \tau) + \mu \{(1 + \theta) \log(1 - \tau) + \theta [\log(1 - \tau) + (1 - \alpha) \log x]\}. \end{aligned} \quad (17)$$

Equation (17) can be interpreted as a probabilistic voting model (Lindbeck and Weibull, 1987; Grossman and Helpman, 1998). The first-order derivatives of Equation (17) are

$$\frac{\partial W_t}{\partial \tau} = -\frac{\theta + \mu(1 + 2\theta)}{1 - \tau} + \frac{(1 - \alpha)\mu\theta}{x} \frac{\partial x}{\partial \tau}, \quad (18a)$$

$$\frac{\partial W_t}{\partial \eta} = \frac{(1 - \alpha)\mu\theta}{x} \frac{\partial x}{\partial \eta}. \quad (18b)$$

A rise in tax rate directly reduces  $W_t$  through a decrease in disposable income of the old and young generation and indirectly raises  $W_t$  through an increase in future disposable income of the young generation by public investment. Therefore, from Equation (18a), we can find the tax rate that maximizes the value of the government's objective function (17) for  $\tau \in (0, 1)$ . We have the following result (See Appendix E for the proof of Proposition 5):

**Proposition 5.** *Suppose that  $\eta$  is fixed. An equilibrium tax rate exists,  $\tau^* \in (0, 1)$ , in the politico economy.*

Proposition 5 shows the existence of the equilibrium tax rate but does not ensure its uniqueness and provides any information about an absolute level. The equilibrium tax rate is not necessarily equal to the growth-maximizing tax rate.

Debt-financed public investment is relevant to the young generation's future disposable income only. Hence, the sign of Equation (18b) determines that of Equation (13b). From Equations (13b) and (18b), the following proposition holds (See Appendix F for the proof of Proposition 6):

**Proposition 6.** *Suppose that  $\tau$  is fixed. (i) In a dynamically inefficient economy, the GRPF,  $\eta = 1$ , is chosen. (ii) In a dynamically efficient economy, a balanced budget scheme is preferable if the income tax rate is sufficiently small. In contrast, debt financing is preferable and the equilibrium degree of debt-financing,  $\eta^*$ , exists in the dynamically efficient economy if the income tax rate is sufficiently high.*

The presence of debt in a dynamically inefficient economy could improve welfare as shown in Diamond (1965). In our model, the debt is used for financing productive public investment. Naturally, the GRPF is preferred. Surprisingly, debt financing could be chosen in a dynamically efficient economy. Sufficiently large productivity effect of public capital ensures low costs of debt financing.

Considering the future generations who do not exist currently will change the outcome concerning the equilibrium rule. Future generations' benefit depends on the economic growth rate because it can be evaluated by the present value of their utility functions. The presence of a negative growth effect of debt financing will decrease the equilibrium degree of debt financing. Therefore, government tend to avoid debt financing if concerned future generations exist.

## 5 Numerical simulation

This section illustrates a numerical simulation for three issues. First, we numerically verify the existence of chaos or economic cycles and a sufficient condition for stable stationary equilibrium to support Proposition 1. Second, the income tax rate and debt financing ratio will be calculated from a system composed of first-order conditions for maximizing the equilibrium growth rate or the government's objective function. Finally, we examine the short-term effects of fiscal policy to provide the further policy insights.

### 5.1 Baseline parameters

We assume that private capital share is  $\alpha = 0.3$  from Mankiw et al. (1992) for numerical simulations.<sup>15</sup> The preference parameter is set to  $\theta = 0.95$ . Then, the value of  $\beta$  approximates 0.483. Based on the parameters we assumed, the saving rate to national income becomes 33.8%. Our calculations from OECD data indicate the 5-year average rates in G5 countries: 49.1%, 50%, 47%, 36.2%, and 35.6% in France, Germany, Japan, the United Kingdom, and the United States, respectively.<sup>16</sup>  $(1 - \alpha)\beta$  is important to determine whether the economy is dynamically efficient or not. Hence, our baseline parameter of  $\beta$  is set below the real values.

$\tau = 0.2$  is used from OECD data. For instance, the 5-year average tax revenue excepts for social security contributions (% of GDP) in OECD average indicates 24.9%. Calculations show 29%, 23.3%, 18.2%, 26.5%, and 19.7% in France, Germany, Japan, the United Kingdom, and the United States, respectively.<sup>17</sup> Furthermore, the different values of  $\eta$  ( $\eta = 0.2$ ,  $\eta = 0.3$ , and  $\eta = 0.4$ ) are tested. In practice, a typical infrastructure has a 50-year lifetime. In the overlapping generations model, one period corresponds to 25-30 years. The realistic value of  $\delta$  should be set to 0.5 or above.

Based on Society at Glance 2016 (OECD), OECD average ratio of the voter turnout rate for age 18-24 to age 25-50 is 0.835. The ratios of the voter turnout rate for age 16-35 to age over 55 are 0.761, 0.787, 0.535, and 0.720 in France, Germany, the United Kingdom, and the United States, respectively. A survey report on 2017 House of Representatives general election in Japan shows that the ratio of age 20-55 to age over 55 in Japan is 0.730. Hence, we will use  $\mu = 0.5$ ,  $\mu = 0.75$ , and  $\mu = 1$ .

<sup>15</sup>The output elasticity of public capital has been reported in the range [0.1, 0.4]. Bom and Ligthart (2014) found that the average output elasticity of public capital in 68 studies is 0.106 after correcting for publication bias and its long-run value goes up to 0.193.

<sup>16</sup>These values are calculated from OECD data: Household Spending (doi: 10.1787/b5f46047-en, Accessed on 24 May 2020) and Gross National Income (doi: 10.1787/8a36773a-en, Accessed on 24 May 2020).

<sup>17</sup>Calculations are made from OECD data of Tax Revenue (doi: 10.1787/d98b8cf5-en, Accessed on 24 May 2020).

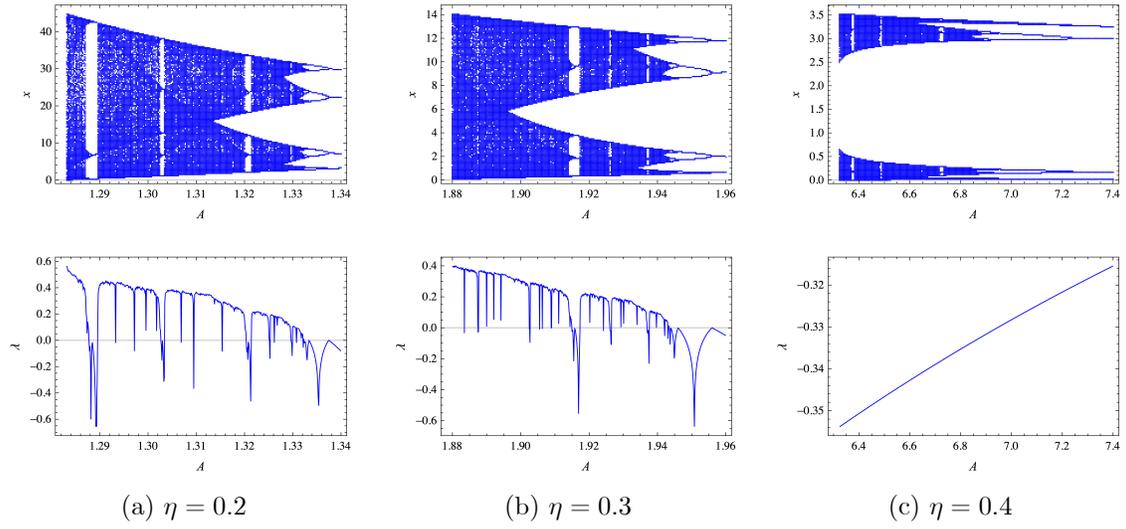


Figure 2. Bifurcation diagrams and Lyapunov exponent ( $\delta = 0$ )

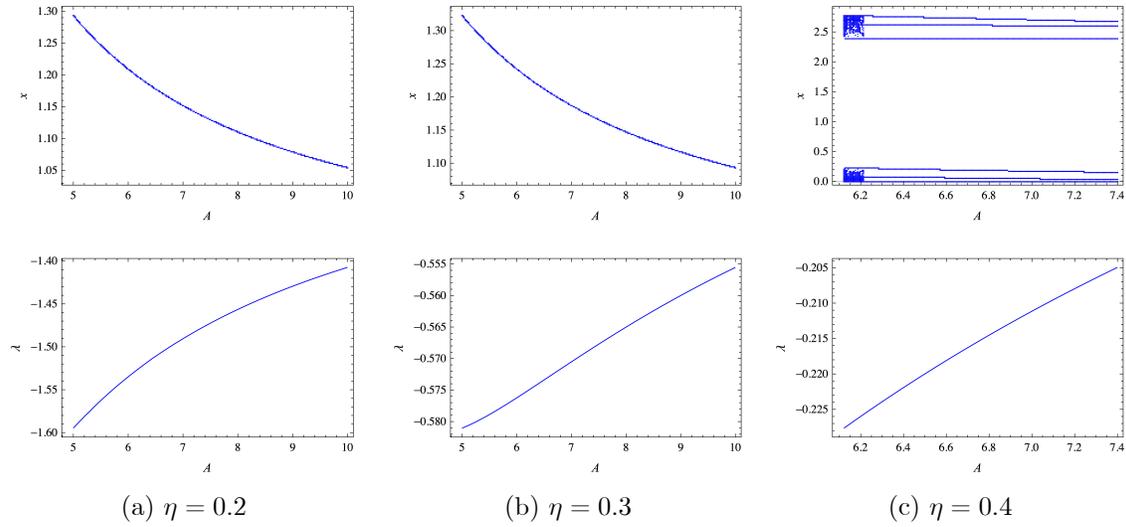


Figure 3. Bifurcation diagrams and Lyapunov exponent ( $\delta = 0.5$ )

## 5.2 Equilibrium dynamics

Proposition 1 shows that a small value of  $\delta$  raises the possibility of chaos. Hence, we test two cases:  $\delta = 0$  (as an extreme case) and  $\delta = 0.5$  (as a realistic case).<sup>18</sup> Our computations illustrate a bifurcation diagram to show whether numerous cycles exist. The values of  $x$  after 10000 steps are plotted in the diagrams for the different ranges of  $A$  with different widths. Furthermore, using values of  $x$ , we calculate Lyapunov exponent such that

$$\lambda = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \log |\psi'(x_t)|.$$

If chaos exists, then the Lyapunov exponent is positive. Hence, it is helpful to distinguish between chaotic and non-chaotic regime.

*No depreciation ( $\delta = 0$ ).* Figure 2 displays bifurcation diagrams with the graphs of the Lyapunov exponent for  $\eta = 0.2$ ,  $\eta = 0.3$ , and  $\eta = 0.4$ . In Figure 2, Panels (a) and (b) show that values of  $x$  are densely distributed in a certain range. Furthermore, the graphs of the Lyapunov exponent in panel (a) and (b) of Figure 2 are positive in most parts of the domains. Therefore, the economic dynamics exhibits chaos for a small value of  $A$  with  $\eta = 0.2$  and  $\eta = 0.3$ . In panel (c) of Figure 2, values of  $x$  split into two bands for  $\eta = 0.4$ . The corresponding values of the Lyapunov exponent are negative in the domain. Hence, there exists non-chaotic economic fluctuations for  $A \in [6.324, 7.4]$ .

The computed results provide examples of chaos for a small value of  $A$ . On the other hand, the productivity parameter  $A$  should be necessarily large to ensure a positive growth rate, depending on other parameters and endogenous variable  $x$ . For example,  $A$  must be at least 5 to exhibit a positive growth rate for the value of  $x$  around unity. When  $\eta = 0.2$  and  $\eta = 0.3$ , numerical simulations show the convergence of the dynamic sequence of  $x$  to the stationary value. On the contrary, a period-4 cycle exists for  $\eta = 0.4$  and  $A \in [7.4, 10]$ . This result implies that debt financing tends to make cyclical equilibrium dynamics.

*Imperfect depreciation ( $\delta = 0.5$ ).* The possibility of chaos when  $\delta = 0.5$  is rarer than that when  $\delta = 0$ . Hereafter, we focus on a realistic situation with a large value of  $A$  to ensure a positive growth rate. In Panels (a) and (b) of Figure 3 show that bifurcation does not exist and the Lyapunov exponents have negative values for  $A \in [5, 10]$ . These results show that  $x_t$  converges to a unique stationary value. In contrast, persistent economic fluctuations occur in panel (c) of Figure 3. The complicated dynamics is not chaos because the corresponding Lyapunov exponents are negative. Numerical analyses imply that larger dependency on debt financing causes economic cycles even if the economy shows a positive growth trend.

## 5.3 Simultaneous determination of fiscal policy variables

Propositions 2–6 do not ensure that there exists a unique pair of income tax rate and debt financing ratio as interior solutions to maximizing equilibrium growth rate or the government's objective function. We calculate the income tax rate and debt-financing ratio to verify the existence or non-existence of the interior solutions, which are consistent with the first-order conditions of Equations (14) and (15) or (18a) and (18b). The parameters are set as  $(A, \alpha, \beta) = (10, 0.3, 0.483)$ . The values of  $\delta$  vary in the range  $[0.4, 1]$ , and those of  $\mu$  are defined over  $[0.5, 1]$ .

*Growth-maximizing tax rate and debt-financing ratio.* The calculated results for  $\delta \in (0.4, 1)$  show that the growth-maximizing tax rate is equal to 0.7 and the debt-financing ratio is almost 0. For example,  $x^* = 7.764$  and  $\tau^* = 0.7$  are obtained when  $\delta = 0.5$ . Therefore, the balanced-budget scheme is good for economic growth within realistic values of parameters. On the other hand, the calculated value of  $x$  is larger than its realistic value, which is expected to be around unity.

*Income tax rate and debt-financing ratio determined by existing generations.* Table 1 reports the calculation results of  $x^*$ ,  $\tau^*$ ,  $\eta^*$ , and the deficit-to-GDP ratio for  $\delta = 0.4$ ,  $\delta = 0.5$ , and  $\delta = 0.95$  with

<sup>18</sup>When  $\delta = 1$  (perfect depreciation),  $\psi(x_t)$  is monotonically decreasing in  $x_t$ . Complicated dynamics do not occur even though a two-period cycle exists or growth path oscillatory diverges from a stationary equilibrium.

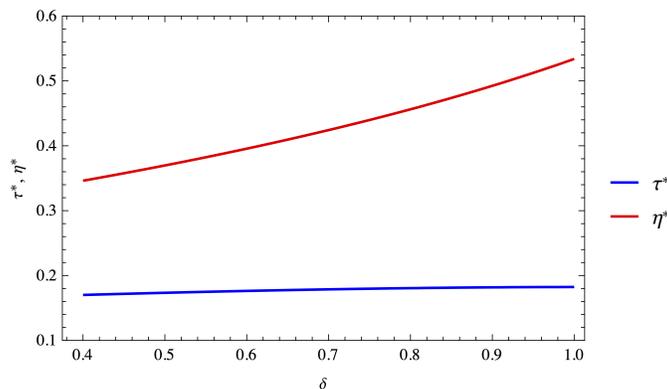


Figure 4.

three different values of  $\mu$  (i.e.,  $\mu = 0.5$ ,  $\mu = 0.75$ , and  $\mu = 1$ ).  $\mu = 0.5$  complies the convergence criterion of the government budget deficit in EU, unlike  $\mu = 0.75$  and  $\mu = 1$ . Figure 4 displays the curves based on the values of  $\tau^*$  and  $\eta^*$  for  $\delta \in (0.4, 1)$  and  $\mu = 0.75$ . It shows that the values of  $x^*$  are between 0.824 and 0.919, those of  $\tau^*$  are between 0.170 and 0.182, and those of  $\eta^*$  are between 0.346 and 0.534 for  $\delta \in (0.4, 1)$ .<sup>19</sup> In reality, public-to-private capital ratio is below unity.<sup>20</sup> The calculated result seems to replicate the actual outcomes.

#### 5.4 Short-run effects of fiscal policy

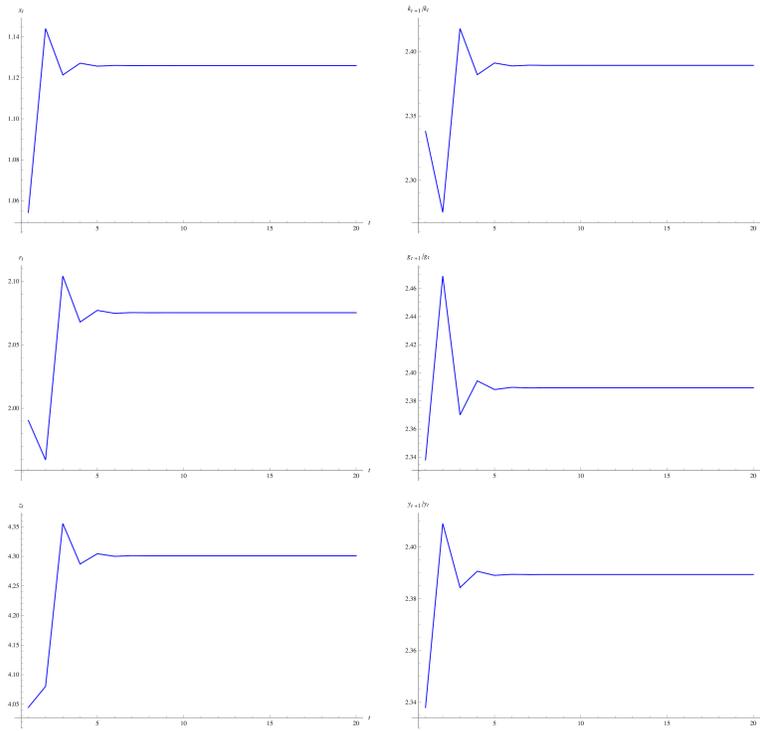
Based on the analysis in the previous subsections, we consider two scenarios. Scenario 1 is the baseline case where  $(A, \alpha, \beta, \delta, \tau, \eta) = (10, 0.3, 0.483, 0.5, 0.2, 0.2)$ . Scenario 2 is the situation where output elasticity of public capital and saving are above the values of the first scenario. We use  $(\alpha, \beta) = (0.2, 0.6)$ . Other parameters are same as those of the baseline case. Suppose that the economy is initially at the stationary equilibrium. At the initial period, the government permanently increases 1%-point in income tax rate or debt financing ratio, and the policy changes are unexpected for households.

**Scenario 1.** At period-0 in the baseline case, public-to-private capital ratio  $x_0$ , interest rate  $r_0$ , and equilibrium growth factor  $\Gamma_0$  are 1.054, 1.990, and 2.338, respectively (i.e., growth rate  $\gamma_0$  is 1.338). Hence,  $r_0 > \gamma_0$  holds and the economy is dynamically efficient. At the period 1, the government unexpectedly raises one percent point in the income tax rate or debt financing ratio. Figure 5 illustrates the dynamics of economic variables after the policy shock in income tax and debt financing ratio. Note that the variable  $z_t$  in the figures is defined as the percent of the interest payment to GDP.

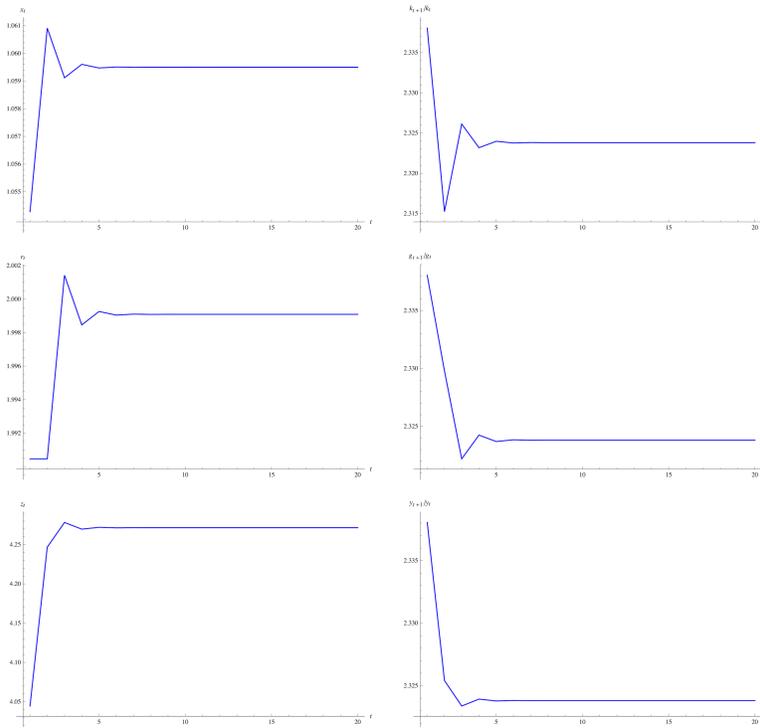
*1%-point increase in income tax rate:* The policy shock induces slowdown of private capital accumulation and a surge in public investment at  $t = 1$ . Public-to-private capital ratio sharply increases at  $t = 1$  through these two effects. Private and public capital accumulations affect economic growth rate differently. The positive productivity effect of public capital overweighs the negative impact on output through slowing down the private capital accumulation. Given that  $\gamma_0 = 1.338$  and  $\gamma_1 = 1.409$ , a 1%-point increase in the tax rate raises economic growth rate at  $t = 1$  by 7.1% point. However, a negative feedback on  $x$  occurs at  $t = 2$ . Public capital accumulation increases bond issuance under Equation (8) and interest payment at the next period after the bond issuance. As a result, public capital accumulation slows down while diminishing the crowding-out effect. Thus, the economic growth rate drops at  $t = 2$ . Economic variables alternately increase and decrease, and finally converge to new stationary values ( $\gamma = 1.389$ ). In the long-run, tax-financed public investment increases economic

<sup>19</sup>If  $\mu = 0.5$ , we have  $x^* \in [0.660, 0.723]$ ,  $\tau^* \in [0.140, 0.155]$  and  $\eta^* \in [0.320, 0.566]$  for  $\delta \in [0.4, 1]$ . When  $\mu = 0.5$ , we have  $x^* \in [0.940, 1.061]$ ,  $\tau^* \in [0.190, 0.200]$  and  $\eta^* \in [0.355, 0.516]$  for  $\delta \in [0.4, 1]$ .

<sup>20</sup>The 5-year average ratios of public to private capital in G5 countries are 0.329, 0.217, 0.529, 0.248, and 0.365 for France, Germany, Japan, the United Kingdom, and United States. These values are calculated based on the data for 2013-2017 in the Investment and Capital Stock Database provided by IMF.

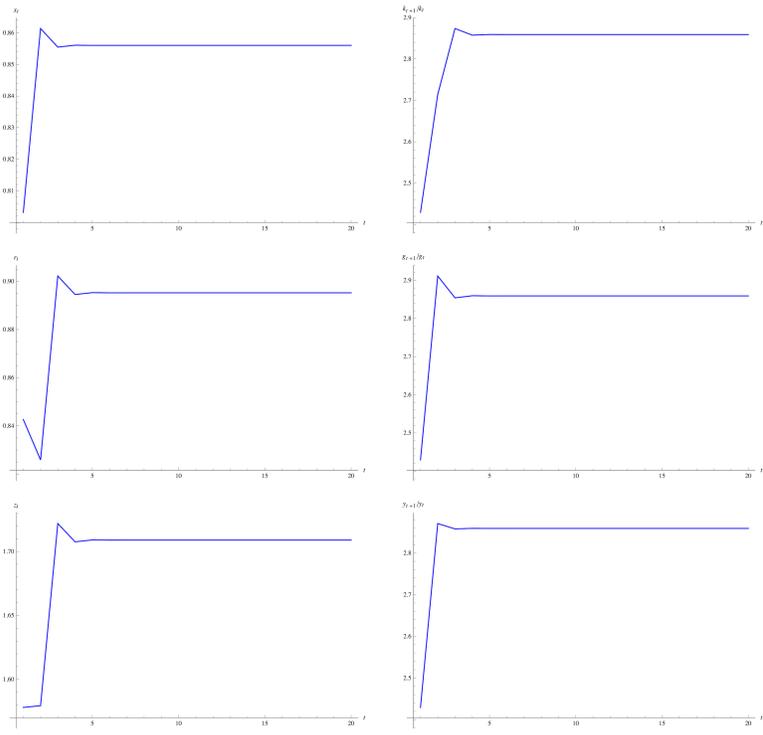


1%-increase in income tax rate

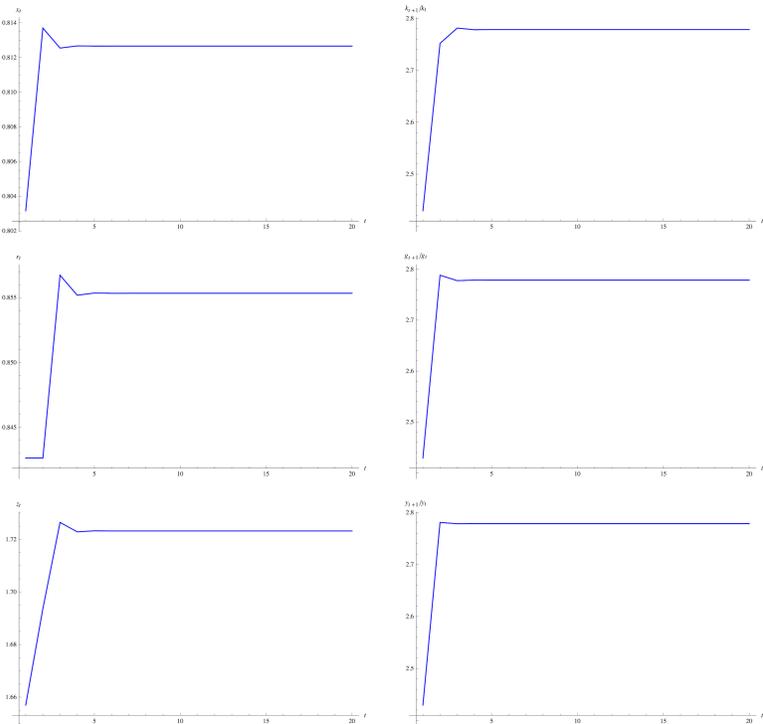


1%-increase in debt financing ratio

Figure 5. Short-run effects of fiscal policy for Scenario 1



1%-increase in income tax rate



1%-increase in debt financing ratio

**Figure 6. Short-run effects of fiscal policy for Scenario 2**

growth rate by 5.1%-point.

*1%-point increase in debt financing ratio:* It leads to investment plunge of private and public capital at the period when the shock occurs. At  $t = 1$ , private capital investment decreases due to the crowding-out effect while a decrease in public investment is caused by an increase in interest payment of public bonds. The impact on public capital accumulation is weaker than the impact on private capital accumulation. Thus, public-to-private capital ratio slightly increases at  $t = 1$ . At the next period, the interest payment increases as well as the interest rate. Public investment has declined for two periods. In contrast, growth rate of private capital is weakly recovered from the bottom at  $t = 2$ . Accordingly, economic growth rate has reduced for two periods: from  $t = 0$  to  $t = 1$ , a 1%-point increase in  $\eta$  reduces 1.3% point in economic growth rate ( $\gamma_0 = 1.338$  to  $\gamma_1 = 1.325$ ). Finally, economic growth rate converges to its new stationary level ( $\gamma = 1.324$ ). Debt-financed public investment decreases economic growth rate by 1.4%-point.

**Scenario 2.** With  $(\alpha, \beta) = (0.2, 0.6)$ , the initial values are  $x_0 = 0.803$ ,  $r_0 = 0.843$ , and  $\Gamma_0 = 2.429$  (i.e.,  $\gamma_0 = 1.429$ ). Then, we have  $r_0 < \gamma_0$  and the economy is dynamically inefficient. Similar to Scenario 1, the policy shocks occur at  $t = 1$ . Figure 6 illustrates the dynamic effects of the policy shocks.

*1%-point increase in income tax rate:* Increased tax enhances private and public capital accumulation with high productivity effects of public capital at  $t = 1$ . Public-to-private capital ratio rises because the effect on public capital accumulation dominates that on private capital accumulation. Although the crowding-out effect and an increasing in interest payment arise, these negative effects on capital accumulation are weak. Therefore, the policy shock raises economic growth rate (and growth rates of private and public capital) to its new stationary level at almost one push. Based on the calculated results, economic growth rate increases 44.1%-point from  $t = 0$  to  $t = 1$  ( $\gamma_0 = 1.429$  to  $\gamma_1 = 1.870$ ) and 43%-point in the long-term ( $\gamma = 1.859$ ).

*1%-point increase in debt financing ratio:* The numerical simulation results are similar to those of 1%-point increase in the income tax rate. The difference between the impacts of different policy shocks depends on the magnitude of the crowding-out effect and interest payment. A rise in  $\eta$  increases these two negative effects on growth in two ways. A rise in  $\eta$  (issuing bonds) directly increases interest payment and strengthens the crowding-out effect, whereas a rise in income tax rate temporally reduces interest payment through a decrease in post-tax interest rate. The calculation shows that economic growth rate rises by 35.2%-point (to  $\gamma_1 = 1.781$ ) from  $t = 0$  to  $t = 1$  and by 35% in the long-run ( $\gamma = 1.779$ ). Hence, the impact of 1%-point increase in the debt financing ratio on economic growth is smaller than its impact of 1%-point increase in the income tax rate.

## 6 Conclusion

The literature on fiscal policy and economic growth has numerous studies examining how public investment, financing methods, and fiscal rules mutually affect macroeconomic outcomes. In particular, the recent literature has focused on the growth effects of debt-financed public investment. Contrary to theoretical findings, empirical studies have found positive growth effects of the debt-financed public investment. Without positive growth effects, debt financing has not been theoretically recommended, though such financing has been adopted in practice. Hence, this paper aims to elucidate growth effects of debt-financed public investment and examine the possibility of choosing borrowing rules by existing generations endogenously.

The answer to the first research question is clear. If interest rate exceeds economic growth rate (i.e., the economy is dynamically efficient), then debt financing for public investment decreases the equilibrium growth rate. In contrast, if the interest rate is below to the economic growth rate (i.e., the economy is dynamically inefficient), then debt-financed public investment may raise the equilibrium growth rate. Particularly, debt-financed public investment has a positive effect on equilibrium growth rate when public capital has sufficiently large productivity effect. Naturally, public debt and economic growth could have a monotonic or non-monotonic inverted-U shaped relationship.

The answer to the second research question is that the existing generations tends to choose debt financing for public investment through an endogenous determination of fiscal policy. The GRPF, as a perfect debt finance, is preferred if the economy is dynamically inefficient. Furthermore, debt financing for public investment could be chosen even if the economy is dynamically efficient. The sufficient condition is that the marginal productivity effect of public capital is sufficiently large to raise capital income.

Our quantitative analysis supports the two main findings of our qualitative analysis. Equilibrium dynamics with debt financed public investment could be complicated and chaotic with low productivity. Calculation based on data shows that equilibrium tax rate and degree of debt finance are close to realistic values. Persistent economic fluctuations are conceivable when the equilibrium degree of debt finance is large, though the economy in the baseline case converges to a unique stationary equilibrium. On the stable growth path, a 1%-point increase in debt financing ratio raises approximately 35% point of economic growth rate as well as the growth rate of private capital and public capital if the growth rate is larger than the interest rate at the initial time. Debt cost is essential to determining the growth effect of debt financed public investment.

Future directions of this research may address peculiar issues to the overlapping generations model. The extensive analyses should include the impact of uncertainty, the role of intergenerational altruism, the effect of early retirement with endogenous labor supply. First, our model has no disparity between the rates of return on assets without uncertainty. To address the real disparity, the impact of uncertainty is worthwhile to examine.

Second, this paper mentioned the importance of the growth effects of public investment in the process of determining fiscal policy endogenously if the authority cares about future generations. Such a government's objective function naturally arises from intergenerational altruistic preferences. Then, a self-enforcing commitment of government policy can be treated in the extensive analysis.

Third, one of the benefits of public investment for existing generations is the future increase in their income through interest and wage rate increase. Endogenous labor supply in the old generation will be essential to evaluate future benefits. This factor affects the growth effects of public investment and endogenous determination of fiscal policy through the change in future benefit of public investment. These extensions will modify our results and give us further insights into the effects of debt-financed public investment. Our main findings will remain to highlight new aspects. This paper provides the analytical basis for future studies.

# Appendix

## A. Proof of Proposition 1

As  $\psi$  is continuous in the closed interval  $[0, \bar{x}]$  with  $\psi(0) > 0$  and  $\psi(\bar{x}) = 0$ ,  $\Omega(x) \equiv \psi(x) - x$  is also continuous in the closed interval  $[0, \bar{x}]$  with  $\Omega(0) = \psi(0) > 0$  and  $\Omega(\bar{x}) = -\bar{x} < 0$ . There exists at least one stationary equilibrium that satisfies  $x = \psi(x)$  ( $x_{t+1} = x_t$ ). Differentiating  $\Omega(x)$  with respect to  $x$  yields

$$\Omega'(x) = \psi'(x) - 1 \gtrless 0 \Leftrightarrow x \lesseqgtr \tilde{x},$$

where  $\tilde{x}$  is  $\{x | \Omega'(x) = 0\}$ . Note that  $\tilde{x} < \hat{x}$  holds. With  $\Omega(0) > 0$  and  $\Omega'(x) > 0$  for  $x \in [0, \tilde{x}]$ , we have  $\Omega(x) > 0$  for  $x \in [0, \tilde{x}]$ . As  $\Omega(\bar{x}) < 0$  and  $\Omega'(x) < 0$  for  $x \in [\tilde{x}, \bar{x}]$ , a unique fixed point exists, such as  $\Omega(x) = 0$  in the range  $[\tilde{x}, \bar{x}]$ . Hence, there is a unique stationary equilibrium. If  $|\psi'(x)| < 1$ , the stationary equilibrium is locally stable.

We now consider the possibility that  $\psi$  is chaotic map in the Li-Yorke sense. The value of  $\psi$  at  $x_t = \hat{x}$  is

$$\psi(\hat{x}) = \frac{[\tau - (1 - \tau)\alpha\eta\hat{x}]A + (1 - \delta)(1 - \eta)\hat{x}^\alpha}{(1 - \alpha)(1 - \eta)(1 - \tau)\beta A - \eta\{[\tau - (1 - \tau)\alpha\eta\hat{x}]A + (1 - \delta)(1 - \eta)\hat{x}^\alpha\}}.$$

Since  $\bar{x}$  satisfies  $\psi(\bar{x}) = 0$ , total differentiation of  $\psi(\bar{x}) = 0$  yields

$$\begin{aligned} 0 &= [\tau - (1 - \tau)\alpha\eta\bar{x}]dA - (1 - \tau)\alpha\eta A d\bar{x} + (1 - \delta)(1 - \eta)\alpha\bar{x}^{\alpha-1}d\bar{x} \\ &\Leftrightarrow \frac{A}{\bar{x}} \frac{\partial \bar{x}}{\partial A} = -\frac{(1 - \alpha)(1 - \delta)(1 - \eta)\bar{x}^\alpha + \tau}{(1 - \delta)(1 - \eta)\bar{x}^\alpha} < 0. \end{aligned}$$

Hence, we have

$$A \rightarrow \infty \Rightarrow \bar{x} = \frac{\tau}{(1 - \tau)\alpha\eta} \text{ and } A \rightarrow 0 \Rightarrow \bar{x} \rightarrow \infty. \quad (\text{A1})$$

A differentiation of  $\hat{x}$  with respect to  $A$  is

$$\frac{A}{\hat{x}} \frac{\partial \hat{x}}{\partial A} \equiv -\frac{1}{1 - \alpha} < 0.$$

Then, we have  $A \rightarrow \infty \Rightarrow \hat{x} = 0$  and  $A \rightarrow 0 \Rightarrow \hat{x} \rightarrow \infty$ . Using these equations, we obtain

$$A \rightarrow \infty \Rightarrow \psi(\hat{x}) = \frac{\tau}{(1 - \alpha)(1 - \eta)(1 - \tau)\beta - \eta\tau} = \psi(0) \geq 0 \text{ and } A \rightarrow \underline{A} \Rightarrow \psi(\hat{x}) \rightarrow \infty. \quad (\text{A2})$$

Equations (A1) and (A2) show that there exists the critical value of  $A$  where  $\psi(\hat{x}) = \bar{x}$  if

$$\frac{\tau}{(1 - \alpha)(1 - \eta)(1 - \tau)\beta - \eta\tau} < \frac{\tau}{(1 - \tau)\alpha\eta}. \quad (\text{A3})$$

If  $\psi(0) \leq \bar{x}$  and  $\psi(\hat{x}) = \bar{x}$ , there exists the Li-Yorke chaos (Li and Yorke, 1975). One of the sufficient condition for  $0 < \psi(0) < \hat{x} < \psi(\hat{x}) = \bar{x}$  is Equation (A3) and

$$\hat{x} > \psi(0) \Leftrightarrow \left[ \frac{(1 - \delta)(1 - \eta)}{(1 - \tau)\eta A} \right]^{\frac{1}{1 - \alpha}} > \frac{\tau}{(1 - \alpha)(1 - \eta)(1 - \tau)\beta - \eta\tau}. \quad (\text{A4})$$

## B. Derivation of equations (13a) and (13b)

Partial differentiation of  $\psi$  with respect to  $\tau$  yields

$$\frac{\partial \psi(x)}{\partial \tau} = \frac{(1 + \alpha\eta x)A + [(1 - \alpha)(1 - \eta)\beta A + (1 + \alpha\eta x)\eta A]x}{\Psi} > 0, \quad (\text{A5})$$

where

$$\begin{aligned}\Phi &\equiv [\tau - (1 - \tau) \alpha \eta x] A + (1 - \delta) (1 - \eta) x^\alpha, \\ \Psi &\equiv (1 - \alpha) (1 - \eta) (1 - \tau) \beta A - \eta \{[\tau - (1 - \tau) \alpha \eta x] A + (1 - \delta) (1 - \eta) x^\alpha\}.\end{aligned}$$

Using Equation (A5), we obtain Equation (13a) from total differentiation of Equation (11) when  $x_t = x$ :

$$\frac{\partial x}{\partial \tau} = \frac{1}{1 - \psi'(x)} \frac{\partial \psi(x)}{\partial \tau} > 0.$$

Partial differentiation of  $\psi$  with respect to  $\eta$  is

$$\frac{\partial \psi(x)}{\partial \eta} = \frac{[(1 - \eta) (1 + \gamma) x - (1 + \eta x) (r - \gamma)] x^\alpha}{\Psi}. \quad (\text{A6})$$

The sign of Equation (A6) is indeterminate. Equation (A6) and total differentiation of Equation (11) when  $x_t = x$  lead to Equation (13b):

$$\frac{\partial x}{\partial \eta} = \frac{1}{1 - \psi'(x)} \frac{\partial \psi(x)}{\partial \eta}.$$

### C. Proof of Proposition 2

Using Equation (13a), there exists a lower bound of  $\tau$  that satisfies  $\Phi = 0$ . Let be  $\underline{\tau}$  as the lower bound. Then, we have  $\lim_{\tau \searrow \underline{\tau}} x = 0$ . Taking the limit of (13a) leads to

$$\lim_{\tau \searrow \underline{\tau}} \left( \frac{1}{x} \frac{\partial x}{\partial \tau} \right) = +\infty. \quad (\text{A7})$$

Equations (14) and (A7) derive

$$\lim_{\tau \searrow \underline{\tau}} \frac{1}{\Gamma} \frac{\partial \Gamma}{\partial \tau} = -\frac{1}{1 - \underline{\tau}} + \left\{ (1 - \alpha) - \lim_{\tau \searrow \underline{\tau}} \left[ \frac{\eta x}{1 + \eta x} \right] \right\} \lim_{\tau \searrow \underline{\tau}} \left( \frac{1}{x} \frac{\partial x}{\partial \tau} \right) = +\infty.$$

By Equation (13a), an upper bound of  $\tau$  exists. Let be  $\bar{\tau}$  as the upper bound. Note that  $\bar{\tau}$  must be less than unity. Then,  $\lim_{\tau \nearrow \bar{\tau}} \Psi = 0$  and  $\lim_{\tau \nearrow \bar{\tau}} x = +\infty$  hold. Equation (13a) yields

$$\lim_{\tau \nearrow \bar{\tau}} \left( \frac{1}{x} \frac{\partial x}{\partial \tau} \right) = \frac{1}{1 - \bar{\tau}} > 0. \quad (\text{A8})$$

Using Equations (15) and (A8) provide

$$\lim_{\tau \nearrow \bar{\tau}} \frac{1}{\Gamma} \frac{\partial \Gamma}{\partial \tau} = -\frac{1}{1 - \bar{\tau}} + (1 - \alpha) \lim_{\tau \nearrow \bar{\tau}} \left( \frac{1}{x} \frac{\partial x}{\partial \tau} \right) - \lim_{\tau \nearrow \bar{\tau}} \left[ \frac{\eta x}{1 + \eta x} \frac{1}{x} \frac{\partial x}{\partial \tau} \right] = -\frac{1 + \alpha}{1 - \bar{\tau}} < 0.$$

These results show that there exists a tax rate to maximize the equilibrium growth rate for  $0 < \tau < 1$ .

### D. Proof of Proposition 3

From Equation (13b), we have

$$\frac{1}{x} \frac{\partial x}{\partial \eta} = \frac{(1 - \eta) (1 + \gamma) x - (1 + \eta x) (r - \gamma)}{(1 - \eta) (1 + \gamma) - [(1 - \delta) (1 - \eta) \alpha - (r + \delta) \eta] (1 + \eta x)}. \quad (\text{A9})$$

Equation (A9) becomes

$$\lim_{\eta \searrow 0} \left( \frac{1}{x} \frac{\partial x}{\partial \eta} \right) = \frac{(1 + \gamma)x - (r - \gamma)}{\gamma + \delta + (1 - \delta)(1 - \alpha)}, \quad (\text{A10})$$

$$\lim_{\eta \nearrow 1} \left( \frac{1}{x} \frac{\partial x}{\partial \eta} \right) = -\frac{r - \gamma}{r + \delta}. \quad (\text{A11})$$

Using Equations (15) and (A9), we obtain

$$\begin{aligned} \frac{\eta}{\Gamma} \frac{\partial \Gamma}{\partial \eta} &= (1 - \alpha - \chi) \frac{\eta}{x} \frac{\partial x}{\partial \eta} - \chi \\ &= -\frac{[(1 - \eta)\alpha + \eta](\gamma + \delta)\chi + (1 - \alpha)(r - \gamma)\eta}{(1 - \eta)(1 - \chi)(1 + \gamma) - [(1 - \delta)(1 - \eta)\alpha - (r + \delta)\eta]}, \end{aligned} \quad (\text{A12})$$

where

$$\chi \equiv \frac{\eta x}{1 + \eta x}.$$

Equations (A10)–(A12) yield

$$\lim_{\eta \searrow 0} \left( \frac{1}{\Gamma} \frac{\partial \Gamma}{\partial \eta} \right) = -\frac{(\gamma + \delta)\alpha x + (1 - \alpha)(r - \gamma)}{(1 + \gamma) - (1 - \delta)\alpha}, \quad (\text{A13})$$

$$\lim_{\eta \nearrow 1} \left( \frac{1}{\Gamma} \frac{\partial \Gamma}{\partial \eta} \right) = -\left( \frac{r - \gamma}{r + \delta} \right) \left( 1 - \alpha - \frac{x}{1 + x} \right) - \frac{x}{1 + x}. \quad (\text{A14})$$

If the economy is dynamically efficient, Equations (A12) and (A13) lead to

$$\frac{\partial \Gamma}{\partial \eta} < 0 \text{ for } 0 \leq \eta \leq 1.$$

If the economy is dynamically inefficient, then the sign of Equation (A12) is ambiguous, though that of Equation (A9) is positive. There exists the growth-maximizing level of  $\eta$  for  $\eta \in (0, 1)$  if Equations (A13) and (A14) have a positive and negative signs, respectively. To ensure the existence of  $\eta^* \in (0, 1)$ , the following is required:

$$x^B < \frac{(1 - \alpha)(\gamma^B - r^B)}{(\gamma^B + \delta)\alpha} \text{ and } \frac{(1 - \alpha)(\gamma^G - r^G)}{\gamma^G + \delta - (1 - \alpha)(\gamma^G - r^G)} < x^G.$$

The sufficient condition indicates that public-to-private capital ratio under a balanced budget scheme is sufficiently small and that under the GRPF is sufficiently large. By Equation (A9), it can be replaced as the condition that the marginal effect of debt financing on  $x$  is sufficiently large.

## E. Proof of Proposition 5

As shown in Appendix C, the lower bound and upper bound of  $\tau$  exist and then  $\tau \in (\underline{\tau}, \bar{\tau})$ . Using Equations (18a) and (A7), the limit of Equation (18a) as  $\tau \rightarrow \underline{\tau}$  is

$$\lim_{\tau \searrow \underline{\tau}} \frac{\partial W_0}{\partial \tau} = -\frac{[\theta + \mu(1 + 2\theta)]}{1 - \underline{\tau}} + \lim_{\tau \searrow \underline{\tau}} \frac{(1 - \alpha)\mu\theta}{x} \frac{\partial x}{\partial \tau} = +\infty.$$

By Equations (18a) and (A8), the limit of Equation (18a) as  $\tau \rightarrow \bar{\tau}$  becomes

$$\lim_{\tau \nearrow \bar{\tau}} \frac{\partial W_0}{\partial \tau} = -\frac{\theta + \mu(1 + 2\theta)}{1 - \bar{\tau}} + \lim_{\tau \nearrow \bar{\tau}} \frac{(1 - \alpha)\mu\theta}{x} \frac{\partial x}{\partial \tau} = -\frac{\theta + \mu + (1 + \alpha)\theta\mu}{1 - \bar{\tau}} < 0.$$

These results show that there exists  $\tau^* = \{\tau \mid \partial W_0 / \partial \tau = 0, \underline{\tau} < \tau < \bar{\tau}\}$ .

## F. Proof of Proposition 6

For  $\delta = 1$  and  $\eta = 0$ , Equations (11) and (A10) yield

$$\lim_{\eta \searrow 0} x = x^B = \frac{\tau}{(1-\alpha)(1-\tau)\beta} \text{ and } \lim_{\eta \searrow 0} \frac{1}{x} \frac{\partial x}{\partial \eta} = \frac{(1-\tau)[(1-\alpha)\beta - \alpha] + \tau}{(1-\alpha)(1-\tau)\beta}. \quad (\text{A15})$$

For  $\delta = 1$  and  $\eta = 1$ , Equations (11) and (A11) provide

$$\lim_{\eta \nearrow 1} x = x^G = \frac{\tau}{(1-\tau)\alpha} \text{ and } \lim_{\eta \nearrow 1} \frac{1}{x} \frac{\partial x}{\partial \eta} = \frac{(1-\tau)[(1-\alpha)\beta - \alpha]}{(1-\alpha)\beta}. \quad (\text{A16})$$

The limits of Equation (18b) with Equations (A15) and (A16) lead to

$$\text{sgn} \lim_{\eta \searrow 0} \frac{\partial W_t}{\partial \eta} = \text{sgn} \lim_{\eta \searrow 0} \frac{\partial x}{\partial \eta} = \text{sgn} \{ (1-\tau)[(1-\alpha)\beta - \alpha] + \tau \}, \quad (\text{A18})$$

$$\text{sgn} \lim_{\eta \nearrow 1} \frac{\partial W_t}{\partial \eta} = \text{sgn} \lim_{\eta \nearrow 1} \frac{\partial x}{\partial \eta} = \text{sgn} [(1-\alpha)\beta - \alpha]. \quad (\text{A19})$$

If the economy is dynamically inefficient, we have  $(1-\alpha)\beta > \alpha$  and

$$\frac{1}{x} \frac{\partial x}{\partial \eta} = \frac{(1-\eta)(1+\gamma)x - (1+\eta x)(r-\gamma)}{(1-\eta)(1+\gamma) + (1+r)(1+\eta x)\eta} > 0 \text{ for } 0 < \eta < 1. \quad (\text{A20})$$

Using Equations (18b) and (A15)–(A20), the following equation holds:

$$\frac{\partial W_t}{\partial \eta} > 0, \text{ where } \lim_{\eta \searrow 0} \frac{\partial W_t}{\partial \eta} > \lim_{\eta \nearrow 1} \frac{\partial W_t}{\partial \eta} > 0.$$

Therefore,  $\eta = 1$  (GRPF) is chosen.

If the economy is dynamically efficient,  $(1-\alpha)\beta < \alpha$  holds. Using Equations (A15)–(A19), we have

$$\lim_{\eta \searrow 0} \frac{\partial W_t}{\partial \eta} \geq 0 \Leftrightarrow \tau \geq \frac{(1-\alpha)(1+\beta) - 1}{(1-\alpha)(1+\beta)} \text{ and } \lim_{\eta \nearrow 1} \frac{\partial W_t}{\partial \eta} < 0.$$

When  $\tau$  is sufficiently large, there exists  $\eta^*$  where  $\partial W_t / \partial \eta = 0$  and  $\eta^* \in (0, 1)$ . On the other hand,  $\eta = 0$  (the balanced budget) is preferable to GRPF or debt financing when  $\tau$  is sufficiently small.

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