SIMPLE-MINDED OBJECTS IN TRIANGULATED CATEGORIES AND COHEN-MACAULAY DIFFERENTIAL GRADED MODULES
(三角圏における単純対象の類似物と COHEN-MACAULAY DG 加群)

HAIBO JIN

Contents

Preface

This thesis is divided into three parts. The first one is aim to introduce a new reduction process of triangulated categories. In the second part, we introduce Cohen-Macaulay differential graded modules over Gorenstein differential graded algebras and study their representation theory. In the third one, we built a bijection between simple-minded systems in negative Calabi-Yau cluster category and silting objects in some subcategory of derived category. The first chapter is devoted to basic definitions and properties used in this thesis.

Part 1. Derived categories and triangulated categories appear in many areas of mathematics, such as algebraic geometry, representation theory and algebraic topology. An important way to construct a new triangulated category is the Verdier quotient. But usually the morphisms in the quotient category are too complicated to understand. It is known in some nice cases (see for example, [AI, IYa1, IYo, CSP]), Verdier quotient can be realized as a reduction process, which is another way to construct a new triangulated category. The reduction can be realized as a certain sub (or subfactor) category of the given one and is easier to study.

The first part of this thesis, which is based on [Ji2], is devoted to introducing a new reduction process of triangulated categories with respect to *simple-minded collections*(or *SMC* for short). Simple modules are one of the most basic and important objects in the representation theory of algebras. There have been many generalizations and studies of simple modules. Among them, SMC in derived categories and SMS (=simple-minded system) in singularity categories have particular importance. SMCs play a key role in Koszul duality [BGS, KN], and bijectively correspond to silting objects (which are generalizations of projective modules in derived categories)[Ric2, KaY1]. SMSs appear naturally in some negative Calabi-Yau (=CY) triangulated categories [KL, D, CS3], including the stable categories of Cohen-Macaulay (=CM) dg modules [Ji1], which will be introduced and studied in the second part of this thesis.

Let k be a field. Let $\mathcal{T}$ be a Krull-Schmidt k-linear triangulated category. We introduce the *SMC reduction* $\mathcal{U}$ *of* $\mathcal{T}$ with respect to some pre-SMC $\mathcal{R}$ (see Definition 0.2.4) as the Verdier quotient $\mathcal{U} = \mathcal{T}/\operatorname{thick}(\mathcal{R})$. Our first result realizes $\mathcal{U}$ as the additive subcategory

$$\mathcal{Z} = \mathcal{R}[\geq 0]^{\perp} \cap {}^{\perp}\mathcal{R}[\leq 0]$$

of $\mathcal{T}$ under some mild assumptions (R1) and (R2) in Section 1.2 (see Section 0.1 for the definition of $(\,)^{\perp}$ and ${}^{\perp}(\,)$). Namely,

Theorem 0.0.1 (Theorem 1.2.1)**.** *Under the setting above, the following results hold.*

(1) *The composition* $\mathcal{Z} \hookrightarrow \mathcal{T} \to \mathcal{U}$ *gives an equivalence* $\mathcal{Z} \xrightarrow{\simeq} \mathcal{U}$*;*
(2) *There is a bijection*

$$\operatorname{SMC}_{\mathcal{R}}\mathcal{T} := \{\textit{SMCs of } \mathcal{T} \textit{ contain } \mathcal{R}\} \longleftrightarrow \operatorname{SMC}\mathcal{U} := \{\textit{SMCs of } \mathcal{U}\}.$$

This result can be regard as a dual of silting reduction [IYa1], where it was necessary to take an ideal quotient of $\mathcal{Z}$. It also plays an important role in the proof of Theorem 2.7.1.

Another aim of Part 1 is to generalize the singularity category of a finite dimensional Gorenstein k-algebra A over a field k. In this case, the singularity category is defined as the Verdier quotient $\mathsf{D}_{\mathrm{sg}}(A) = \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)/\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ by [B, O]. Buchweitz's equivalence states that $\mathsf{D}_{\mathrm{sg}}(A)$ is triangle equivalence to the stable category $\underline{\mathsf{CM}}A$ of Cohen-Macaulay A-modules. A key observation in our context is that $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$ has a SMC consisting of simple A-modules, and there is a relative Serre functor $\nu = ? \otimes^{\mathbf{L}}_{A} DA$.

To generalize the notion of singularity categories and Buchweitz's equivalence, we work on a *SMC quadruple* $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathbb{S}, \mathcal{S})$, where $\mathcal{T}^{\mathrm{p}}$ is a thick subcategory of a triangulated category $\mathcal{T}$, $\mathbb{S}$ is a relative Serre functor, $\mathcal{S}$ is a SMC of $\mathcal{T}$ and they satisfy some conditions (see Definition 1.3.1). We define the *singularity category* as the Verdier quotient

$$\mathcal{T}_{\mathrm{sg}} := \mathcal{T}/\mathcal{T}^{\mathrm{p}}.$$

In this setting, we can define some subcategories $\mathcal{F}$ and $\mathcal{P}$ of $\mathcal{T}$ (see Section 1.3 for the details), where in the algebra case above, $\mathcal{F} = \mathsf{CM}\,A$ and $\mathcal{P} = \mathsf{proj}\,A$. Our second result realizes $\mathcal{T}_{\rm sg}$ as a subfactor category of $\mathcal{T}$.

Theorem 0.0.2 (Theorem 1.3.5 (1), (2))**.** *Let $(\mathcal{T}, \mathcal{T}^{\rm p}, \mathbb{S}, \mathcal{S})$ be a SMC quadruple and let $\mathcal{T}_{\rm sg}$, $\mathcal{F}$, $\mathcal{P}$ be defined as above. Then*

(1) *$\mathcal{F}$ is a Frobenius extriangulated category with* $\mathsf{Proj}\,\mathcal{F} = \mathcal{P}$ (in the sense of [NP]);
(2) The composition
$$\mathcal{F} \subset \mathcal{T} \to \mathcal{T}_{\rm sg}$$
induces an equivalence $\pi : \frac{\mathcal{F}}{[\mathcal{P}]} \xrightarrow{\simeq} \mathcal{T}_{\rm sg}$. Moreover, $\mathcal{T}_{\rm sg}$ has a Serre functor $\mathbb{S}[-1]$.

Theorem 1.3.5 can be regard as a dual of the equivalence between the fundamental domain and the cluster category [Am, Gu, IYa1], where it was not necessary to take ideal quotient.

We end the first part by studying the relation between the SMC reduction of $\mathcal{T}$ and the SMS reduction introduced by [CSP] of $\mathcal{T}_{\rm sg}$ for a SMC quadruple $(\mathcal{T}, \mathcal{T}^{\rm p}, \mathbb{S}, \mathcal{S})$ with $\mathbb{S} = [d]$ for some $d \geq 1$. We show the SMS reduction is the shadow of SMC reduction (see Theorem 1.5.4), that is, the following diagram of operations commute.

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\text{sing. category}} & \mathcal{T}_{\rm sg} \\ {\scriptstyle \text{SMC reduction}}\downarrow & & \rightsquigarrow {\scriptstyle \text{SMS reduction}} \\ \mathcal{U} & \xrightarrow{\text{sing. category}} & \mathcal{U}_{\rm sg} \cong (\mathcal{T}_{\rm sg})_{\mathcal{R}} \end{array}$$

The results we obtain here are parallel to the connection between silting reductions and CY reductions given in [IYa1].

Part 2. The second part of this thesis is devoted to introducing Cohen-Macaulay (CM) differential graded (dg) modules and study their representation theory. This part is based on the paper [Ji1]. The notion of Cohen-Macaulay (CM) modules is classical in commutative algebra [Ma, BH], and has natural generalizations for non-commutative algebras [B, H2, IW]. The category of CM modules has been studied by many researchers in representation theory (see, for example, [CR, Yo, Si, LW]). On the other hand, the derived categories of differential graded (dg) categories introduced by Bondal-Kapranov [BK] and Keller [K1, K3] is an active subject appearing in various areas of mathematics [Min, T, Ye1]. We are aimed to introduce CM dg modules to connect these two subjects.

We work on a nice class of dg k-algebras A called Gorenstein (see Assumption 2.1.1) and introduce the category $\mathsf{CM}\,A$ (see Definition 2.2.1) of CM dg A-modules. For the case of Gorenstein algebras, the category of CM modules is a Frobenius exact category, and we have Buchweitz's equivalence as mentioned above. In our setting, $\mathsf{CM}\,A$ does not necessarily have a natural structure of exact category. Instead, the following result shows it has a natural structure of extriangulated category, which is introduced by Nakaoka and Palu [NP] as a common generalization of exact category and triangulated category.

Theorem 0.0.3 (Theorems 2.2.4, 2.3.1 and 2.3.7)**.** *Let A be a non-positive proper Gorenstein dg algebra. Then*

(1) *$\mathsf{CM}\,A$ is functorially finite in $\mathsf{D}^{\rm b}(A)$;*
(2) *$\mathsf{CM}\,A$ is a Frobenius extriangulated category with $\mathsf{Proj}(\mathsf{CM}\,A) = \mathsf{add}\,A$;*
(3) *The stable category $\underline{\mathsf{CM}}A := (\mathsf{CM}\,A)/[\mathsf{add}\,A]$ is a triangulated category;*
(4) *The composition $\mathsf{CM}\,A \hookrightarrow \mathsf{D}^{\rm b}(A) \to \mathsf{D}^{\rm b}(A)/\,\mathsf{per}\,A$ induces a triangle equivalence*
$$\underline{\mathsf{CM}}A = (\mathsf{CM}\,A)/[\mathsf{add}\,A] \simeq \mathsf{D}^{\rm b}(A)/\,\mathsf{per}\,A = \mathsf{D}_{\rm sg}(A);$$
(5) *$\underline{\mathsf{CM}}A$ admits a Serre functor and $\mathsf{CM}\,A$ admits almost split extensions.*

One of the traditional subjects is the classification of Gorenstein rings which are representation-finite in the sense that they have only finitely many indecomposable Cohen-Macaulay modules. Riedtmann [Rie2, Rie3] and Wiedemann [W] considered the classification of representation-finite self-injective algebras and Gorenstein orders respectively in the 1980s. In both classifications, configurations play an important role. We introduce *negative Calabi-Yau (CY) configuration* (see Definition 2.5.1), which recovers Riedtmann's configuration as (-1)-CY configuration, to classify representation-finite Gorenstein dg algebras A.

Let d be a positive integer. We say a Gorenstein dg k-algebra A is *d-self-injective* (resp. *d-symmetric*) if $\mathsf{add}\, A = \mathsf{add}\, DA[d-1]$ in $\mathsf{D}(A)$ (resp. $\mathsf{D}(A^{\mathrm{e}})$). We characterize simple dg A-modules as a $(-d)$-CY configuration (see Theorem 2.5.5), which generalizes [Rie2, Proposition 2.4].

Let Δ be a Dynkin digram. For a subset C of vertices of $\mathbb{Z}\Delta$, we define a translation quiver $(\mathbb{Z}\Delta)_C$ by adding to $\mathbb{Z}\Delta$ a vertex p_c and two arrows $c \to p_c \to \tau^{-1}(c)$ for each $c \in C$ (see Definition 0.5.1). Our main result of Part 2 states that the Auslander-Reiten quivers of representation-finite d-symmetric dg algebras are classified by $(-d)$-CY configurations in the following sense.

Theorem 0.0.4 (Theorem 2.7.1)**.** *Let Δ be a Dynkin digram. Let C be a subset of vertices of $\mathbb{Z}\Delta/\mathbb{S}[d]$. The following are equivalent.*

(1) *C is a $(-d)$-CY configuration;*

(2) *There exists a d-symmetric dg k-algebra A with AR quiver of $\mathsf{CM}\, A$ being $(\mathbb{Z}\Delta)_C/\mathbb{S}[d]$.*

For the case $\Delta = A_n$, we give another proof of Theorem 0.0.4 (see Theorem 2.8.32) by introducing some combinatorial objects called *maximal d-Brauer relations* (see Definition 2.8.2). In this case, for any given $(-d)$-CY configuration C, the corresponding d-symmetric dg k-algebra is given explicitly by *Brauer tree dg algebra* (see Section 2.8.3 for the details). The following table explains the comparison among different configurations.

$(-d)$-CY ($d \geq 1$)	(-1)-CY	0-CY
$(-d)$-CY configuration	Riedtmann's configuration	Wiedemann's configuration
maximal d-Brauer relation	Brauer relation	2-Brauer relation
d-self-injective dg algebras	self-injective algebras	Gorenstein orders

We point out that for representation-finite d-self-injective dg algebras A, the notation '$(-d)$-CY configuration' of $\underline{\mathsf{CM}}A$ coincides with d-SMS. We will study d-SMS in negative cluster category in the third part.

Part 3. Fomin and Zelevinsky [FZ1] showed that cluster algebras of finite type correspond bijectively with finite root systems Φ. As a generalization of their combinatorial structure, Fomin and Reading [FR] introduced generalized cluster complex $\Delta^d(\Phi)$ for each positive integer d. It is a simplicial complex whose ground set is the disjoint union of d copies of the set Φ^+ of positive roots and the set of negative simple roots, and studied actively in combinatorics [Ar, STW]. It is known that $\Delta^d(\Phi)$ is categorified by $(d+1)$-Calabi-Yau ($(d+1)$-CY) cluster categories $\mathcal{C}_{d+1}(kQ)$ [1] for the corresponding Dynkin quiver Q [K2, T]. The category $\mathcal{C}_{d+1}(kQ)$ has special objects called cluster tilting objects, which correspond bijectively with maximal simplices in $\Delta^d(\Phi)$ [Z] and with silting objects contained in some subcategory of $\mathsf{D}^{\mathrm{b}}(kQ)$ [BRT1]. Culster tilting objects also play a key role in Cohen-Macaulay representations [I2].

Recently there is increasing interest in negative CY triangulated categories (see [CS1, CS2, CS3, CSP, HJY, Ji1, Ji2, Jo1, KYZ]), including $(-d)$-CY cluster categories $\mathcal{C}_{-d}(kQ)$. These categories often contain special objects called d-simple-minded systems (or d-SMS) [CS2] (see Definition 0.2.7). It plays a key role in the study of Cohen-Macaulay dg modules [Ji1], as shown in Part 2.

The aim of Part 3 is to show that there is a bijection between d-SMSs and maximal simplices in $\Delta^d(\Phi)$ without negative simple roots (Theorem 3.1.1). In particular, the number of d-SMSs in $\mathcal{C}_{-d}(kQ)$ is precisely the positive Fuss-Catalan number (see Section 3.1.1 for the definition). Our method is based on a refined version of silting-t-structure correspondence.

[1] $(d+1)$-CY cluster categories are usually called d-cluster categories in hereditary setting, and $(d+1)$-cluster categories in non-hereditary setting.

Preliminary

0.1. Notations

Throughout this thesis, k will be a field. All algebras, modules and categories are over k. We denote by $D = \mathrm{Hom}_k(?, k)$ the k-dual. When we consider graded k-module, D means the graded dual. Let $\mathcal{T}$ be an additive category. Let $\mathcal{S}$ be a full subcategory of $\mathcal{T}$. For an object X in $\mathcal{T}$, a morphism $f : S \to X$ is called a *right $\mathcal{S}$-approximation* if $S \in \mathcal{S}$ and $\mathrm{Hom}_{\mathcal{T}}(S', f)$ is surjective for any $S' \in \mathcal{S}$. We say $\mathcal{S}$ is *contravariantly finite* if every object in $\mathcal{T}$ has a right $\mathcal{S}$-approximation. Dually, we define *left $\mathcal{S}$-approximation* and *covariantly finite* subcategories. We say $\mathcal{S}$ is *functorially finite* if it is both contravariantly finite and covariantly finite. If $\mathcal{T}$ is a Krull-Schmidt k-linear category, we denote by $\mathsf{ind}\,\mathcal{T}$ the set of indecomposable objects in $\mathcal{T}$.

We denote by $\mathsf{add}\,\mathcal{S}$ the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{S}$ and closed under isomorphism, finite direct sums, and direct summands. Denote by $[\mathcal{S}]$ the ideal of $\mathcal{T}$ consisting of morphisms which factor through an object in $\mathsf{add}\,\mathcal{S}$ and denote by $\frac{\mathcal{T}}{[\mathcal{S}]}$ the additive quotient of $\mathcal{T}$ by $\mathcal{S}$. Define subcategories

$$\begin{aligned} {}^{\perp}\mathcal{S} &:= \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(X, \mathcal{S}) = 0\}, \\ \mathcal{S}^{\perp} &:= \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(\mathcal{S}, X) = 0\}. \end{aligned}$$

We denote by $[1]$ (or $\langle 1\rangle$) the suspension functors for triangulated categories. Let $\mathcal{T}$ be a triangulated category. For any $X, Y \in \mathcal{T}$ and $n \in \mathbb{Z}$, when we write $\mathrm{Hom}_{\mathcal{T}}(X, Y[>n]) = 0$ (resp. $\mathrm{Hom}_{\mathcal{T}}(X, Y[<n]) = 0$, $\mathrm{Hom}_{\mathcal{T}}(X, Y[\geq n]) = 0$, $\mathrm{Hom}_{\mathcal{T}}(X, Y[\leq n]) = 0$), we mean $\mathrm{Hom}_{\mathcal{T}}(X, Y[i]) = 0$ for all $i > n$ (resp. $i < n$, $i \geq n$, $i \leq n$).

Let $\mathcal{S}$ be a full subcategory of $\mathcal{T}$. We denote by $\mathsf{thick}(\mathcal{S})$ the smallest thick subcategory containing $\mathcal{S}$. Let $\mathcal{S}'$ be another full subcategory of $\mathcal{T}$. Define a new subcategory of $\mathcal{T}$ as follows.

$$\mathcal{S} * \mathcal{S}' := \{X \in \mathcal{T} \mid \text{there is a triangle } S \to X \to S' \to S[1] \text{ with } S \in \mathcal{S} \text{ and } S' \in \mathcal{S}'\}.$$

If $\mathrm{Hom}_{\mathcal{T}}(\mathcal{S}, \mathcal{S}') = 0$, that is, if $\mathrm{Hom}_{\mathcal{T}}(S, S') = 0$ for any $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$, we write $\mathcal{S} * \mathcal{S}' = \mathcal{S} \perp \mathcal{S}'$. For subcategory $\mathcal{S}_1, \cdots, \mathcal{S}_n$ of $\mathcal{T}$, we define $\mathcal{S}_1 * \cdots * \mathcal{S}_n$ and $\mathcal{S}_1 \perp \cdots \perp \mathcal{S}_n$ inductively. We say $\mathcal{S}$ is *extension-closed* if $\mathcal{S} * \mathcal{S} = \mathcal{S}$. We denote by $\mathsf{Filt}(\mathcal{S})$ the smallest extension-closed subcategory of $\mathcal{T}$ containing $\mathcal{S}$. It is easy to see $\mathsf{Filt}(\mathcal{S}) = \bigcup_{n\geq 1} \underbrace{\mathcal{S} * \cdots * \mathcal{S}}_{n}$. We write $\mathsf{Filt}(\mathcal{S}[\geq n]) = \mathsf{Filt}(\bigcup_{i\geq n} \mathcal{S}[i])$ and $\mathsf{Filt}(\mathcal{S}[\leq n]) = \mathsf{Filt}(\bigcup_{i\leq n} \mathcal{S}[-i])$. If $\mathcal{S} = \{S\}$ has only one object, we write $\mathsf{thick}(\{S\})$ as $\mathsf{thick}(S)$, and we use same convention for $\mathsf{add}, \mathsf{Filt}$ and $*$.

Here we recall some well-known results on additive closures and approximations for later use.

Lemma 0.1.1. *Let $\mathcal{T}$ be a Krull-Schmidt triangulated category. Let $\mathcal{X}$ and $\mathcal{Y}$ be two extension-closed subcategories of $\mathcal{T}$. Then*

(1) *If $\mathcal{Y} * \mathcal{X} \subset \mathcal{X} * \mathcal{Y}$, then $\mathcal{X} * \mathcal{Y}$ is also extension-closed;*
(2) *If $\mathrm{Hom}_T(\mathcal{X}, \mathcal{Y}) = 0$, then $\mathsf{add}(\mathcal{X} * \mathcal{Y}) = \mathcal{X} * \mathcal{Y}$;*
(3) *Let $T \in \mathcal{T}$. Let $R_T \xrightarrow{f} T \to T' \to X[1]$ be the triangle extended by the minimal right $\mathcal{X}$-approximation f of T. Then $T' \in \mathcal{X}^{\perp}$. If moreover, $\mathrm{Hom}_{\mathcal{T}}(\mathcal{Y}, T) = 0$, then f is also a minimal right $(\mathcal{Y} * \mathcal{X})$-approximation of T.*

Proof. (1) follows from $(\mathcal{X} * \mathcal{Y}) * (\mathcal{X} * \mathcal{Y}) = \mathcal{X} * (\mathcal{Y} * \mathcal{X}) * \mathcal{Y} \subset \mathcal{X} * \mathcal{X} * \mathcal{Y} * \mathcal{Y} = \mathcal{X} * \mathcal{Y}$.

(2) See [IYo, Proposition 2.1 (1)].

(3) The first assertion follows from the proof of [IYo, Proposition 2.3 (1)] and the second one is easy to check. □

0.2. Simple-minded objects in triangulated categories

0.2.1. ***t*-structure and co-*t*-structures.** Let $\mathcal{T}$ be a triangulated category. Let $\mathcal{X}$ and $\mathcal{Y}$ be two full subcategories of $\mathcal{T}$. If $\mathcal{T} = \mathcal{X} \perp \mathcal{Y}$, $\mathcal{X}^{\perp} = \mathcal{Y}$ and ${}^{\perp}\mathcal{Y} = \mathcal{X}$ hold, we say $\mathcal{T} = \mathcal{X} \perp \mathcal{Y}$ is a *torsion pair.* If a torsion pair $\mathcal{T} = \mathcal{X} \perp \mathcal{Y}$ satisfies $\mathcal{X}[1] \subset \mathcal{X}$ (resp. $\mathcal{Y}[1] \subset \mathcal{Y}$), we call it a *t-structure* (resp. *co-t-structure*), in this case, we denote by $\mathcal{H} = \mathcal{X} \cap \mathcal{Y}[1]$ (resp. $\mathcal{P} = \mathcal{X} \cap \mathcal{Y}[-1]$) the *heart* (resp. *co-heart*). We say a t-structure $\mathcal{T} = \mathcal{X} \perp \mathcal{Y}$ is *stable* if $\mathcal{X}[1] = \mathcal{X}$. We say a t-structure $\mathcal{T} = \mathcal{X} \perp \mathcal{Y}$ is *bounded*, if $\bigcup_{i\in\mathbb{Z}} \mathcal{X}[i] = \mathcal{T} = \bigcup_{i\in\mathbb{Z}} \mathcal{Y}[i]$. A bounded t-structure is determined by its heart.

Lemma 0.2.1. *Let $\mathcal{T} = \mathcal{X} \perp \mathcal{Y}$ be a bounded t-structure with heart $\mathcal{H}$. Then $\mathcal{X} = \mathsf{Filt}(\mathcal{H}[\geq 0])$ and $\mathcal{Y} = \mathsf{Filt}(\mathcal{H}[<0])$.*

On the set of t-structures on $\mathcal{T}$, there is a natural partial order defined by

$$(\mathcal{X}, \mathcal{Y}) \geq (\mathcal{X}', \mathcal{Y}') :\Leftrightarrow \mathcal{X} \supset \mathcal{X}' \Leftrightarrow \mathcal{Y} \subset \mathcal{Y}',$$

where $\mathcal{T} = \mathcal{X} \perp \mathcal{Y} = \mathcal{X}' \perp \mathcal{Y}'$ are t-structures with hearts $\mathcal{H}$ and $\mathcal{H}'$ respectively. It induces a partial order on the set of hearts of bounded t-structures by Lemma 0.2.1, that is

$$\mathcal{H} \geq \mathcal{H}' :\Leftrightarrow \mathcal{X} \supset \mathcal{X}' \Leftrightarrow \mathcal{Y} \subset \mathcal{Y}' \Leftrightarrow \mathrm{Hom}_{\mathcal{T}}(\mathcal{H}', \mathcal{H}[<0]) = 0. \tag{0.2.1}$$

An object $P \in \mathcal{T}$ is called *silting object* if $\mathrm{Hom}_{\mathcal{T}}(P, P[>0]) = 0$ and $\mathcal{T} = \mathsf{thick}\, P$. Two silting objects P and Q are *equivalent* if $\mathsf{add}\, P = \mathsf{add}\, Q$. We denote by $\mathsf{silt}\, \mathcal{T}$ the set of equivalence classes of silting objects in $\mathcal{T}$. If $P \in \mathcal{T}$ is silting, then we have a natural co-t-structure

$$\mathcal{T} = \mathcal{T}^P_{\geq 0} \perp \mathcal{T}^P_{<0}, \text{ where } \mathcal{T}^P_{\geq 0} := \mathsf{Filt}(P[\leq 0]) \text{ and } \mathcal{T}^P_{<0} := \mathsf{Filt}(P[>0]). \tag{0.2.2}$$

We have a partial order on $\mathsf{silt}\, \mathcal{T}$, that is, for $P, Q \in \mathsf{silt}\, \mathcal{T}$,

$$P \geq Q :\Leftrightarrow \mathcal{T}^P_{<0} \supset \mathcal{T}^Q_{<0} \Leftrightarrow \mathcal{T}^P_{\geq 0} \subset \mathcal{T}^Q_{\geq 0} \Leftrightarrow \mathrm{Hom}_{\mathcal{T}}(P, Q[>0]) = 0. \tag{0.2.3}$$

Let $\mathcal{S}$ be a thick subcategory of $\mathcal{T}$. Let us recall a sufficient condition for the Verdier quotient $\mathcal{T}/\mathcal{S}$ to be realized as an ideal quotient given in [IYa2]. We consider the following setting.

(T0) $\mathcal{T}$ is a triangulated category, $\mathcal{S}$ is a thick subcategory of $\mathcal{T}$ and $\mathcal{U} = \mathcal{T}/\mathcal{S}$;
(T1) $\mathcal{S}$ has a torsion pair $\mathcal{S} = \mathcal{X} \perp \mathcal{Y}$;
(T2) $\mathcal{T}$ has torsion pairs $\mathcal{T} = \mathcal{X} \perp \mathcal{X}^\perp = {}^\perp\mathcal{Y} \perp \mathcal{Y}$.

Let $\mathcal{Z} := \mathcal{X}^\perp \cap {}^\perp\mathcal{Y}[1]$ and $\mathcal{P} := \mathcal{X}[1] \cap \mathcal{Y}$. Then

Proposition 0.2.2. [IYa2, Theorem 1.1] *Under the assumptions (T0), (T1) and (T2), the composition $\mathcal{Z} \subset \mathcal{T} \to \mathcal{U}$ induces an equivalence of additive category $\frac{\mathcal{Z}}{[\mathcal{P}]} \cong \mathcal{U}$. In particular, the category $\frac{\mathcal{Z}}{[\mathcal{P}]}$ has a structure of a triangulated category.*

Remark 0.2.3. If (T0), (T1) and (T2) hold, we may regard $\mathcal{Z}$ as a Frobenius extriangulated category with $\mathsf{Proj}\, \mathcal{Z} = \mathcal{P}$ in the sense of [NP] (see [IYa2, Section 1.2]).

0.2.2. **Simple-minded collections and simple-minded systems.** Let $\mathcal{T}$ be a Krull-Schmidt triangulated category and let $\mathcal{S}$ be a subcategory of $\mathcal{T}$.

Definition 0.2.4. We call $\mathcal{S}$ a *pre-simple-minded collection* (*pre-SMC*) if for any $X, Y \in \mathcal{S}$, the following conditions hold.

(1) $\mathrm{Hom}_{\mathcal{T}}(X, Y[<0]) = 0$;
(2) $\dim_k \mathrm{Hom}_{\mathcal{T}}(X, Y) = \delta_{X,Y}$.

We call $\mathcal{S}$ a *simple-minded collection* (*SMC*) if $\mathcal{S}$ is a pre-SMC and moreover, $\mathsf{thick}(\mathcal{S}) = T$.

For any pre-SMC, there is a standard t-structure associated to it in the following sense, see [Al, Corollary 3 and Proposition 4] or [KaY1, Proposition 5.4].

Proposition 0.2.5. *Let $\mathcal{R}$ be a pre-SMC of $\mathcal{T}$. Let $\mathcal{X}_{\mathcal{R}} := \mathsf{Filt}(\mathcal{R}[\geq 0])$ and $\mathcal{Y}_{\mathcal{R}} := \mathsf{Filt}(\mathcal{R}[<0])$. $\mathcal{H}_{\mathcal{R}} = \mathsf{Filt}(R)$. Then*

(1) *We have a bounded t-structure $\mathcal{T} = \mathcal{X}_{\mathcal{R}} \perp \mathcal{Y}_{\mathcal{R}}$ with heart $\mathcal{H}_{\mathcal{R}}$.*
(2) *We have $\mathcal{X}_{\mathcal{R}} = \bigcup_{n\geq 0} \mathcal{H}_{\mathcal{R}}[n] * \mathcal{H}_{\mathcal{R}}[n-1] * \cdots * \mathcal{H}_{\mathcal{R}}$ and $\mathcal{Y}_{\mathcal{R}} = \bigcup_{n\geq 1} \mathcal{H}_{\mathcal{R}}[-1] * \cdots * \mathcal{H}_{\mathcal{R}}[-n+1] * \mathcal{H}_{\mathcal{R}}[-n]$.*

Let $\mathcal{S}$ be a SMC in $\mathcal{T}$ and let $\mathcal{H} = \mathsf{Filt}(\mathcal{S})$. We write $\mathcal{T}^{\leq n} = \mathsf{Filt}(\mathcal{S}[\geq n])$ and $\mathcal{T}^{\geq n} = \mathsf{Filt}(\mathcal{S}[\leq n])$. The following result is directly from Proposition 0.2.5.

Lemma 0.2.6. *Let $\mathcal{T}$ be a triangulated category. Let $\mathcal{S}$ be a SMC of $\mathcal{T}$ and $\mathcal{H} = \mathsf{Filt}(\mathcal{S})$. Then*
(1) *We have a bounded t-structure $\mathcal{T} = \mathcal{T}^{\leq 0} \perp \mathcal{T}^{\geq 1}$ with heart $\mathcal{H}$;*
(2) *For any $X, Y \in \mathcal{T}$, we get $\mathrm{Hom}_{\mathcal{T}}(X[\gg 0], Y) = 0$;*
(3) *For any $Y \in \mathcal{T}$, we get $\mathrm{Hom}_{\mathcal{T}}(\mathcal{H}[\gg 0], Y) = 0$ and $\mathrm{Hom}_{\mathcal{T}}(Y[\gg 0], \mathcal{H}) = 0$.*

Next we recall the notion of simple-minded systems, which is introduced in [KL] and generalized in [CS1].

Definition 0.2.7. [CS1, Definition 2.1] Let $d \geq 1$. We call $\mathcal{S}$ a d-Simple-minded system (or d-SMS) if for any $X, Y \in \mathcal{S}$, the following conditions hold.
(1) $\dim\mathrm{Hom}_{\mathcal{T}}(X, Y) = \delta_{X,Y}$;
(2) $\mathrm{Hom}_{\mathcal{T}}(X[i], Y) = 0$ for any $1 \leq i \leq d-1$;
(3) $\mathcal{T} = \mathsf{add}\,\mathsf{Filt}(\{\mathcal{S}[d-1], \cdots, \mathcal{S}\})$.

By [CSP, Lemma 2.8], the condition (3) above is equivalent to say that $\mathcal{T} = \mathcal{H}[d-1] * \mathcal{H}[d-2] * \cdots * \mathcal{H}$.

0.3. DG algebras

0.3.1. **DG algebras and the Nakayama functor.** Let A be a dg k-algebra, that is, a graded algebra endows with a compatible structure of a complex. A (right) dg A-module is a graded A-module endows with a compatible structure of a complex. Let $\mathsf{D}(A)$ be the derived category of right dg A-modules (see [K1, K3]). It is a triangulated category obtained from the category of dg A-modules by formally inverting all quasi-isomorphisms. The shift functor is given by the shift of complexes.

Let $\mathsf{per}\,A = \mathsf{thick}(A_A) \subset \mathsf{D}(A)$ be the perfect category and let $\mathsf{D}^{\mathsf{b}}(A)$ be the full subcategory of $\mathsf{D}(A)$ consisting of the objects whose total cohomology is finite-dimensional.

We consider the derived dg functor

$$\nu := ? \otimes_A^{\mathbf{L}} DA : \mathsf{D}(A) \to \mathsf{D}(A),$$

called the *Nakayama functor*. We have the following Auslander-Reiten formula.

Lemma 0.3.1. [K1, Section 10.1] *There is a bifunctorial isomorphism*

$$D\mathrm{Hom}_{\mathsf{D}A}(X, Y) \cong \mathrm{Hom}_{\mathsf{D}A}(Y, \nu(X)) \tag{0.3.1}$$

for $X \in \mathsf{per}\,A$ and $Y \in \mathsf{D}(A)$.

Proof. For any $Y \in \mathsf{D}(A)$, taking $X = A[n]$, then we have isomorphisms

$$D\mathrm{Hom}_{\mathsf{D}A}(A[n], Y) = D\mathrm{H}^{-n}(Y) \cong \mathrm{H}^n(DY) \cong \mathrm{Hom}_{\mathsf{D}A}(Y, DA[n]).$$

By "devissage", we know the isomorphism holds for any $X \in \mathsf{per}\,A$. □

It is clear that ν restricts to a triangle functor

$$\nu : \mathsf{per}\,A \to \mathsf{thick}(DA). \tag{0.3.2}$$

By Lemma 0.3.1, (0.3.2) is a triangle equivalence provides that A has finite-dimensional cohomology in each degree. In this case, if we have $\mathsf{per}\,A = \mathsf{thick}(DA)$ (for example, A is a finite-dimensional Gorenstein k-algebra), then ν defines a Serre functor on $\mathsf{per}\,A$. Immediately, we have the following result.

Lemma 0.3.2. *Assume A has finite-dimensional cohomology in each degree and* $\mathsf{per}\,A = \mathsf{thick}(DA)$ *in $\mathsf{D}(A)$. Let X, Y be two dg A-modules with finite-dimensional cohomology in each degree. Then the isomorphism* (0.3.1) *also holds for $Y \in \mathsf{per}\,A$ and $X \in \mathsf{D}(A)$.*

Let A be a dg k-algebra and let M be a dg A-module. Then $\mathrm{H}^0(A)$ is the usual k-algebra and we regard $\mathrm{H}^n(M)$ as a $\mathrm{H}^0(A)$-module for $n \in \mathbb{Z}$.

Lemma 0.3.3. *Let A be a dg k-algebra and $M \in \mathsf{D}(A)$. Then*

(1) [KN, Lemma 4.4] *For $P \in \mathsf{add}\, A$, the morphism of k-modules*

$$\mathrm{Hom}_{\mathsf{D}(A)}(P, M) \to \mathrm{Hom}_{\mathrm{H}^0(A)}(\mathrm{H}^0(P), \mathrm{H}^0(M))$$

induced by H^0 is an isomorphism;

(2) *Dually, for $I \in \mathsf{add}\, DA$, the morphism of k-modules*

$$\mathrm{Hom}_{\mathsf{D}(A)}(M, I) \to \mathrm{Hom}_{\mathrm{H}^0(A)}(\mathrm{H}^0(M), \mathrm{H}^0(I))$$

induced by H^0 is an isomorphism.

We need the following lemma for later use.

Lemma 0.3.4. *Let A be a dg k-algebra and $M \in \mathsf{D}(A)$. Then*

(1) *Let $P \in \mathsf{add}\, A$ and $f \in \mathrm{Hom}_{\mathsf{D}(A)}(M, P)$. If the induced map $\mathrm{H}^0(f) : \mathrm{H}^0(M) \to \mathrm{H}^0(P)$ is surjective, then f is a retraction in $\mathsf{D}(A)$;*

(2) *Let $I \in \mathsf{add}\, DA$ and $g \in \mathrm{Hom}_{\mathsf{D}(A)}(I, M)$. If the induced map $\mathrm{H}^0(g) : \mathrm{H}^0(I) \to \mathrm{H}^0(M)$ is injective, then g is a section in $\mathsf{D}(A)$.*

Proof. We only prove (1), since (2) is a dual. Because $\mathrm{H}^0(P)$ is a projective $\mathrm{H}^0(A)$-module and $\mathrm{H}^0(f)$ is surjective, then $\mathrm{H}^0(f)$ it is a retraction. Then by Lemma 0.3.3, there is $p \in \mathrm{Hom}_{\mathsf{D}(A)}(P, M)$, such that $\mathrm{H}^0(f) \circ \mathrm{H}^0(p) = \mathrm{Id}_{\mathrm{H}^0(\mathrm{P})}$. By Lemma 0.3.3 again, we have $f \circ g = \mathrm{Id}_{\mathrm{P}}$. Therefore f is a retraction in $\mathsf{D}(A)$. □

0.3.2. **Non-positive dg algebras.** We call a dg k-algebra A *non-positive* if it satisfies $\mathrm{H}^i(A) = 0$ for $i > 0$. Let A be a non-positive dg k-algebra and M be a dg A-module. We define the *standard truncation* $\tau^{\leq i}$ and $\tau^{>i}$ by

$$(\tau^{\leq i} M)^j := \begin{cases} M^j & \text{for } j < i, \\ \ker d_M^i & \text{for } j = i, \\ 0 & \text{for } j > i. \end{cases} \qquad (\tau^{> i} M)^j := \begin{cases} 0 & \text{for } j < i, \\ M^i / \ker d_M^i & \text{for } j = i, \\ M^j & \text{for } j > i. \end{cases}$$

Since A is non-positive, then $\tau^{\leq i} M$ and $\tau^{>i} M$ are also dg A-modules. Moreover, we have a triangle

$$\tau^{\leq i} M \to M \to \tau^i M \to \tau^{\leq i} M[1]$$

in $\mathsf{D}(A)$. Notice that $A' = \tau^{\leq 0} A$ has a natural structure of a dg algebra and the inclusion $A' \hookrightarrow A$ is a quasi-isomorphism of dg k-algebras. Thus in this thesis, when we mention non-positive dg k-algebra A, we always assume that $A^i = 0$ for $i > 0$. In this case, the canonical projection $A \to \mathrm{H}^0(A)$ is a homomorphism of dg k-algebras (here we regard $\mathrm{H}^0(A)$ as a dg algebra concentrated in degree 0). Then we can regard a module over $\mathrm{H}^0(A)$ as a dg module over A via this homomorphism. This induces a natural functor $\mathsf{mod}\, \mathrm{H}^0(A) \to \mathsf{D}(A)$.

Denote by $\mathcal{S}_A$ the set of simple $\mathrm{H}^0(A)$-modules and we may also regard S_A as the set of simple dg A-modules (concentrated in degree 0). Now we introduce the radical of A, which will be used later. Let $P \in \mathsf{add}\, A$. We have a short exact sequence in $\mathsf{mod}\, \mathrm{H}^0(A)$

$$0 \to \mathrm{rad}\mathrm{H}^0(P) \to \mathrm{H}^0(P) \xrightarrow{f} \mathrm{top}\, \mathrm{H}^0(P) \to 0.$$

By Lemma 0.3.3, there is a morphism $f' \in \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(P, \mathrm{top}\, \mathrm{H}^0(P))$ which is sent to f by H^0. Then we define the radical of P in $\mathsf{D}^{\mathrm{b}}(A)$ as the third term of the following triangle.

$$\mathrm{rad}P \to P \xrightarrow{f'} \mathrm{top}\, \mathrm{H}^0(P) \to \mathrm{rad}P[1].$$

The following results are well-known.

Lemma 0.3.5. *Let A be a non-positive dg k-algebra. Assume $\mathrm{H}^i(A)$ is finite-dimensional for any $i \in \mathbb{Z}$. Then*

(1) *$\mathsf{D}^{\mathrm{b}}(A)$ is Hom-finite.*

(2) *$\mathsf{D}^{\mathrm{b}}(A) = \mathsf{thick}(S_A)$ and S_A is a SMC of $\mathsf{D}^{\mathrm{b}}(A)$.*

Proof. (1) is a corollary of [K1, Theorem 3.1], see also [AMY, Proposition 6.12].
(2) is directly from [KaY2, Proposition 2.1]. □

We need the following useful observation that any SMC of $\mathsf{D}^{\mathrm{b}}(A)$ can be regarded as simple dg B-modules for some non-positive dg algebra B.

Proposition 0.3.6. *Let A be a non-positive proper dg k-algebra and let $\mathcal{S}$ be a SMC of $\mathsf{D}^{\mathrm{b}}(A)$. Then there exists a non-positive dg k-algebra B and a triangle equivalence $F : \mathsf{D}^{\mathrm{b}}(B) \xrightarrow{\simeq} \mathsf{D}^{\mathrm{b}}(A)$ such that $F(\mathcal{S}_B) = \mathcal{S}$, where S_B is the set of simple dg B-modules.*

Proof. There is a bijection

$$\{\text{SMCs of } \mathsf{D}^{\mathrm{b}}(A)\} \longleftrightarrow \{\text{silting objects of } \mathsf{per}\, A\}$$

by [KaY1, Theorem 6.1] (see also [SY, Theorem 1.2]). Then there is a silting object $P \in \mathsf{per}\, A$ corresponding to S. Considering the dg algebra $B' := \mathscr{E}\mathrm{nd}_A(P)$, then we have $\mathrm{H}^i(B') = 0$ for $i > 0$ and the truncation $B := \tau^{\leq 0} B'$ also has a structure of dg k-algebra, which is quasi-isomorphic to B'. Notice that the functor $\mathbf{R}\mathscr{H}\mathrm{om}_A(P, ?) : \mathsf{D}(A) \to \mathsf{D}(B')$ is a triangle equivalence by [K1, Lemma 4.2], so there a triangle equivalence $F : \mathsf{D}(B) \to \mathsf{D}(A)$ which restricts to per and D^{b}. Moreover, by [SY, Theorem 1.1], we have that $F(\mathcal{S}_B) = \mathcal{S}$. □

For $i \in \mathbb{Z}$, Let $\mathsf{D}^{\mathrm{b}}_{\leq i}$ (respectively, $\mathsf{D}^{\mathrm{b}}_{\geq i}$) denote the full subcategory of $\mathsf{D}^{\mathrm{b}}(A)$ consisting of those dg A-modules whose cohomologies are concentrated in degree $\leq i$ (respectively, $\geq i$). This gives us a standard t-structure as the following result shows.

Proposition 0.3.7. [KaY1, Proposition 2.1] *Let A be a non-positive dg algebra. Then*
(1) *The pair $(\mathsf{D}^{\mathrm{b}}_{\leq 0}, \mathsf{D}^{\mathrm{b}}_{\geq 0})$ is a t-structure on $\mathsf{D}^{\mathrm{b}}(A)$;*
(2) *For $n \leq m$, we have the following, where $\mathcal{H}$ is the heart of $(\mathsf{D}^{\mathrm{b}}_{\leq 0}, \mathsf{D}^{\mathrm{b}}_{\geq 0})$.*

$$\mathsf{D}^{\mathrm{b}}_{\geq n} \cap \mathsf{D}^{\mathrm{b}}_{\leq m} = \mathcal{H}[-n] * \mathcal{H}[-n+1] * \cdots * \mathcal{H}[-m];$$

(3) *Moreover, taking H^0 is an equivalence from the $\mathcal{H}$ to $\mathsf{mod}\, \mathrm{H}^0(A)$, and the natural functor $\mathsf{mod}\, \mathrm{H}^0(A) \to \mathsf{D}^{\mathrm{b}}(A)$ is a quasi-inverse to this equivalence.*

We call a dg algebra A *proper*, if $A \in \mathsf{D}^{\mathrm{b}}(A)$. We end this section by a proposition, which plays an important role in the proof of Theorem 2.8.32.

Proposition 0.3.8. *Let A be a non-positive proper dg k-algebra whose underlying graded algebra is a quotient kQ/I of the path algebra of a graded quiver Q. Let j and j' be vertices in Q.*
(1) *If $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(S_j, S_{j'}[l]) \neq 0$ for some $l > 0$, then there exists a path from j to j' with degree bigger than $-l$;*
(2) *Assume that the differential of A is zero and I is an admissible ideal of kQ. If there is an arrow $j \to j'$ with degree $-l \leq 0$, then $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(S_j, S_{j'}[l+1]) \neq 0$.*

Proof. (1) Let $X_0 := S_j$. For each $i \geq 0$, we take the following triangle,

$$X_{i+1} \to Q_i \to X_i \to X_{i+1}[1],$$

such that $Q_i \in \mathsf{add}\, A[\geq 0]$ and the induced map $\mathrm{H}^*(Q_i) \to \mathrm{H}^*(X_i)$ is the projective cover of $\mathrm{H}^*(X_i)$ as a graded $\mathrm{H}^*(A)$-module. Then we have an exact sequence

$$0 \to \mathrm{H}^*(X_{i+1}) \to \mathrm{H}^*(Q_i) \to \mathrm{H}^*(X_i) \to 0.$$

Thus the composition $Q_{i+1} \to X_{i+1} \to Q_i$ is non-zero. For each direct summand $P_{a_i}[s_i]$ of Q_i with a vertex a_i of Q and $s_i \in \mathbb{Z}$, there exists a direct summand $P_{a_{i-1}}[s_{i-1}]$ of Q_{i-1} with a vertex a_{i-1} in Q and $s_{i-1} \in \mathbb{Z}$, such that $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(P_{a_i}[s_i], P_{a_{i-1}}[s_{i-1}]) \neq 0$. Then there is a path $a_{i-1} \rightsquigarrow a_i$ with degree $s_{i-1} - s_i$. Repeating this, we obtain a path $j = a_0 \rightsquigarrow a_1 \rightsquigarrow \cdots \rightsquigarrow a_i$ with degree $\sum_{k=1}^{i}(s_{k-1} - s_k) = s_0 - s_i = -s_i$.

By the construction above, we have $S_j \in Q_0 * Q_1[1] * Q_2[2] * \cdots * Q_l[l] * \mathsf{D}^{\mathrm{b}}_{\leq -l-1}$. Since $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(Y, S_{j'}[l]) = 0$ for any $Y \in \mathsf{D}^{\mathrm{b}}_{\leq -l-1}$ and by our assumption $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(S_j, S_{j'}[l]) \neq 0$, then there exists a non-zero map from some object in $\mathsf{add}(Q_0 * Q_1[1] * Q_2[2] * \cdots * Q_l[l])$ to $S_{j'}[l]$, which implies $P_{j'}[l] \in \mathsf{add}\, Q_k[k]$ for some $0 \le k \le l$. Since $Q_0 = P_j$ and l is positive, we have $1 \le k \le l$. Since $P_{j'}[l-k] \in \mathsf{add}\, Q_k$, by our argument above, there is a path from j to j' with degree $k - l$, which is bigger than $-l$.

(2) Consider the following triangle,

$$\mathrm{rad}P_j \to P_j \to S_j \to \mathrm{rad}P_j[1].$$

Since the differential of A is zero, I is admissible, and there is an arrow $j \to j'$ with degree $-l$, then $S_{j'} \in \mathsf{add}\, \mathrm{H}^{-l}(\frac{\mathrm{rad}P_j}{\mathrm{rad}^2 P_j})$. Then the composition $\mathrm{rad}P_j \to \frac{\mathrm{rad}P_j}{\mathrm{rad}^2 P_j} \to \mathrm{H}^{-l}(\frac{\mathrm{rad}P_j}{\mathrm{rad}^2 P_j})[l] \to S_{j'}[l]$ is non-zero. Thus $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(\mathrm{rad}P_j, S_{j'}[l]) \neq 0$. Applying the functor $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(?, S_{j'}[l+1])$ to the triangle above, we obtain an exact sequence

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(S_j, S_{j'}[l]) \to \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(P_j, S_{j'}[l]) \to \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(\mathrm{rad}P_j, S_{j'}[l]) \to \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(S_j, S_{j'}[l+1]).$$

By dividing into two cases, $(l, j) = (0, j')$ or not, one can check that the left map is always surjective. Then $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(S_j, S_{j'}[l+1]) \neq 0$. □

0.4. Extriangulated categories

0.4.1. **Extriangulated categories.** In this section, we briefly recall the definition and basic properties of extriangulated categories from [NP]. We omit some details here, but the reader can find them in [NP].

Let $\mathscr{C}$ be an additive category equipped with an additive bifunctor $\mathbb{E} : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \to Ab$. For any pair of objects $A, C \in \mathscr{C}$, an element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-*extension*. Let $\mathfrak{s}$ be a correspondence which associates an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ to any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. This $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$ if it makes the diagrams in [NP, Definition 2.9] commutative. A triple $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is called an *extriangulated category* if it satisfies the following conditions.

(1) $\mathbb{E} : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \to Ab$ is an additive bifunctor;
(2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$;
(3) $\mathbb{E}$ and $\mathfrak{s}$ satisfy the compatibility conditions in [NP, Definition 2.12].

Extriangulated categories is a generalization of exact categories and triangulated categories. Let us see some easy examples.

Example 0.4.1. (1) Let $\mathscr{C}$ be an exact category. Then $\mathscr{C}$ is extriangulated by taking $\mathbb{E}$ as the bifunctor $\mathrm{Ext}^1_{\mathscr{C}}(?, ?) : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \to Ab$ and for any $\delta \in \mathrm{Ext}^1_{\mathscr{C}}(C, A)$, taking $\mathfrak{s}(\delta)$ as the equivalence class of short exact sequences (=conflations) correspond to δ;
(2) Let $\mathscr{C}$ be a triangulated category. Then $\mathscr{C}$ is extriangulated by taking $\mathbb{E}$ as the bifunctor $\mathrm{Hom}_{\mathscr{C}}(?, ?[1]) : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \to Ab$, and for any $\delta \in \mathrm{Hom}_{\mathscr{C}}(C, A[1])$, taking $\mathfrak{s}(\delta)$ as the equivalence class of the triangle $A \to B \to C \xrightarrow{\delta} A[1]$;
(3) Let $\mathscr{C}$ be a triangulated category and let $\mathscr{D}$ be an extension-closed (that is, for any triangle $X \to Y \to Z \to X[1]$ in $\mathscr{C}$, if $X, Z \in \mathscr{D}$, then $Y \in \mathscr{D}$) subcategory of $\mathscr{C}$. Then $\mathscr{D}$ has an extriangulated structure given by restricting the extriangulated structure of $\mathscr{C}$ on $\mathscr{D}$.

Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. An object X in $\mathscr{C}$ is called *projective* if $\mathbb{E}(X, Y) = 0$ for any $Y \in \mathscr{C}$. We say $\mathscr{C}$ has *enough projective objects* if for any $Y \in \mathscr{C}$, there exists $Z \in \mathscr{C}$ and $\delta \in \mathbb{E}(Y, Z)$, such that the middle term of the realization $\mathfrak{s}(\delta)$ is projective. We denote by $\mathscr{P}$ (resp. $\mathscr{I}$) the subcategory of projective (resp. injectvie) objects. When $\mathscr{C}$ has enough projective (resp. injective) objects, we define *the stable (resp. costable) category of* $\mathscr{C}$ as the ideal quotient $\underline{\mathscr{C}} := \mathscr{C}/[\mathscr{P}]$ (resp. $\overline{\mathscr{C}} := \mathscr{C}/[\mathscr{I}]$). We call $\mathscr{C}$ *Frobenius* if it has enough projective objects and enough injective objects, and projective objects coincide with injective ones. In this case $\underline{\mathscr{C}}$ coincides with $\overline{\mathscr{C}}$, and we call $\underline{\mathscr{C}}$ *the stable category of* $\mathscr{C}$.

Proposition 0.4.2. [NP, Corollary 7.4] *Let* $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ *be a Frobenius extriangulated category and let* $\mathcal{I}$ *be subcategory of injective objects. Then* $\underline{\mathscr{C}}$ *is a triangulated category.*

0.4.2. **Auslander-Reiten theory in extriangulated categories.** Let us briefly recall Auslander-Reiten theory in extriangulated categories form [INP]. In this subsection, let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category.

Definition 0.4.3. [INP, Definition 2.1] A non-split $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$ is said to be *almost split* if it satisfies the following conditions

(1) $\mathbb{E}(C, a)(\delta) = 0$ for any non-section $a \in \mathscr{C}(A, A')$;
(2) $\mathbb{E}(c, A)(\delta) = 0$ for any non-retraction $c \in \mathscr{C}(C', C)$.

We say that $\mathscr{C}$ *has right almost split extensions* if for any endo-local non-projective object $A \in \mathscr{C}$, there exists an almost split extension $\delta \in \mathbb{E}(A, B)$ for some $B \in \mathscr{C}$. Dually, we say that $\mathscr{C}$ *has left almost split extensions* if for any endo-local non-projective object $B \in \mathscr{C}$, there exists an almost split extension $\delta \in \mathbb{E}(A, B)$ for some $A \in \mathscr{C}$. We say that $\mathscr{C}$ *has almost split extension* if it has right and left almost split extensions.

Let $A \in \mathscr{C}$. If there exists an almost split extension $\delta \in \mathbb{E}(A, B)$, then it is unique up to isomorphism of $\mathbb{E}$-extensions.

Definition 0.4.4. [INP, Definition 3.2] Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be a k-linear extriangulated category.

(1) A *right Auslander-Reiten-Serre (ARS) duality* is a pair (τ, η) of an additive functor $\tau : \underline{\mathscr{C}} \to \overline{\mathscr{C}}$ and a binatural isomorphism
$$\eta_{A,B} : \underline{\mathscr{C}}(A, B) \simeq D\mathbb{E}(B, \tau A) \text{ for any } A, B \in \mathscr{C};$$
(2) If moreover τ is an equivalence, we say that (τ, η) is an *Auslander-Reiten-Serre (ARS) duality.*

We say k-linear extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is *Ext-finite*, if $\dim_k \mathbb{E}(A, B) < \infty$ for any $A, B \in \mathscr{C}$.

Proposition 0.4.5. [INP, Theorem 3.4] *Let $\mathscr{C}$ be a k-linear Ext-finite Krull-Schmidt extriangulated category. Then the following are equivalent.*

(1) *$\mathscr{C}$ has almost split extensions;*
(2) *$\mathscr{C}$ has an Auslander-Reiten-Serre duality.*

The following characterization of almost split extensions are analogous to the corresponding result on Auslander-Reiten triangles (see [RV, Proposition I.2.1]) and on almost split sequences (see [ARS]).

Proposition 0.4.6. *Assume $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ has Auslander-Reiten extensions. Assume $A \in \mathscr{C}$ is an end-local object and $\delta \in \mathbb{E}(A, B)$. Then the following are equivalent.*

(1) *δ is an almost split extension;*
(2) *δ is in the socle of $\mathbb{E}(A, B)$ as right $\mathrm{End}_{\mathscr{C}}(A)$-module and $B \cong \tau(A)$;*
(3) *δ is in the socle of $\mathbb{E}(A, B)$ as left $\mathrm{End}_{\mathscr{C}}(B)$-module and $B \cong \tau(A)$.*

0.5. Translation quivers

We recall some definitions and notations concerning quivers. A *quiver* $Q = (Q_0, Q_1, s, t)$ is given by the set Q_0 of its vertices, the set Q_1 of its arrows, a source map s and a target map t. If $x \in Q_0$ is a vertex, we denote by x^+ the set of direct successors of x, and by x^- the set of its direct predecessors. We say that Q is *locally finite* if for each vertex $x \in Q_0$, there are finitely many arrows ending at x and starting at x. An automorphism group G of Q is said to be *weakly admissible* if for each $g \in G \backslash \{1\}$ and for each $x \in Q_0$, we have $x^+ \cap (gx)^+ = \emptyset$.

A *stable translation quiver* (Q, τ) is a locally finite quiver Q without double arrows with a bijection $\tau : Q_0 \to Q_0$ such that $(\tau x)^+ = x^-$ for each vertex x. For each arrow $\alpha : x \to y$, we denote by $\sigma\alpha$ the unique arrow $\tau y \to x$.

Definition 0.5.1. Let Q be a stable translation quiver and C be a subset of Q_0. We define a translation quiver Q_C by adding to Q_0 a vertex p_c and two arrows $c \to p_c \to \tau^{-1}(c)$ for each $c \in C$. The translation of Q_C coincides with the translation of Q on Q_0 and is not defined on $\{p_c \mid c \in C\}$.

Let Δ be an oriented tree, then the *repetition quiver of* Δ is defined as follows:

(1) $(\mathbb{Z}\Delta)_0 = \mathbb{Z} \times \Delta_0$

(2) $(\mathbb{Z}\Delta)_1 = \mathbb{Z}\times\Delta_1 \cup \sigma(\mathbb{Z}\times\Delta_1)$ with arrows $(n,\alpha) : (n,x) \to (n,y)$ and $\sigma(n,\alpha) : (n-1,y) \to (n,x)$ for each arrow $\alpha : x \to y$ of Δ.

The quiver $\mathbb{Z}\Delta$ with the translation $\tau(n,x) = (n-1,x)$ is a stable translation quiver which does not depend (up to isomorphism) on the orientation of Δ which does not depend on the orientation of Δ (see [Rie1]).

From now on, we assume Δ is a Dynkin diagram. Let us fix a numbering and an orientation of the simply-laced Dynkin trees.

$$A_n(n \geq 1): \qquad 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n$$

$$D_n(n \geq 4): \qquad 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-2 \longleftarrow n-1, \qquad n-2 \longleftarrow n$$

$$E_n(n = 6,7,8): \qquad 1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 5 \longrightarrow \cdots \longrightarrow n, \qquad 3 \longrightarrow 4$$

We define the "Nakayama permutation" $\mathbb{S}$ of $\mathbb{Z}\Delta$ as follows:

- if $\Delta = A_n$, then $\mathbb{S}(p,q) = (p+q-1, n+1-q)$;
- if $\Delta = D_n$ with n even, then $\mathbb{S} = \tau^{-n+2}$;
- if $\Delta = D_n$ with n odd, then $\mathbb{S} = \tau^{-n+2}\phi$, where ϕ is the automorphism which exchanges n and $n-1$;
- if $\Delta = E_6$, then $\mathbb{S} = \phi\tau^{-5}$, where ϕ is the automorphism which exchanges 2 and 5, and 1 and 6;
- if $\Delta = E_7$, then $\mathbb{S} = \tau^{-8}$;
- if $\Delta = E_8$, then $\mathbb{S} = \tau^{-14}$.

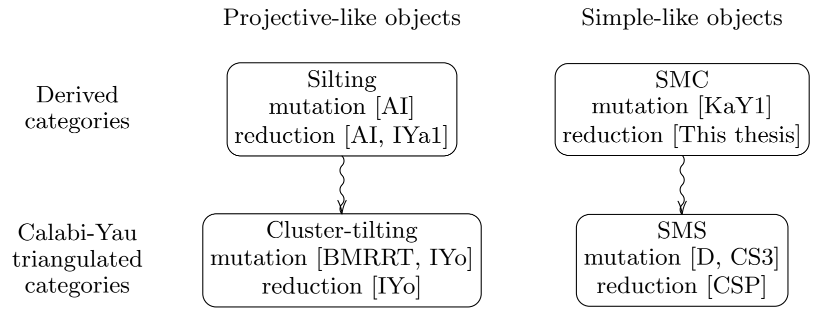
By [Ga, Proposition 6.5], when we identify the Auslander-Reiten quiver of $k\Delta$ as the full subquiver of $\mathbb{Z}\Delta$, the Nakayama functor is related to $\mathbb{S}$ defined above. We can also define "shift permutation" [1] of $\mathbb{Z}\Delta$ by $\mathbb{S}\tau^{-1}$.

Part 1. Reductions of triangulated categories and simple-minded collections

1.1. Introduction

Triangulated categories appear in many branches of mathematics, such as algebraic geometry, representation theory and algebraic topology. In derived categories, there are two important classes of objects: projective objects and simple objects. Projective objects (or more generally, tilting objects) play a central role in tilting theory, which is one of the standard tools for studying triangulated categories. Their variants, silting objects and cluster tilting objects, have been used to study positive Calabi-Yau (= CY) triangulated categories [BMRRT, IYo, KR, KMV] and the categorification of cluster category [FZ1]. On the other hand, simple objects, or more generally, simple-minded collections (=SMCs) are also well-studied in derived categories. They are important in Koszul duality [BGS, KN], and bijectively correspond to silting objects [Ric2, KaY1]. Simple-minded systems (= SMSs) in stable module categories were introduced in [KL] and studied for negative CY triangulated categories in [D, CS3]. Recently, there is increasing interest in negative CY triangulated categories (see, for example [CS1, CS2, CS3, CSP]), including the stable categories of Cohen-Macaulay (= CM) dg modules [Ji1].

There are two useful tools to study the class of silting (resp. cluster-tilting, SMC, SMS) objects in a triangulated category $\mathcal{T}$. One is mutation, which gives a new object in this class from a given one. Another is reduction, which is a new triangulated category $\mathcal{U}$ realized as a certain sub (or subfactor) category of $\mathcal{T}$. There is a bijection between silting (resp. cluster-tilting, SMS, SMC) objects in $\mathcal{U}$ and those in $\mathcal{T}$ with some properties. The following picture shows some works on these subjects, where the reduction of SMC was not studied before.



Thus our first aim of Part 1 is to introduce the *SMC reduction*. For a pre-SMC $\mathcal{R}$ (which is a SMC without generating condition) of a Krull-Schmidt triangulated category $\mathcal{T}$, the corresponding SMC reduction is the Verdier quotient $\mathcal{U} = \mathcal{T}/\operatorname{\mathsf{thick}}(\mathcal{R})$. One can realize $\mathcal{U}$ as the additive subcategory

$$\mathcal{Z} = \mathcal{R}[\geq 0]^{\perp} \cap {}^{\perp}\mathcal{R}[\leq 0]$$

of $\mathcal{T}$ under certain assumptions (R1) and (R2) in Section 1.2. Namely,

Theorem 1.1.1 (Theorem 1.2.1)**.** *Under the setting above, the following results hold.*

(1) *The composition $\mathcal{Z} \hookrightarrow \mathcal{T} \to \mathcal{U}$ gives an equivalence $\mathcal{Z} \xrightarrow{\simeq} \mathcal{U}$;*
(2) *There is a bijection*

$$\mathrm{SMC}_{\mathcal{R}}\mathcal{T} := \{\textit{SMCs in } \mathcal{T} \textit{ contain } \mathcal{R}\} \longleftrightarrow \mathrm{SMC}\,\mathcal{U} := \{\textit{SMCs in } \mathcal{U}\}.$$

Since $\mathcal{Z}$ is not closed under $[\pm 1]$, it dose not have a triangulated structure a priori. Nevertheless, the theorem above shows that it has a canonical triangulated structure induced by $\mathcal{U}$. Also notice that, Theorem 1.1.1 can be regarded as a dual of silting reduction [IYa1], where it was necessary to take an ideal quotient of $\mathcal{Z}$. Theorem 1.1.1 will be used to construct SMCs and it plays an important role in the proof of Theorem 2.7.1.

The second aim of Part 1 is to generalize the singularity category of a finite dimensional Gorenstein k-algebra A over a field k. In this case, the singularity category is defined as the Verdier quotient $\mathsf{D}_{\mathrm{sg}}(A) = \mathsf{D}^{\mathrm{b}}(\operatorname{\mathsf{mod}} A)/\mathsf{K}^{\mathrm{b}}(\operatorname{\mathsf{proj}} A)$ by [B, O]. Buchweitz's equivalence states that $\mathsf{D}_{\mathrm{sg}}(A)$ is triangle equivalence to the stable category $\underline{\mathsf{CM}}A$ of Cohen-Macaulay A-modules. A key observation in our context is that $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$ has a SMC consisting of simple A-modules, and there is a relative Serre functor $\nu = ? \otimes^{\mathbf{L}}_A DA$.

To generalize the notion of singularity categories and Buchweitz's equivalence, we work on a *SMC quadruple* $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathbb{S}, \mathcal{S})$, where $\mathcal{T}^{\mathrm{p}}$ is a thick subcategory of a triangulated category $\mathcal{T}$, $\mathbb{S}$ is a relative Serre functor, $\mathcal{S}$ is a SMC of $\mathcal{T}$ and they satisfy some conditions (see Definition 1.3.1). We define the *singularity category* as the Verdier quotient

$$\mathcal{T}_{\mathrm{sg}} := \mathcal{T}/\mathcal{T}^{\mathrm{p}}.$$

In this setting, we have a co-t-structure $\mathcal{T} = \mathcal{T}_{>0} \perp \mathcal{T}_{\leq 0}$, where $\mathcal{T}_{>0} = {}^{\perp}\mathcal{S}[\geq 0]$ and $\mathcal{T}_{\leq 0} = {}^{\perp}\mathcal{S}[<0]$. Using them we define subcategories

$$\mathcal{F} = \mathcal{T}_{>0}^{\perp} \cap {}^{\perp}(\mathcal{T}_{\leq -1} \cap \mathcal{T}^{\mathrm{p}}),\ \mathcal{P} = \mathcal{T}_{\geq 0} \cap \mathcal{T}_{\leq 0},$$

where in the algebra case above, $\mathcal{F} = \mathsf{CM}\, A$ and $\mathcal{P} = \mathsf{proj}\, A$. Our second result realizes $\mathcal{T}_{\mathrm{sg}}$ as a subfactor category of $\mathcal{T}$.

Theorem 1.1.2 (Theorem 1.3.5 (1), (2))**.** *Let $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathbb{S}, \mathcal{S})$ be a SMC quadruple and let $\mathcal{T}_{\mathrm{sg}}$, $\mathcal{F}$, $\mathcal{P}$ be defined as above. Then*

(1) *$\mathcal{F}$ is a Frobenius extriangulated category with* $\mathsf{Proj}\, \mathcal{F} = \mathcal{P}$ (in the sense of [NP]);
(2) The composition

$$\mathcal{F} \subset \mathcal{T} \to \mathcal{T}_{\mathrm{sg}}$$

induces an equivalence $\pi : \frac{\mathcal{F}}{[\mathcal{P}]} \xrightarrow{\simeq} \mathcal{T}_{\mathrm{sg}}$. Moreover, $\mathcal{T}_{\mathrm{sg}}$ has a Serre functor $\mathbb{S}[-1]$.

Theorem 1.3.5 can be regard as a dual of the equivalence between the fundamental domain and the cluster category [Am, Gu, IYa1], where it was not necessary to take ideal quotient.

An important case of Serre quadruple is non-positive *CY triple*, which is a Serre quadruple $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathbb{S}, \mathcal{S})$ with $\mathbb{S} = [1-d]$ for $d \geq 1$. In the rest part of introduction, we will focus on $(1-d)$-CY triple. In this case, there is a nice description of $\mathcal{F}$ as follows.

Proposition 1.1.3 (Theorem 1.3.5 (3))**.** *Let $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathcal{S})$ be a $(1-d)$-CY triple. Then $\mathcal{F} = \mathcal{H}[d-1] * \mathcal{H}[d-2] * \cdots * \mathcal{H}$ and $\mathcal{S}$ is a d-SMS in $\mathcal{T}_{\mathrm{sg}}$, where $\mathcal{H} = \mathsf{Filt}\mathcal{S}$ is the extension-closed subcategory generated by $\mathcal{S}$.*

The third aim of Part 1 is to connect our SMC reductions and the SMS reductions defined by Coelho Simões and Pauksztello [CSP]. We first show that the SMC reduction of a CY triple gives rise to a new CY triple.

Theorem 1.1.4 (Theorem 1.5.1)**.** *Let $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathcal{S})$ be a $(1-d)$-CY triple. Let $\mathcal{R}$ be a subset of $\mathcal{S}$ such that the extension-closed subcategory $\mathcal{H}_{\mathcal{R}}$ generated by $\mathcal{R}$ is functorially finite in $\mathcal{T}$. Let $\mathcal{U}$ be the SMC reduction of $\mathcal{T}$ with respect to $\mathcal{R}$. Then the triple $(\mathcal{U}, \mathcal{U}^{\mathrm{p}}, \mathcal{S})$ is also a $(1-d)$-CY triple, where one can regard $\mathcal{U}^{\mathrm{p}} := \mathcal{T}^{\mathrm{p}} \cap (\mathsf{thick}\, \mathcal{R})^{\perp}$ as a subcategory of $\mathcal{U}$.*

For a $(1-d)$-CY triple $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathcal{S})$, we know $\mathcal{T}_{\mathrm{sg}}$ is a $(-d)$-CY triangulated category by Theorem 1.1.2 (2), and we can consider the SMS reduction $(\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}$ in $\mathcal{T}_{\mathrm{sg}}$ with respect to $\mathcal{R}$ introduced in [CSP]. Our main theorem of Part 1 shows that SMS reduction is the shadow of SMC reduction in the following sense.

Theorem 1.1.5 (Theorem 1.5.4)**.** *Keep the assumption in Theorem* 1.1.4*. Then there is a triangle equivalence from the singularity category $\mathcal{U}_{\mathrm{sg}}$ to the SMS reduction $(\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}$ of the singularity category $\mathcal{T}_{\mathrm{sg}}$ with respect to $\mathcal{R}$.*

This can be illustrated by the following commutative diagram of operations.

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\text{sing. category}} & \mathcal{T}_{\mathrm{sg}} \\ {\scriptstyle\text{SMC reduction}}\big\downarrow & & \wr\big\downarrow{\scriptstyle\text{SMS reduction}} \\ \mathcal{U} & \xrightarrow{\text{sing. category}} & \mathcal{U}_{\mathrm{sg}} \cong (\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}} \end{array}$$

The diagram above induces a commutative diagram of maps

$$\begin{array}{ccc} \text{SMCs in } \mathcal{T} \text{ contains } \mathcal{R} & \longrightarrow & \text{SMSs in } \mathcal{T}_{\mathrm{sg}} \text{ contains } \mathcal{R} \\ \big\downarrow & & \big\downarrow \\ \text{SMCs in } \mathcal{U} & \longrightarrow & \text{SMSs in } (\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}} \end{array}$$

where the horizontal two maps above are well-defined under mild conditions (see Theorem 1.3.14). The results we obtain here are parallel to the connection between silting reductions and CY reductions given in [IYa1].

In Appendix A, we give a triangle equivalence induced by derived Schur functors. It provides us an important class of examples on SMC reduction and it is also useful itself.

1.2. SMC reductions of triangulated categories

The aim of this section is to introduce the SMC reduction. It is an operation to construct a new triangulated category form the given triangulated category and one of its pre-Simple-minded collections (pre-SMCs). One important property is that, under mild conditions, there is a bijection between the SMCs of the new category and the SMCs of the original one containing the given pre-SMC.

1.2.1. **SMC reductions.** Let $\mathcal{T}$ be a Krull-Schmidt triangulated category and $\mathcal{R}$ be a pre-SMC of $\mathcal{T}$ (see Definition 0.2.4). We denote by $\mathsf{SMC}\,\mathcal{T}$ the set of SMCs of $\mathcal{T}$ and by $\mathsf{SMC}_{\mathcal{R}}\,\mathcal{T}$ the set of SMCs of $\mathcal{T}$ containing $\mathcal{R}$. We define the *SMC reduction* of $\mathcal{T}$ with respect to $\mathcal{R}$ as the Verdier quotient

$$\mathcal{U} := \mathcal{T}/\,\mathsf{thick}(\mathcal{R}).$$

By Proposition 0.2.5, $\mathsf{thick}(\mathcal{R})$ admits a natural t-structure $\mathsf{thick}(\mathcal{R}) = \mathcal{X}_{\mathcal{R}} \perp \mathcal{Y}_{\mathcal{R}}$, where $\mathcal{X}_{\mathcal{R}} = \mathsf{Filt}(\mathcal{R}[\geq 0])$ and $\mathcal{Y}_{\mathcal{R}} = \mathsf{Filt}(\mathcal{R}[<0])$, whose heart is denote by

$$\mathcal{H}_{\mathcal{R}} = \mathsf{Filt}(\mathcal{R}).$$

Consider the following mild conditions.

- (R1) $\mathcal{H}_{\mathcal{R}}$ is contravariantly finite in $\mathcal{R}[>0]^{\perp}$ and convariantly finite in ${}^{\perp}\mathcal{R}[<0]$;
- (R2) For any $X \in \mathcal{T}$, we have $\mathrm{Hom}_{\mathcal{T}}(X, \mathcal{H}_{\mathcal{R}}[i]) = 0 = \mathrm{Hom}_{\mathcal{T}}(\mathcal{H}_{\mathcal{R}}, X[i])$ for $i \ll 0$.

Notice that by Lemma 0.2.6, (R2) holds if there is a SMC of $\mathcal{T}$ containing $\mathcal{R}$. Let

$$\mathcal{Z} := \mathcal{R}[\geq 0]^{\perp} \cap {}^{\perp}\mathcal{R}[\leq 0].$$

Similar to silting reduction (see [IYa1, Theorems 3.1 and 3.7]), we have the following results.

Theorem 1.2.1. *Assume the assumptions (R1) and (R2) hold. Then*

(1) *The composition* $\mathcal{Z} \hookrightarrow \mathcal{T} \to \mathcal{U}$ *is an additive equivalence* $\mathcal{Z} \xrightarrow{\simeq} \mathcal{U}$*;*
(2) *There is a bijection*

$$\mathsf{SMC}_{\mathcal{R}}\,\mathcal{T} \longleftrightarrow \mathsf{SMC}\,\mathcal{U},$$

sending $\mathcal{S} \in \mathsf{SMC}_{\mathcal{R}}\,\mathcal{T}$ *to* $\mathcal{S}\backslash\mathcal{R} \in \mathsf{SMC}\,\mathcal{U}$*.*

The rest of this section is devoted to the proof of Theorem 1.2.1. We start with the following observation, which is the 'dual' of [IYa1, Proposition 3.2].

Proposition 1.2.2. *The following are equivalent.*
(1) $\mathcal{T} = \mathcal{X}_{\mathcal{R}} \perp \mathcal{X}_{\mathcal{R}}^{\perp} = {}^{\perp}\mathcal{Y}_{\mathcal{R}} \perp \mathcal{Y}_{\mathcal{R}}$ *are two t-structures;*
(2) $\mathcal{H}_{\mathcal{R}}$ *satisfies the conditions (R1) and (R2).*
In this case, the heart of t-structures in (1) are $\mathcal{H}_{\mathcal{R}}$.

Proof. We first claim that $\mathcal{X}_{\mathcal{R}} \cap \mathcal{X}_{\mathcal{R}}^{\perp}[1] = \mathcal{H}_{\mathcal{R}} = {}^{\perp}\mathcal{Y}_{\mathcal{R}} \cap \mathcal{Y}_{\mathcal{R}}[1]$. We only show the first equality since the second one is dual. Since $\mathcal{R}$ is a pre-SMC, then $\mathrm{Hom}_{\mathcal{T}}(\mathcal{X}_{\mathcal{R}}[1], \mathcal{H}_{\mathcal{R}}) = 0$ and thus $\mathcal{H}_{\mathcal{R}} \subset \mathcal{X}_{\mathcal{R}} \cap \mathcal{X}_{\mathcal{R}}^{\perp}[1]$. Now assume $X \in \mathcal{X}_{\mathcal{R}} \cap \mathcal{X}_{\mathcal{R}}^{\perp}[1]$. Since we know $X \in \mathcal{X}_{\mathcal{R}} = \mathcal{X}_{\mathcal{R}}[1] * \mathcal{H}_{\mathcal{R}}$ by Proposition 0.2.5 and $\mathrm{Hom}_{\mathcal{T}}(\mathcal{X}_{\mathcal{R}}[1], X) = 0$, then it is clear that $X \in \mathsf{add}\, \mathcal{H}_{\mathcal{R}} = \mathcal{H}_{\mathcal{R}}$ (Since $\mathcal{H}_{\mathcal{R}}$ is the heart of a t-structure, so $\mathsf{add}\, \mathcal{H}_{\mathcal{R}} = \mathcal{H}_{\mathcal{R}}$). Therefore $\mathcal{X}_{\mathcal{R}} \cap \mathcal{X}_{\mathcal{R}}^{\perp}[1] = \mathcal{H}_{\mathcal{R}}$.

(1) $\Rightarrow$ (2) We show (R1). For any $X \in \mathcal{R}[>0]^{\perp} = \mathcal{X}_{\mathcal{R}}^{\perp}[1]$, there is a triangle

$$\mathcal{Z}[-1] \to Y \xrightarrow{f} X \to Z$$

with $Y \in \mathcal{X}_{\mathcal{R}}$ and $Z \in \mathcal{X}_{\mathcal{R}}^{\perp}$. We claim $Y \in \mathcal{H}_{\mathcal{R}}$ and f is a right $\mathcal{H}_{\mathcal{R}}$-approximation of X. Since $\mathcal{X}_{\mathcal{R}}^{\perp}[-1] \subset \mathcal{X}_{\mathcal{R}}^{\perp}[1]$, then $Y \in Z[-1] * X \in \mathcal{X}_{\mathcal{R}}^{\perp}[1]$ and thus $Y \in \mathcal{X}_{\mathcal{R}}^{\perp}[1] \cap \mathcal{X}_{\mathcal{R}} = \mathcal{H}_{\mathcal{R}}$. Since $\mathrm{Hom}_{\mathcal{T}}(\mathcal{H}_{\mathcal{R}}, Z) = 0$, then it follows that f is a right $\mathcal{H}_{\mathcal{R}}$-approximation. So $\mathcal{H}_{\mathcal{R}}$ is contravariantly finite in $\mathcal{R}[>0]^{\perp}$. Dually, $\mathcal{H}_{\mathcal{R}}$ is convariantly finite in ${}^{\perp}\mathcal{R}[<0]$.

We show (R2). For any $T \in \mathcal{T}$, consider the triangle $T' \to T \to T'' \to T'[1]$ with $T' \in \mathcal{X}_{\mathcal{R}}$ and $T'' \in \mathcal{X}_{\mathcal{R}}^{\perp}$. Since $\mathrm{Hom}_{\mathcal{T}}(\mathcal{H}_{\mathcal{R}}[\gg 0], T') = 0$ by Lemma 0.2.6 and $\mathrm{Hom}_{\mathcal{T}}(\mathcal{H}_{\mathcal{R}}[\geq 0], T'') = 0$, then we know $\mathrm{Hom}_{\mathcal{T}}(\mathcal{H}_{\mathcal{R}}[\gg 0], T) = 0$. The dual argument shows $\mathrm{Hom}_{\mathcal{T}}(T[\gg 0], \mathcal{H}_{\mathcal{R}}) = 0$.

(2) $\Rightarrow$ (1) We only show $\mathcal{T} = \mathcal{X}_{\mathcal{R}} \perp \mathcal{X}_{\mathcal{R}}^{\perp}$ is a t-structure, because the other assertion can be shown similarly. Since $\mathcal{X}_{\mathcal{R}}[1] \subset \mathcal{X}_{\mathcal{R}}$, it is enough to show $\mathcal{T} = \mathcal{X}_{\mathcal{R}} * \mathcal{X}_{\mathcal{R}}^{\perp}$. Let $X \in \mathcal{T}$. We have $\mathrm{Hom}_{\mathcal{T}}(\mathcal{H}_{\mathcal{R}}[\geq l], X) = 0$ for some $l \in \mathbb{Z}$ by (R2). Notice that by Proposition 0.2.5, $\mathcal{X}_{\mathcal{R}} = \bigcup_{n \geq 0} \mathcal{H}_{\mathcal{R}}[n] * \cdots * \mathcal{H}_{\mathcal{R}}$. If $l \leq 0$, then we get $X \in \mathcal{H}_{\mathcal{R}}[\geq l]^{\perp} \subset \mathcal{H}_{\mathcal{R}}[\geq 0]^{\perp} = \mathcal{X}_{\mathcal{R}}^{\perp}$, and thus $X \in \mathcal{X}_{\mathcal{R}} * \mathcal{X}_{\mathcal{R}}^{\perp}$.

Next we use the induction on l to prove $X \in \mathcal{X}_{\mathcal{R}} * \mathcal{X}_{\mathcal{R}}^{\perp}$ generally. We assume $\mathcal{H}_{\mathcal{R}}[\geq l-1]^{\perp} \subset \mathcal{X}_{\mathcal{R}} * \mathcal{X}_{\mathcal{R}}^{\perp}$ for some $l > 0$. By assumption (R1), there exists a triangle $H[l-1] \xrightarrow{f} X \to X' \to H[l]$ such that f is a minimal right $(\mathcal{H}_{\mathcal{R}}[l-1])$-approximation of X. Since $X \in \mathcal{H}_{\mathcal{R}}[\geq l]^{\perp}$, then f is also a minimal right $(\mathcal{H}_{\mathcal{R}}[\geq l-1])$-approximation and $\mathrm{Hom}_{\mathcal{T}}(\mathcal{H}_{\mathcal{R}}[\geq l-1], X') = 0$ by Lemma 0.1.1. By our assumption, $X' \in \mathcal{X}_{\mathcal{R}} * \mathcal{X}_{\mathcal{R}}^{\perp}$. Thus $X \in H[l-1] * X' \subset \mathcal{H}_{\mathcal{R}}[l-1] * \mathcal{X}_{\mathcal{R}} * \mathcal{X}_{\mathcal{R}}^{\perp} = \mathcal{X}_{\mathcal{R}} * \mathcal{X}_{\mathcal{R}}^{\perp}$ holds since $\mathcal{X}_{\mathcal{R}}$ is extension-closed. □

The following proposition shows the first statement of Theorem 1.2.1.

Proposition 1.2.3. (1) *The natural functor* $\mathcal{Z} \hookrightarrow \mathcal{T} \to \mathcal{U}$ *gives an equivalence* $\mathcal{Z} \xrightarrow{\simeq} \mathcal{U}$*;*
(2) *We have* $\mathcal{T} = \mathcal{X}_{\mathcal{R}} \perp \mathcal{Z} \perp \mathcal{Y}_{\mathcal{R}}[1]$.

Proof. (1) By Propositions 0.2.5 and 1.2.2, we have t-structures $\mathsf{thick}(\mathcal{R}) = \mathcal{X}_{\mathcal{R}} \perp \mathcal{Y}_{\mathcal{R}}$ and $\mathcal{T} = \mathcal{X}_{\mathcal{R}} \perp \mathcal{X}_{\mathcal{R}}^{\perp} = {}^{\perp}\mathcal{Y}_{\mathcal{R}} \perp \mathcal{Y}_{\mathcal{R}}$. Notice that $\mathcal{X}[1] \cap \mathcal{Y} = 0$, then the assertion holds by Proposition 0.2.2.

(2) It suffices to show $\mathcal{X}_{\mathcal{R}}^{\perp} = \mathcal{Z} \perp \mathcal{Y}_{\mathcal{R}}[1]$. For any $M \in \mathcal{X}_{\mathcal{R}}^{\perp}$, there is a triangle $M''[-1] \to M' \to M \to M''$ with $M' \in {}^{\perp}\mathcal{Y}_{\mathcal{R}}[1]$ and $M'' \in \mathcal{Y}_{\mathcal{R}}[1]$ by Proposition 1.2.2. Applying $\mathrm{Hom}_{\mathcal{T}}(\mathcal{X}_{\mathcal{R}}, ?)$ to this triangle, it is easy to see $M' \in \mathcal{X}_{\mathcal{R}}^{\perp}$. Then $M' \in \mathcal{X}_{\mathcal{R}}^{\perp} \cap {}^{\perp}\mathcal{Y}_{\mathcal{R}}[1] = \mathcal{Z}$. So $\mathcal{X}_{\mathcal{R}}^{\perp} = \mathcal{Z} \perp \mathcal{Y}_{\mathcal{R}}[1]$. □

In the next part, we study the triangulated structure of $\mathcal{Z}$, which will be used later. Since $\mathcal{U}$ has a natural structure of triangulated category, then by using the additive equivalence $\mathcal{Z} \xrightarrow{\simeq} \mathcal{U}$, we may also regard $\mathcal{Z}$ as a triangulated category. Now we describe the shift functor $\langle 1 \rangle$ in $\mathcal{Z}$.

We define $\langle 1 \rangle$ on objects of $\mathcal{Z}$ first. For any $X \in \mathcal{Z}$, we have $X[1] \in \mathcal{R}[>0]^{\perp}$ and by (R1), there exists a $\mathcal{H}_{\mathcal{R}}$-approximation of $X[1]$. Define $X\langle 1 \rangle$ as the third term of the following triangle.

$$R_X \xrightarrow{f_X} X[1] \to X\langle 1 \rangle \to R_X[1] \qquad (1.2.1)$$

where f_X is the minimal right $\mathcal{H}_\mathcal{R}$-approximation of $X[1]$. Notice that $X\langle 1\rangle$ is defined uniquely up to isomorphism. Similarly, we can define $X\langle -1\rangle$. Immediately, we have the following observation.

Lemma 1.2.4. *Let $\langle 1\rangle$ be defined as above. Then*

(1) *For any $X \in \mathcal{Z}$, we have $X\langle 1\rangle \in Z$;*
(2) *For $X \in \mathbb{Z}$ and $n \geq 1$, we have $X\langle n\rangle \in X[n] * \mathcal{H}_\mathcal{R}[n] * \cdots * \mathcal{H}_\mathcal{R}[1]$.*

Proof. (1) Since $X \in \mathcal{Z}$, then $\mathrm{Hom}_\mathcal{T}(\mathcal{R}[>0], X[1]) = 0$. Notice that $\mathcal{X}_\mathcal{R}[1] = \mathsf{Filt}(\mathcal{R}[>0])$ and $\mathcal{X}_\mathcal{R} = \mathcal{X}_\mathcal{R}[1] * \mathcal{H}_\mathcal{R}$ by Proposition 0.2.5, then f_X in triangle (1.2.1) is also a minimal right $\mathcal{X}_\mathcal{R}$-approximation of $X[1]$ and $X\langle 1\rangle \in \mathcal{X}_\mathcal{R}^\perp$ by Lemma 0.1.1 (3).

On the other hand, since $X \in {}^\perp\mathcal{Y}_\mathcal{R}[1]$ and $R_X \in \mathcal{H}_\mathcal{R} \subset {}^\perp\mathcal{Y}_\mathcal{R}$, then $X[1] \in {}^\perp\mathcal{Y}_\mathcal{R}[2] \subset {}^\perp\mathcal{Y}_\mathcal{R}[1]$ and $R_X[1] \in {}^\perp\mathcal{Y}_\mathcal{R}[1]$. Therefore, $X\langle 1\rangle \in {}^\perp\mathcal{Y}_\mathcal{R}[1]$ by triangle (1.2.1). So $X\langle 1\rangle \in \mathcal{X}_\mathcal{R}^\perp \cap {}^\perp\mathcal{Y}_\mathcal{R}[1] = \mathcal{Z}$.

(2) For $n \geq 1$, consider the following triangle.

$$R_{X\langle n-1\rangle} \to X\langle n-1\rangle[1] \to X\langle n\rangle \to R_{X\langle n-1\rangle}[1], \tag{1.2.2}$$

where $R_{X\langle n-1\rangle} \to X\langle n-1\rangle[1]$ is the minimal right $\mathcal{H}_\mathcal{R}$-approximation of $X\langle n-1\rangle[1]$, then we have $X\langle n\rangle \in X\langle n-1\rangle[1] * \mathcal{H}_\mathcal{R}[1]$. By induction, it easy to see $X\langle n\rangle \in X[n] * \mathcal{H}_\mathcal{R}[n] * \cdots * \mathcal{H}_\mathcal{R}[1]$. □

Next we define $\langle 1\rangle$ on morphisms of $\mathcal{Z}$. Let $s \in \mathrm{Hom}_\mathcal{Z}(X, Y)$ for any X, Y in $\mathcal{Z}$. Consider the following diagram.

$$\begin{array}{ccccccc} R_X & \xrightarrow{f_X} & X[1] & \xrightarrow{g_X} & X\langle 1\rangle & \longrightarrow & R_X[1] \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle s[1]} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle h[1]} \\ R_Y & \xrightarrow{f_Y} & Y[1] & \xrightarrow{g_Y} & Y\langle 1\rangle & \longrightarrow & R_Y[1] \end{array} \tag{1.2.3}$$

Since $\mathrm{Hom}_\mathcal{T}(R_X, Y\langle 1\rangle) = 0$, then there exists a morphism $h \in \mathrm{Hom}_\mathcal{T}(R_X, R_Y)$, such that $s[1] \circ f_X = f_Y \circ h$. Let $t : X\langle 1\rangle \to Y\langle 1\rangle$ be a morphism such that diagram (1.2.3) is commutative. We define $s\langle 1\rangle := t$. The following lemma shows $s\langle 1\rangle$ is well defined.

Lemma 1.2.5. *Let $X, Y \in \mathcal{Z}$. For any $s \in \mathrm{Hom}_\mathcal{Z}(X, Y)$, $s\langle 1\rangle$ defined above is determined by s uniquely.*

Proof. We first claim the morphism h in diagram (1.2.3) is uniquely determined by s. If there exists $h' \in \mathrm{Hom}_\mathcal{T}(R_X, R_Y)$ such that $s[1] \circ f_X = f_Y \circ h'$, then $f_Y \circ (h - h') = 0$ and moreover, $h - h'$ factors through $Y\langle 1\rangle[-1]$. But $R_X \in \mathcal{H}_\mathcal{R} \subset \mathcal{X}_\mathcal{R}$ and $Y\langle 1\rangle \in \mathcal{X}_\mathcal{R}^\perp$, so $\mathrm{Hom}_\mathcal{T}(R_X, Y\langle 1\rangle[-1]) = 0$. Thus $h = h'$.

Next we show t is unique. If there exists $t' : X\langle 1\rangle \to Y\langle 1\rangle$ such that the diagram (1.2.3) commutes. Then we have $(t-t')\circ g_X = 0$, so $t-t'$ factors through $R_X[1]$. But $\mathrm{Hom}_T(R_X[1], Y\langle 1\rangle) = 0$, then $t = t'$. □

By Lemma 1.2.4 and Lemma 1.2.5, it is easy to check that $\langle 1\rangle : \mathcal{Z} \to \mathcal{Z}$ is a well-defined functor. Notice that the triangle (1.2.1) gives an isomorphism $X[1] \cong X\langle 1\rangle$ in $\mathcal{U}$.

Next we describe the triangles in $\mathcal{Z}$. Let $X, Y \in \mathcal{Z}$ and $s \in \mathrm{Hom}_\mathcal{Z}(X, Y)$. Then s induces a triangle $X \xrightarrow{s} Y \to Z \to X[1]$ in $\mathcal{T}$. Consider the right $\mathcal{H}_\mathcal{R}$-approximations of Z and $X[1]$. We have the following commutative diagrams.

$$\begin{array}{ccccccc} & & & & R_Z & \longrightarrow & R_X \\ & & & & \downarrow & & \downarrow \\ X & \xrightarrow{s} & Y & \xrightarrow{t} & Z & \longrightarrow & X[1] \\ & & & & \downarrow{\scriptstyle u} & & \downarrow \\ & & & & W & \longrightarrow & X\langle 1\rangle \end{array} \tag{1.2.4}$$

In this case, the following result holds.

Proposition 1.2.6. *Consider the triangulated structure of $\mathcal{Z}$ induced by $\mathcal{U}$. Then*

(1) *The suspension functor of $\mathcal{Z}$ is given by $\langle 1\rangle$;*

(2) *Let $s : X \to Y$ be a morphism in $\mathcal{Z}$. Then the triangle in $\mathcal{Z}$ induced by s is $X \xrightarrow{s} Y \xrightarrow{ut} W \to X\langle 1\rangle$.*

Proof. (1) Directly form Lemma 1.2.4 and Lemma 1.2.5.

(2) Notice we have isomorphism $Z \cong W$ and $X[1] \cong X\langle 1\rangle$ in $\mathcal{U}$. Moreover, $X \xrightarrow{s} Y \xrightarrow{ut} W \to X\langle 1\rangle$ is a triangle in $\mathcal{U}$. Since we have $W \in \mathcal{Z}$ (similar to the proof of Lemma 1.2.4). Then the assertion holds by the equivalence $\mathcal{Z} \simeq \mathcal{U}$. □

Now we are ready to prove Theorem 1.2.1.

Proof of Theorem 1.2.1. (1) is directly from Proposition 1.2.3 (1).

(2) Let $\mathcal{S} \in \mathsf{SMC}_{\mathcal{R}}\,\mathcal{T}$. We first show that $\mathcal{S}\backslash\mathcal{R} \in \mathsf{SMC}\,\mathcal{U}$. Since $\mathsf{thick}_{\mathcal{T}}(\mathcal{S}) = \mathcal{T}$, then $\mathsf{thick}_{\mathcal{U}}(\mathcal{S}\backslash\mathcal{R}) = \mathcal{U}$. Let $X, Y \in \mathcal{S}\backslash\mathcal{R}$. It is clear from Definition 0.2.4 that $\mathcal{S}\backslash\mathcal{R} \subset \mathcal{Z}$. So by (1), we have

$$\dim\mathrm{Hom}_{\mathcal{Z}}(X, Y) = \dim\mathrm{Hom}_{\mathcal{T}}(X, Y) = \delta_{X,Y}.$$

Since $X\langle n\rangle \in X[n] * \mathcal{H}_{\mathcal{R}}[n] * \cdots * \mathcal{H}_{\mathcal{R}}[1]$ for $n > 0$ by Lemma 1.2.4 and $\mathrm{Hom}_{\mathcal{T}}(\mathcal{H}_{\mathcal{R}}[\geq 0], Y) = 0$, then by (1) again,

$$\mathrm{Hom}_{\mathcal{U}}(X[n], Y) = \mathrm{Hom}_{\mathcal{Z}}(X\langle n\rangle, Y) = \mathrm{Hom}_{\mathcal{T}}(X\langle n\rangle, Y) = \mathrm{Hom}_{\mathcal{T}}(X[n], Y) = 0.$$

So $\mathcal{S}\backslash\mathcal{R} \in \mathsf{SMC}\,\mathcal{U}$. Therefore, sending $\mathcal{S} \in \mathsf{SMC}_{\mathcal{R}}\,\mathcal{T}$ to $\mathcal{S}\backslash\mathcal{R} \in \mathsf{SMC}\,\mathcal{U}$ gives us a well-defined map $\mathsf{SMC}_{\mathcal{R}}\,\mathcal{T} \to \mathsf{SMC}\,\mathcal{U}$, which is clearly injective.

We show the map is also surjective. Let $\mathcal{S}_{\mathcal{U}}$ be a SMC of $\mathcal{U}$. By (1), we may assume $\mathcal{S}_{\mathcal{U}} \subset \mathcal{Z}$. In this case, $\mathcal{S}_{\mathcal{U}}$ is also a SMC of $\mathcal{Z}$. Let $\mathcal{S} = \mathcal{S}_{\mathcal{U}} \cup \mathcal{R}$. We claim $\mathcal{S} \in \mathsf{SMC}_{\mathcal{R}}\,\mathcal{T}$. Since $\mathcal{R}$ is a pre-SMC and $\mathcal{Z} = \mathcal{R}[\geq 0]^{\perp} \cap {}^{\perp}\mathcal{R}[\leq 0]$, it is clear that $\dim\mathrm{Hom}_{\mathcal{T}}(X, Y) = \delta_{X,Y}$ for any $X, Y \in \mathcal{S}$, and $\mathrm{Hom}_{\mathcal{T}}(X[>0], Y) = 0$ for $X \in \mathcal{R}$, $Y \in \mathcal{S}$ or $X \in \mathcal{S}$, $Y \in \mathcal{R}$. Next we show $\mathrm{Hom}_{\mathcal{T}}(X[>0], Y) = 0$ for any $X, Y \in \mathcal{S}_{\mathcal{U}}$. Notice that Lemma 1.2.4 (2) implies $X[n] \in \mathcal{H}_{\mathcal{R}}[n-1] * \cdots * \mathcal{H}_{\mathcal{R}} * X\langle n\rangle$ for $n > 0$. Since $\mathrm{Hom}_{\mathcal{T}}(\mathcal{H}_{\mathcal{R}}[\geq 0], Y) = 0$, then $\mathrm{Hom}_{\mathcal{T}}(X[n], Y) = \mathrm{Hom}_{\mathcal{Z}}(X\langle n\rangle, Y) = 0$ for $n > 0$.

To show $\mathcal{S}$ is a SMC of $\mathcal{T}$, we are left to show $\mathcal{T} = \mathsf{thick}_{\mathcal{T}}(\mathcal{S})$. Since $X\langle m\rangle \in \mathsf{thick}_{\mathcal{T}}(S)$ for any $X \in \mathcal{S}_{\mathcal{U}}$ and $\mathsf{thick}_{\mathcal{Z}}(\mathcal{S}_{\mathcal{U}}) = \mathcal{Z}$, then $\mathcal{Z} \subset \mathsf{thick}_{\mathcal{T}}(\mathcal{S})$. So $\mathsf{thick}_{\mathcal{T}}(\mathcal{Z} \cup \mathcal{R}) \subset \mathsf{thick}_{\mathcal{T}}(\mathcal{S}) \subset \mathcal{T}$. But by Proposition 1.2.3 (2), we have $\mathsf{thick}_{\mathcal{T}}(\mathcal{Z} \cup \mathcal{R}) = \mathcal{T}$, so $\mathsf{thick}_{\mathcal{T}}(\mathcal{S}) = \mathcal{T}$ and thus $\mathcal{S} \in \mathsf{SMC}_{\mathcal{R}}\,\mathcal{T}$. Then the map $\mathsf{SMC}_{\mathcal{R}}\,\mathcal{T} \to \mathsf{SMC}\,\mathcal{U}$ is bijective. □

1.2.2. **Examples.** In this subsection, we consider the application of Theorem 1.2.1 to non-positive dg algebras. We call a dg k-algebra *proper*, if $A \in \mathsf{D}^{\mathrm{b}}(A)$. We first give the following result.

Proposition 1.2.7. *Let A be a non-positive proper dg k-algebra. Let $\mathcal{S}$ be a SMC of $\mathsf{D}^{\mathrm{b}}(A)$ and $\mathcal{R}$ be a subset of $\mathcal{S}$. Then $\mathcal{H}_{\mathcal{R}} = \mathsf{Filt}(\mathcal{R})$ satisfies the conditions (R1) and (R2) in Section 1.2.1.*

Proof. We know (R2) is true by Lemma 0.2.6. So we only need to show (R1). In fact, we show that $\mathcal{H}_{\mathcal{R}}$ is functorially finite in $\mathsf{D}^{\mathrm{b}}(A)$. By Proposition 0.3.6, we may assume $\mathcal{S} = \mathcal{S}_A$ is the set of simple dg A-modules. In this case, $\mathcal{H} = \mathsf{Filt}(\mathcal{S})$ is equivalent to $\mathsf{mod}\,\mathrm{H}^0(A)$ (see Lemma 0.3.5).

We first claim that $\mathcal{H}$ is functorially finite in $\mathsf{D}^{\mathrm{b}}(A)$. Let $M \in \mathsf{D}^{\mathrm{b}}(A)$. Considering the P-resolution ${}_PM$ of M, then ${}_PM \cong M$ in $\mathsf{D}^{\mathrm{b}}(A)$ and for any $N \in \mathsf{D}^{\mathrm{b}}(A)$, we have $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(M, N) = \mathrm{Hom}_{\mathscr{H}(A)}({}_PM, N)$, where $\mathscr{H}(A)$ is the homotopy category (see [K1, Section 3]). We write ${}_PM$ as a k-complex and consider the following diagram.

$$
\begin{array}{lccccccc}
{}_PM: & \cdots \longrightarrow & P^{-1} & \xrightarrow{d^{-1}} & P^0 & \xrightarrow{d^0} & P^1 & \longrightarrow \cdots \\
 & & & & \downarrow{\scriptstyle g} & \searrow{\scriptstyle f} & & \\
 & & & & L & \overset{h}{\longleftarrow\cdots} & N &
\end{array}
$$

where $N := \frac{P^0}{\mathrm{Im}\,d^{-1} + K}$ and $K := P^0 \cap A(\oplus_{i\geq 1} P^i)$. Then $N \in \mathcal{H}$ and the map $f : {}_PM \to N$ above is a morphism of dg A-modules. For any $L \in \mathcal{H}$ and a morphism $g : {}_PM \to L$ of dg A-modules, we have $g(K) = 0$, so there exists $h : N \to L$ such that $g = h \circ f$. Then f is a left $\mathcal{H}$-approximation of M. Thus $\mathcal{H}$ is a covariantly finite subcategory of $\mathsf{D}^{\mathrm{b}}(A)$. Dually, by using I-resolutions, we can show that $\mathcal{H}$ is contravariantly finite. Therefore the claim is true.

To show $\mathcal{H}_{\mathcal{R}}$ is functorially finite in $\mathsf{D}^{\mathrm{b}}(A)$, it is enough to show $\mathcal{H}_{\mathcal{R}}$ is functorially finite in $\mathcal{H} = \mathsf{mod}\,\mathrm{H}^0(A)$. By Lemma 1.2.8 below, we know it is true. Then we finish the proof. □

We need the following well-known fact.

Lemma 1.2.8. *Let Λ be a finite-dimensional k-algebra and let $\mathcal{R}$ be a subset of simple Λ-modules. Then $\mathcal{H}_{\mathcal{R}} = \mathsf{Filt}(\mathcal{R})$ is functorially finite in $\mathsf{mod}\,\Lambda$.*

Proof. There exists an idempotent $e \in \Lambda$ such that $\mathcal{R} = \mathrm{top}(1-e)\Lambda$. It is well-known that we have a standard recollement of abelian categories (see, for example [PV, Example 2.10])

$$\mathsf{mod}\,\Lambda/\Lambda e\Lambda \xrightarrow{i_*=\mathrm{inc.}} \mathsf{mod}\,\Lambda \xrightarrow{j^*=?\otimes_\Lambda \Lambda e} \mathsf{mod}\,e\Lambda e\ .$$
$$i^*=?\otimes_\Lambda \Lambda/\Lambda e\Lambda,\quad i^!=\mathrm{Hom}_\Lambda(\Lambda/\Lambda e\Lambda,?),\quad j_!=?\otimes_{e\Lambda e} e\Lambda,\quad j_*=\mathrm{Hom}_{e\Lambda e}(\Lambda e,?)$$

Then one can show $i_*(\mathsf{mod}\,\Lambda/\Lambda e\Lambda) = \mathcal{H}_{\mathcal{R}}$ and by [PV, Proposition 2.8], for any $M \in \mathsf{mod}\,\Lambda$, we have two exact sequences

$$j_!j^*(M) \to M \xrightarrow{f} i_*i^*(M) \to 0,$$

$$0 \to i_*i^!(M) \xrightarrow{g} M \to j_*j^*(M).$$

It is easy to check that f (resp. g) is a left (resp. right) $\mathcal{H}_{\mathcal{R}}$-approxiamtion of M. So $\mathcal{H}_{\mathcal{R}}$ is functorially finite. □

Next we give some useful observations, which allow us to realize the SMC reduction of bounded derived categories as new bounded derived categories.

Proposition 1.2.9. *Let A be a non-positive proper dg k-algebra. Let e be an idempotent. Assume $e \in A^0$.*

(1) *Let $\mathcal{R} = \mathrm{top}(1-e)\mathrm{H}^0(A)$. Then the SMC reduction $\mathsf{D}^{\mathrm{b}}(A)/\mathsf{thick}(\mathcal{R})$ is triangle equivalent to $\mathsf{D}^{\mathrm{b}}(eAe)$;*

(2) *Assume eA is a pre-SMC in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$. Then the SMC reduction $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)/\mathsf{thick}(eA)$ is triangle equivalent to $\mathsf{D}^{\mathrm{b}}(B)$, where B is the dg k-algebra $\mathscr{E}\mathrm{nd}_{\mathsf{per}\,A/\,\mathsf{thick}\,eA}(A)$.*

Proof. We have a natural derived Schur functor $F = ?\otimes^{\mathbf{L}}_A Ae : \mathsf{D}(A) \to \mathsf{D}(eAe)$, which restricts to a functor $F^{\mathrm{b}} = ?\otimes^{\mathbf{L}}_A Ae : \mathsf{D}^{\mathrm{b}}(A) \to \mathsf{D}^{\mathrm{b}}(eAe)$. It is well-known that F admits a left adjoint $G = ?\otimes^{\mathbf{L}}_{eAe} eA$, which is fully faithful (see for example [K1, Lemma 4.2]).

(1) By Proposition 1.5.12, the functor F^{b} induces a triangle equivalence $F^{\mathrm{b}} : \mathsf{D}^{\mathrm{b}}(A)/\ker F^{\mathrm{b}} \xrightarrow{\simeq} \mathsf{D}^{\mathrm{b}}(eAe)$. Since $\ker F^{\mathrm{b}} = \{M \in \mathsf{D}^{\mathrm{b}}(A) \mid Me = 0 \text{ in } \mathsf{D}^{\mathrm{b}}(eAe)\}$, then by standard truncation, we have $\ker F^{\mathrm{b}} = \mathsf{thick}(\mathcal{R})$. So the SMC reduction $\mathsf{D}^{\mathrm{b}}(A)/\mathsf{thick}(\mathcal{R})$ is equivalent to $\mathsf{D}^{\mathrm{b}}(eAe)$.

(2) We claim under our assumption, G also restricts to D^{b}. Since eA is a pre-SMC, then $\mathrm{End}(eA) = eAe$ is a division ring. Thus $\mathsf{D}^{\mathrm{b}}(eAe) = \mathsf{per}\,eAe$ and eA has finite projective dimension as left eAe-module, so G also restricts to $G^{\mathrm{b}} : \mathsf{D}^{\mathrm{b}}(eAe) \to \mathsf{D}^{\mathrm{b}}(A)$. Then we have an adjoint pair $(G^{\mathrm{b}}, F^{\mathrm{b}})$ between D^{b} and moreover, we have a t-structure $(G^{\mathrm{b}}(\mathsf{D}^{\mathrm{b}}(eAe)), \ker F^{\mathrm{b}})$ of $\mathsf{D}^{\mathrm{b}}(A)$. So there is a triangle equivalent $\mathsf{D}^{\mathrm{b}}(A)/G^{\mathrm{b}}(\mathsf{D}^{\mathrm{b}}(eAe)) \xrightarrow{\simeq} \ker F^{\mathrm{b}}$. Notice that $G^{\mathrm{b}}(\mathsf{D}^{\mathrm{b}}(eAe)) = G^{\mathrm{b}}(\mathsf{per}\,eAe) = \mathsf{thick}_A(eA)$ and by [KaY2, Corollary 2.12] (b), we have $\ker F^{\mathrm{b}} \cong \mathsf{D}^{\mathrm{b}}(B)$, where B is the dg k-algebra $\mathscr{E}\mathrm{nd}_{\mathsf{per}\,A/\,\mathsf{thick}\,eA}(A)$. Then the SMC reduction $\mathsf{D}^{\mathrm{b}}(A)/\mathsf{thick}(eA)$ is equivalent to $\mathsf{D}^{\mathrm{b}}(B)$. □

Let us consider a concrete example.

Example 1.2.10. Let A be a finite-dimensional k-algebra presented by a quiver $1 \underset{\alpha}{\overset{\beta}{\rightleftarrows}} 2$ with relations $\alpha\beta = 0 = \beta\alpha$. Let P_i (resp. S_i), $i = 1, 2$, be the indecomposable projective (simple) module which corresponds to the vertex i. It is easy to check P_1 is a pre-SMC in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$. Then by Proposition 1.2.9 (2), the SMC reduction $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)/\,\mathsf{thick}(P_1)$ is equivalent to $\mathsf{D}^{\mathrm{b}}(B)$, where B is the dg algebra $k[X]/(X^2)$ with $\deg X = -1$ and zero differential. Then by Theorem 1.2.1, we have the following bijection,

$$\begin{array}{ccc} \mathsf{SMC}_{P_1}\, \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A) & \longleftrightarrow & \mathsf{SMC}\ \mathsf{D}^{\mathrm{b}}(B)\ . \\ \| & & \| \\ \{\cdots, S_1[-2], S_1[-1], S_2[1], S_2[2], \cdots\} & \longleftrightarrow & \{k[i] \mid \in \mathbb{Z}\} \end{array}$$

We mention that in [AI, Example 2.47], the silting quiver of $\mathsf{per}\, A$ is given and by using silting-SMC correspondence (see [KaY1, Theorem 6.1]), one gets the description of $\mathsf{SMC}\ \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$ and thus the description of $\mathsf{SMC}_{P_1}\, \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$.

1.3. Singularity category of SMC quadruple

1.3.1. **Main results.** In this subsection, we introduce the singularity category of a SMC quadruple and show some basic properties of this category. We give the definition of a SMC quadruple first.

Definition 1.3.1. We say a quadruple $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathbb{S}, \mathcal{S})$ is a *SMC quadruple* if the following conditions are satisfied.

(RS0) $\mathcal{T}$ is a k-linear Hom-finite Krull-Schmidt triangulated category and $\mathcal{T}^{\mathrm{p}}$ is a thick subcategory of $\mathcal{T}$;

(RS1) $\mathbb{S} : \mathcal{T} \to \mathcal{T}$ is a triangle equivalence restricting to an equivalence $\mathbb{S} : \mathcal{T}^{\mathrm{p}} \to \mathcal{T}^{\mathrm{p}}$ and satisfying a bifunctorial isomorphism for any $X \in \mathcal{T}^{\mathrm{p}}$ and $Y \in \mathcal{T}$:

$$D\mathrm{Hom}_{\mathcal{T}}(X, Y) \simeq \mathrm{Hom}_{\mathcal{T}}(Y, \mathbb{S}X);$$

(RS2) $\mathcal{S}$ is a SMC in $\mathcal{T}$ and $\mathcal{T} = {}^{\perp}\mathcal{S}[\geq 0] \perp {}^{\perp}\mathcal{S}[<0] = \mathcal{S}[\geq 0]^{\perp} \perp \mathcal{S}[<0]^{\perp}$ are co-t-structures of $\mathcal{T}$ satisfying ${}^{\perp}\mathcal{S}[\geq 0] \subset \mathcal{T}^{\mathrm{p}}$ and $\mathcal{S}[<0]^{\perp} \subset \mathcal{T}^{\mathrm{p}}$;

Moreover, If $\mathbb{S} = [1-d]$ for some $d \geq 1$, we call $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathcal{S})$ a $(1-d)$-*CY triple*.

The definition above is inspired from the following example and we will see (RS2) plays an important role later.

Example 1.3.2. Let A be a finite-dimensional Gorenstein k-algebra. Then one can show that the quadruple $(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A), \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A), \nu, \mathcal{S})$ is a SMC quadruple, where $\nu = ? \otimes^{\mathbf{L}}_A DA$ is the Nakayama functor and $\mathcal{S}$ is the set of simple A-modules.

For simplicity, we introduce the following notations for $i \in \mathbb{Z}$.

$$\mathcal{T}_{>i} = \mathcal{T}_{\geq i+1} := {}^{\perp}\mathcal{S}[>-1-i], \quad \mathcal{T}_{<i} = \mathcal{T}_{\leq i-1} := {}^{\perp}S[<1-i];$$

$$\mathcal{T}^{>i} = \mathcal{T}^{\geq i+1} := \mathsf{Filt}(\mathcal{S}[<-i]), \quad \mathcal{T}^{<i} = \mathcal{T}^{\leq i-1} := \mathsf{Filt}(\mathcal{S}[\geq 1-i]).$$

Let $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathbb{S}, \mathcal{S})$ be a SMC quadruple. Then we have co-t-structures $\mathcal{T} = \mathcal{T}_{>i} \perp \mathcal{T}_{\leq i}$ and moreover, $\mathcal{T}_{>i} \subset \mathcal{T}^{\mathrm{p}}$ by (RS2). Also notice that we have bounded t-structures $\mathcal{T} = \mathcal{T}^{\leq i} \perp \mathcal{T}^{>i}$ and $\mathcal{T}^{\leq i} = \mathcal{T}_{\leq i}$ by Lemma 0.2.6. Immediately, we have the following useful observation.

Lemma 1.3.3. *Let $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathbb{S}, \mathcal{S})$ be a SMC quadruple. Then $(\mathcal{T}^{\mathrm{p}})^{\perp} = 0$ in $\mathcal{T}$.*

Proof. For any $X \in \mathcal{T}$ and $i \in \mathbb{Z}$, there exists a triangle $X_{>i} \to X \to X_{\leq i} \to X_{>i}[1]$, such that $X_{>i} \in \mathcal{T}_{>i} \subset \mathcal{T}^{\mathrm{p}}$ and $X_{\leq i} \in \mathcal{T}_{\leq i} = \mathcal{T}^{\leq i}$ by (RS2). If $X \in (\mathcal{T}^{\mathrm{p}})^{\perp}$, then $\mathrm{Hom}_{\mathcal{T}}(X_{>i}, X) = 0$ and thus $X_{\leq i} \cong X \oplus X_{>i}[1]$ in $\mathcal{T}$. So $X \in \mathcal{T}^{\leq i}$ for any $i \in \mathbb{Z}$. Since $\mathcal{T} = \mathcal{T}^{\leq 0} \perp \mathcal{T}^{>0}$ is a bounded t-structure by Lemma 0.2.6, then $X \in \bigcap_{i \in \mathbb{Z}} \mathcal{T}^{\leq i} = 0$. □

Now we introduce a new class of triangulated categories, which is a generalization of Buchweitz and Orlov's construction of singularity categories.

Definition 1.3.4. For a SMC quadruple $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathbb{S}, \mathcal{M})$, we define the *singularity category* as the Verdier quotient

$$\mathcal{T}_{\mathrm{sg}} := \mathcal{T}/\mathcal{T}^{\mathrm{p}}.$$

One important property of $\mathcal{T}_{\mathrm{sg}}$ is that $\mathcal{T}_{\mathrm{sg}}$ can be realized as a subfactor category of $\mathcal{T}$. To make it clear, let us introduce the following subcategories of $\mathcal{T}$.

$$\mathcal{F} = \mathcal{T}_{>0}^{\perp} \cap {}^{\perp}(\mathcal{T}_{\leq -1} \cap \mathcal{T}^{\mathrm{p}}), \quad \mathcal{P} = \mathcal{T}_{\geq 0} \cap \mathcal{T}_{\leq 0}, \quad \mathcal{H} = \mathcal{T}^{\geq 0} \cap \mathcal{T}^{\leq 0}.$$

It is clear $\mathcal{P}$ is just the co-heart of the co-t-structure of $\mathcal{T} = \mathcal{T}_{>0} \perp \mathcal{T}_{\leq 0}$ and $\mathcal{H}$ is the heart of the t-structure $\mathcal{T} = \mathcal{T}^{\leq 0} \perp \mathcal{T}^{>0}$. Our main results in this section is as follows.

Theorem 1.3.5. *Let $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathbb{S}, \mathcal{S})$ be a SMC quadruple. Then we have*

(1) *$\mathcal{F}$ is a Frobenius extriangulated category with $\mathsf{Proj}\,\mathcal{F} = \mathcal{P}$ in the sense of* [NP]*;*

(2) *The composition*

$$\mathcal{F} \subset \mathcal{T} \to \mathcal{T}_{\mathrm{sg}}$$

induces an equivalence $\pi : \frac{\mathcal{F}}{[\mathcal{P}]} \xrightarrow{\simeq} \mathcal{T}_{\mathrm{sg}}$. Moreover, $\mathcal{T}_{\mathrm{sg}}$ has a Serre functor $\mathbb{S}[-1]$;

(3) *If $\mathbb{S} = [1-d]$, then $\mathcal{F} = \mathcal{H}[d-1] * \mathcal{H}[d-2] * \cdots * \mathcal{H}$ and $\pi(\mathcal{S})$ is a d-SMS in $\mathcal{T}_{\mathrm{sg}}$.*

Proof. (1) and (2) We want to apply Proposition 0.2.2. Let $\mathcal{X} = \mathcal{T}_{>0}$ and $\mathcal{Y} = \mathcal{T}_{\leq 0} \cap \mathcal{T}^{\mathrm{p}}$. Then it is easy to check $\mathcal{F} = \mathcal{X}^{\perp} \cap {}^{\perp}\mathcal{Y}[1]$ and $\mathcal{P} = \mathcal{X}[1] \cap \mathcal{Y}$. We claim that we have co-t-structures $\mathcal{T}^{\mathrm{p}} = \mathcal{X} \perp \mathcal{Y}$ and $\mathcal{T} = \mathcal{X} \perp \mathcal{X}^{\perp} = {}^{\perp}\mathcal{Y} \perp \mathcal{Y}$. In fact, we know $\mathcal{X}^{\perp} = \mathcal{T}_{\leq 0}$ and $\mathcal{T} = \mathcal{X} \perp \mathcal{X}^{\perp}$ is a co-t-structure by (RS2). For any $T \in \mathcal{T}^{\mathrm{p}}$, there exists a triangle $T_{>0} \to T \to T_{\leq 0} \to T_{>0}[1]$ such that $T_{>0} \in \mathcal{T}_{>0}$ and $T_{\leq 0} \in \mathcal{T}_{\leq 0}$. Since $T_{>0} \in \mathcal{T}^{\mathrm{p}}$ by (RS2), so $\mathcal{T}_{\leq 0}$ is also in $\mathcal{T}^{\mathrm{p}}$. Then the co-t-structure $\mathcal{T} = \mathcal{X} \perp \mathcal{X}^{\perp}$ restricts to a co-t-structure $\mathcal{T}^{\mathrm{p}} = \mathcal{X} \perp (\mathcal{X}^{\perp} \cap \mathcal{T}^{\mathrm{p}}) = \mathcal{X} \perp \mathcal{Y}$ of $\mathcal{T}^{\mathrm{p}}$.

Now we show $\mathcal{T} = {}^{\perp}\mathcal{Y} \perp \mathcal{Y}$ is also a co-t-structure. Since $\mathcal{Y} = {}^{\perp}\mathcal{S}[<0] \cap \mathcal{T}^{\mathrm{p}}$, then $\mathcal{Y} \subset \mathbb{S}^{-1}\mathcal{S}[<0]^{\perp}$ by (RS1). Notice that $\mathcal{S}[<0]^{\perp} \subset \mathcal{T}^{\mathrm{p}}$ by (RS2), then it is easy to see $\mathbb{S}^{-1}\mathcal{S}[<0]^{\perp} \subset \mathcal{Y}$ by (RS1). So $\mathcal{Y} = \mathbb{S}^{-1}\mathcal{S}[<0]^{\perp}$. By (RS2), there is a co-t-structure $\mathcal{T} = \mathcal{S}[\geq 0]^{\perp} \perp \mathcal{S}[<0]^{\perp}$. Then $\mathcal{T} = {}^{\perp}\mathcal{Y} \perp \mathcal{Y}$ is also a co-t-structure with ${}^{\perp}\mathcal{Y} = \mathbb{S}^{-1}\mathcal{S}[\geq 0]^{\perp}$.

By Proposition 0.2.2 and Remark 0.2.3, we know $\mathcal{F}$ is a Frobenius extriangulated category with $\mathsf{Proj}\,\mathcal{F} = \mathcal{P}$ and the composition $\mathcal{F} \subset \mathcal{T} \to \mathcal{T}_{\mathrm{sg}}$ induces an equivalence $\pi : \frac{\mathcal{F}}{[\mathcal{P}]} \xrightarrow{\simeq} \mathcal{T}_{\mathrm{sg}}$.

We are left to show the existence of Serre functor in $\mathcal{T}_{\mathrm{sg}}$. Let $X, Y \in \mathcal{T}$. There exist $i \in \mathbb{Z}$ such that $Y \in \mathcal{T}^{>i}$ (because $\mathcal{T} = \mathcal{T}^{\leq 0} \perp \mathcal{T}^{>0}$ is a bounded t-structure by Lemma 0.2.6). By (RS2), there is a triangle

$$X_{>i} \to X \to X_{\leq i} \to X_{>i}[1],$$

with $X_{>i} \in \mathcal{T}_{>i}$ and $X_{\leq i} \in \mathcal{T}_{\leq i} = \mathcal{T}^{\leq i}$. Since $\mathrm{Hom}_{\mathcal{T}}(X_{\leq i}, Y) = 0$ and $X_{>i} \in \mathcal{T}^{\mathrm{p}}$, then the morphism $X_{>i} \to X$ is a local $\mathcal{T}^{\mathrm{p}}$-cover of X relative to Y in the sense of [Am, Definition 1.2]. Then by [Am, Lemma1.1, Theorem 1.3 and Proposition 1.4], we know $\mathbb{S}[-1]$ is a Serre functor of $\mathcal{T}_{\mathrm{sg}}$.

(3) For the case $\mathbb{S} = [1-d]$, we have ${}^{\perp}\mathcal{Y} = \mathbb{S}^{-1}\mathcal{S}[\geq 0]^{\perp} = \mathcal{T}^{>1-d}$. On the other hand, $\mathcal{X}^{\perp} = \mathcal{T}_{\leq 0} = \mathcal{T}^{\leq 0}$. So $\mathcal{F} = \mathcal{X}^{\perp} \cap {}^{\perp}\mathcal{Y}[1] = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 1-d} = \mathcal{H}[d-1] * \mathcal{H}[d-2] * \cdots * \mathcal{H}$ by Proposition 0.2.5.

Next we show $\pi(\mathcal{S})$ is a d-SMS in $\mathcal{T}_{\mathrm{sg}}$. Let $X, Y \in \mathcal{S}$. We may assume $\pi(X)$ and $\pi(Y)$ are non-zero objects in $\mathcal{T}_{\mathrm{sg}} \cong \frac{\mathcal{F}}{[\mathcal{P}]}$. Since

$$\mathrm{dimHom}_{\mathcal{T}_{\mathrm{sg}}}(\pi(X), \pi(Y)) = \mathrm{dimHom}_{\frac{\mathcal{F}}{[\mathcal{P}]}}(X, Y) \leq \mathrm{dimHom}_{\mathcal{T}}(X, Y),$$

and $\mathrm{dimHom}_{\mathcal{T}}(X, Y) = \delta_{X,Y}$, then we have $\mathrm{dimHom}_{\mathcal{T}_{\mathrm{sg}}}(\pi(X), \pi(Y)) = \delta_{\pi(X),\pi(Y)}$. If $d \geq 1$, since $\mathrm{Hom}_{\mathcal{T}}(X[i], Y) = 0$ for any $1 \leq i \leq d-1$, then $\mathrm{Hom}_{\mathcal{T}_{\mathrm{sg}}}(\pi(X)[i], \pi(Y)) = \mathrm{Hom}_{\frac{\mathcal{F}}{[\mathcal{P}]}}(X[i], Y) = 0$. The fact $\mathcal{F} = \mathcal{H}[d-1] * \mathcal{H}[d-2] * \cdots * \mathcal{H}$ implies that $\frac{\mathcal{F}}{[\mathcal{P}]} = \mathcal{H}[d-1] * \mathcal{H}[d-2] * \cdots * \mathcal{H}$. Then $\mathcal{T}_{\mathrm{sg}} = \pi(\mathcal{H})[d-1] * \pi(\mathcal{H})[d-2] * \cdots * \pi(\mathcal{H})$. So $\pi(S)$ is a d-SMS of $\mathcal{T}_{\mathrm{sg}}$. $\square$

We apply Theorem 1.3.5 to Example 1.3.2 and then we have the following well-known result.

Example 1.3.6. Let A be a finite-dimensional Gorenstein k-algebra. Then $\mathcal{P} = \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ and $\mathcal{F} = \mathsf{CM}\,A$. By theorem 1.3.5, the natural functor $\mathsf{CM}\,A \subset \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{sg}}(A)$ gives an equivalence $\underline{\mathsf{CM}}A \simeq \mathsf{D}^{\mathrm{b}}_{\mathrm{sg}}(A)$ and moreover, $\mathsf{D}^{\mathrm{b}}_{\mathrm{sg}}(A)$ has a Serre functor $? \otimes^{\mathbf{L}}_{A} DA[-1]$.

1.3.2. **Further properties.** In this subsection, we continue to study the properties of a SMC quadruple. This part is technical and abstract, but we will see it is useful. Let $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathbb{S}, \mathcal{S})$ be a SMC quadruple. Let $\mathcal{P}$ be the co-heart of the co-t-structure $\mathcal{T} = \mathcal{T}_{>0} \perp \mathcal{T}_{\leq 0}$. It is clear that $\mathcal{P}$ is a subcategory of $\mathcal{T}^{\mathrm{p}}$. We mainly study the properties of $\mathcal{P}$. First we point out that $\mathcal{P}$ is silting in $\mathcal{T}^{\mathrm{p}}$.

Proposition 1.3.7. (1) *$\mathcal{P}$ is a silting subcategory in $\mathcal{T}^{\mathrm{p}}$;*
(2) *We have a co-t-structure $\mathcal{T}^{\mathrm{p}} = \mathsf{Filt}(\mathcal{P}[\leq 0]) \perp \mathsf{Filt}(\mathcal{P}[>0])$. Moreover, $\mathsf{Filt}(\mathcal{P}[\leq 0]) = \mathcal{T}_{\geq 0}$ and $\mathsf{Filt}(\mathcal{P}[>0]) = \mathcal{T}_{<0} \cap \mathcal{T}^{\mathrm{p}}$.*

To prove this proposition, we give two lemmas first.

Lemma 1.3.8. *For $X \in \mathcal{T}$, if there exist $i \leq j \in \mathbb{Z}$ such that $X \in \mathcal{T}_{\geq i} \cap \mathcal{T}_{\leq j}$, then $X \in \mathsf{thick}\,\mathcal{P}$.*

Proof. We apply the induction on $j - i$. If $j - i = 0$, then $\mathcal{T}_{\geq i} \cap \mathcal{T}_{\leq j} = \mathcal{P}[-i]$, the assertion is clear. Assume it holds for $j - i < n$, $n > 0$. Now consider the case $j - i = n$. There exists a triangle

$$X_{<j}[-1] \to X_{\geq j} \to X \to X_{<j}$$

such that $X_{\geq j} \in \mathcal{T}_{\geq j}$ and $X_{<j} \in \mathcal{T}_{<j}$. Since $X, X_{<j}[-1] \in \mathcal{T}_{\leq j}$, then $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}_{>j}, X_{<j}[-1]) = 0 = \mathrm{Hom}_{\mathcal{T}}(\mathcal{T}_{>j}, X)$. By the triangle above, we have $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}_{>j}, X_{\geq j}) = 0$. So $X_{\geq j} \in \mathcal{T}_{\leq j} \cap \mathcal{T}_{\geq j} = \mathcal{P}[-j]$. Since $X_{<j} \in \mathcal{T}_{\geq i} \cap \mathcal{T}_{\leq j-1}$, by assumption, $X_{<j} \in \mathsf{thick}\,\mathcal{P}$. Then $X \in \mathsf{thick}\,\mathcal{P}$. So the statement is true. □

Lemma 1.3.9. *For any $P \in \mathcal{T}^{\mathrm{p}}$, $\mathrm{Hom}_{\mathcal{T}}(P, \mathcal{S}[n]) \neq 0$ for only finite many $n \in \mathbb{Z}$.*

Proof. We know $\mathrm{Hom}_{\mathcal{T}}(P, X[\ll 0]) = 0$ by Lemma 0.2.6. On the other hand, we have $\mathrm{Hom}_{\mathcal{T}}(P, \mathcal{S}[n]) = D\mathrm{Hom}_{\mathcal{T}}(\mathcal{S}[n], \mathbb{S}P)$ by (RS1), which vanishes for big enough n. So the statement holds. □

Now we are ready to prove Proposition 1.3.7.

Proof of Proposition 1.3.7. (1) Since $\mathcal{P}$ is the co-heart of a co-t-structure, then $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, \mathcal{P}[>0]) = 0$. To show $\mathcal{P}$ is silting in $\mathcal{T}^{\mathrm{p}}$, it suffices to show $\mathcal{T}^{\mathrm{p}} = \mathsf{thick}\,\mathcal{P}$. For any $P \in \mathcal{T}^{\mathrm{p}}$, there are only finite many $n \in \mathbb{Z}$ such that $\mathrm{Hom}_{\mathcal{T}}(P, \mathcal{S}[n]) \neq 0$ by lemma 1.3.9. Then there exist $i, j \in \mathbb{Z}$ such that $P \in \mathcal{T}_{\geq i} \cap \mathcal{T}_{\leq j}$. By Lemma 1.3.8, $P \in \mathsf{thick}\,\mathcal{P}$. So $\mathcal{P}$ is a silting object in $\mathcal{T}^{\mathrm{p}}$.

(2) Since $\mathcal{P}$ is silting in $\mathcal{T}^{\mathrm{p}}$, then it is known that $\mathcal{P}$ gives us a standard co-t-structure $\mathcal{T}^{\mathrm{p}} = \mathsf{Filt}(\mathcal{P}[\leq 0]) \perp \mathsf{Filt}(\mathcal{P}[>0])$ (see [IYa1, Proposition 2.8]). In the proof of Theorem 1.3.5, we showed the co-t-structure $\mathcal{T} = \mathcal{T}_{\geq 0} \perp \mathcal{T}_{<0}$ of $\mathcal{T}$ restricts to a co-t-structure $\mathcal{T}^{\mathrm{p}} = \mathcal{T}_{\geq 0} \perp (\mathcal{T}_{<0} \cap T^{\mathrm{p}})$ of $\mathcal{T}^{\mathrm{p}}$. Since $\mathsf{Filt}(\mathcal{P}[\leq 0]) \subset \mathcal{T}_{\geq 0}$ and $\mathsf{Filt}(\mathcal{P}[>0]) \subset \mathcal{T}_{<0}$, it turns out that these two co-t-structure coincide with each other. In particular, $\mathcal{T}_{\geq 0} = \mathsf{Filt}(\mathcal{P}[\leq 0])$. □

Next we study the relation between $\mathcal{P}$ and the standard t-structure of $\mathcal{T} = \mathcal{T}^{\leq 0} \perp \mathcal{T}^{>0}$.

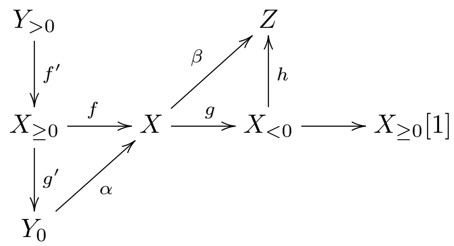
Proposition 1.3.10. (1) *We have $\mathcal{P}[\geq 0]^{\perp} = \mathcal{T}^{>0}$ and $\mathcal{P}[\leq 0]^{\perp} = \mathcal{T}^{<0}$ in $\mathcal{T}$;*
(2) *The functor $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, ?) : \mathcal{T} \to \mathsf{mod}\,\mathcal{P}$ restricts to an equivalence form the heart $\mathcal{H}$ to $\mathsf{mod}\,\mathcal{P}$.*

We first show a lemma.

Lemma 1.3.11. (1) *$\mathcal{P}$ is a contravariantly finite subcategory of $\mathcal{T}$;*
(2) *$\mathsf{mod}\,\mathcal{P}$ is an abelian category.*

Proof. (1) Because $\mathcal{T} = \mathcal{T}^{\leq 0} \perp \mathcal{T}^{>0}$ is a t-structure and $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, \mathcal{T}^{>0}) = 0$, it suffices to show there exists a right $\mathcal{P}$-approximation for any $X \in T^{\leq 0}$. Let $X \in T^{\leq 0}$. There is a triangle $X_{\geq 0} \to X \to X_{<0} \to X_{\geq 0}[1]$ with $X_{\geq 0} \in \mathcal{T}_{\geq 0}$ and $X_{<0} \in \mathcal{T}_{<0}$. Notice that $X_{\geq 0} \in \mathcal{T}_{\geq 0} = \mathsf{Filt}(\mathcal{P}[\leq 0])$ by

Proposition 1.3.7. Then there is a triangle $Y_{>0} \to X_{\geq 0} \to Y_0$ such that $Y_{>0} \in \mathsf{Filt}(\mathcal{P}[<0])$ and $Y_0 \in \mathsf{Filt}(\mathcal{P}[\geq 0])$. It is easy to check $Y_0 \in \mathcal{P}$. We have the following diagram.



Since $X \in \mathcal{T}^{\leq 0} = \mathcal{T}_{\leq 0}$, then $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}[<0], X) = 0$. Then $f \circ f' = 0$ and there is $\alpha \in \mathrm{Hom}_{\mathcal{T}}(Y_0, X)$ such that $f = \alpha \circ g'$.

We claim $\alpha : Y_0 \to X$ is a right $\mathcal{P}$-approximation of X. Let Z be the third term of the triangle extended by α. Since $\beta \circ f = \beta \circ \alpha \circ g' = 0$, then there exists $h \in \mathrm{Hom}_{\mathcal{T}}(X_{<0}, Z)$ such that $\beta = h \circ g$. Since $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, X_{<0}) = 0$, then $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, \beta) = 0$. So α is right $\mathcal{P}$-approximation of X.

(2) See [IYa1, Lemma 4.7]. □

Now let us prove Proposition 1.3.10.

Proof of Proposition 1.3.10. (1) We only show $\mathcal{P}[\geq 0]^{\perp} = \mathcal{T}^{>0}$, since $\mathcal{P}[\leq 0]^{\perp} = \mathcal{T}^{<0}$ is directly induced from Proposition 1.3.7. Since $\mathcal{P}[\geq 0] \subset \mathcal{T}_{\leq 0} \cap \mathcal{T}^{\mathrm{p}} \subset \mathcal{T}_{\leq 0} = \mathcal{T}^{\leq 0}$, then $\mathcal{T}^{>0} \subset \mathcal{P}[\geq 0]^{\perp}$.

We claim $\mathcal{P}[\geq 0]^{\perp} \subset \mathcal{T}^{<0}$. Let $X \in \mathcal{P}[\geq 0]^{\perp}$. Consider the following triangle

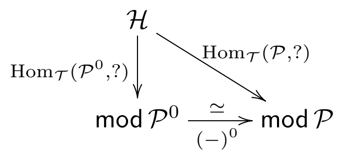
$$X^{>0}[-1] \to X^{\leq 0} \to X \to X^{>0}$$

with $X^{\leq 0} \in \mathcal{T}^{\leq 0}$ and $X^{>0} \in \mathcal{T}^{>0}$. Since $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}[\geq 0], X^{>0}[-1]) = 0$ and $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}[\geq 0], X) = 0$, then by applying $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}[\geq 0], ?)$ to the triangle above, we have $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}[\geq 0], X^{\leq 0}) = 0$. On the other hand, by the definition of co-heart, we know $\mathcal{P} = {}^{\perp}\mathcal{S}[\neq 0]$, thus $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}[<0], X^{\leq 0}) = 0$. So $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}[n], X^{\leq 0}) = 0$ for any $n \in \mathbb{Z}$. Thus $X^{\leq 0} = 0$ by Proposition 1.3.7 and Lemma 1.3.3. So $X \cong X^{>0} \in \mathcal{T}^{>0}$. Then $\mathcal{P}[\geq 0]^{\perp} = \mathcal{T}^{>0}$ holds.

(2) We have $\mathcal{H} = \mathsf{Filt}(\mathcal{S}) = \mathcal{P}[\neq 0]^{\perp}$ by (1). For any $P \in \mathcal{P}$, consider the following triangle.

$$P^{<0} \to P \to P^0 \to P^{<0}[1] \tag{1.3.1}$$

with $P^{<0} \in \mathcal{T}^{<0}$ and $P^0 \in \mathcal{T}^{\geq 0}$. Since $\mathrm{Hom}_{\mathcal{T}}(P, \mathcal{S}[<0]) = 0$ and $\mathrm{Hom}_{\mathcal{T}}(P^{<0}[1], \mathcal{S}[<0]) = 0$, then $\mathrm{Hom}_{\mathcal{T}}(P^0, \mathcal{S}[<0]) = 0$ and $P^0 \in \mathcal{H}$. Let $\mathcal{P}^0 = \{P^0 \mid P \in \mathcal{P}\} \subset \mathcal{H}$ be a subcategory of $\mathcal{H}$. It is easy to check that the functor $(-)^0 : \mathcal{P} \to \mathcal{P}^0$ is an equivalence. Since $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}^{<0}, \mathcal{H}) = 0$, then $\mathrm{Hom}_{\mathcal{T}}(P, \mathcal{H}) = \mathrm{Hom}_{\mathcal{T}}(P^0, \mathcal{H})$ for any $P \in \mathcal{P}$. So we have the following commutative diagram.



To show $\mathcal{H}$ is equivalent to $\mathsf{mod}\,\mathcal{P}$, it suffices to show that $\mathcal{P}^0$ forms a class of projective generators of $\mathcal{H}$. For any $X \in \mathcal{H}$ and $P \in \mathcal{P}$, applying $\mathrm{Hom}_{\mathcal{T}}(?, X)$ to the triangle (1.3.1), we get $\mathrm{Hom}_{\mathcal{T}}(P^0, X[1]) = 0$ by $\mathrm{Hom}_{\mathcal{T}}(P^{<0}, X) = 0$ and $\mathrm{Hom}_{\mathcal{T}}(P[-1], X) = 0$. So P^0 is projective in $\mathcal{H}$.

For any $X \in \mathcal{H}$. Consider the minimal right $\mathcal{P}$-approximation of X ($\mathcal{P}$ is a contravariantly finite subcategory of $\mathcal{T}$ by Lemma 1.3.11).

$$Y_{\mathcal{P}} \to X_{\mathcal{P}} \to X \to Y_{\mathcal{P}}[1].$$

Applying $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, ?)$ to the triangle, we have long exact sequence

$$\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, X_{\mathcal{P}}[i]) \to \mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, X[i]) \to \mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, Y_{\mathcal{P}}[i+1]) \to \mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, X_{\mathcal{P}}[i+1]).$$

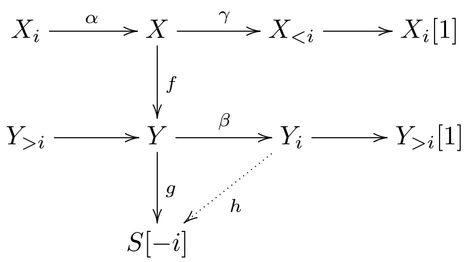
Since $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, X_{\mathcal{P}}[i]) = \mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, X[i]) = 0$ for $i > 0$, then $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, Y_{\mathcal{P}}[>1]) = 0$. For the case $i = 0$, since $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, X_{\mathcal{P}}) \to \mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, X)$ is surjective, then $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, Y_{\mathcal{P}}[1]) = 0$. So

$Y_{\mathcal{P}}[1] \in \mathcal{P}[\geq 0]^{\perp} = \mathsf{Filt}\mathcal{S}[<0]$. Taking 0-th cohomology, we have an exact sequence $(X_{\mathcal{P}})^0 \to X \to 0$. So $\mathcal{P}^0$ is a projective generator of $\mathcal{H}$. □

The following Proposition is important in the sequel.

Proposition 1.3.12. *Let $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathbb{S}, \mathcal{S})$ be a SMC quadruple. Let $X \in \mathcal{T}_{\leq i}$ and $Y \in \mathcal{T}_{\geq i}$ for some $i \in \mathbb{Z}$. Then for any $f \in \mathrm{rad}(X, Y)$ and $S \in \mathcal{S}$, the induced map $\mathrm{Hom}_{\mathcal{T}}(f, S[-i]) : \mathrm{Hom}_{\mathcal{T}}(Y, S[-i]) \to \mathrm{Hom}_{\mathcal{T}}(X, S[-i])$ is zero.*

Proof. Let $g \in \mathrm{Hom}_{\mathcal{T}}(Y, S[-i])$. We show $g \circ f = 0$. Consider the following diagram,



where $X_i \in \mathcal{T}_{\geq i}, X_{<i} \in \mathcal{T}_{<i}$ and $Y_{>i} \in \mathcal{T}_{>i}, Y_i \in \mathcal{T}_{\leq i}$. Since $X \in \mathcal{T}_{\leq i}$ and $Y \in \mathcal{T}_{\geq i}$, it is easy to check that $X_i, Y_i \in \mathcal{T}_{\leq i} \cap \mathcal{T}_{\geq i} = \mathcal{P}[-i]$. Notice that $\mathrm{Hom}_{\mathcal{T}}(Y_{>i}, S[-i]) = 0$ for $Y_{>i} \in \mathcal{T}_{>i} = {}^{\perp}\mathcal{S}[\geq -i]$, then there exists $h \in \mathrm{Hom}_{\mathcal{T}}(Y_0, S[-i])$ such that $g = h \circ \beta$. Since $f \in \mathrm{rad}(X, Y)$, then $\beta \circ f \circ \alpha \in \mathrm{rad}(X_i, Y_i)$, and moreover, we have $g \circ f \circ \alpha = h \circ \beta \circ f \circ \alpha = 0$ by the following Lemma 1.3.13. So $g \circ f$ factors through γ. But $\mathrm{Hom}_{\mathcal{T}}(X_{<i}, S[-i]) = 0$ by $X_{<i} \in \mathcal{T}_{\leq i} = {}^{\perp}\mathcal{S}[\leq -i]$, then $g \circ f = 0$. □

The following lemma is a generalization of a well-known result: for a finite-dimensional k-algebra A, the radical map $f : Q \to P$ induces a zero map $\mathrm{Hom}_A(f, S) = 0$, where P, Q are projective A-modules and S is simple.

Lemma 1.3.13. *Let $P, Q \in \mathcal{P}$ and $S \in \mathcal{S}$. Let $f \in \mathrm{rad}(Q, P)$, then the induced morphism $\mathrm{Hom}_{T}(f, S) : \mathrm{Hom}_{\mathcal{T}}(P, S) \to \mathrm{Hom}_{\mathcal{T}}(Q, S)$ is zero.*

Proof. By Proposition 1.3.10, the functor $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, ?) : \mathcal{H} \to \mathsf{mod}\,\mathcal{P}$ is an equivalence. Since $\mathcal{S}$ is the set of simples of $\mathcal{H}$, then $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, S)$ is simple in $\mathsf{mod}\,\mathcal{P}$ for any $S \in \mathcal{S}$. Since f is a radical map, then $\mathrm{Hom}_{\mathcal{T}}(Q, f) : \mathrm{Hom}_{\mathcal{T}}(Q, Q) \to \mathrm{Hom}_{\mathcal{T}}(Q, P)$ is a radical map as $\mathrm{End}_{\mathcal{T}}(Q)$-module. Then the composition $\mathrm{Hom}_{\mathcal{T}}(Q, Q) \to \mathrm{Hom}_{\mathcal{T}}(Q, P) \to \mathrm{Hom}_{\mathcal{T}}(Q, S)$ is zero. Consider the image of $1_Q \in \mathrm{Hom}_{\mathcal{T}}(Q, Q)$ in the composition, we get that the induced morphism $\mathrm{Hom}_{T}(f, S) : \mathrm{Hom}_{\mathcal{T}}(P, S) \to \mathrm{Hom}_{\mathcal{T}}(Q, S)$ is also zero. □

1.3.3. **Independence of SMC quadruple.** The aim of this subsection is to show under certain conditions, being a SMC quadruple is independent of the choice of SMC. Let $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathbb{S}, \mathcal{S})$ be a SMC quadruple. Let $\mathcal{H} = \mathsf{Filt}(\mathcal{S})$. We show the following result.

Theorem 1.3.14. *Let $\mathcal{S}'$ be another SMC of $\mathcal{T}$. Assume that*
(1) $\mathcal{H}' = \mathsf{Filt}(\mathcal{S}')$ is functorially finite;
(2) There exists $n \in \mathbb{Z}$ such that $\mathcal{S}' \subset \mathcal{H}[n] * \mathcal{H}[n-1] * \cdots * \mathcal{H}[-n]$.
Then $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathbb{S}, \mathcal{S}')$ is also a SMC quadruple.

Proof. To show $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathbb{S}, \mathcal{S}')$ is a SMC quadruple, we only need to check (RS2) in Definition 1.3.1 holds, that is, $\mathcal{T} = {}^{\perp}\mathcal{S}'[\geq 0] \perp {}^{\perp}\mathcal{S}'[<0] = \mathcal{S}'[\geq 0]^{\perp} \perp \mathcal{S}'[<0]^{\perp}$ are co-t-structures of $\mathcal{T}$, satisfying ${}^{\perp}\mathcal{S}'[\geq 0] \subset \mathcal{T}^{\mathrm{p}}$ and $\mathcal{S}'[<0]^{\perp} \subset \mathcal{T}^{\mathrm{p}}$. We may assume, up to shift, that

$$\mathcal{H}' \quad \subset \quad \mathcal{H}[n] * \mathcal{H}[n-1] * \cdots * \mathcal{H}. \tag{1.3.2}$$

Then $\mathrm{Hom}_{\mathcal{T}}(\mathcal{H}, \mathcal{H}'[<-n]) = 0$ and $\mathrm{Hom}_{\mathcal{T}}(\mathcal{H}', \mathcal{H}[<0]) = 0$. So in this case, we also have

$$\mathcal{H} \quad \subset \quad \mathcal{H}' * \mathcal{H}'[-1] * \cdots * \mathcal{H}'[-n]. \tag{1.3.3}$$

We prove $\mathcal{T} = {}^{\perp}\mathcal{S}'[\geq 0] \perp {}^{\perp}\mathcal{S}'[<0]$ is a co-t-structure. By Proposition 0.2.5, we have

$$ {}^{\perp}\mathcal{S}'[<0] \quad = \quad \bigcup_{i\geq 0} \mathcal{H}'[i] * \mathcal{H}'[i-1] * \cdots * \mathcal{H}'. \tag{1.3.4}$$

Then (1.3.2), (1.3.3) and (1.3.4) imply the following equality.

$$ {}^{\perp}\mathcal{S}'[<0] \quad = \quad \bigcup_{i\geq n} \mathcal{H}[i] * \cdots * \mathcal{H}[n] * \mathcal{H}'[n-1] * \cdots * \mathcal{H}'. \tag{1.3.5}$$

Now fix an integer $l \geq 2n$. Let $\mathcal{X} := \mathcal{H}'[l] * \mathcal{H}'[l-1] * \cdots * \mathcal{H}'$ and $\mathcal{Y} := {}^{\perp}\mathcal{X}$ be two subcategories of $\mathcal{T}$. Since $\mathcal{H}'$ is convariantly finite, then $\mathcal{X}$ is also convariantly finite (see [C1, Theorem 1.4]) and thus $\mathcal{T} = \mathcal{Y} * \mathcal{X}$ is a torsion pair by [IYo, Proposition 2.3]. We claim that $\mathcal{Y} \subset {}^{\perp}\mathcal{S}'[\geq 0] * {}^{\perp}\mathcal{S}'[<0]$. Then $\mathcal{T} = \mathcal{Y} * \mathcal{X} \subset {}^{\perp}\mathcal{S}'[\geq 0] * {}^{\perp}\mathcal{S}'[<0] \subset \mathcal{T}$ and therefore, $\mathcal{T} = {}^{\perp}\mathcal{S}'[\geq 0] * {}^{\perp}\mathcal{S}'[<0]$ is a co-t-structure.

Now we show the claim. For any $Y \in \mathcal{Y}$, there exists a triangle

$$ Y_{<-l}[-1] \xrightarrow{f} Y_{\geq -l} \to Y \to Y_{<-l} \tag{1.3.6}$$

such that $Y_{\geq -l} \in \mathcal{T}_{\geq -l} = {}^{\perp}\mathcal{S}[\geq l+1]$ and $Y_{<-l} \in \mathcal{T}_{<-l} = {}^{\perp}\mathcal{S}[\leq l]$. Since $\mathcal{T}_{<-l} = \mathsf{Filt}\mathcal{S}[>l] \subset {}^{\perp}\mathcal{S}'[<0]$ by (1.3.5), then to prove the claim, it suffices to show $Y_{\geq -l} \in {}^{\perp}\mathcal{S}'[\geq 0]$. With (1.3.2), we only need to check the following cases.

(i) $\mathrm{Hom}_{\mathcal{T}}(Y_{\geq -l}, \mathcal{S}[i]) = 0$ for $l < i$;
(ii) $\mathrm{Hom}_{\mathcal{T}}(Y_{\geq -l}, \mathcal{S}[i]) = 0$ for $n \leq i \leq l$;
(iii) $\mathrm{Hom}_{\mathcal{T}}(Y_{\geq -l}, \mathcal{S}'[i]) = 0$ for $0 \leq i \leq n-1$.

Notice that (i) is clear since $Y_{\geq -l} \in {}^{\perp}\mathcal{S}[\geq l+1]$. We show (ii). For any $n \leq i \leq l$, since $\mathcal{S}[i] \subset \mathcal{X}$ by (1.3.3), then $\mathrm{Hom}_{\mathcal{T}}(Y, \mathcal{S}[i]) = 0$. On the other hand, notice that $Y_{<-l}[-1] \in \mathcal{T}_{\leq -l} = \mathcal{T}^{\leq -l}$ and $Y_{\geq -l} \in \mathcal{T}_{\geq -l}$. Then $\mathrm{Hom}_{\mathcal{T}}(Y_{<-l}[-1], S[i]) = 0$ for $n \leq i < l$ and by Proposition 1.3.12, $\mathrm{Hom}_{\mathcal{T}}(f, S[l]) = 0$. Then (ii) is true by triangle (1.3.6).

We show (iii). For $0 \leq i \leq n-1$, (1.3.2) implies $S'[i] \subset \mathcal{H}[2n-1] * \cdots * \mathcal{H} \subset \mathcal{T}^{\geq -l+1}$ (Because $l \geq 2n$ by our assumption). So in this case, $\mathrm{Hom}_{\mathcal{T}}(Y_{-l}[-1], \mathcal{S}'[i]) = 0$. Since $Y \in \mathcal{Y}$, then $\mathrm{Hom}_{\mathcal{T}}(Y, \mathcal{S}'[i]) = 0$ for $0 \leq i \leq l$. Then by triangle (1.3.6), (iii) is true.

So our claim above holds and thus $\mathcal{T} = {}^{\perp}\mathcal{S}'[\geq 0] \perp {}^{\perp}\mathcal{S}'[<0]$ is a co-t-structure. By (1.3.3), ${}^{\perp}\mathcal{S}'[\geq 0] \subset {}^{\perp}\mathcal{S}[\geq n]$. Since ${}^{\perp}\mathcal{S}[\geq n] \subset \mathcal{T}^{\mathrm{p}}$, then ${}^{\perp}\mathcal{S}'[\geq 0] \subset \mathcal{T}^{\mathrm{p}}$. Similarly, one can show $\mathcal{T} = \mathcal{S}'[\geq 0]^{\perp} \perp \mathcal{S}'[<0]^{\perp}$ is a co-t-structure of $\mathcal{T}$ and $\mathcal{S}'[\leq 0]^{\perp} \subset \mathcal{T}^{\mathrm{p}}$. So $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathbb{S}, \mathcal{S}')$ is also a SMC quadruple. □

Immediately form Theorem 1.3.14 above and Theorem 1.3.5 (3), we have the following observation.

Corollary 1.3.15. *Let $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathbb{S}, \mathcal{S})$ be a SMC quadruple. Assume there are only finitely many indecomposable objects in $\mathcal{T}$ (up to isomorphism). Then the functor $\mathcal{T} \to \mathcal{T}_{\mathrm{sg}}$ induces a well-defined map*

$$\{\textit{SMCs of } \mathcal{T}\} \longrightarrow \{d\textit{-SMSs of } \mathcal{T}_{\mathrm{sg}}\}.$$

1.4. Application to Gorenstein dg algebras

In this section, we consider the applications of Theorem 1.3.5 to Gorenstein dg k-algebras, which are our main study object in Part 2. Let A be a dg k-algebra. Assume A satisfies the following conditions (see Assumption 2.1.1).

(1) A is *non-positive*, *i.e.* $\mathrm{H}^i(A) = 0$ for $i > 0$ (without loss of generality, we may assume $A^i = 0$ for $i > 0$, see Section 0.3.2);
(2) A is *proper*, *i.e.* $\dim_k \bigoplus_{i\in\mathbb{Z}} \mathrm{H}^i(A) < \infty$;
(3) A is *Gorenstein*, *i.e.* the thick subcategory $\mathsf{per}\,A$ of the derived category $\mathsf{D}(A)$ generated by A coincides with the thick subcategory generated by DA, where $D = \mathrm{Hom}_k(?, k)$ is the k-dual.

Let $\mathbb{S} := ? \otimes^{\mathbf{L}}_A DA$ be the Nakayama functor. Let $\mathcal{S} = \{S_i, 1 \leq i \leq n\}$ be the set of simple $\mathrm{H}^0(A)$-modules. We may also regard $\mathcal{S}$ as the set of simple dg A-modules concentrated in degree 0. In this case, we have the following observation.

Proposition 1.4.1. *The quadruple* $(\mathsf{D}^{\mathrm{b}}(A), \mathsf{per}\, A, \mathbb{S}, \mathcal{S})$ *is a SMC quadruple.*

To show this proposition, we need prepare some lemmas first.

Lemma 1.4.2. *Let* $X \in \mathsf{D}^{\mathrm{b}}(A)$*. Then the following are equivalent.*
(1) $X \in \mathsf{per}\, A$*;*
(2) *For all* $Y \in \mathsf{D}^{\mathrm{b}}(A)$*, the space* $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X, Y[i])$ *vanishes for almost all* $i \in \mathbb{Z}$*.*

Remark 1.4.3. This lemma is known for finite dimensional k-algebras (see [AKLY, Lemma 2.4]). Here we generalize it to any non-positive proper dg k-algebras.

Proof. (1) $\Rightarrow$ (2) Since for any $Y \in \mathsf{D}^{\mathrm{b}}(A)$ and $i \in \mathbb{Z}$, we have $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A, Y[i]) = \mathrm{H}^i(Y)$, then it is clear (2) holds for A. Thus by dévissage, (2) holds for any $X \in \mathsf{per}\, A = \mathsf{thick}(A)$.

(2) $\Rightarrow$ (1) Assume $X \in \mathsf{D}^{\mathrm{b}}(A)$ satisfies (2). We construct the following triangles inductively.

$$P_n[l_n] \xrightarrow{f_n} X_n \to X_{n+1} \to P_n[l_n+1], \tag{1.4.1}$$

such that $X_0 = X$, $P_n \in \mathsf{add}\, A$ and $l_n = -\sup\{l \in \mathbb{Z} \mid \mathrm{H}^l(X_n) \neq 0\}$. In addition, the induced map $\mathrm{H}^{-l_n}(P_n) \to \mathrm{H}^{-l_n}(X_n)$ is the projective cover of $\mathrm{H}^{-l_n}(X_n)$. By our construction, it is easy to see that $l_0 < l_1 < l_2 < \cdots$. We only need to show $X_n = 0$ for big enough n and then $X \in \mathsf{per}\, A$.

We claim

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X_m, S[l_m]) = \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X, S[l_m]),$$

for any $S \in \mathcal{S}$. Notice that $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(P_i[t], S[l_m]) = 0$ for any i and $t < l_m$. We consider two cases $l_{m-1}+1 < l_m$ and $l_{m-1}+1 = l_m$. For the first case, we know $l_n + 1 < l_m$ for all $n < m$, then we have

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X_m, S[l_m]) = \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X_{m_1}, S[l_m]) = \cdots = \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X, S[l_m])$$

by applying $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(?, S[l_m])$ to triangles (1.4.1) for $n < m$. For the second case, we consider the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X_{m-1}[1], S[l_m]) & \longrightarrow & \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(P_{m-1}[l_m], S[l_m]) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Hom}_A(\mathrm{H}^{-l_{m-1}}(X_{m-1}), S) & \xrightarrow{\simeq} & \mathrm{Hom}_A(\mathrm{H}^{-l_{m-1}}(P_{m-1}), S). \end{array}$$

The left and right arrows are bijective (see for example, [KN, Lemma 4.4]). Since the lower map is isomorphic by our construction of P_{m-1}, so is the upper one. Then we have $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X_m, S[l_m]) = \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X_{m-1}, S[l_m])$ by triangle (1.4.1) (taking $n = m-1$). Moreover the claim holds by triangle (1.4.1).

By our assumption, there exists $N > 0$, such that for any $n > N$ and $S \in \mathcal{S}$, we have $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X, S[n]) = 0$. Since there exists m such that $l_m > N$. Then by the claim above, $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X_m, S[l_m]) = 0$ for all $S \in \mathcal{S}$. Then it is easy to check

$$\mathrm{Hom}_A(\mathrm{H}^{-l_m}(X_m), S) = \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X_m, S[l_m]) = 0.$$

It suggests X_m must be zero. Thus $X \in P_0[l_0] * P_1[l_1] * \cdots * P_m[l_m] \subset \mathsf{per}\, A$. □

Lemma 1.4.4. (1) *There is a standard co-t-structure of* $\mathsf{per}\, A$ *given by* $\mathsf{per}\, A = \mathsf{Filt}(A[<0]) \perp \mathsf{Filt}(A[\geq 0])$*. Moreover, we have*

$$\begin{aligned} \mathsf{Filt}(A[<0]) &= \bigcup_{n>0} \mathsf{Filt}(A[-n]) * \mathsf{Filt}(A[-n+1] * \cdots * \mathsf{Filt}(A[-1])); \\ \mathsf{Filt}(A[\geq 0]) &= \bigcup_{n\geq 0} \mathsf{Filt}(A) * \cdots * \mathsf{Filt}(A[n-1]) * \mathsf{Filt}(A[n]). \end{aligned}$$

(2) $\mathsf{Filt}(A[<0])$ *is a contravariantly finite subcategory of* $\mathsf{D}^{\mathrm{b}}(A)$ *and* $\mathsf{Filt}(A[\geq 0])$ *is a covariantly finite subcategory of* $\mathsf{D}^{\mathrm{b}}(A)$.

Proof. (1) is well-known, see for example [IYa1, Proposition 2.8].

(2) We only show $\mathsf{Filt}(A[<0])$ is contravariantly finite, since the other statement can be show in a dual way. Notice that $\mathsf{Filt}(A) = \mathsf{add}\, A$. Then $\mathsf{Filt}(A[n])$ is a functorially finite subcategory of $\mathsf{D}^{\mathrm{b}}(A)$ for any $n \in \mathbb{Z}$ and thus, $\mathsf{Filt}(A[-n]) * \mathsf{Filt}(A[-n+1]) * \cdots * \mathsf{Filt}(A[-1])$ is contravariantly finite for $n > 0$ by the dual of [C1, Theorem 1.4]. Let $M \in \mathsf{D}^{\mathrm{b}}(A)$. There exists $n > 0$, such that $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A[<-n], M) = 0$. Since

$$\mathsf{Filt}(A[<0]) = \mathsf{Filt}(A[<-n]) * (\mathsf{Filt}(A[-n]) * \mathsf{Filt}(A[-n+1]) * \cdots * \mathsf{Filt}(A[-1])),$$

then Lemma 0.1.1 (2) suggests that there is a right $\mathsf{Filt}(A[<0])$-approximation of M. Therefore $\mathsf{Filt}(A[<0])$ is contravariantly finite. □

Now we prove Proposition 1.4.1.

Proof of Proposition 1.4.1. We check the conditions (RS0), (RS1) and (RS2) in Definition 1.3.1 hold. (RS0) is clear and in our setting, (RS1) is well-known (see for example [K1, Section 10.1]).

We show (RS2). We claim $\mathsf{D}^{\mathrm{b}}(A) = {}^{\perp}\mathcal{S}[\geq 0] \perp {}^{\perp}\mathcal{S}[<0]$ is a co-t-structure with ${}^{\perp}\mathcal{S}[\geq 0] = \mathsf{Filt}(A[<0])$. In fact, we have a co-$t$-structure $\mathsf{D}^{\mathrm{b}}(A) = \mathsf{Filt}(A[<0]) \perp \mathsf{Filt}(A[<0])^{\perp}$ by Lemma 1.4.4 and [IYo, Proposition 2.3]. Since $\mathsf{Filt}(A[<0])^{\perp} = \{M \in \mathsf{D}^{\mathrm{b}}(A) \mid \mathrm{H}^{>0}(M) = 0\}$, then we have

$$\mathsf{Filt}(A[<0])^{\perp} = \mathsf{Filt}(\mathcal{S}[\geq 0]) = {}^{\perp}\mathcal{S}[<0]$$

by Proposition 0.2.5. Thus the claim is ture.

Notice that we have another co-t-structure $\mathsf{D}^{\mathrm{b}}(A) = {}^{\perp}\mathsf{Filt}(A[\geq 0]) \perp \mathsf{Filt}(A[\geq 0])$. Since by (RS1), we have a triangle equivalence $\mathbb{S} : \mathsf{D}^{\mathrm{b}}(A) \simeq \mathsf{D}^{\mathrm{b}}(A)$, then $\mathbb{S}$ induces a new co-t-structure

$$\mathsf{D}^{\mathrm{b}}(A) = {}^{\perp}\mathsf{Filt}(\mathbb{S}A[\geq 0]) \perp \mathsf{Filt}(\mathbb{S}A[\geq 0]),$$

and ${}^{\perp}\mathsf{Filt}(\mathbb{S}A[\geq 0]) = \mathsf{Filt}(\mathcal{S}[<0]) = \mathcal{S}[\geq 0]^{\perp}$ by (RS1). Then we have co-t-structure $\mathsf{D}^{\mathrm{b}}(A) = \mathcal{S}[\geq 0]^{\perp} \perp \mathcal{S}[<0]^{\perp}$ with $\mathcal{S}[<0]^{\perp} = \mathsf{Filt}(\mathbb{S}A[\geq 0]) \subset \mathsf{per}\, A$. So $(\mathsf{D}^{\mathrm{b}}(A), \mathsf{per}\, A, \mathbb{S}, \mathcal{S})$ is a SMC quadruple. □

We point out that in this case, $\mathcal{F} = A[<0]^{\perp} \cap {}^{\perp}A[>0]$ is the category $\mathsf{CM}\, A$ of Cohen-Macaulay dg A-modules, which will be introduced and studied in Part 2. Then by Theorem 1.3.5, we have the following result.

Corollary 1.4.5. *Let A be a Gorenstein proper non-positive dg k-algebra.*

(1) *The composition* $\mathsf{CM}\, A \hookrightarrow \mathsf{D}^{\mathrm{b}}(A) \to \mathsf{D}^{\mathrm{b}}(A)/\mathsf{per}\, A$ *induces a triangle equivalence* $\underline{\mathsf{CM}}A \xrightarrow{\simeq} \mathsf{D}^{\mathrm{b}}(A)/\mathsf{per}\, A$. *Moreover,* $\underline{\mathsf{CM}}A$ *admits a Serre functor* $? \otimes^{\mathbf{L}}_A DA[-1]$*;*

(2) *If $\mathbb{S} = [1-d]$, then the set of simple dg A-modules is a d-SMS in* $\underline{\mathsf{CM}}A$.

1.5. SMC reduction Versus SMS reduction

1.5.1. **The SMC reduction of a Calabi-Yau triple.** Let $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathcal{S})$ be a $(1-d)$-CY triple for $d \geq 1$. Let $\mathcal{R}$ be subset of $\mathcal{S}$ such that $\mathcal{H}_{\mathcal{R}} = \mathsf{Filt}(\mathcal{R})$ is functorially finite subcategory of $\mathcal{T}$. Then $\mathcal{R}$ is a pre-SMC of $\mathcal{T}$ and the conditions (R1) and (R2) in Section 1.2 hold. Let

$$\mathcal{U} = \mathcal{T}/\mathsf{thick}(\mathcal{R})$$

be the SMC reduction of $\mathcal{T}$ with respect to $\mathcal{R}$. By relative Serre property (RS1), we have $\mathcal{T}^{\mathrm{p}} \cap \mathsf{thick}(\mathcal{R})^{\perp} = \mathcal{T}^{\mathrm{p}} \cap {}^{\perp}\mathsf{thick}(\mathcal{R})$, which will be denoted by $\mathcal{U}^{\mathrm{p}}$, that is,

$$\mathcal{U}^{\mathrm{p}} := \mathcal{T}^{\mathrm{p}} \cap \mathsf{thick}(\mathcal{R})^{\perp} = \mathcal{T}^{\mathrm{p}} \cap {}^{\perp}\mathsf{thick}(\mathcal{R}).$$

This category can be regarded as a full subcategory of $\mathcal{U}$ (see [Ne, Lemma 9.1.5]).

Our aim in this subsection is to show the SMC reduction of a Calabi-Yau triple gives us a new Calabi-Yau triple.

Theorem 1.5.1. *The triple $(\mathcal{U}, \mathcal{U}^{\mathrm{p}}, \mathcal{S})$ is a $(1-d)$-CY triple.*

To prove the theorem above, we need the description of $\mathcal{U}$ obtained in Section 1.2. Let

$$\mathcal{Z} := \mathcal{R}[\geq 0]^{\perp} \cap {}^{\perp}\mathcal{R}[\leq 0].$$

Then by Theorem 1.2.1, there is an equivalence $\mathcal{Z} \cong \mathcal{U}$ and the SMC $\mathcal{S}$ in $\mathcal{U}$ corresponds to SMC $\mathcal{S}' := \mathcal{S}\backslash\mathcal{R}$ in $\mathcal{Z}$. The following lemma implies the triple $(\mathcal{U}, \mathcal{U}', \mathcal{S})$ is equivalent to the triple $(\mathcal{Z}, \mathcal{T}^{\mathrm{p}} \cap \mathcal{Z}, \mathcal{S}')$. So to prove Theorem 1.5.1, it is equivalent to show $(\mathcal{Z}, \mathcal{T}^{\mathrm{p}} \cap \mathcal{Z}, \mathcal{S}')$ is a $(-d)$-CY triple.

Lemma 1.5.2. *We have $\mathcal{U}^{\mathrm{p}} = \mathcal{T}^{\mathrm{p}} \cap \mathcal{Z}$ as subcategories of $\mathcal{T}$.*

Proof. Let $X \in \mathcal{T}^{\mathrm{p}}$. Then $X \in \mathcal{Z}$ if and only if $\mathrm{Hom}_{\mathcal{T}}(\mathcal{R}[\geq 0], X) = 0 = \mathrm{Hom}_{\mathcal{T}}(X, \mathcal{R}[\leq 0])$. By the relative Serre duality (RS1), we have $\mathrm{Hom}_{\mathcal{T}}(\mathcal{R}[\geq 0], X) = D\mathrm{Hom}_{\mathcal{T}}(X, \mathcal{R}[\geq 1-d])$. Then $X \in \mathcal{T}^p \cap \mathcal{Z}$ if and only if $X \in \mathcal{T}^{\mathrm{p}} \cap {}^{\perp}\,\mathsf{thick}(\mathcal{R})$. □

By (RS2), we have co-t-structures $\mathcal{T} = {}^{\perp}\mathcal{S}[\geq 0] \perp {}^{\perp}\mathcal{S}[<0] = \mathcal{S}[\geq 0]^{\perp} \perp \mathcal{S}[<0]^{\perp}$. Recall we denote by $\mathcal{T}_{>0} = {}^{\perp}\mathcal{S}[\geq 0]$ and $\mathcal{T}_{\leq 0} = {}^{\perp}\mathcal{S}[<0]$. For $X \in \mathcal{T}$, there is a triangle

$$X_{\leq 0}[-1] \xrightarrow{f} X_{>0} \to X \to X_{\leq 0}, \tag{1.5.1}$$

with $X_{>0} \in \mathcal{T}_{>0}$ and $X_{\leq 0} \in \mathcal{T}_{\leq 0} = \mathcal{T}^{\leq 0}$. We may assume that $f \in \mathrm{rad}(X_{\leq 0}[-1], X_{>0})$. There is also a triangle

$$X'_{\geq 0} \to X \to X'_{<0} \to X'_{\geq 0}[1],$$

with $X'_{\geq 0} \in S[>0]^{\perp}$ and $X'_{<0} \in S[\leq 0]^{\perp}$. Then we have the following results.

Lemma 1.5.3. *Let $X \in \mathcal{Z}$. Then*

(1) *$X_{>0} \in \mathcal{T}^{\mathrm{p}} \cap \mathcal{Z}$ and $X_{\leq 0} \in \mathcal{Z}$;*
(2) *$X'_{<0} \in \mathcal{T}^{\mathrm{p}} \cap \mathcal{Z}$ and $X'_{\geq 0} \in \mathcal{Z}$.*

Proof. We only prove (1), since the second one can be shown in a similar way. We first show $X_{>0} \in \mathcal{T}^{\mathrm{p}} \cap \mathcal{Z}$. Since $X_{>0} \in \mathcal{T}^{\mathrm{p}}$ by (RS2) and $\mathcal{T}^{\mathrm{p}} \cap \mathcal{Z} = \mathcal{T}^{\mathrm{p}} \cap {}^{\perp}\,\mathsf{thick}(\mathcal{R})$ by Lemma 1.5.2, it suffices to show $X_{>0} \in {}^{\perp}\,\mathsf{thick}(\mathcal{R})$. Since $X_{>0} \in \mathcal{T}_{>0} = {}^{\perp}\mathcal{S}[\geq 0]$, then $\mathrm{Hom}_{\mathcal{T}}(X_{>0}, \mathcal{R}[\geq 0]) = 0$. Because $\mathrm{Hom}_{\mathcal{T}}(X, R[<-1]) = 0$ and $\mathrm{Hom}_{\mathcal{T}}(X_{\leq 0}, R[<0]) = 0$, then we have $\mathrm{Hom}_{\mathcal{T}}(X_{>0}, R[<-1]) = 0$ by triangle (1.5.1).

We are left to show $\mathrm{Hom}_{\mathcal{T}}(X_{>0}, R[-1]) = 0$ for any $R \in \mathcal{R}$. Since $X_{\leq 0}[-1] \in \mathcal{T}_{\leq 1}$, $X_{>0} \in \mathcal{T}_{\geq 1}$ and $f \in \mathrm{rad}(X_{\leq 0}[1], X_{>0})$, then the induced map $\mathrm{Hom}_{\mathcal{T}}(f, R[-1])$ is zero by Proposition 1.3.12. Since $\mathrm{Hom}_{\mathcal{T}}(X, R[-1]) = 0$, then $\mathrm{Hom}_{\mathcal{T}}(X_{>0}, R[-1]) = 0$ by the triangle (1.5.1). So $X_{>0} \in {}^{\perp}\,\mathsf{thick}(\mathcal{R})$ and therefore, $X_{>0} \in \mathcal{T}^{\mathrm{p}} \cap \mathcal{Z}$.

Since $X_{>0} \in \mathcal{T}^{\mathrm{p}} \cap {}^{\perp}\,\mathsf{thick}(\mathcal{R}) = \mathcal{T}^{\mathrm{p}} \cap \mathsf{thick}(\mathcal{R})^{\perp}$ and $X \in \mathcal{Z}$, then it is easy to check $X_{\leq 0} \in \mathcal{Z}$ by applying $\mathrm{Hom}_{\mathcal{T}}(\mathcal{R}[\geq 0], ?)$ and $\mathrm{Hom}_{\mathcal{T}}(?, \mathcal{R}[\leq 0])$ to (1.5.1). Thus the assertion is true. □

Now we are ready to prove Theorem 1.5.1.

Proof of Theorem 1.5.1. It is enough to prove $(\mathcal{Z}, \mathcal{Z} \cap \mathcal{T}^{\mathrm{p}}, \mathcal{S}')$ is a $(1-d)$-CY triple.

By Lemma 1.5.2, we know $\mathcal{T}^{\mathrm{p}} \cap \mathcal{Z}$ is a thick subcategory of $\mathcal{Z}$ and moreover, $P\langle 1\rangle = P[1]$ for any $P \in \mathcal{T}^{\mathrm{p}} \cap Z$. So the conditions (RS0) and (RS1) in Definition 1.3.1 hold directly. Next we show there is a co-t-structure $\mathcal{Z} = {}^{\perp}\mathcal{S}'\langle \geq 0\rangle \perp {}^{\perp}\mathcal{S}'\langle <0\rangle$ and ${}^{\perp}\mathcal{S}'\langle \geq 0\rangle \subset \mathcal{T}^{\mathrm{p}} \cap \mathcal{Z}$.

Let $X \in \mathcal{Z}$. Consider the triangle (1.5.1), we claim $X_{>0} \in {}^{\perp}\mathcal{S}'\langle \geq 0\rangle$ and $X_{\leq 0} \in {}^{\perp}\mathcal{S}'\langle <0\rangle$. Notice that for any $S \in \mathcal{S}'$ and $n \geq 1$, we have

$$S\langle n\rangle \in S[n] * \mathcal{H}_{\mathcal{R}}[n] * \cdots * \mathcal{H}_{\mathcal{R}}[1]$$

by Lemma 1.2.4. Then $\mathrm{Hom}_{\mathcal{T}}(X_{>0}, \mathcal{S}[\geq 0]) = 0$ implies $\mathrm{Hom}_{\mathcal{Z}}(X_{>0}, \mathcal{S}'\langle \geq 0\rangle) = 0$, that is $X_{>0} \in {}^{\perp}\mathcal{S}'\langle \geq 0\rangle$. Similarly, $X_{\leq 0} \in {}^{\perp}\mathcal{S}'\langle <0\rangle$ by the fact that $S\langle -m\rangle = \mathcal{H}_{\mathcal{R}}[\leq -1] * \cdots * \mathcal{H}_{\mathcal{R}}[\leq -m] * S[-m]$ for $m > 0$ and $X_{\leq 0} \in \mathcal{T}^{\leq 0}$. Thus we have

$$\mathcal{Z} = {}^{\perp}\mathcal{S}'\langle \geq 0\rangle * {}^{\perp}\mathcal{S}'\langle <0\rangle.$$

Notice that $\mathcal{S}'$ is a SMC in $\mathcal{Z}$ by Theorem 1.2.1, then ${}^{\perp}\mathcal{S}'\langle <0\rangle = \mathsf{Filt}(\mathcal{S}'\langle \geq 0\rangle)$ and therefore, $\mathrm{Hom}_{\mathcal{Z}}({}^{\perp}\mathcal{S}'\langle \geq 0\rangle, {}^{\perp}\mathcal{S}'\langle <0\rangle) = 0$. So the claim holds and $\mathcal{Z} = {}^{\perp}\mathcal{S}'\langle \geq 0\rangle \perp {}^{\perp}\mathcal{S}'\langle <0\rangle$ is a co-t-structure.

Assume $X \in {}^{\perp}\mathcal{S}'\langle \geq 0\rangle$, consider the triangle (1.5.1), since $X_{>0} \in \mathcal{T}^{\mathrm{p}}$ by (RS2) and we have shown $X_{\leq 0} \in {}^{\perp}\mathcal{S}'\langle <0\rangle$ above, then $\mathrm{Hom}_{\mathcal{Z}}(X, X_{\leq 0}) = 0$ and thus X is a direct summand of $X_{>0}$. So $X \in \mathcal{T}^{\mathrm{p}}$ and ${}^{\perp}\mathcal{S}'\langle \geq 0\rangle \subset \mathcal{T}^{\mathrm{p}} \cap \mathcal{Z}$.

Similarly, one can show $\mathcal{Z} = \mathcal{S}'\langle \geq 0\rangle^{\perp} \perp \mathcal{S}'\langle <0\rangle^{\perp}$ is also a co-t-structure with $\mathcal{S}'\langle <0\rangle^{\perp} \subset \mathcal{Z} \cap \mathcal{T}^{\mathrm{p}}$. Thus $(\mathcal{Z}, \mathcal{Z} \cap \mathcal{T}^{\mathrm{p}}, \mathcal{S}')$ is a $(1-d)$-CY triple and so is $(\mathcal{U}, \mathcal{U}^{\mathrm{p}}, \mathcal{S})$. □

1.5.2. **SMC reduction reduces SMS reduction.** In this section, we study the relation between SMC reduction and SMS reduction introduced in [CSP]. Let $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathcal{S})$ be a $(1-d)$-CY triple for $d \geq 1$. Let $\mathcal{H} = \mathsf{Filt}(S)$. Let $\mathcal{R}$ be a subset of $\mathcal{S}$ such that $\mathcal{H}_{\mathcal{R}} = \mathsf{Filt}(\mathcal{R})$ is functorially finite subcategory of $\mathcal{T}$.

The singularity category $\mathcal{T}_{\mathrm{sg}}$ is a $(-d)$-CY triangulated category and $\mathcal{S}$ is a d-SMS in $\mathcal{T}_{\mathrm{sg}}$ by Theorem 1.3.5. Moreover, we may regard $\mathcal{T}_{\mathrm{sg}}$ as a subfactor category of $\mathcal{T}$, that is

$$\frac{\mathcal{F}}{[\mathcal{P}]} \simeq \mathcal{T}_{\mathrm{sg}},$$

where $\mathcal{F} = \mathcal{H}[d-1] * \mathcal{H}[d-2] * \cdots * \mathcal{H}$, and $\mathcal{P} = \mathcal{T}_{\geq 0} \cap \mathcal{T}_{\leq 0}$. By this description, it is easy to check $\mathcal{H}_{\mathcal{R}}$ is also functorially finite in $\mathcal{T}_{\mathrm{sg}}$. Let

$$(\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}} = \{X \in \mathcal{T}_{\mathrm{sg}} \mid \mathrm{Hom}_{\mathcal{T}_{\mathrm{sg}}}(\mathcal{R}[i], X) = \mathrm{Hom}_{\mathcal{T}_{\mathrm{sg}}}(X, \mathcal{R}[-i]) = 0, \text{ for } 0 \leq i \leq d-1\}.$$

Then we regard $(\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}$ as the SMS reduction of $\mathcal{T}_{\mathrm{sg}}$ with respect to $\mathcal{R}$ in the sense of [CSP]. By [CSP, Theorems 4.1 and 5.1], $(\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}$ has a structure of triangulated category.

In Section 1.5.1, we have shown the triple $(\mathcal{U}, \mathcal{U}^{\mathrm{p}}, \mathcal{S})$ of the reduction of $(\mathcal{T}, \mathcal{T}^{\mathrm{p}}, \mathcal{S})$ is still a $(1-d)$-CY triple (Theorem 1.5.1). Our main result of this subsection is that the SMS reduction of the singularity category coincides with the singularity category of the SMC reduction in the following sense.

Theorem 1.5.4. *There is a triangle equivalence from $\mathcal{U}_{\mathrm{sg}} = \mathcal{U}/\mathcal{U}^{\mathrm{p}}$ to $(\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}$.*

Recall we may regard the triple $(\mathcal{U}, \mathcal{U}^{\mathrm{p}}, \mathcal{S})$ as $(\mathcal{Z}, \mathcal{Z} \cap \mathcal{T}^{\mathrm{p}}, \mathcal{S}')$. Let $\mathcal{H}' = \mathsf{Filt}_{\mathcal{Z}}\mathcal{S}'$. Then $\mathcal{U}_{\mathrm{sg}} \cong \mathcal{Z}/(\mathcal{Z} \cap \mathcal{T}^{\mathrm{p}})$ is equivalent to $\frac{\mathcal{F}_{\mathcal{Z}}}{[\mathcal{P}_{\mathcal{Z}}]}$ by Theorem 1.3.5, where $\mathcal{F}_{\mathcal{Z}} = \mathcal{H}'\langle d-1\rangle * \mathcal{H}'\langle d-2\rangle * \cdots * \mathcal{H}'$ and $\mathcal{P}_{\mathcal{Z}} = {}^{\perp}\mathcal{H}'[\neq 0]$.

We first show the functor $\mathcal{Z} \hookrightarrow \mathcal{T} \to \mathcal{T}_{\mathrm{sg}}$ induces a well-defined functor $\mathcal{Z} \to (\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}$. Before this, we give some general results, which will be used later.

Lemma 1.5.5. (1) *Let $X \in \mathcal{T}$ and $Y \in \mathcal{T}^{\leq 0}$. Then any morphism in $\mathrm{Hom}_{\mathcal{T}_{\mathrm{sg}}}(X, Y)$ has a representative of the form $X \xrightarrow{f} Z \xleftarrow{s} Y$ such that the cocone of s belongs to $\mathcal{T}^{\mathrm{p}} \cap \mathcal{T}^{\leq 0}$;*

(2) *Let $X \in \mathcal{T}^{\geq 0}$ and $Y \in \mathcal{T}$. Then any morphism in $\mathrm{Hom}_{\mathcal{T}_{\mathrm{sg}}}(X, Y)$ has a representative of the form $X \xleftarrow{t} Z \xrightarrow{f} Y$ such that the cone of t belongs to $\mathcal{T}^{\mathrm{p}} \cap \mathcal{T}^{\geq 0}$.*

Proof. We only show (1), since (2) can be shown in a similar way. Any morphism $X \to Y$ in $\mathcal{T}_{\mathrm{sg}}$ can be written as $X \xrightarrow{f} Z \xleftarrow{s} Y$, such that there is a triangle

$$W \xrightarrow{g} Y \xrightarrow{s} Z \longrightarrow W[1]$$

with $W \in \mathcal{T}^{\mathrm{p}}$. Consider the triangle $W_{>0} \to W \to W_{\leq 0} \to W_{>0}[1]$ with $W_{>0} \in \mathcal{T}_{>0}$ and $W_{\leq 0} \in T_{\leq 0}$. Notice that $W_{\leq 0} \in \mathcal{T}^{\mathrm{p}}$ by the fact $\mathcal{T}_{>0} \subset \mathcal{T}^{\mathrm{p}}$ and the triangle above. Since $Y \in \mathcal{T}^{\leq 0} = \mathcal{T}_{\leq 0}$, then $\mathrm{Hom}_{\mathcal{T}}(W_{>0}, Y) = 0$ and g factors though $W \to W_{\leq 0}$. Thus we obtain the following commutative diagram of triangles.

$$\begin{array}{ccccccc} W & \xrightarrow{g} & Y & \xrightarrow{s} & Z & \longrightarrow & W[1] \\ \downarrow & & \| & & \downarrow{\scriptstyle h} & & \downarrow \\ W_{\leq 0} & \longrightarrow & Y & \xrightarrow{hs} & Z' & \longrightarrow & W_{\leq 0}[1] \end{array}$$

The morphism $X \xrightarrow{f} Z \xleftarrow{s} Y$ is equivalent to $X \xrightarrow{hf} Z' \xleftarrow{hs} Y$, and in this case, the cocone $W_{\leq 0}$ of hs belongs to $\mathcal{T}^{\mathrm{p}} \cap \mathcal{T}^{\leq 0}$, so the assertion follows. □

The following observation is useful.

Proposition 1.5.6. (1) *The functor $\mathcal{T} \to \mathcal{T}_{\mathrm{sg}}$ induces a bijection (resp. surjection) $\mathrm{Hom}_{\mathcal{T}}(X, Y) \to \mathrm{Hom}_{\mathcal{T}_{\mathrm{sg}}}(X, Y)$ for $X \in \mathcal{T}^{\geq 2-d}$ (resp. $X \in \mathcal{T}^{\geq 1-d}$) and $Y \in \mathcal{T}^{\leq 0}$;*

(2) *The functor $\mathcal{T} \to \mathcal{T}_{\mathrm{sg}}$ induces a bijection (resp. surjection) $\mathrm{Hom}_{\mathcal{T}}(X, Y) \to \mathrm{Hom}_{\mathcal{T}_{\mathrm{sg}}}(X, Y)$ for $X \in \mathcal{T}^{\geq 0}$ and $Y \in \mathcal{T}^{\leq d-2}$ (resp. $T \in \mathcal{T}^{\leq d-1}$).*

Proof. We only prove the first statement and (2) is similar by using Lemma 1.5.5 (2). We first show $\mathrm{Hom}_{\mathcal{T}}(X, Y) \to \mathrm{Hom}_{\mathcal{T}_{\mathrm{sg}}}(X, Y)$ is surjective for $X \in \mathcal{T}^{\geq 1-d}$ and $Y \in \mathcal{T}^{\leq 0}$. By Lemma 1.5.5 (1), any morphism in $\mathrm{Hom}_{\mathcal{T}_{\mathrm{sg}}}(X, Y)$ has a representative $X \xrightarrow{f} Z \xleftarrow{s} Y$ such that the cocone W of s is in $\mathcal{T}^{\mathrm{p}} \cap \mathcal{T}^{\leq 0}$, then we have the following exact sequence

$$\mathrm{Hom}_{\mathcal{T}}(X, Y) \to \mathrm{Hom}_{\mathcal{T}}(X, Z) \to \mathrm{Hom}_{\mathcal{T}}(X, W[1]).$$

Since $X \in \mathcal{T}^{\geq 1-d}$ and $W \in \mathcal{T}^{\mathrm{p}} \cap \mathcal{T}^{\leq 0}$, then by relative Serre duality (RS1), we have $\mathrm{Hom}_{\mathcal{T}}(X, W[1]) = D\mathrm{Hom}_{\mathcal{T}}(W, X[\leq -d]) = 0$. So there exists $g \in \mathrm{Hom}_{\mathcal{T}}(X, Y)$ such that $f = s \circ g$. Then the morphism $X \xrightarrow{f} Z \xleftarrow{s} Y$ is equivalent to $X \xrightarrow{g} Y$ in $\mathcal{T}_{\mathrm{sg}}$ and moreover, $\mathrm{Hom}_{\mathcal{T}}(X, Y) \to \mathrm{Hom}_{\mathcal{T}_{\mathrm{sg}}}(S, T)$ is surjective.

Next we show $\mathrm{Hom}_{\mathcal{T}}(X, Y) \to \mathrm{Hom}_{\mathcal{T}_{\mathrm{sg}}}(X, Y)$ is injective if $X \in \mathcal{T}^{\geq 2-d}$. Assume $f \in \mathrm{Hom}_{\mathcal{T}}(X, Y)$ is zero in $\mathcal{T}_{\mathrm{sg}}$, then it factors though some $P \in \mathcal{T}^{\mathrm{p}}$. We may assume $P \in \mathcal{T}^{\mathrm{p}} \cap \mathcal{T}^{\leq 0}$ by the proof of Lemma 1.5.5 (1). Then by (RS1), $\mathrm{Hom}_{\mathcal{T}}(X, P) = D\mathrm{Hom}_{\mathcal{T}}(P, X[1-d]) = 0$ since $X \in \mathcal{T}^{\geq 2-d}$. Thus f is zero in $\mathcal{T}$. So the statement follows. □

The following lemma suggests the existence of functor from $\mathcal{Z}$ to $(\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}$ directly.

Lemma 1.5.7. *Let $X \in \mathcal{Z}$, then*

(1) *The map $\mathrm{Hom}_{\mathcal{T}}(\mathcal{R}[i], X) \to \mathrm{Hom}_{\mathcal{T}_{\mathrm{sg}}}(\mathcal{R}[i], X)$ is bijective (resp. surjective) for $i \leq d-2$ (resp. $i \leq d-1$). In particular, $\mathrm{Hom}_{\mathcal{T}_{\mathrm{sg}}}(\mathcal{R}[i], X) = 0$ for $0 \leq i \leq d-1$;*

(2) *The map $\mathrm{Hom}_{\mathcal{T}}(X, \mathcal{R}[-i]) \to \mathrm{Hom}_{\mathcal{T}_{\mathrm{sg}}}(X, \mathcal{R}[-i])$ is bijective (resp. surjective) for $i \leq d-2$ (resp. $i \leq d-1$). In particular, $\mathrm{Hom}_{\mathcal{T}_{\mathrm{sg}}}(X, \mathcal{R}[-i]) = 0$ for $0 \leq i \leq d-1$.*

Proof. We only shown (1), since (2) is similar by using Lemma 1.5.3 (2) and Proposition 1.5.6 (2). The triangle (1.5.1) induces a commutative diagram as follows,

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{T}}(\mathcal{R}[i], X) & \longrightarrow & \mathrm{Hom}_{\mathcal{T}}(\mathcal{R}[i], X_{\leq 0}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{T}_{\mathrm{sg}}}(\mathcal{R}[i], X) & \longrightarrow & \mathrm{Hom}_{\mathcal{T}_{\mathrm{sg}}}(\mathcal{R}[i], X_{\leq 0}) \end{array}$$

The upper map is bijective since $X_{>0} \in \mathcal{T}^{\mathrm{p}} \cap \mathcal{Z} \subset \mathsf{thick}(\mathcal{R})^{\perp}$ by Lemma 1.5.3 (1) and the lower map is bijective since $X \to X_{\leq 0}$ becomes an isomorphism in $\mathcal{T}_{\mathrm{sg}}$. Since $X_{\leq 0} \in \mathcal{T}_{\leq 0} = \mathcal{T}^{\leq 0}$, then the right map is bijective (resp. surjective) for $i \leq d-2$ (resp. $i \leq d-1$) by Proposition 1.5.6 (1), so is the left one. Since $X \in \mathcal{Z}$, then $\mathrm{Hom}_{\mathcal{T}}(\mathcal{R}[\geq 0], X) = 0$. So $\mathrm{Hom}_{\mathcal{T}_{\mathrm{sg}}}(\mathcal{R}[i], X) = 0$ for $0 \leq i \leq d-1$. □

The following proposition shows we have a triangle functor $\mathcal{Z} \to (\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}$.

Proposition 1.5.8. *The composition of functors $\mathcal{Z} \hookrightarrow \mathcal{T} \xrightarrow{\pi} \mathcal{T}_{\mathrm{sg}}$ induces a well-defined triangle functor $\rho : \mathcal{Z} \to (\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}$.*

Proof. By Lemma 1.5.7, it is easy to see $\rho(\mathcal{Z}) \subset (\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}$. So $\rho : \mathcal{Z} \to (\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}$ is well-defined. We show it is a triangle functor.

First we claim ρ commutes with shift functors. Let $X \in \mathcal{Z}$. Then $X\langle 1\rangle$ is defined by the following triangle (see Section 1.2).

$$R_X \xrightarrow{f_X} X[1] \to X\langle 1\rangle \to R_X[1], \tag{1.5.2}$$

where $R_X \xrightarrow{f_X} X[1]$ is the right $\mathcal{H}_{\mathcal{R}}$-approximation of $X[1]$ in $\mathcal{T}$. Now we consider the triangle (1.5.2) in $\mathcal{T}_{\mathrm{sg}}$. By the equivalence $\frac{\mathcal{F}}{[\mathcal{P}]} \simeq \mathcal{T}_{\mathrm{sg}}$ (Theorem 1.3.5), it is clear that $R_X \xrightarrow{f_X} X[1]$ is also a right $\mathcal{H}_{\mathcal{R}}$-approximation of $X[1]$ in $\mathcal{T}_{\mathrm{sg}}$. Then $\rho(X\langle 1\rangle) = \rho(X)\langle 1\rangle$ in $(\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}$ (see [CSP, Definition 4.2] for the shift functor of $(\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}$).

Next we show ρ sends triangles in $\mathcal{Z}$ to triangles in $(\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}$. Let $s : X \to Y$ be a morphism in $\mathcal{Z}$. Consider the commutative diagrams (1.2.4), then $X \xrightarrow{s} Y \to W \to X\langle 1\rangle$ is the triangle induced by s in $\mathcal{Z}$ by Proposition 1.2.6. In fact, every triangle in $\mathcal{Z}$ can be obtained in this way. Now we consider the diagrams (1.2.4) in $\mathcal{T}_{\mathrm{sg}}$. We have shown that $R_Z \to Z$ and $R_X \to X[1]$ are right $\mathcal{H}_{\mathcal{R}}$-approximations in $\mathcal{T}_{\mathrm{sg}}$ above. Then by the construction of triangles of $(\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}$, we know $X \xrightarrow{s} Y \to W \to X\langle 1\rangle$ is the triangle given by s in $(\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}$ (see [CSP, Theorem 4.1 and Definition 4.4]). Then ρ sends triangles to triangles.

So ρ is a triangle functor and the assertion is true. □

Now we are ready to prove our main result.

Proof of Theorem 1.5.4. The natural functor $\rho : \mathcal{Z} \to (\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}$ is a triangle functor by Proposition 1.5.8. Since $\rho(\mathcal{T}^{\mathrm{p}}) = 0$, then ρ induces a triangle functor $\tilde{\rho} : \mathcal{Z}/(\mathcal{Z} \cap \mathcal{T}^{\mathrm{p}}) \to (\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}$. Since $\mathcal{Z}/(\mathcal{Z} \cap \mathcal{T}^{\mathrm{p}})$ is equivalent to $\frac{\mathcal{F}_{\mathcal{Z}}}{[\mathcal{P}_{\mathcal{Z}}]}$ by Theorem 1.3.5, we have a functor $\frac{\mathcal{F}_{\mathcal{Z}}}{[\mathcal{P}_{\mathcal{Z}}]} \to (\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}$, which is also denoted by $\tilde{\rho}$. We claim $\tilde{\rho}$ is fully faithful and dense.

Let $\mathcal{S}' = \mathcal{S}\backslash\mathcal{R}$ and $\mathcal{H}' = \mathsf{Filt}_{\mathcal{Z}}(\mathcal{S}')$. Then $\mathcal{S}'$ is a SMC in $\mathcal{Z}$ by Theorem 1.2.1 and moreover, $\mathcal{S}'$ is a d-SMS in $\frac{\mathcal{F}_{\mathcal{Z}}}{[\mathcal{P}_{\mathcal{Z}}]}$ by Theorem 1.3.5. So

$$\frac{\mathcal{F}_{\mathcal{Z}}}{[\mathcal{P}_{\mathcal{Z}}]} = \mathcal{H}'\langle d-1\rangle * \mathcal{H}'\langle d-2\rangle * \cdots * \mathcal{H}'.$$

On the other hand, $\rho(\mathcal{S}')$ is a d-SMS in $(\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}$ by [CSP, Theorem 6.6] and thus by [CSP, Lemma 2.8], we have

$$(\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}} = \rho(\mathcal{H}')\langle d-1\rangle * \rho(\mathcal{H}')\langle d-2\rangle * \cdots * \rho(\mathcal{H}').$$

Then it is clear that $\tilde{\rho}$ is dense. We are left to show $\tilde{\rho}$ is fully faithful. Let $X, Y \in \mathcal{S}'$. We may assume $X, Y \notin \mathcal{P}_{\mathcal{Z}}$. It is enough to show

$$\mathrm{Hom}_{\frac{\mathcal{F}_{\mathcal{Z}}}{[\mathcal{P}_{\mathcal{Z}}]}}(X\langle i\rangle, Y\langle j\rangle) = \mathrm{Hom}_{(\mathcal{T}_{\mathrm{sg}})_{\mathcal{R}}}(\tilde{\rho}(X)\langle i\rangle, \tilde{\rho}(Y)\langle j\rangle) \tag{1.5.3}$$

for any $i, j \in \mathbb{Z}$. Let $t = j - i$. Notice that if $t < 0$, then the both sides of equation (1.5.3) are zero. If $t = 0$. Since

$$\mathrm{dimHom}_{\frac{\mathcal{F}_{\mathcal{Z}}}{[\mathcal{P}_{\mathcal{Z}}]}}(X, Y) = \mathrm{dimHom}_{\mathcal{Z}}(X, Y) = \delta_{X,Y},$$

and $\mathrm{Hom}_{\mathcal{Z}}(X, Y) = \mathrm{Hom}_{\mathcal{T}_{\mathrm{sg}}}(\rho(X), \rho(Y))$ by Proposition 1.5.6, then (1.5.3) holds.

If $t > 0$. Notice that $Y\langle t\rangle \in Y[t] * \mathcal{H}_{\mathcal{R}}[t] * \cdots * \mathcal{H}_{\mathcal{R}}[1]$ by Lemma 1.2.4. Then there is a triangle $Y[t] \to Y\langle t\rangle \to Z \to Y[t+1]$ in $\mathcal{T}$ such that $Z \in \mathcal{H}_{\mathcal{R}}[t] * \cdots * \mathcal{H}_{\mathcal{R}}[1] \subset \mathcal{T}^{\leq -1}$. Then by Proposition 1.5.6 and five lemma, one can show

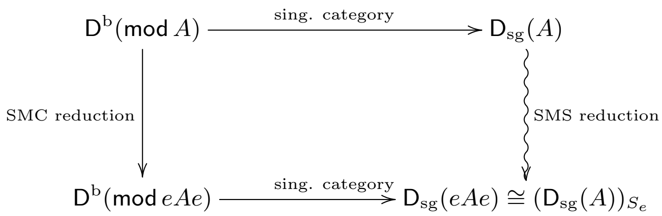
$$\mathrm{Hom}_{\mathcal{T}}(X, Y\langle t\rangle) = \mathrm{Hom}_{\mathcal{T}_{\mathrm{sg}}}(X, Y\langle t\rangle).$$

Because $\mathrm{Hom}_{\mathcal{Z}}(\mathcal{P}_{\mathcal{Z}}, Y\langle t\rangle) = 0$ by our constructionof $\mathcal{P}_{\mathcal{Z}}$, then $\mathrm{Hom}_{\frac{\mathcal{F}_{\mathcal{Z}}}{[\mathcal{P}_{\mathcal{Z}}]}}(X, Y\langle t\rangle) = \mathrm{Hom}_{\mathcal{Z}}(X, Y\langle t\rangle)$. So the equation (1.5.3) is true. Then $\tilde{\rho}$ is fully faithful.

Thus $\tilde{\rho} : \frac{\mathcal{F}_{\mathcal{Z}}}{[\mathcal{P}_{\mathcal{Z}}]} \to (\mathcal{T}_{\text{sg}})_{\mathcal{R}}$ gives a triangle equivalence and the theorem holds. □

We finish Part 1 by consider some examples.

Example 1.5.9. Let A be a finite-dimensional symmetric k-algebra and let e be an idempotent. Let $S_e = \text{top}(1-e)A$. Then by Proposition 1.2.9 (1), the SMC reduction of $\mathsf{D}^{\text{b}}(\mathsf{mod}\, A)$ with respect to S_e is triangle equivalent to $\mathsf{D}^{\text{b}}(\mathsf{mod}\, eAe)$. Then by Theorem 1.5.4, we have the following commutative diagram,

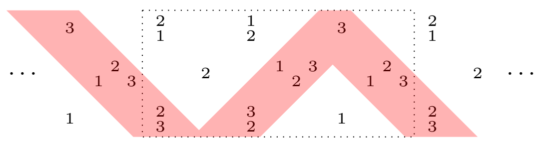


We point out that the left map is given by the functor $? \otimes^{\mathbf{L}}_A Ae : \mathsf{D}^{\text{b}}(\text{mod}A) \to \mathsf{D}^{\text{b}}(\text{mod}eAe)$ and, the upper and lower maps are given by the Verdier quotient. But the right map is usually not given by functors.

Next we consider a concrete algebra A and check the equivalence $\mathsf{D}_{\text{sg}}(eAe) \cong (\mathsf{D}_{\text{sg}}(A))_{S_e}$ by comparing the AR quivers of them.

Example 1.5.10. Let A be the k-algebra given by the quiver $1 \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\beta_1}{\rightleftarrows}} 3$, with relations $\{\alpha_1\alpha_2\alpha_1, \beta_2\beta_1\beta_2, \alpha_1\beta_1, \beta_2\alpha_2, \alpha_2\alpha_1 - \beta_1\beta_2\}$. Let S_i be the simple A-modules at vertices i $(i = 1, 2, 3)$ and let $P_1 = \begin{smallmatrix}1\\2\\1\end{smallmatrix}$ (resp. $P_2 = \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix}$, $P_3 = \begin{smallmatrix}3\\2\\3\end{smallmatrix}$) be the indecomposable projective A-module at the vertex 1 (resp. 2, 3). Let $e = e_1 + e_2$ be an idempotent. Consider the SMC reduction $\mathsf{D}^{\text{b}}(\mathsf{mod}\, A)/\mathsf{thick}(S_3)$ of $\mathsf{D}^{\text{b}}(\mathsf{mod}\, A)$ with respect to S_3. It is equivalent to $\mathsf{D}^{\text{b}}(\mathsf{mod}\, B)$ by Proposition 1.2.9 (1), where $B = eAe$ is given by the quiver $1 \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} 2$ with relations $\{\alpha_1\alpha_2\alpha_1, \alpha_2\alpha_1\alpha_2\}$.

Since A is symmetric, then it is well-know that $\mathsf{D}_{\text{sg}}(A) \cong \underline{\mathsf{mod}}A$ and the AR quiver of $\mathsf{D}_{\text{sg}}(A)$ is given by $\mathbb{Z}A_3/\nu[1]$. In fact, we can describe it specifically as follows,



where the arrows are omitted and a fundamental domain is outlined in dotted line. By the definition of SMS reduction, we know that

$$(\mathsf{D}^{\text{b}}_{\text{sg}}(A))_{S_3} = \{X \in \mathsf{D}^{\text{b}}_{\text{sg}}(A) \mid \text{Hom}_{\mathsf{D}^{\text{b}}_{\text{sg}}(A)}(X[i], S_3) = 0 = \text{Hom}_{\mathsf{D}^{\text{b}}_{\text{sg}}(A)}(S_3[i], X) \text{ with } i = 0, 1\}.$$

So the indecomposable objects of $(\mathsf{D}^{\text{b}}_{\text{sg}}(A))_{S_3}$ are given by the AR quiver above without the shaded part. The AR quiver of $(\mathsf{D}^{\text{b}}_{\text{sg}}(A))_{S_3}$ is $\mathbb{Z}A_2/\nu[1]$, which is the same as the AR quiver of $\mathsf{D}^{\text{b}}_{\text{sg}}(B)$.

Appendix A. An equivalence induced by derived Schur functor

Let A be a non-positive proper dg algebra. Let e be an idempotent of A. Assume $e \in A^0$. Then eA (resp. Ae) is a right (resp. left) dg A-module. We have a natural derived Schur functor $F = ? \otimes^{\mathbf{L}}_A Ae : \mathsf{D}(A) \to \mathsf{D}(eAe)$, which restricts to a functor $F^{\text{b}} = ? \otimes^{\mathbf{L}}_A Ae : \mathsf{D}^{\text{b}}(A) \to \mathsf{D}^{\text{b}}(eAe)$. It is well-known that F admits a left adjoint $G = ? \otimes^{\mathbf{L}}_{eAe} eA$. We first give an easy observation.

Lemma 1.5.11. *Let $M \in \mathsf{D}^{\mathrm{b}}(eAe)$. Then $G(M) \in \mathsf{D}(A)$ is upper bounded and $\mathrm{H}^i(G(M))$ is finite-dimensional for any $i \in \mathbb{Z}$.*

Proof. Since $M \in \mathsf{D}^{\mathrm{b}}(eAe)$, we may assume $M^{\gg 0} = 0$. We have

$$D\mathrm{H}^i(G(M)) = \mathrm{Hom}_{\mathsf{D}(A)}(G(M)[i], DA) = \mathrm{Hom}_{\mathsf{D}(eAe)}(M, F(DA)[i]).$$

Since $M, F(DA)[i] \in \mathsf{D}^{\mathrm{b}}(eAe)$ and by Lemma 0.3.5, $\mathsf{D}^{\mathrm{b}}(eAe)$ is Hom-finite, then $\mathrm{H}^i(M)$ is finite dimensional for any $i \in \mathbb{Z}$. Notice that M and eA are both upper bounded, so $G(M) = M \otimes^{\mathbf{L}}_{eAe} eA$ is also upper bounded. □

The following result should be well-known, but we could not find a reference. So we include a complete proof for the convenience of the reader.

Proposition 1.5.12. *Let A be a non-positive proper dg algebra and $e \in A$ be an idempotent. Let F, G be defined as above. Then F induces a triangle equivalence $\overline{F}^{\mathrm{b}} : \mathsf{D}^{\mathrm{b}}(A)/\ker F^{\mathrm{b}} \simeq \mathsf{D}^{\mathrm{b}}(eAe)$.*

Remark 1.5.13. We point out that Proposition 1.5.12 is known for finite-dimensional k-algebra (see [C2, Lemma 2.2]). But the approach in [C2] fails in dg setting, so here we prove it in a more direct way.

Proof. Notice that G is fully faithful (see for example, [K1, Lemma 4.2]), then $\mathsf{D}(A)$ has a stable t-structure $(\mathrm{Im}\, G, \ker F)$ and moreover, there is a triangle equivalence $\overline{F} : \mathsf{D}(A)/\ker F \simeq \mathsf{D}(eAe)$. Considering the following commutative diagram.

$$\begin{array}{ccc} \mathsf{D}(A)/\ker F & \xrightarrow[\simeq]{\overline{F}} & \mathsf{D}(eAe) \\ {\scriptstyle H}\uparrow & & \uparrow \\ \mathsf{D}^{\mathrm{b}}(A)/\ker F^{\mathrm{b}} & \xrightarrow{\overline{F}^{\mathrm{b}}} & \mathsf{D}^{\mathrm{b}}(eAe) \end{array}$$

where $H : \mathsf{D}^{\mathrm{b}}(A)/\ker F^{\mathrm{b}} \to \mathsf{D}(A)/\ker F$ is the natural functor. To show $\overline{F}^{\mathrm{b}}$ is fully faithful, it is enough to show H is fully faithful and $\overline{F}^{\mathrm{b}}$ is dense.

(1) H is full. Let $X, Y \in \mathsf{D}^{\mathrm{b}}(A)$. Any morphism $X \to Y$ in $\mathsf{D}(A)/\ker F$ can be written as $X \stackrel{s}{\leftarrow} Z \stackrel{f}{\rightarrow} Y$, such that there is a triangle

$$K \to Z \stackrel{s}{\to} X \to K[1]$$

with $K \in \ker F$. In this case, $F(Z) \cong F(X)$ in $\mathsf{D}(eAe)$ and thus $F(Z) \in \mathsf{D}^{\mathrm{b}}(eAe)$. Let $Z' := GF(Z)$. Then we have a natural triangle $Z' \stackrel{t}{\to} Z \to K' \to Z'[1]$ in $\mathsf{D}(A)$ given by the adjoint pair such that $K' \in \ker F$. It is easy to check that the morphism $X \stackrel{st}{\longleftarrow} Z' \stackrel{ft}{\longrightarrow} Y$ is equivalent to $X \stackrel{s}{\leftarrow} Z \stackrel{f}{\rightarrow} Y$ in $\mathsf{D}(A)/\ker F$. By Lemma 1.5.11, we know that Z' is upper bounded and $\mathrm{H}^n(Z')$ is finite dimensional for any $n \in \mathbb{Z}$.

Now we consider the standard truncation of Z'. Since $X, Y \in \mathsf{D}^{\mathrm{b}}(A)$, we can find small enough m such that

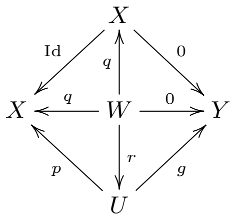
$$\mathrm{Hom}_{\mathsf{D}(A)}(\tau^{<m} Z', X) = 0 = \mathrm{Hom}_{\mathsf{D}(A)}(\tau^{<m} Z', Y).$$

Since $F(Z) = Ze$, which acts on cohomology, we may also assume $\tau^{<m} Z' \in \ker F$. Then we have the following diagram.

$$\begin{array}{ccccc} & & \tau^{<m} Z' & & \\ & & \downarrow & & \\ & \swarrow & Z' & \searrow & \\ & & \downarrow & & \\ X & \longleftarrow & \tau^{\geq m} Z' & \longrightarrow & Y \end{array}$$

By our construction, $\tau^{\geq m}Z' \in \mathsf{D}^{\mathrm{b}}(A)$ and the morphism $X \leftarrow \tau^{\geq m}Z' \rightarrow Y$ is equivalent to $X \leftarrow Z' \rightarrow Y$ in $\mathsf{D}(A)/\ker F$. So the functor $H : \mathsf{D}^{\mathrm{b}}(A)/\ker F^{\mathrm{b}} \rightarrow \mathsf{D}(A)/\ker F$ is full.

(2) H is faithful. Let $X \overset{p}{\leftarrow} U \overset{g}{\rightarrow} Y$ be any morphism in $\mathsf{D}^{\mathrm{b}}(A)/\ker F^{\mathrm{b}}$, which sends to zero map in $\mathsf{D}(A)/\ker F$. Then the morphism is equivalent to $X \overset{\mathrm{Id}}{\longleftarrow} X \overset{0}{\rightarrow} Y$ in $\mathsf{D}(A)/\ker F$. So we have commutative diagram.



where $W \in \mathsf{D}(A)$, $\mathrm{cone}(q) \in \ker F$, $gr = 0$ and $pr = q$. By the same strategy in (1), we can take $W \in \mathsf{D}^{\mathrm{b}}(A)$. Then $X \overset{p}{\leftarrow} U \overset{g}{\rightarrow} Y$ is also zero map in $\mathsf{D}^{\mathrm{b}}(A)$. So H is faithful.

(3) $\overline{F}^{\mathrm{b}}$ is dense. Let $M \in \mathsf{D}^{\mathrm{b}}(eAe)$. We know $G(M) \in \mathsf{D}(A)$ is upper bounded and $\mathrm{H}^i(G(M))$ is finite dimensional for any i by Lemma 1.5.11. Notice that we have

$$F(\tau^{\geq n}G(M)) = (\tau^{\geq n}G(M))e = \tau^{\geq n}(G(M)e) = \tau^{\geq n}(FG(M)) = \tau^{\geq n}(M).$$

Since $M \in \mathsf{D}^{\mathrm{b}}(eAe)$, we may take $n \ll 0$ such that $\tau^{\geq n}(M) \cong M$. Then $F(\tau^{\geq n}G(M)) = M$ and $\tau^{\geq n}(G(M)) \in \mathsf{D}^{\mathrm{b}}(A)$. So $\widetilde{F}$ is dense. $\square$

Part 2. Cohen-Macaulay differential graded modules and negative Calabi-Yau configurations

2.1. Introduction

The notion of Cohen-Macaulay (CM) modules is classical in commutative algebra [Ma, BH], and has natural generalizations for non-commutative algebras [B, H2, IW], often called Gorenstein projective modules [ABr, C3, EJ]. The category of CM modules has been studied by many researchers in representation theory (see, for example, [CR, Yo, Si, LW]). On the other hand, the derived categories of differential graded (dg) categories introduced by Bondal-Kapranov [BK] and Keller [K1, K3] is an active subject appearing in various areas of mathematics [Min, T, Ye1]. Among others, we refer to [An, Jo1, Ji1, KaY2, Schm] for the representation theory of dg categories.

In this part, we introduce Cohen-Macaulay dg modules over dg algebras and develop their representation theory to build a connection between these two subjects. One of the main properties of the category of Cohen-Macaulay dg modules is that it has a structure of extriangulated category and the stable category is equivalent to the singularity category, which is an analogue of Buchweitz's equivalence. Moreover, it admits almost split extensions and we can study it by Auslander-Reiten theory. In fact, there are many nice dg algebras (including those given in this part), whose categories of Cohen-Macaulay dg modules can be well understood, while the derived category of dg algebras are usually wild and it is hopeless to classify all the indecomposable objects.

To make everything work well, we need to add some restrictions on dg algebras. More precisely, we work on dg algebras A over a field k satisfying the following assumption.

Assumption 2.1.1. (1) A is *non-positive*, *i.e.* $\mathrm{H}^i(A) = 0$ for $i > 0$ (without loss of generality, we may assume $A^i = 0$ for $i > 0$, see Section 0.3.2);

(2) A is *proper*, *i.e.* $\dim_k \bigoplus_{i\in\mathbb{Z}} \mathrm{H}^i(A) < \infty$;

(3) A is *Gorenstein*, *i.e.* the thick subcategory $\mathsf{per}\, A$ of the derived category $\mathsf{D}(A)$ generated by A coincides with the thick subcategory generated by DA, where $D = \mathrm{Hom}_k(?, k)$ is the k-dual.

In this case, we define Cohen-Macaulay dg A-modules as follows, where we denote by $\mathsf{D}^{\mathrm{b}}(A)$ the full subcategory of $\mathsf{D}(A)$ consisting of the dg A-modules whose total cohomology is finite-dimensional.

Definition 2.1.2 (Definition 2.2.1)**.** (1) A dg A-module M in $\mathsf{D}^{\mathrm{b}}(A)$ is called a *Cohen-Macaulay dg A-module* if $\mathrm{H}^i(M) = 0$ and $\mathrm{Hom}_{\mathsf{D}A}(M, A[i]) = 0$ for $i > 0$;

(2) We denote by $\mathsf{CM}\, A$ the subcategory of $\mathsf{D}^{\mathrm{b}}(A)$ consisting of Cohen-Macaulay dg A-modules.

Definition 2.1.2 is motived by the fact that if A is concentrated in degree zero, then the condition (1) above gives an alternative description of classical Cohen-Macaulay modules [IYa1, Theorem 3.10]. Moreover, in this case the category $\mathsf{CM}\, A$ forms a Frobenius category in the sense of [H1] and the stable category $\underline{\mathsf{CM}}A$ is a triangulated category which is triangle equivalent to the singularity category $\mathsf{D}_{\mathrm{sg}}(A) = \mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)/\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ introduced by Buchweitz [B] and Orlov [O]. However, $\mathsf{CM}\, A$ does not necessarily have a natural structure of exact category in our setting. Instead, the following result shows it has a natural structure of extriangulated category introduced by Nakaoka and Palu [NP].

Theorem 2.1.3 (Theorems 2.2.4, 2.3.1 and 2.3.7)**.** *Let A be a non-positive proper Gorenstein dg algebra. Then*

(1) $\mathsf{CM}\, A$ *is functorially finite in* $\mathsf{D}^{\mathrm{b}}(A)$*;*

(2) $\mathsf{CM}\, A$ *is a Frobenius extriangulated category with* $\mathsf{Proj}(\mathsf{CM}\, A) = \mathsf{add}\, A$*;*

(3) *The stable category* $\underline{\mathsf{CM}}A := (\mathsf{CM}\, A)/[\mathsf{add}\, A]$ *is a triangulated category;*

(4) *The composition* $\mathsf{CM}\, A \hookrightarrow \mathsf{D}^{\mathrm{b}}(A) \to \mathsf{D}^{\mathrm{b}}(A)/\mathsf{per}\, A$ *induces a triangle equivalence*

$$\underline{\mathsf{CM}}A = (\mathsf{CM}\, A)/[\mathsf{add}\, A] \simeq \mathsf{D}^{\mathrm{b}}(A)/\mathsf{per}\, A = \mathsf{D}_{\mathrm{sg}}(A);$$

(5) $\underline{\mathsf{CM}}A$ *admits a Serre functor and* $\mathsf{CM}\, A$ *admits almost split extensions.*

The main examples we consider in this part are trivial extension dg algebras and truncated polynomial dg algebras. We determine all indecomposable Cohen-Macaulay dg modules over truncated polynomial dg algebras concretely and give their AR quivers (see Theorem 2.4.2 for the details). We also show that, in this case, the stable category is a cluster category by using a criterion given by Keller and Reiten [KR] (see Theorem 2.4.9).

One of the traditional subjects is the classification of Gorenstein rings which are representation-finite in the sense that they have only finitely many indecomposable Cohen-Macaulay modules. Riedtmann [Rie2, Rie3] and Wiedemann [W] considered the classification of representation-finite self-injective algebras and Gorenstein orders respectively. In both classifications, configurations play an important role. We may regard Wiedemann's configurations as "0-Calabi-Yau" since they are preserved by Serre functor $\mathbb{S}$ and regard Riedtmann's configurations as "(-1)-Calabi-Yau" since they are preserved by $\mathbb{S}\circ[1]$. Inspired by this, we introduce the negative Calabi-Yau configurations to study the AR quivers of $\mathsf{CM}\,A$.

Definition 2.1.4 (Definitions 2.5.1)**.** Let $\mathcal{T}$ be a k-linear Hom-finite Krull-Schmidt triangulated category and let C be a set of indecomposable objects of $\mathcal{T}$. We call C a *$(-d)$-Calabi-Yau configuration* (or *$(-d)$-CY configuration* for short) for $d \geq 1$ if the following conditions hold.
(1) $\dim_k \mathrm{Hom}_{\mathcal{T}}(X,Y) = \delta_{X,Y}$ for $X, Y \in C$;
(2) $\mathrm{Hom}_{\mathcal{T}}(X, Y[-j]) = 0$ for any two objects X, Y in C and $0 < j \leq d-1$;
(3) For any indecomposable object M in $\mathcal{T}$, there exists $X \in C$ and $0 \leq j \leq d-1$, such that $\mathrm{Hom}_{\mathcal{T}}(X, M[-j]) \neq 0$.

It is precisely Riedtmann's configuration if $d = 1$ and $\mathcal{T}$ is the mesh category of $\mathbb{Z}\Delta$ for a Dynkin diagram Δ (see [Rie2, Definition 2.3] for the details). It is easy to see d-SMS (see Definition 0.2.7) implies $(-d)$-CY configuration and the converse is also true if $\mathsf{Filt}(C)$ is functorially finite in $\mathcal{T}$ due to [CSP, Proposition 2.13]. "$(-d)$-Calabi-Yau configuration" are also introduced as "left d-Riedtmann configuration" in [CS2], and further studied in [CS3, CSP]. When the AR quiver of $\mathcal{T}$ is $\mathbb{Z}\Delta/G$ for some Dynkin diagram Δ and some group G, Calabi-Yau configuration can be characterized combinatorially (see Section 2.5.3). Our name "$(-d)$-Calabi-Yau configuration" is motivated by the following theorem, which is new even for $d = 1$ (see Remark 2.5.3 for the detail).

Theorem 2.1.5 (Theorem 2.5.2)**.** *Let $\mathcal{T}$ be a k-linear Hom-finite Krull-Schmidt triangulated category with a Serre functor $\mathbb{S}$. Let C be a $(-d)$-CY configuration in $\mathcal{T}$, then $\mathbb{S}[d]C = C$.*

We say a dg k-algebra A in Assumption 2.1.1 is *d-self-injective* (resp. *d-symmetric*) if $\mathsf{add}\,A = \mathsf{add}\,DA[d-1]$ in $\mathsf{D}(A)$ (resp. $\mathsf{D}(A^{\mathrm{e}})$). The following result, characterizing simple dg A-modules as a $(-d)$-CY configuration, generalizes [Rie2, Proposition 2.4].

Theorem 2.1.6 (Theorem 2.5.5)**.** *Let A be a d-self-injective dg algebra. Then the set of simple dg A-modules is a d-SMS, and hence a $(-d)$-CY configuration in $\underline{\mathsf{CM}}A$.*

Let Δ be a Dynkin digram. For a subset C of vertices of $\mathbb{Z}\Delta$, we define a translation quiver $(\mathbb{Z}\Delta)_C$ by adding to $\mathbb{Z}\Delta$ a vertex p_c and two arrows $c \to p_c \to \tau^{-1}(c)$ for each $c \in C$ (see Definition 0.5.1). Our main result in Part 2 states that the converse of Theorem 2.1.6 also holds in the following sense.

Theorem 2.1.7 (Theorem 2.7.1)**.** *Let Δ be a Dynkin digram. Let C be a subset of vertices of $\mathbb{Z}\Delta/\mathbb{S}[d]$. The following are equivalent.*
(1) *C is a $(-d)$-CY configuration;*
(2) *There exists a d-symmetric dg k-algebra A with AR quiver of $\mathsf{CM}\,A$ being $(\mathbb{Z}\Delta)_C/\mathbb{S}[d]$.*

To study the classification of configurations, Riedtmann [Rie2] gave a geometrical description of configurations by Brauer relations, and Luo [L] gave a description of Wiedemann's configuration by 2-Brauer relations. Similarly, we introduce *maximal d-Brauer relations* (see Definition 2.8.2). It gives a nice description of $(-d)$-CY configurations of type A_n. This geometric model has been studied by Coelho Simões [CS2, Theorem 6.5]. By using this model, we show the number of

$(-d)$-CY configurations in $\mathbb{Z}A_n/\mathbb{S}[d]$ is $\frac{1}{n+1}\binom{(d+1)n+d-1}{n}$ (Corollary 2.8.20). We develop several technical concepts and results on maximal d-Brauer relations and by using them we give another proof of Theorem 2.1.7 for the case $\Delta = A_n$ (Theorem 2.8.32). In this case, for any given $(-d)$-CY configuration C, the corresponding d-symmetric dg k-algebra is given explicitly by *Brauer tree dg algebra* (see Section 2.8.3 for the details). The following table explains the comparison among different configurations.

$(-d)$-CY ($d \geq 1$)	(-1)-CY	0-CY
$(-d)$-CY configuration	Riedtmann's configuration	Wiedemann's configuration
maximal d-Brauer relation	Brauer relation	2-Brauer relation
d-self-injective dg algebras	self-injective algebras	Gorenstein orders

2.2. Cohen-Macaulay dg modules

Let A be a dg k-algebra. In this section, we assume A satisfies Assumption 2.1.1.

Definition 2.2.1. (1) A dg A-module M is called *Cohen-Macaulay* if $M \in \mathsf{D}^{\mathrm{b}}_{\leq 0}(A)$ and $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(M, A[i]) = 0$ for $i > 0$;
(2) We denote by $\mathsf{CM}\,A$ the subcategory of $\mathsf{D}^{\mathrm{b}}(A)$ consisting of Cohen-Macaulay dg A-modules.

If A is an ordinary k-algebra, then $\mathsf{CM}\,A$ defined here is canonically equivalent to the usual one. We mention that Yekutieli also introduced Cohen-Macaulay dg modules (see [Ye2, Section 8]), but it is different from ours. Now we introduce some special dg algebras which are the main objects in this part.

Definition 2.2.2. Let A be a non-positive dg k-algebra and let d be a positive integer.
(1) We call A *d-self-injective* if $\mathsf{add}\,A = \mathsf{add}(DA[d-1])$ in $\mathsf{D}(A)$;
(2) We call A *d-symmetric* if $\mathsf{add}\,A = \mathsf{add}(DA[d-1])$ in $\mathsf{D}(A^{\mathrm{e}})$.

Since in our setting A is Gorenstein, then the equivalence (0.3.2) induces the following triangle auto-equivalence.

$$\nu : \mathsf{per}\,A \simeq \mathsf{per}\,A,$$

and moreover we get another triangle auto-equivalence $\nu : \mathsf{D}^{\mathrm{b}}(A) \simeq \mathsf{D}^{\mathrm{b}}(A)$. In particular, ν is a Serre functor on $\mathsf{per}\,A$. We give another description of $\mathsf{CM}\,A$ as follows.

Proposition 2.2.3. (1) $\mathsf{CM}\,A = \mathsf{D}^{\mathrm{b}}_{\leq 0} \cap \nu^{-1}(\mathsf{D}^{\mathrm{b}}_{\geq 0})$;
(2) *In particular, If A is a d-self-injective dg algebra, then* $\mathsf{CM}\,A = \mathsf{D}^{\mathrm{b}}_{\leq 0} \cap \mathsf{D}^{\mathrm{b}}_{\geq -d+1}$.

Proof. (1) By definition,

$$A[<0]^{\perp} = \{X \in \mathsf{D}^{\mathrm{b}}(A) \mid \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A[<0], X) = 0\} = \mathsf{D}^{\mathrm{b}}_{\leq 0}.$$

By Lemma 0.3.2, $\mathrm{H}^{<0}(\nu(X)) = \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A[>0], \nu(X)) = D\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X, A[>0])$, then

$${}^{\perp}A[>0] = \{X \in \mathsf{D}^{\mathrm{b}}(A) \mid \mathrm{H}^{<0}(\nu(X)) = 0\} = \{X \in \mathsf{D}^{\mathrm{b}}(A) \mid \nu(X) \in \mathsf{D}^{\mathrm{b}}_{\geq 0}\}.$$

Then $\mathsf{CM}\,A = \mathsf{D}^{\mathrm{b}}_{\leq 0} \cap \nu^{-1}(\mathsf{D}^{\mathrm{b}}_{\geq 0})$.

(2) Let $X \in \mathsf{CM}\,A$. If A is d-self-injective, then we have

$$\mathrm{Hom}_{\mathsf{D}A}(X, A[>0]) = \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X, DA[>d-1]) = D\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A[>d-1], X) = 0,$$

which implies $X \in \mathsf{D}^{\mathrm{b}}_{\geq -d+1}$. So $\mathsf{CM}\,A = \mathsf{D}^{\mathrm{b}}_{\leq 0} \cap \mathsf{D}^{\mathrm{b}}_{\geq -d+1}$. □

The first properties of $\mathsf{CM}\,A$ are the following, which are analogues of the well-known properties of Cohen-Macaulay modules. We refer to Section 0.4.1 for the notion of extriangulated category. We call the Verdier quotient $\mathsf{D}^{\mathrm{b}}(A)/\,\mathsf{per}\,A$ the *singularity category* $\mathsf{D}_{\mathrm{sg}}(A)$ of A.

Theorem 2.2.4. *Let A be a dg k-algebra satisfying Assumption 2.1.1. Then*
(1) $\mathsf{CM}\,A$ *is an Ext-finite Frobenius extriangulated category with* $\mathsf{Proj}(\mathsf{CM}\,A) = \mathsf{add}\,A$*;*
(2) *The stable category* $\underline{\mathsf{CM}}A := (\mathsf{CM}\,A)/[\mathsf{add}\,A]$ *is a triangulated category;*

(3) *The composition* $\mathsf{CM}\,A \hookrightarrow \mathsf{D}^{\mathrm{b}}(A) \to \mathsf{D}^{\mathrm{b}}(A)/\operatorname{\mathsf{per}} A$ *induces a triangle equivalence*

$$\underline{\mathsf{CM}}A = (\mathsf{CM}\,A)/[\operatorname{\mathsf{add}} A] \simeq \mathsf{D}^{\mathrm{b}}(A)/\operatorname{\mathsf{per}} A = \mathsf{D}_{\mathrm{sg}}(A).$$

Proof. See Corollary 1.4.5. Here we give a concrete proof for the reader. By our definition, $\mathsf{CM}\,A$ is an extension-closed subcategory of $\mathsf{D}^{\mathrm{b}}(A)$, then it has a natural extriangulated category structure by restricting the triangles of $\mathsf{D}^{\mathrm{b}}(A)$ on $\mathsf{CM}\,A$ (see [NP, Remark 2.18]). By Lemma 0.3.5, $\mathsf{D}^{\mathrm{b}}(A)$ is Hom-finite, then so is $\mathsf{CM}\,A$. It implies that $\mathsf{CM}\,A$ is Ext-finite and $\operatorname{\mathsf{add}} A$ is functorially finite in $\mathsf{CM}\,A$.

Since $\mathrm{Hom}_{\mathsf{CM}\,A}(P, X[1]) = 0 = \mathrm{Hom}_{\mathsf{CM}\,A}(X, P[1])$ for any $P \in \operatorname{\mathsf{add}} A$ and $X \in \mathsf{CM}\,A$, then we have $\operatorname{\mathsf{add}} A \subset \mathsf{Proj}(\mathsf{CM}\,\mathsf{A}) \cap \mathsf{Inj}(\mathsf{CM}\,\mathsf{A})$. For any $X \in \mathsf{CM}\,A$, we consider the right $(\operatorname{\mathsf{add}} A)$-approximation $P \to X$, which extends to the following triangle in $\mathsf{D}^{\mathrm{b}}(A)$.

$$Y \to P \to X \to Y[1].$$

It is easy to check $Y \in \mathsf{CM}\,A$ by applying the functors $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A[<0], ?)$ and $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(?, A[>0])$ to the triangle above. So $\mathsf{CM}\,A$ has enough projectives. Similarly, it also has enough injectives.

Finally, we show $\mathsf{Proj}(\mathsf{CM}\,A) = \operatorname{\mathsf{add}} A = \mathsf{Inj}(\mathsf{CM}\,A)$. Assume $X \in \mathsf{CM}\,A$ is projective, and take a right $(\operatorname{\mathsf{add}} A)$-approximation $P \to X$. As we have shown above, we have a triangle $Y \to P \to X \to Y[1]$ where $Y \in \mathsf{CM}\,A$. Since X is projective, then $\mathrm{Hom}_{\mathsf{CM}\,A}(X, Y[1]) = 0$. Then the triangle splits and thus $X \in \operatorname{\mathsf{add}} A$. So $\mathsf{Proj}(\mathsf{CM}\,\mathsf{A}) = \operatorname{\mathsf{add}}\mathsf{A}$. Similarly, one can show $\mathsf{Inj}(\mathsf{CM}\,\mathsf{A}) = \operatorname{\mathsf{add}}\mathsf{A}$. Then by Proposition 0.4.2, $\underline{\mathsf{CM}}A$ is a triangulated category.

For the last statement, applying [IYa2, Corollary 2.1] to $\mathcal{T} = \mathsf{D}^{\mathrm{b}}(A)$ and $\mathcal{P} = \operatorname{\mathsf{add}} A$, we have $\underline{\mathsf{CM}}A$ is triangle equivalent to $\mathsf{D}^{\mathrm{b}}(A)/\operatorname{\mathsf{per}} A$. ∎

Immediately, we have the following.

Corollary 2.2.5. *In Theorem 2.2.4,* $\mathsf{D}^{\mathrm{b}}(A) = \operatorname{\mathsf{per}} A$ *if and only if* $\mathsf{CM}\,A = \operatorname{\mathsf{add}} A$.

Recall from [NP], the suspension functor in $\underline{\mathsf{CM}}A$ is given by the cone of a left $(\operatorname{\mathsf{add}} A)$-approximation $X \to P \to \Omega^{-1}X \to X[1]$ for $X \in \mathsf{CM}\,A$. The following result is an analogue of the well-known property for classical Gorenstein rings.

Proposition 2.2.6. *Let A be a dg algebra satisfying Assumption 2.1.1. Then*

(1) *There is a duality* $(\ )^* = \mathbf{R}\mathscr{H}\mathrm{om}_A(?, A) : \mathsf{D}^{\mathrm{b}}(A) \xrightarrow{\simeq} \mathsf{D}^{\mathrm{b}}(A^{\mathrm{op}})$, *which restricts to a duality* $\mathsf{CM}\,A \xrightarrow{\simeq} \mathsf{CM}(A^{\mathrm{op}})$;

(2) *For* $X, Y \in \mathsf{CM}\,A$ *and* $i > 0$, *we have* $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X, Y[i]) = \mathrm{Hom}_{\underline{\mathsf{CM}}A}(X, \Omega^{-i}Y)$.

Proof. (1) The functor $(\ )^* : \mathsf{D}(A) \to \mathsf{D}(A^{\mathrm{op}})$ restricts to a duality $\operatorname{\mathsf{per}} A \xrightarrow{\simeq} \operatorname{\mathsf{per}}(A^{\mathrm{op}})$. For any $M \in \mathsf{D}^{\mathrm{b}}(A)$, it is clear that $\mathbf{R}\mathscr{H}\mathrm{om}_A(M, DA) = DM \in \mathsf{D}^{\mathrm{b}}(A^{\mathrm{op}})$. Since A is Gorenstein, then $A \in \operatorname{\mathsf{thick}}(DA)$ and moreover, $(\ )^*$ also induces a duality $\mathsf{D}^{\mathrm{b}}(A) \xrightarrow{\simeq} \mathsf{D}^{\mathrm{b}}(A^{\mathrm{op}})$. Since $\mathsf{CM}\,A = A[<0]^{\perp} \cap {}^{\perp}A[>0]$, it is clear that $(\ )^*$ restricts to a functor $(\ )^* : \mathsf{CM}\,A \to \mathsf{CM}(A^{\mathrm{op}})$ and it is a duality. We have the following diagram.

$$\begin{array}{ccc} \mathsf{D}^{\mathrm{b}}(A) & \xrightarrow{\simeq} & \mathsf{D}^{\mathrm{b}}(A^{\mathrm{op}}) \\ \uparrow & & \uparrow \\ \mathsf{CM}\,A & \xrightarrow{\simeq} & \mathsf{CM}(A^{\mathrm{op}}). \end{array}$$

(2) Consider the following triangle induced by the left $(\operatorname{\mathsf{add}} A)$-approximation of Y,

$$Y \to Q \to \Omega^{-1}Y \to Y[1].$$

Applying $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X, ?)$ to the triangle above, since $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X, A[>0]) = 0$, we see

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X, \Omega^{-1}Y[t]) = \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X, Y[t+1])$$

for $t \geq 1$. Moreover, we have the following exact sequence.

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X, Q) \to \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X, \Omega^{-1}Y) \to \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X, Y[1]) \to 0.$$

Since $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A, Y[1]) = 0$, then for $P \in \mathsf{add}\, A$, every map $P \to \Omega^{-1}Y$ has a decomposition $P \to Q \to \Omega^{-1}Y$. Then

$$\mathrm{Hom}_{\underline{\mathsf{CM}}A}(X, \Omega^{-1}Y) = \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X, \Omega^{-1}Y)/\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X, Q),$$

which is isomorphic to $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X, Y[1])$. Then by induction, one can show $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X, Y[i]) = \mathrm{Hom}_{\underline{\mathsf{CM}}A}(X, \Omega^{-i}Y)$ holds for any $i \geq 1$. □

The following proposition tells us that when $\mathsf{CM}\, A$ is an ordinary Frobenius category for a d-self-injective dg algebra A.

Proposition 2.2.7. *Assume A is a d-self-injective dg k-algebra. Then $\mathsf{CM}\, A$ is a Frobenius category with $\mathsf{add}\, A$ as projective objects if and only if $d = 1$ (that is, A has total cohomology concentrated in degree 0).*

Proof. If A has total cohomology concentrated in degree 0, then A is quasi-isomorphic to $\mathrm{H}^0(A)$. In this case, $\mathrm{H}^0(A)$ is a Gorenstein k-algebra and $\mathsf{CM}\, A$ is equivalent to $\mathsf{CM}\, \mathrm{H}^0(A)$, which is Frobenius.

On the other hand, suppose X is a non-zero object of $\mathsf{CM}\, A$. If $\mathsf{CM}\, A$ is a Frobenius category with $\mathsf{add}\, A$ as projective objects, then

$$\mathrm{Hom}_{\mathsf{CM}\, A}(A, X) = \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A, X) = \mathrm{H}^0(X) \neq 0$$

which implies $X \notin \mathsf{D}^{\mathrm{b}}_{\leq -1}$. So $\mathsf{CM}\, A \cap \mathsf{D}^{\mathrm{b}}_{\leq -1} = 0$. But by Proposition 2.2.3, $\mathsf{CM}\, A = \mathsf{D}^{\mathrm{b}}_{\leq 0} \cap \mathsf{D}^{\mathrm{b}}_{\geq -d+1}$. Then $d = 1$, which implies that A has total cohomology concentrated in degree 0. □

2.3. Auslander-Reiten theory in $\mathsf{CM}\, A$

We assume that all the dg k-algebras considered in this section satisfy Assumption 2.1.1.

2.3.1. **Serre duality and almost split extensions.** The aim of this section is to prove the following theorem.

Theorem 2.3.1. (1) *$\underline{\mathsf{CM}}A$ admits a Serre functor $\nu[-1] = ? \otimes^{\mathbf{L}}_A DA[-1]$;*
(2) *$\mathsf{CM}\, A$ admits almost split extensions.*

We first show $\underline{\mathsf{CM}}A$ admits a Serre functor. We will consider it in a general setting given in [Am, Section 1.2]. Let $\mathcal{T}$ be a k-linear Hom-finite triangulated category and $\mathcal{N}$ be a thick subcategory of $\mathcal{T}$. Assume $\mathcal{T}$ has an auto-equivalence S, which gives a relative Serre duality in the sense that $S(\mathcal{N}) \subset \mathcal{N}$ and there exists a functorial isomorphism for any $X \in \mathcal{N}$ and $Y \in \mathcal{T}$

$$D\mathrm{Hom}_{\mathcal{T}}(X, Y) \simeq \mathrm{Hom}_{\mathcal{T}}(Y, SX).$$

Definition 2.3.2. [Am, Definition 1.2] Let X and Y be objects in $\mathcal{T}$. A morphism $p : P \to X$ is called a *local $\mathcal{N}$-cover of X relative to Y* if P is in $\mathcal{N}$ and it induces an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{T}}(X, Y) \xrightarrow{p^*} \mathrm{Hom}_{\mathcal{T}}(P, Y).$$

Dually, let Y and Z be objects in $\mathcal{T}$. A morphism $q : Y \to Q$ is called a *local $\mathcal{N}$-envelop of Y relative to Z* if Q is in $\mathcal{N}$ and it induces an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{T}}(Z, Y) \xrightarrow{q_*} \mathrm{Hom}_{\mathcal{T}}(Z, Q).$$

Amiot gave the following sufficient condition for $\mathcal{T}/\mathcal{N}$ to admit a Serre functor.

Proposition 2.3.3. [Am, Theorem 1.3] *Assume for any $X, Y \in \mathcal{T}$, there is a local $\mathcal{N}$-cover of X relative to Y and a local $\mathcal{N}$-envelop of SX relative to Y. Then the quotient category $\mathcal{T}/\mathcal{N}$ admits a Serre functor given by $S[-1]$.*

To check the condition in Proposition 2.3.3, the following lemma is useful.

Lemma 2.3.4. [Am, Proposition 1.4] *Let X and Y be two objects in $\mathcal{T}$. If for any $P \in \mathcal{N}$ the vector space $\mathrm{Hom}_{\mathcal{T}}(P, X)$ and $\mathrm{Hom}_{\mathcal{T}}(Y, P)$ are finite-dimensional, then the existence of a local $\mathcal{N}$-cover of X relative to Y is equivalent to the existence of a local $\mathcal{N}$-envelop of Y relative to X.*

In our setting, to apply Proposition 2.3.3, we need the following observation.

Lemma 2.3.5. *For any $X, Y \in \mathsf{D}^{\mathrm{b}}(A)$, there exists an object $P_X \in \mathsf{per}\, A$ with a morphism $P_X \xrightarrow{p} X$ such that we have the following exact sequence.*

$$0 \to \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X, Y) \xrightarrow{p^*} \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(P_X, Y).$$

Proof. Since A is non-positive and $X, Y \in \mathsf{D}^{\mathrm{b}}(A)$, by truncation, we may assume

$$X := [\cdots 0 \to X^m \xrightarrow{d_m} X^{m+1} \xrightarrow{d_{m+1}} \cdots \xrightarrow{d_{n-1}} X^n \to 0 \to \cdots],$$
$$Y := [\cdots 0 \to Y^s \xrightarrow{d_s} Y^{s+1} \xrightarrow{d_{s+1}} \cdots \xrightarrow{d_{t-1}} Y^t \to 0 \to \cdots].$$

Apply induction on $n - s$.

If $n - s < 0$, then $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(X, Y) = 0$, we can take any object in $\mathsf{per}\, A$ as P_X.

Now assume the result is true for $n - s = k$. Consider the case $n - s = k + 1$.

There exists $Q_X \in \mathsf{add}\, A[-n]$ and a morphism $p : Q_X \to X$ such that $\mathrm{H}^n(p)$ is surjective. Then $\mathrm{H}^{i \geq n}(\mathrm{cone(p)}) = 0$. Let $Z = \mathrm{cone(p)}$. By our assumption, there exists $P_Z \in \mathsf{per}\, A$ with a morphism $r : P_Z \to Z$ satisfies our condition. By the Octahedral Axiom, we have the following diagram.

$$\begin{array}{ccccccc}
Q_X & \longrightarrow & P_X & \longrightarrow & P_Z & \longrightarrow & Q_X[1] \\
\| & & \downarrow & & \downarrow & & \| \\
Q_X & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & Q_X[1] \\
 & & \downarrow & & \downarrow & & \\
 & & T & = & T & & \\
 & & \downarrow & & \downarrow & & \\
 & & P_X[1] & \longrightarrow & P_Z[1] & &
\end{array}$$

Then it is easy to check P_X is the cover we want. □

Now we are ready to prove Theorem 2.3.1.

Proof of Theorem 2.3.1. (1) Since $\mathsf{per}\, A = \mathsf{thick}(DA)$, then ν induces triangle equivalences $\mathsf{D}^{\mathrm{b}}(A) \simeq \mathsf{D}^{\mathrm{b}}(A)$ and $\mathsf{per}\, A \simeq \mathsf{per}\, A$. Moreover, ν gives a relative Serre duality by Lemma 0.3.1. We only need to show the conditions in Proposition 2.3.3 hold in our setting. Because $\mathsf{D}^{\mathrm{b}}(A)$ is Hom-finite by Lemma 0.3.5, then by Lemma 2.3.4, it suffices to check the existence of local $\mathsf{per}\, A$-cover. This has been proved in Lemma 2.3.5. So the assertion is true.

(2) By Lemma 0.3.5, $\mathsf{D}^{\mathrm{b}}(A)$ is Hom-finite, then $\mathsf{CM}\, A$ is Ext-finite (see Section 0.4.2). It is clear that $\mathsf{CM}\, A$ is a k-linear Krull-Schmidt extriangulated category. Moreover, $\underline{\mathsf{CM}}A$ admits a Serre functor by (1), then by Proposition 0.4.5, $\mathsf{CM}\, A$ admits almost split extensions. □

We give the following lemma for later use.

Lemma 2.3.6. *Let X be an non-projective indecomposable object in $\mathsf{CM}A$. Let τ be the Auslander-Reiten translation. If $\mathrm{End}_{\mathsf{CM}A}(X) = k$, then any non-split extension*

$$\tau(X) \xrightarrow{f} Y \xrightarrow{g} X$$

is an almost split extension.

Proof. By Proposition 0.4.6, it is clear. □

2.3.2. **Cohen-Macaulay approximation.** In this subsection, we show the following result.

Theorem 2.3.7. (1) $\mathsf{CM}\,A$ *is functorially finite in* $\mathsf{D}^{\mathrm{b}}(A)$*;*

(2) *More precisely, we have the following result, where* $(\ )^* = \mathbf{R}\mathscr{H}\mathrm{om}_{A^{\mathrm{op}}}(?, A^{\mathrm{op}}) : \mathsf{D}^{\mathrm{b}}(A^{\mathrm{op}}) \xrightarrow{\simeq} \mathsf{D}^{\mathrm{b}}(A)$.

$$\mathsf{D}^{\mathrm{b}}(A) = \mathsf{CM}\,A \perp \mathsf{add}(\mathsf{Filt}A[>0]) \perp \mathsf{D}^{\mathrm{b}}_{>0}(A) = \mathsf{D}^{\mathrm{b}}_{>0}(A^{\mathrm{op}})^* \perp \mathsf{add}(\mathsf{Filt}A[<0]) \perp \mathsf{CM}\,A.$$

Immediately, $\mathsf{CM}\,A$ admits a property analogous to the usual Cohen-Macaulay approximation (see [ABu]) in the following sense.

Corollary 2.3.8. *Let* $M \in \mathsf{D}^{\mathrm{b}}_{\leq 0}(A)$*, then there is a triangle*

$$P \to T \to M \to P[1],$$

such that $T \to M$ *is a right* $(\mathsf{CM}\,A)$*-approximation of* M *and* $P \in \mathsf{per}\,A$*.*

To show the theorem, we consider the t-structures and co-t-structures on $\mathsf{D}^{\mathrm{b}}(A)$ first. Let

$$\begin{aligned} A_{\geq l} = A_{>l-1} &:= \mathsf{add} \bigcup_{i\geq 0} A[-l-i] * \cdots * A[-l-1] * A[-l], \\ A_{\leq l} = A_{<l+1} &:= \mathsf{add} \bigcup_{i\geq 0} A[-l] * A[-l+1] * \cdots * A[-l+i]. \end{aligned}$$

There are two t-structures and two co-t-structures [P] (also called weight structures [Bo]) in $\mathsf{D}^{\mathrm{b}}(A)$ induced by A.

Lemma 2.3.9. (1) *The two paris* $(A[<0]^{\perp}, A[>0]^{\perp})$ *and* $({}^{\perp}A[<0], {}^{\perp}A[>0])$ *are t-structures on* $\mathsf{D}^{\mathrm{b}}(A)$*;*

(2) *The two pairs* $({}^{\perp}A[>0], A_{\leq 0})$ *and* $(A_{\geq 0}, A[<0]^{\perp})$ *are co-t-structures on* $\mathsf{D}^{\mathrm{b}}(A)$*.*

Proof. (1) By Proposition 0.3.7, $(A[<0]^{\perp}, A[>0]^{\perp}) = (\mathsf{D}^{\mathrm{b}}_{\leq 0}, \mathsf{D}^{\mathrm{b}}_{\geq 0})$ is a t-structure on $\mathsf{D}^{\mathrm{b}}(A)$. Since the Nakayama functor ν induces a triangle equivalence $\nu : \mathsf{D}^{\mathrm{b}}(A) \simeq \mathsf{D}^{\mathrm{b}}(A)$, then applying ν^{-1} to this t-structure, we get a new t-structure $({}^{\perp}A[<0], {}^{\perp}A[>0])$ on $\mathsf{D}^{\mathrm{b}}(A)$.

(2) See [IYa1, Propsition 3.2]. □

Now we show the theorem.

Proof of Theorem 2.3.7. We only show (2), since (1) is directly from (2).

By Lemma 2.3.9, we have $\mathsf{D}^{\mathrm{b}}(A) = \mathsf{D}^{\mathrm{b}}_{\leq 0} \perp \mathsf{D}^{\mathrm{b}}_{>0}$. We claim that $\mathsf{D}^{\mathrm{b}}_{\leq 0} = \mathsf{CM}\,A \perp \mathsf{Filt}A[>0]$. Let $M \in \mathsf{D}^{\mathrm{b}}_{\leq 0} = A[<0]^{\perp}$. Considering the co-$t$-structure $({}^{\perp}A[>0], A_{\leq 0})$, we have the following decomposition of M.

$$T \to M \to S \to T[1],$$

where $T \in {}^{\perp}A[>0]$ and $S \in A_{<0} = \mathsf{Filt}A[>0]$. Applying $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A[<0], ?)$ to the triangle above, we have $T \in \mathsf{CM}\,A$. So the claim holds and $\mathsf{D}^{\mathrm{b}}(A) = \mathsf{CM}\,A \perp \mathsf{Filt}A[>0] \perp \mathsf{D}^{\mathrm{b}}_{>0}(A)$. By the duality $(\ )^* : \mathsf{D}^{\mathrm{b}}(A^{\mathrm{op}}) \xrightarrow{\simeq} \mathsf{D}^{\mathrm{b}}(A)$, we have $\mathsf{D}^{\mathrm{b}}(A) = \mathsf{D}^{\mathrm{b}}_{>0}(A^{\mathrm{op}})^* \perp \mathsf{Filt}A[<0] \perp \mathsf{CM}\,A$. □

We end this section by giving a result analogous to famous results of Auslander and Yamagata [ARS, Theorem VI.1.4][Ya] on the first Brauer-Thrall theorem. For an object $X \in \mathsf{D}^{\mathrm{b}}(A)$, there is an integer t such that $\mathrm{H}^{<t}(X) = 0$ and $\mathrm{H}^t(X) \neq 0$. By the standard truncation, we may assume $X^i = 0$ for $i < t$ and in this case, we have a natural inclusion $\mathrm{H}^t(X) \hookrightarrow X$. So we may regard $\mathrm{soc}\,\mathrm{H}^t(X)$ as the socle $\mathrm{soc}\,X$ of X.

Proposition 2.3.10. *Let* $\mathcal{S}$ *be a finite subset of* $\mathsf{ind}(\mathsf{CM}\,A)$*. Then* $\mathcal{S} = \mathsf{ind}(\mathsf{CM}\,A)$ *if* $\mathcal{S}$ *is closed under successors in the AR quiver of* $\mathsf{CM}\,A$ *and for any* $i \geq 0$*, there exists a left* $(\mathsf{CM}\,A)$*-approximation* $A[i] \to X$ *in* $\mathsf{D}^{\mathrm{b}}(A)$ *such that* $X \in \mathsf{add}\,\mathcal{S}$*.*

Proof. Notice that $A \xrightarrow{\mathrm{id}} A$ is the minimal left $(\mathsf{CM}\,A)$-approximation of A, then $A \in \mathsf{add}\,\mathcal{S}$ by our assumption. Let $M \in \mathsf{ind}(\mathsf{CM}\,A)$. Then there exists $i \geq 0$ such that $\mathrm{Hom}_{\mathsf{CM}\,A}(A[i], M) \neq 0$. Let N be the left $(\mathsf{CM}\,A)$-approximation of $A[i]$ such that $N \in \mathsf{add}\,\mathcal{S}$. Then $\mathrm{Hom}_{\mathsf{CM}\,A}(N, M) \neq 0$. Let X_1 be an indecomposable direct summand of N with $0 \neq f \in \mathrm{Hom}_{\mathsf{CM}\,A}(X_1, M)$. If f is a section, since $X_1, M \in \mathsf{ind}(\mathsf{CM}\,A)$, then $X_1 \cong M$ and we are done. If f is not a section. Considering the left almost split morphism $g : X_1 \to Y$ (If $X_1 \in \mathsf{add}\,A$, g is given by $X_1 \to X_1/\operatorname{soc} X_1$), then there is a $h \in \mathrm{Hom}_{\mathsf{CM}\,A}(Y, M)$ such that $f = h \circ g$. So we can find an indecomposable module $X_2 \in \mathsf{add}\,Y$, such that the composition $X_1 \to X_2 \to M$ is non-zero. Repeat this step, we may construct a series of indecomposable modules $X_1 \to X_2 \to \cdots \to M$, such that the composition is non-zero. Since $\mathcal{S}$ is closed under successors, then $X_i \in \mathcal{S}$. Since $\mathcal{S}$ is finite and $\mathsf{CM}\,A$ is Hom-finite, then $\mathrm{rad}(\mathcal{S}, \mathcal{S})^N = 0$ for big enough N. So there exist $n \geq 1$ such that $X_n = M$ and $M \in \mathsf{add}\,\mathcal{S}$. Therefore $\mathcal{S} = \mathsf{ind}(\mathsf{CM}\,A)$. □

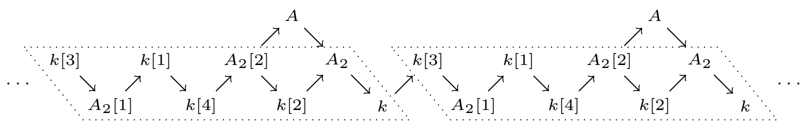
2.4. Example: Truncated polynomial dg algebras

In this section, we give some examples. The reader may skip this section, since results here will not be used in this part. Consider a truncated polynomial dg k-algebra.

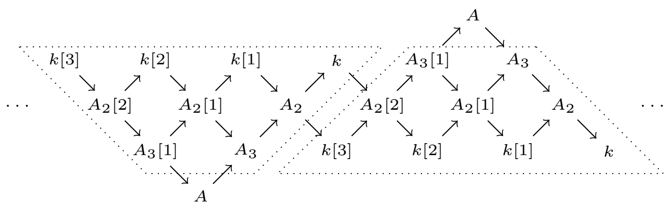
$$A := k[X]/(X^{n+1}), n \geq 0,$$

with $\deg X = -d \leq 0$ and zero differential. We determine the indecomposable Cohen-Macaulay modules explicitly and draw the AR quiver of $\mathsf{CM}\,A$. Then we show $\underline{\mathsf{CM}}A$ is a $(d+1)$-cluster category by using a criterion given by Keller and Reiten [KR]. Let A_i be the dg A-module $k[X]/(X^i)$, $i = 1, 2, \cdots, n$. We give two small examples first.

Example 2.4.1. (1) Let $n = 2$ and $d = 2$. Then the AR quiver of $\mathsf{CM}\,A$ is as follows.



(2) Let $n = 3$ and $d = 1$. Then the AR quiver of $\mathsf{CM}\,A$ is as follows.



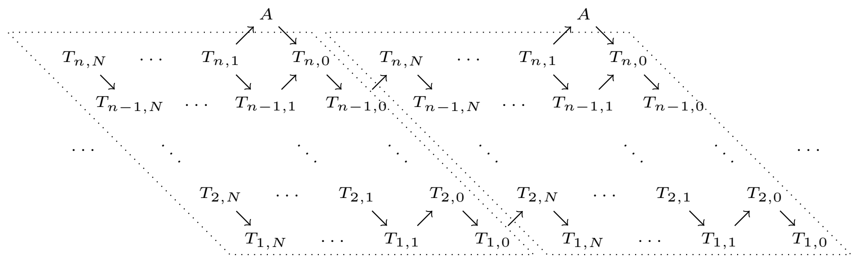
By Proposition 2.3.8, for any A_i, $1 \leq i \leq n$, and $t \geq 0$, we have the following triangle.

$$T_{i,t} \to A_i[td] \to P_{i,t} \to T_{i,t}[1], \tag{2.4.1}$$

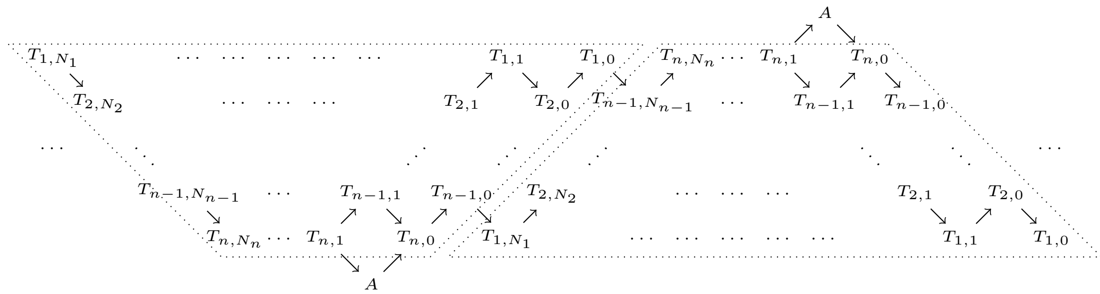
such that $T_{i,t} \to A_i[td]$ is a right $(\mathsf{CM}\,A)$-approximation of $A_i[td]$ and $P_{i,t} \in \mathsf{per}\,A$. We assume $T_{i,t}$ is minimal. Then $T_{i,t}$ is unique up to isomorphism and if $A_i[td] \in \mathsf{CM}\,A$ (for example, $t = 0$ or 1), we have $T_{i,t} = A_i[td]$. We give the first result of this section.

Theorem 2.4.2. *Let A be the dg algebra $k[X]/(X^{n+1})$, $n \geq 0$ with $\deg X = -d \leq 0$ and zero differential.*

(1) *Assume d is even. Let $N := \frac{(n+1)d}{2}$. Then the AR quiver of* $\mathsf{CM}A$ *is as follows.*



(2) *Assume d is odd. Let $N_i := \frac{(n+1)d+n-2i+1}{2}$, $1 \leq i \leq n$. Then the AR quiver of $\mathsf{CM}A$ is as follows.*



Before proving Theorem 2.4.2, we consider AR triangles in $\underline{\mathsf{CM}}A$ first. It is easy to see that A is an $(nd+1)$-symmetric dg algebra. Then by Proposition 2.2.3, $\mathsf{CM}\,A = \mathsf{D}^{\mathrm{b}}_{\leq 0} \cap \mathsf{D}^{\mathrm{b}}_{\geq -nd}$. The Nakayama functor $\nu : \mathsf{D}^{\mathrm{b}}(A) \to \mathsf{D}^{\mathrm{b}}(A)$ is given by $\nu = ? \otimes^{\mathbf{L}}_A DA = [-nd]$. By Theorem 2.3.1, $\underline{\mathsf{CM}}A$ admits a Serre functor $\nu[-1] = [-nd-1]$. Moreover, the Auslander-Reiten translation on $\underline{\mathsf{CM}}A$ is $\tau = [-nd-2]$. The following lemma shows A_i and $T_{i,t}$ are indecomposable.

Lemma 2.4.3. *Let $A_i, T_{i,t}$ be defined as above. Then*
(1) $\mathrm{End}_{\underline{\mathsf{CM}}A}(A_i) = \mathrm{End}_{\mathsf{CM}\,A}(A_i) = k$. *Moreover, each A_i is indecomposable in $\underline{\mathsf{CM}}A$;*
(2) *$T_{i,t}$ is indecomposable in $\mathsf{CM}\,A$.*

Proof. (1) For any A_i, $1 \leq i \leq n$, there is a natural triangle in $\mathsf{D}^{\mathrm{b}}(A)$.

$$A_{n+1-i}[id] \to A \to A_i \to A_{n+1-i}[id+1]. \tag{2.4.2}$$

Since $A_{n+1-i}[id] \in \mathsf{D}_{\leq -id}$ and $A_i \in \mathsf{D}_{\geq -(i-1)d}$, then $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A_{n+1-i}[\geq id], A_i) = 0$. Applying $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(?, A_i)$ to triangle (2.4.2), we have

$$\mathrm{End}_{\mathsf{D}^{\mathrm{b}}(A)}(A_i) \cong \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A, A_i) = k.$$

So A_i is indecomposable in $\mathsf{CM}\,A$. Since A itself is indecomposable in $\mathsf{CM}\,A$ by $\mathrm{End}_{\mathsf{CM}\,A}(A) = k$, and $A_i \neq A$ by cohomology. Then $A_i \notin \mathsf{add}\,A$, and A_i is a non-zero object in $\mathsf{CM}\,A$.

(2) It is clear $T_{i,t} \cong A_i[td]$ in $\underline{\mathsf{CM}}A$. So $T_{i,t}$ is indecomposable in $\underline{\mathsf{CM}}A$ by (1). Because if $t = 0$, $T_{i,0} = A_i$ and if $t > 0$, $\mathrm{Hom}_{\mathsf{CM}\,A}(A, A_i[td]) = 0$, then $T_{i,t}$ does not contain $P \in \mathsf{add}\,A$ as a direct summand. Thus $T_{i,t}$ is also indecomposable in $\mathsf{CM}\,A$. □

We point out the periodicity of $\underline{\mathsf{CM}}A$.

Lemma 2.4.4. *The functor $[(n+1)d+2] : \underline{\mathsf{CM}}A \to \underline{\mathsf{CM}}A$ is isomorphic to the identity functor. In particular, $\tau \cong [d]$ as functors on $\underline{\mathsf{CM}}A$ and $\underline{\mathsf{CM}}A$ is $(d+1)$-Calabi-Yau.*

Proof. Consider the following sequence in the category of dg $A \otimes A^{\mathrm{op}}$-modules.

$$0 \to A[(n+1)d] \xrightarrow{f} A \otimes A[d] \xrightarrow{g} A \otimes A \xrightarrow{h} A \to 0$$

where f is given by $f(1) := \sum_{i=0}^{n} X^i \otimes X^{n-i}$, g is given by $g(1 \otimes 1) := 1 \otimes X - X \otimes 1$ and h is given by $h(1 \otimes 1) := 1$. Since it is an exact sequence of graded modules, we get two natural triangles in

$\mathsf{D}(A \otimes A^{\mathrm{op}})$.

$$\operatorname{Ker} h \xrightarrow{h} A \otimes A \to A \to \operatorname{Ker} h[1]$$

$$A[(n+1)d] \xrightarrow{f} A \otimes A[d] \to \operatorname{Ker} h \to A[(n+1)d+1]$$

Let $M \in \mathsf{D}^{\mathrm{b}}(A)$. Apply the functor $M \otimes^{\mathbf{L}}_A ?$ to the triangles above, we get two triangles in $\mathsf{D}^{\mathrm{b}}(A)$:

$$M \otimes^{\mathbf{L}}_A \operatorname{Ker} h \to M \otimes A \to M \to M \otimes^{\mathbf{L}}_A \operatorname{Ker} h[1],$$

$$M[(n+1)d] \to M \otimes A[d] \to M \otimes^{\mathbf{L}}_A \operatorname{Ker} h \to M[(n+1)d+1].$$

Notice that $M \otimes A \in \operatorname{per} A$, then we have natural isomorphisms $M \xrightarrow{\sim} M \otimes^{\mathbf{L}}_A \operatorname{Ker} h[1]$ and $M \otimes^{\mathbf{L}}_A \operatorname{Ker} h \xrightarrow{\sim} M[(n+1)d+1]$ in $\mathsf{D}^{\mathrm{b}}(A)/\operatorname{per} A \cong \underline{\mathsf{CM}}A$, which give us the desired isomorphism. □

Remark 2.4.5. The integer $(n+1)d+2$ is the smallest natural number r such that $[r] \simeq \mathrm{id}$ on $\underline{\mathsf{CM}}A$. In fact one may consider the shifts of simple dg A-module $A_1 = k$. It is clear that $k[i] \in \mathsf{CM}\, A$ for $0 \leq i \leq nd$. By the triangle (2.4.2), we have $k[nd+1] \cong A_n$ in $\underline{\mathsf{CM}}A$ (notice that $A_n[0,1,\ldots,d] \in \mathsf{CM}\, A$). Then it is easy to check that k can not be isomorphic to $k[i]$ in $\underline{\mathsf{CM}}A$ for $0 \leq i \leq (n+1)d+1$.

Now we describe the AR-triangles in $\underline{\mathsf{CM}}A$.

Proposition 2.4.6. *Let $1 \leq i \leq n$. Let π_i be the natural surjective map $\pi_i : A_i \to A_{i-1}$ and let ι_i be the natural injective map $\iota_i : A_i[d] \to A_{i+1}$ given by $\iota_i(1) := X$. Let $A_0 = 0$. Then the AR triangle in $\underline{\mathsf{CM}}A$ ending in A_i is given by*

$$A_i[d] \xrightarrow{\begin{pmatrix} \pi_i[d] \\ -\iota_i[d] \end{pmatrix}} A_{i-1}[d] \oplus A_{i+1} \xrightarrow{(\iota_{i-1}\ \pi_{i+1})} A_i.$$

Proof. First notice that the given sequence is a short exact sequence of graded modules and it gives a triangle in $\mathsf{D}^{\mathrm{b}}(A)$ (so in $\underline{\mathsf{CM}}A$). By Lemma 2.4.3, $A_j[m]$ is indecomposable in $\underline{\mathsf{CM}}A$ for any $m \in \mathbb{Z}$, so the given triangle can not be split. Since $\operatorname{End}_{\underline{\mathsf{CM}}A}(A_i) = k$, then by Lemma 2.3.6, the triangle above is an AR triangle. □

It is easy to see that the AR quiver of $\underline{\mathsf{CM}}A$ is of the form $\mathbb{Z}A_n/\phi$ by Proposition 2.4.6 and the fact that $T_{i,t}$ are shifts of A_i in $\underline{\mathsf{CM}}A$. Now we determine the fundamental domain. Notice that by Lemma 2.4.4, $A_i = A_i[(n+1)d+2]$ and by the triangle (2.4.2), $A_{n+1-i} = A_i[-id-1]$ in $\underline{\mathsf{CM}}A$. We need to find the smallest positive integer m such that $A_i = A_i[md]$ or $A_{n+1-i} = A_i[md]$ holds.

Lemma 2.4.7. (1) *Assume d is even. Let $N := \frac{(n+1)d}{2}$. Then N is the smallest positive integer such that $A_i = A_i[(N+1)d]$;*
(2) *Assume d is odd. Let $N_i := \frac{(n+1)d+n-2i+1}{2}$. Then N_i is the smallest positive integer such that $A_{n+1-i} = A_i[(N_i+1)d]$.*

Proof. (1) It is obvious $A_i = A_i[(N+1)d]$ by Lemma 2.4.4. By Remark 2.4.5, we know $(n+1)d+2$ is the smallest natural number r such that $[r] \simeq \mathrm{id}$ on $\underline{\mathsf{CM}}A$. Let $d = 2e$. If $l > 0$ satisfies $A_i = A_i[ld]$, then $(n+1)d+2 \mid ld$, that is $(n+1)e+1 \mid le$. Since $(n+1)e+1$ and e are coprime, then $(n+1)e+1 \mid l$ and $l \geq (n+1)e+1 = N+1$.

(2) Assume positive integer s satisfies $A_{n+1-i} = A_i[sd]$. Then by the fact that $A_{n+1-i} = A_i[-id-1]$, we have $(n+1)d+2 \mid sd+id+1$. Since $sd+id+1 = \frac{d+1}{2}((n+1)d+2)+(s-\frac{(n+1)d+n+3-2i}{2})$, then we need $(n+1)d+2 \mid s-\frac{(n+1)d+n+3-2i}{2}$. So the smallest s is N_i+1. □

We can prove Theorem 2.4.2 now.

Proof of Theorem 2.4.2. The AR triangle given in Proposition 2.4.6 is induced by some conflation

$$T_{i,1} \to T_{i-1,1} \oplus T_{i+1,0} \to T_{i,0} \tag{2.4.3}$$

up to taking projective direct sums in $\mathsf{CM}\, A$ (see [NP]). Notice that for the projective-injective object A, the only right almost split morphism is given by the natural injection $T_{n,1} = A_n[d] \to A$ and the only left almost split morphism is given by the natural surjection $A \to A_n = T_{n,0}$. Then the extension (2.4.3) is an almost split extension in $\mathsf{CM}\,A$ for $i \neq n$. Then by Lemma 2.4.7, the AR sub-quiver $\mathcal{S}$ of $\mathsf{CM}\,A$ consisting of $T_{i,d}$ is given as in Theorem 2.4.2. We only need to show $T_{i,d}$ gives all indecomposable CM A-modules.

By Proposition 2.3.10, it suffices to show that every left $(\mathsf{CM}\,A)$-approximation of $A[p]$, $p \geq 0$, belongs to $\mathsf{add}\,\mathcal{S}$. First notice that if $p > nd$ or $p = 0$, the assertion is obvious (because for the case $p > nd$, we have a natural approximation $A[p] \to 0$ and for the case $p = 0$, the approximation is given by $A \xrightarrow{\text{id}} A$). We only consider other cases and we may assume $td < p \leq (t+1)d$ for some $0 \leq t \leq n-1$. Then the left $(\mathsf{CM}\,A)$-approximation of $A[p]$ is given by the natural map $A[p] \to A_{n-t}[p]$. To show $A_{n-t}[p] \in \mathsf{add}\,\mathcal{S}$, it is enough to show $A_{n-t}[p] = T_{i,s} = A_i[sd]$ in $\underline{\mathsf{CM}}A$ for some $1 \leq i \leq n$ and $s \geq 0$. It can be shown in the following way.

- Assume $(t+1)d-p$ is even and $q = ((t+1)d-p)/2$. Then $A_{n-t}[p] = A_{n-t}[((n+1)d+2)q+p] = A_{n-t}[sd]$, where $s = (n+1)q+t+1$. So $A_{n-t}[p] = T_{n-t,s} \in \mathsf{add}\,\mathcal{S}$.
- Assume $(t+1)d-p$ is odd and $q = ((t+1)d-p-1)/2$. By triangle (2.4.2), we have $A_{n-t}[p] = A_{t+1}[(n-t)d+p+1]$ in $\underline{\mathsf{CM}}A$. Then
$$A_{n-t}[p] = A_{n-t}[((n+1)d+2)q+p] = A_{t+1}[sd],$$
where $s = (n+1)(q+1)$. So $A_{n-t}[p] = T_{t+1,s} \in \mathsf{add}\,\mathcal{S}$.

Then the result is true. □

Before giving the second result of this section, we give another description of $T_{i,t}$, which will be used later.

Proposition 2.4.8. *For any $1 \leq i \leq n$ and $t \geq 0$, there exist $1 \leq j \leq n$ and $0 \leq s \leq (n+1-j)d$ such that $T_{i,t} = A_j[s]$ in $\mathsf{CM}\,A$.*

Proof. First notice that $T_{i,0} = A_i$ and $T_{i,t}$ can be defined by applying induction on t by the following triangle in $\mathsf{D}^{\mathrm{b}}(A)$.
$$T_{i,t+1} \xrightarrow{f_t} T_{i,t}[d] \to P_{i,t} \to T_{i,t+1}[1],$$
where f_t is the minimal right $(\mathsf{CM}\,A)$-approximation of $T_{i,t}[d]$ and $P_{i,t} \in \mathsf{per}\,A$. Also notice that for any $A_j[s] \in \mathsf{CM}\,A$, the minimal right $(\mathsf{CM}\,A)$-approximation of $A_j[s+d]$ also has the form of $A_{j'}[s']$ for some $1 \leq j' \leq n$ and $0 \leq s' \leq (n+1-j')d$. Then the assertion can be shown inductively. □

Theorem 2.4.2 implies that $\underline{\mathsf{CM}}A$ is a cluster category. In fact we have the following result.

Theorem 2.4.9. *The stable category $\underline{\mathsf{CM}}A$ is triangle equivalent to $\mathcal{C}_{d+1}(A_n)$.*

The key ingredient of the proof is Keller and Reiten's result [KR]. We first show $\underline{\mathsf{CM}}A$ admits a $(d+1)$-cluster tilting object.

Proposition 2.4.10. *Let $T := \bigoplus_{i=1}^{n} A_i$, then T is a $(d+1)$-cluster-tilting object in $\underline{\mathsf{CM}}A$, that is, $\mathsf{add}\,T$ is functorially finite in $\underline{\mathsf{CM}}A$ and $X \in \mathsf{add}\,T$ if and only if $\mathrm{Hom}_{\underline{\mathsf{CM}}A}(T, X[m]) = 0$ for all $1 \leq m \leq d$.*

We show the following lemma first.

Lemma 2.4.11. (1) *T is a $(d+1)$-rigid object in $\underline{\mathsf{CM}}A$, i.e. $\mathrm{Hom}_{\underline{\mathsf{CM}}A}(A_i, A_j[s]) = 0$ for any $1 \leq i,j \leq n$ and $1 \leq s \leq d$;*
(2) *$\mathrm{Hom}_{\underline{\mathsf{CM}}A}(A_i, A_j[-s]) = 0$ for any $1 \leq i,j \leq n$ and $1 \leq s \leq d-1$;*
(3) *Let $M \in \mathsf{CM}\,A$. Assume $\mathrm{Hom}_{\underline{\mathsf{CM}}A}(A_i, M[m]) = 0$ for any $1 \leq i \leq n$ and $1 \leq m \leq d$. If $\mathrm{H}^0(M) = 0$, then $M = 0$.*

Proof. By Proposition 2.2.6, we have $\mathrm{Hom}_{\underline{\mathsf{CM}}A}(A_i, A_j[s]) = \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A_i, A_j[s])$. Consider the following two triangles.
$$A_{n-i+1}[id] \longrightarrow A \longrightarrow A_i \longrightarrow A_{n-i+1}[id+1] \longrightarrow A[1], \tag{2.4.4}$$
$$A_i[(n-i+1)d] \longrightarrow A \longrightarrow A_{n-i+1} \longrightarrow A_i[(n-i+1)d+1] \longrightarrow A[1]. \tag{2.4.5}$$

If $s > 1$, by applying the functor $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(?, A_j[s])$ to triangle (2.4.4), we have

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A_i, A_j[s]) = \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A_{n-i+1}[id+1], A_j[s]).$$

Apply the functor $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(?, A_j[s-id-1])$ to the triangle (2.4.5). Since

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A_i[(n-i+1)d+1], A_j[s-id-1]) = 0,$$

then

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A_{n-i+1}, A_j[s-id-1]) = 0.$$

Thus $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A_i, A_j[s]) = 0$.

If $s = 1$. Apply the functor $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(?, A_j[1])$ to triangle (2.4.4). Notice that the induced map

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A[1], A_j[1]) \to \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A_{n-i+1}[id+1], A_j[1])$$

is surjective. Then $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A_i, A_j[1]) = 0$. So (1) is true. The proof of statement (2) is similar to (1).

For (3), let $M \in \mathsf{CM}\,A$. If $M \neq 0$ in $\underline{\mathsf{CM}}A$, let $t := \min\{s \in \mathbb{Z} \mid \mathrm{H}^s(M) \neq 0\}$. We may assume $-id \leq t < -(i-1)d$. We will show $\mathrm{Hom}_{\underline{\mathsf{CM}}A}(A_i, M[t+id+1]) \neq 0$, which is a contradiction.

Since $\mathrm{H}^{\geq 0}M = 0$, then $\mathrm{H}^0 M[id+t] = 0$. Apply the functor $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(?, M[id+t+1])$ to (2.4.4), we have

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A_i, M[id+t+1]) = \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A_{n-i+1}[id+1], M[id+t+1]).$$

Apply the functor $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(?, M[t])$ to (2.4.5), then

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A_{n-i+1}, M[t]) = \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A, M[t]) = \mathrm{H}^0(M[t]).$$

Then $\mathrm{Hom}_{\underline{\mathsf{CM}}A}(A_i, M[id+t+1]) = \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(A_i, M[id+t+1]) = \mathrm{H}^0(M[t]) \neq 0$. It is contradictory to our assumption. So $M = 0 \in \underline{\mathsf{CM}}A$. Since $M \notin \mathsf{add}\,A$ by $\mathrm{H}^0(M) = 0$, then $M = 0$. □

Proof of Proposition 2.4.10. Since $\underline{\mathsf{CM}}A$ is Hom-finite, then $\mathsf{add}\,T$ is a functorially finite subcategory of $\underline{\mathsf{CM}}A$. Let $0 \neq M \in \mathsf{ind}\,\mathsf{CM}\,A$. Then $M = A$ or $M = A_j[s]$ for some $1 \leq j \leq n$ and $0 \leq s \leq (n+1-j)d$ by Theorem 2.4.2 and Proposition 2.4.8. Assume $\mathrm{Hom}_{\underline{\mathsf{CM}}A}(A_i, M[m]) = 0$ for any $1 \leq i \leq n$ and $1 \leq m \leq d$. Then by Lemma 2.4.11, we have $\mathrm{H}^0(M) \neq 0$. So $M \in \mathsf{add}\,T$. Then T is a $(d+1)$-cluster-tilting object in $\underline{\mathsf{CM}}A$. □

Now we are ready to prove Theorem 2.4.9.

Proof of Theorem 2.4.9. Since $\dim\mathrm{Hom}_{\underline{\mathsf{CM}}A}(A_i, A_j) = 0$ for $i > j$ and $\dim\mathrm{Hom}_{\underline{\mathsf{CM}}A}(A_i, A_j) = 1$ for $i \leq j$, the endomorphism algebra $\mathrm{End}_{\underline{\mathsf{CM}}A}(T)$ is isomorphic to a path algebra kQ, where Q is a quiver of type A with vertices $\{A_i \mid 1 \leq i \leq n\}$ and arrows $A_{i+1} \to A_i$ for $1 \leq i \leq n-1$. By Proposition 2.4.10, T is a $(d+1)$-cluster-tilting object in $\underline{\mathsf{CM}}A$. Moreover, by Lemma 2.4.11, $\mathrm{Hom}_{\underline{\mathsf{CM}}A}(A_i, A_j[-k]) = 0$ for any $1 \leq i, j \leq n$ and $1 \leq k \leq d-1$. Then by [KR, Theorem 4.2], there is a triangle equivalence $\underline{\mathsf{CM}}A \xrightarrow{\sim} \mathcal{C}_{d+1}(A_n)$. □

2.5. Negative Calabi-Yau configurations and combinatorial configurations

2.5.1. **Negative Calabi-Yau configurations.** In this subsection, we introduce negative Calabi-Yau configurations in the categorical framework.

Definition 2.5.1. Let $\mathcal{T}$ be a k-linear Hom-finite Krull-Schmidt triangulated category and let C be a set of indecomposable objects of $\mathcal{T}$. For $d \geq 1$, we call C a $(-d)$-*Calabi-Yau configuration* (or $(-d)$-*CY configuration*) if the following conditions hold.

(1) $\dim_k\mathrm{Hom}_{\mathcal{T}}(X, Y) = \delta_{X,Y}$ for $X, Y \in C$;
(2) $\mathrm{Hom}_{\mathcal{T}}(X, Y[-j]) = 0$ for any two objects X, Y in C and $0 < j \leq d-1$;
(3) For any indecomposable object M in $\mathcal{T}$, there exists $X \in C$ and $0 \leq j \leq d-1$, such that $\mathrm{Hom}_{\mathcal{T}}(X, M[-j]) \neq 0$.

If $\mathcal{T}$ admits a Serre functor $\mathbb{S}$, then by Serre duality, (3) is equivalent to the following condition.

(3^{op}) For any indecomposable object N in $\mathcal{T}$, there exists $X \in C$ and $0 \leq j \leq d-1$, such that $\mathrm{Hom}_{\mathcal{T}}(N, X[-j]) \neq 0$.

It is easy to see that "d-SMS" (see Definition 0.2.7) implies "$(-d)$-CY configuration". We show that if $\mathcal{T}$ admits a Serre functor $\mathbb{S}$, then any $(-d)$-CY configuration in $\mathcal{T}$ is preserved by the functor $\mathbb{S}[d]$. This property motivates the name "$(-d)$-CY configuration".

Theorem 2.5.2. *Let $\mathcal{T}$ be a k-linear Hom-finite Krull-Schmidt triangulated category with Serre functor $\mathbb{S}$. Let C be a $(-d)$-CY configuration in $\mathcal{T}$, then $\mathbb{S}C[d] = C$.*

Remark 2.5.3. For the case $d = 1$, Riedtmann showed the periodicity of configurations for type A_n and D_n in combinatorial setting (see [Rie2, Rie4]).

To prove the theorem above, we need the following well-known property.

Lemma 2.5.4. *Let $\mathcal{T}$ be a k-linear Hom-finite triangulated category with Serre functor $\mathbb{S}$. Let $X \in \mathcal{T}$ with $\mathrm{End}_{\mathcal{T}}(X) = k$ and $f \in \mathrm{Hom}_{\mathcal{T}}(X, \mathbb{S}X)$. Then for any $Y \in \mathcal{T}$ and $g \in \mathrm{Hom}_{\mathcal{T}}(\mathbb{S}X, Y)$ which is not a section, we have $g \circ f = 0$.*

Proof. We have the following commutative diagram.

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{T}}(X, \mathbb{S}X) & \xrightarrow{\mathrm{Hom}_{\mathcal{T}}(X,g)} & \mathrm{Hom}_{\mathcal{T}}(X, Y) \\
\downarrow \simeq & & \downarrow \simeq \\
D\mathrm{Hom}_{\mathcal{T}}(\mathbb{S}X, \mathbb{S}X) & \xleftarrow{D\mathrm{Hom}_{\mathcal{T}}(g,\mathbb{S}X)} & D\mathrm{Hom}_{\mathcal{T}}(Y, \mathbb{S}X)
\end{array} .$$

Since the lower map is zero by $\mathrm{End}_{\mathcal{T}}(\mathbb{S}X) = k$ and the assumption that g is not a section, so is the upper one. □

Proof of Theorem 2.5.2. The proof falls into two parts.

(a) We first prove $\mathbb{S}[d]C \subset C$. For any $X \in C$, we only need to show $\mathbb{S}[d]X \in C$. By condition (3), there exist $Y \in C$ and $0 \leq i \leq d-1$ such that $\mathrm{Hom}_{\mathcal{T}}(Y, \mathbb{S}[d]X[-i]) \neq 0$. Since

$$\mathrm{Hom}_{\mathcal{T}}(Y, \mathbb{S}[d]X[-i]) = D\mathrm{Hom}_{\mathcal{T}}(X, Y[-d+i])$$

If $0 < i \leq d-1$, it is zero by condition (2). So we must have $i = 0$. Let $f : Y \to \mathbb{S}[d]X$ be a non-zero morphism and consider the triangle extended by f,

$$\mathbb{S}X[d-1] \xrightarrow{h} N \xrightarrow{g} Y \xrightarrow{f} \mathbb{S}X[d].$$

We claim that $N = 0$.

If $N \neq 0$, then there exist $Z \in C$ and $0 \leq j \leq d-1$, such that $\mathrm{Hom}_{\mathcal{T}}(Z, N[-j]) \neq 0$. Let $p \in \mathrm{Hom}_{\mathcal{T}}(Z[j], N)$ be a non-zero morphism. If $g \circ p \neq 0$, then $j = 0$ and $g \circ p$ is an isomorphism by Definition 2.5.1(1)(2). Thus g is a retraction and $f = 0$, a contradiction. So $g \circ p = 0$. Then there exists a morphism $q : Z[j] \to \mathbb{S}X[d-1]$, such that $p = h \circ q$.

$$\begin{array}{ccccccc}
 & & Z[j] & & & & \\
 & \swarrow^{q} & \downarrow p & & & & \\
\mathbb{S}X[d-1] & \xrightarrow{h} & N & \xrightarrow{g} & Y & \xrightarrow{f} & \mathbb{S}X[d]
\end{array}$$

Since $p \neq 0$, then $q \neq 0$, which implies that $j = d-1$ and $Z \cong X$ by the fact $\mathrm{Hom}_{\mathcal{T}}(Z[j], \mathbb{S}X[d-1]) = D\mathrm{Hom}_{\mathcal{T}}(X, Z[j-d+1])$ and Definition 2.5.1(1)(2). Then by Lemma 2.5.4, we know h is a section. Thus $f = 0$, a contradiction. So $N = 0$ and $\mathbb{S}X[d] \cong Y \in C$.

(b) We prove $\mathbb{S}[d]C \supset C$. By considering conditions (1), (2), and (3^{op}), one can show the statement easily, which is similar to the proof in part (a). We leave it to the reader. □

2.5.2. **CM dg modules and CY configurations.** In this subsection, we study CY configurations in the stable categories of Cohen-Macaulay dg modules over d-self-injective dg algebras. In this case, we show that the set of simple dg A-modules forms a $(-d)$-CY configuration in $\underline{\mathsf{CM}}A$, which generalizes Riedtmann's result [Rie2, Proposition 2.4].

Recall from Section 0.3.2, for a non-positive dg k-algebra A with $A^{>0} = 0$, we may regard $\mathrm{H}^0(A)$-modules as dg A-modules via the homomorphism $A \to \mathrm{H}^0(A)$. Let $\{S_1, \dots, S_r\}$ be the set of simple $\mathrm{H}^0(A)$-modules. We also regard them as simple dg A-modules (when we talk about simple modules, we always assume they are concentrated in degree zero part). Recall that if A is a d-self-injective dg algebra, then $\mathsf{CM}\,A = \mathsf{D}^{\mathrm{b}}_{\leq 0} \cap \mathsf{D}^{\mathrm{b}}_{\geq -d+1}$ (see Proposition 2.2.3).

The main result in this subsection is the following.

Theorem 2.5.5. *Let A be a d-self-injective dg k-algebra with $d \geq 0$. Then the set of simple modules $\{S_i \mid 1 \leq i \leq r\}$ is a d-SMS of $\underline{\mathsf{CM}}A$, and hence a $(-d)$-CY configuration of $\underline{\mathsf{CM}}A$.*

To prove this theorem, we start with the following lemma first.

Lemma 2.5.6. *Let $M \in \mathsf{CM}\,A$. If $d > 1$, then for $1 \leq i \leq r$ and $0 \leq t \leq d-2$, we have*

(1) $\mathrm{Hom}_{\mathsf{CM}\,A}(S_i[t], A) = 0$
(2) $\mathrm{Hom}_{\underline{\mathsf{CM}}A}(S_i[t], M) = \mathrm{Hom}_{\mathsf{CM}\,A}(S_i[t], M)$.

Proof. We only prove (1), since (2) is immediately from (1). Since $DA = A[-d+1]$ in $\mathsf{D}A$, then

$$\mathrm{Hom}_{\mathsf{CM}\,A}(S_i[t], A) = \mathrm{Hom}_{\mathsf{CM}\,A}(S_i[t-d+1], DA) = D\mathrm{H}^{t-d+1}(S_i) = 0$$

for $0 \leq t \leq d-2$. □

Now we prove Theorem 2.5.5.

Proof of Theorem 2.5.5. If $d = 1$, then A is an ordinary self-injective k-algebra and the assertion is known. We only prove it for $d > 1$. By Lemma 2.5.6 and Proposition 0.3.7, we have

$$\mathrm{Hom}_{\underline{\mathsf{CM}}A}(S_i, S_j) = \mathrm{Hom}_{\mathsf{CM}\,A}(S_i, S_j) = \mathrm{Hom}_{\mathrm{H}^0(A)}(S_i, S_j),$$

where $1 \leq i, j \leq r$. So the condition (1) in Definition 2.5.1 holds. Since $\mathsf{CM}\,A = \mathsf{D}^{\mathrm{b}}_{\leq 0} \cap \mathsf{D}^{\mathrm{b}}_{\geq -d+1}$ and $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A)}(S_i[>0], S_j) = 0$, then we have $S_i[t] \in \mathsf{CM}\,A$ for $1 \leq t \leq d-1$ and $\mathrm{Hom}_{\mathsf{CM}\,A}(S_i[t], S_j) = 0$. Thus $\mathrm{Hom}_{\underline{\mathsf{CM}}A}(S_i[t], S_j) = 0$ and the condition (2) in Definition 2.5.1 is true. Now we show the condition (3′) holds. By Propositions 2.2.3 and 0.3.7, we have

$$\mathsf{CM}\,A = \mathsf{Filt}S[d-1] * \mathsf{Filt}S[d-2] * \cdots * \mathsf{Filt}S,$$

where $S = \bigoplus_{i=1}^{n} S_i$. Thus $\underline{\mathsf{CM}}A = \mathsf{add}\,\mathsf{Filt}\{S, S[1], \cdots, S[d-1]\}$ and (3′) is true. Then the set of simples forms a d-SMS of $\underline{\mathsf{CM}}A$. □

To recover the AR quiver $\mathfrak{A}(\mathsf{CM}\,A)$ of $\mathsf{CM}\,A$ from $\mathfrak{A}(\underline{\mathsf{CM}}A)$, we need the following result.

Proposition 2.5.7. *Let A be a d-self-injective dg k-algebra. Let $P \in \mathsf{add}\,A$ be an indecomposable dg A-module. Then* $\mathrm{rad}P$ *is an indecomposable object in* $\mathsf{CM}\,A$ *and it does not belong to* $\mathsf{add}\,A$.

Proof. Since A is d-self-injective, then $\mathsf{CM}\,A = \mathsf{D}^{\mathrm{b}}_{\leq 0}(A) \cap \mathsf{D}^{\mathrm{b}}_{\geq -d+1}(A)$. So $\mathrm{rad}P \in \mathsf{CM}\,A$. We have a natural functor $\mathrm{H} : \mathsf{D}(A) \to \mathsf{Mod}\,\mathrm{H}(A)$ by taking cohomology, where $\mathsf{Mod}\,\mathrm{H}(A)$ is the category of graded $\mathrm{H}(A)$-modules. Notice that $\mathrm{H}(A)$ is a self-injective graded algebra and $\mathrm{H}(P)$ is an indecomposable projective-injective graded $\mathrm{H}(A)$-module. So $\mathrm{H}(\mathrm{rad}P) = \mathrm{rad}(\mathrm{H}(P))$ is an indecomposable graded $\mathrm{H}(A)$-module and it does not belong to $\mathsf{add}\,\mathrm{H}(A)$. Then the assertion is true. □

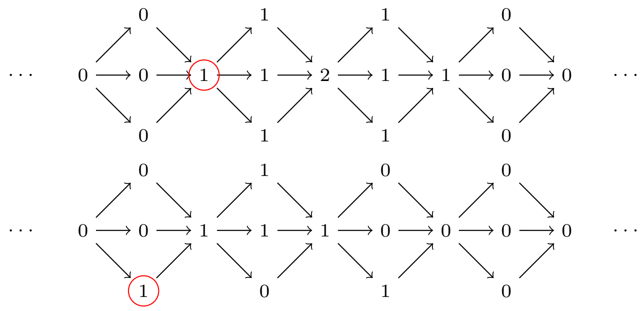
2.5.3. **Combinatorial configurations.** We give a combinatorial interpretation of Calabi-Yau configurations of Dynkin type in our combinatorial framework.

Let Δ be a Dynkin diagram. Recall from [Ga] that a *slice* of $\mathbb{Z}\Delta$ (see Section 0.5 for the definition of $\mathbb{Z}\Delta$) is a connected full subquiver which contains a unique representatives of the vertices (r, q), $r \in \mathbb{Z}$ for each $q \in \Delta_0$. For each vertex $x = (p, q)$ of $\mathbb{Z}\Delta$, there is a unique slice admitting x as its unique source. We call this slice *the slice starting at* x. An integer-valued function f on the vertices of $\mathbb{Z}\Delta$ is *additive* if it satisfies the equation $f(x) + f(\tau x) = \sum_{y \to x \in (\mathbb{Z}\Delta)_1} f(y)$. It is easy to see that f is determined by its value on a slice. Now we define f_x as the additive function which has value 1 on the slice starting at x for each vertex x. Let Q_x be the connected component of the full subquiver $\{y \in (\mathbb{Z}\Delta)_0 \mid f_x(y) > 0\}$ of $\mathbb{Z}\Delta$ containing x. We define a map h_x by

$$h_x(y) = \begin{cases} f_x(y) & \text{if } y \in (Q_x)_0; \\ 0 & \text{otherwise.} \end{cases}$$

Notice that h_x is no longer an additive map. Let us see an example of type D.

Example 2.5.8. Let x be the marked vertex in $(\mathbb{Z}D_4)_0$. Then the value of h_x is given as follows



Let ϕ be a weakly admissible automorphism (see Section 0.5) of $\mathbb{Z}\Delta$. Let $\pi : \mathbb{Z}\Delta \to \mathbb{Z}\Delta/\phi$ be the natural projection. For $x \in \mathbb{Z}\Delta$, we define h_x^ϕ as follows

$$h_x^\phi(y) = \sum_{\pi(z)=y} h_x(z) \text{ for } y \in (\mathbb{Z}\Delta/\phi)_0$$

If ϕ is identity, then h^ϕ is exactly h. Recall we have defined the "shift permutation" [1] in section 0.5. Now we use h_x^ϕ and [1] to define combinatorial configurations.

Definition 2.5.9. Let Δ be a Dynkin diagram and let ϕ be a weakly admissible group. Let C be a subset of $(\mathbb{Z}\Delta/\phi)_0$. For $d \geq 1$, if the following conditions hold

- $h_x^\phi(y) = \delta_{x,y}$ for $x, y \in C$;
- $h_x^\phi(y[-j]) = 0$ for $x, y \in C$ and $0 < j \leq d-1$;
- For any vertex z in $(\mathbb{Z}\Delta/\phi)_0$, there exists $x \in C$ and $0 \leq j \leq d-1$, such that $h_x^\phi(z[-j]) \neq 0$.

we call C a $(-d)$-*combinatorial configuration*.

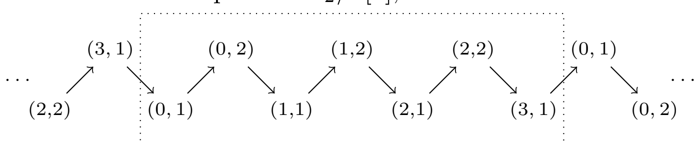
The connection between configurations of $\mathbb{Z}\Delta$ and configurations of $\mathbb{Z}\Delta/\phi$ is given as follows.

Proposition 2.5.10. *Let C be a subset of $(\mathbb{Z}\Delta/\phi)_0$. Then C is a $(-d)$-combinatorial configuration of $\mathbb{Z}\Delta/\phi$ if and only if $\pi^{-1}(C)$ is a $(-d)$-combinatorial configuration of $\mathbb{Z}\Delta$.*

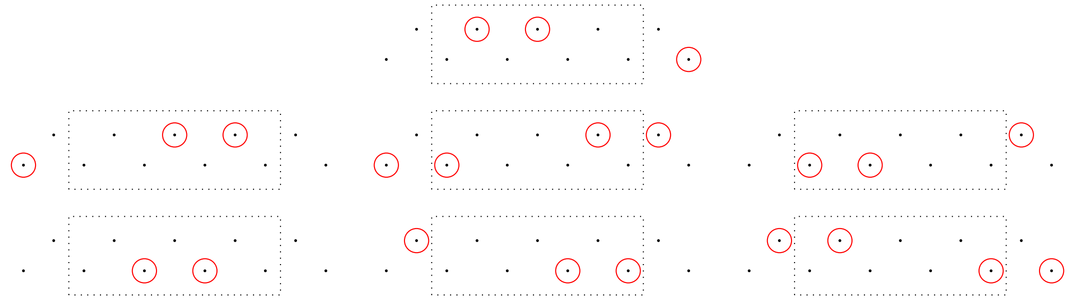
Proof. Using the definition $h_x^G(y) = \sum_{\pi(z)=y} h_x(z)$, it is easy to show the statement. □

Here is a simple example:

Example 2.5.11. We consider the quiver $\mathbb{Z}A_2/\mathbb{S}[2]$,



One can check that there only exist seven (-2)-combinatorial configurations. We give all of them



2.5.4. **Calabi-Yau configurations VS. combinatorial configurations.** In this section we study the connection between Calabi-Yau configurations and combinatorial configurations. Let $\mathcal{T}$ be a Hom-finite Krull-Schmidt triangulated category with AR quiver isomorphic to $\mathbb{Z}\Delta/\mathbb{S}[d]$. We identify the elements in $\mathsf{ind}\,\mathcal{T}$ as the vertices in $\mathbb{Z}\Delta/\mathbb{S}[d]$. Let $\pi : \mathbb{Z}\Delta \to \mathbb{Z}\Delta/\mathbb{S}[d]$ be the natural surjection. We denote by $\bar{h}$ for the map $h^{\mathbb{S}[d]}$. We first show that

Proposition 2.5.12. *For any* $X, Y \in \mathsf{ind}\,\mathcal{T}$*, we have* $\dim\mathrm{Hom}_{\mathcal{T}}(X,Y) = \bar{h}_X(Y)$.

To prove this, we consider the free Abelian monoid $\mathbb{N}_{\geq 0}(\mathbb{Z}\Delta)$ generated by $(\mathbb{Z}\Delta)_0$. For any $n \in \mathbb{N}_{\geq 0}$ and $x \in (\mathbb{Z}\Delta)_0$, we define a map $f_n(x) : \mathbb{N}_{\geq 0}(\mathbb{Z}\Delta) \to \mathbb{N}_{\geq 0}(\mathbb{Z}\Delta)$ by

$$f_n(x) = \begin{cases} x & \text{if } n = 0; \\ \sum_{x \to y \in (\mathbb{Z}\Delta)_0} y & \text{if } n = 1; \\ f_1(f_{n-1}(x)) - \tau^{-1}(f_{n-2}(x)) & \text{if } n \geq 2. \end{cases}$$

By the definition, we have the following lemma immediately.

Lemma 2.5.13. *For any vertices* x, y *in* $(\mathbb{Z}\Delta)_0$*, the multiplicity of* y *in* $\bigcup_{i\geq 0} \mathrm{supp} f_i(x)$ *is* $h_x(y)$.

For any module $M \cong \bigoplus_{i=1}^{l} M_i^{t_i}$ in $\mathcal{T}$, we identify it as the element $\sum_{i=1}^{l} t_i M_i$ in $\mathbb{N}_{\geq 0}\mathbb{Z}\Delta$, and vice versa.

Proposition 2.5.14. [I1, Theorems 4.1 and 7.1] *Let* $X \in \mathsf{ind}\,\mathcal{T}$*, then we have a surjective morphism*

$$\mathrm{Hom}_{\mathcal{T}}(f_n(X), ?) \to \mathrm{rad}^n_{\mathcal{T}}(X, ?)$$

of functors which induces an isomorphism

$$\mathrm{Hom}_{\mathcal{T}}(f_n(X), ?)/\mathrm{rad}_{\mathcal{T}}(f_n(X), ?) \cong \mathrm{rad}^n_{\mathcal{T}}(X, ?)/\mathrm{rad}^{n+1}_{\mathcal{T}}(X, ?).$$

Proof of Proposition 2.5.12. Since $\mathcal{T}$ is representation-finite, then $\mathrm{rad}^n_{\mathcal{T}}(X, ?) = 0$ for n large enough. For any $X, Y \in \mathsf{ind}\,\mathcal{T}$, we have

$$\begin{aligned} \dim_k \mathrm{Hom}_{\mathcal{T}}(X,Y) &= \sum_{i\geq 0} \dim_k(\mathrm{rad}^i_{\mathcal{T}}(X,Y)/\mathrm{rad}^{i+1}_{\mathcal{T}}(X,Y)) \\ &= \sum_{i\geq 0} \dim_k(\mathrm{Hom}_{\mathcal{T}}(f_i(X),Y)/\mathrm{rad}_{\mathcal{T}}(f_i(X),Y)) \\ &= \sum_{\pi(y)=Y}\sum_{i\geq 0}(\text{multiplicity of } y \text{ in } f_i(X)) = \sum_{\pi(y)=Y} h_X(y) = \bar{h}_X(Y) \end{aligned}$$

So the assertion is true. □

The following theorem shows that Calabi-Yau configurations in $\mathcal{T}$ coincide with combinatorial configurations.

Theorem 2.5.15. *Let* $C \subset \mathsf{ind}\,\mathcal{T}$ *be a subset. Then the following are equivalent:*

(1) *C is a $(-d)$-CY configuration in $\mathcal{T}$;*
(2) *C is a $(-d)$-combinatorial configuration in $\mathbb{Z}\Delta/\mathbb{S}[d+1]$.*

Proof. It directly follows from Proposition 2.5.12. □

Thanks to the theorem above, by abuse of notation, we may use the name "Calabi-Yau configuration" even in the combinatorial context.

2.6. Trivial extension dg algebras

In this section, we consider a class of self-injective dg algebras given by trivial extension. Some results here will be used to prove a certain converse of Theorem 2.5.5 (see Theorem 2.7.1). Let B be a non-positive proper dg k-algebra. Let $\inf(B)$ be the smallest integer i such that $\mathrm{H}^i(B) \neq 0$. Clearly, $\inf(B) \leq 0$. For $d \in \mathbb{Z}$, we consider the complex $A := B \oplus DB[d]$. We regard A as a dg k-algebra whose multiplication is given by

$$(a, f)(b, g) := (ab, ag + fb)$$

where $a, b \in B$ and $f, g \in DB$, and the differential of A inherits from B and DB. If $d \geq -\inf(B)$, then A is non-positive. Moreover, we have an isomorphism $DA \simeq A[-d]$ in $\mathsf{D}A^{\mathrm{e}}$. If $\inf(B) = 0$ and $d = 0$, A is the usual trivial extension.

We give a result analogies to [Ric1, Theorem 3.1].

Proposition 2.6.1. *Let B be a non-positive proper dg k-algebra and let X be a silting object in $\mathsf{per}\, B$. Let $B' := \mathscr{E}\mathrm{nd}_B(X)$. Consider the trivial extension dg algebras $A = B \oplus DB[d]$ and $A' = B' \oplus DB'[d]$, then $\mathsf{per}\, A$ is triangle equivalent to $\mathsf{per}\, A'$.*

Proof. We may regard A as a dg B-module through the injection $B \hookrightarrow A$. Consider the functor

$$? \otimes^{\mathsf{L}}_B A : \mathsf{per}\, B \longrightarrow \mathsf{per}\, A.$$

It sends B to A. Since $\mathsf{thick}_B(X) = \mathsf{per}\, B$, then $\mathsf{thick}_A(X \otimes_B A) = \mathsf{per}\, A$. Then $X \otimes_B A$ is a compact generator of $\mathsf{D}A$ and we have a triangle equivalence between $\mathsf{per}\, \mathscr{E}\mathrm{nd}(X \otimes_B A)$ and $\mathsf{per}\, A$ (see for example [K1, Lemma 4.2]) . Next we consider the dg algebra $\mathscr{E}\mathrm{nd}(X \otimes_B A)$. Notice that, as k-complexes, we have the following isomorphisms.

$$\mathscr{H}\mathrm{om}_A(X \otimes_B A, X \otimes_B A) \simeq \mathscr{H}\mathrm{om}_B(X, X \oplus (X \otimes_B DB[d])) \simeq \mathscr{E}\mathrm{nd}_B(X) \oplus D\mathscr{E}\mathrm{nd}_B(X)[d].$$

In fact these isomorphisms also induce an isomorphism between dg algebras $\mathscr{E}\mathrm{nd}(X \otimes_B A)$ and $\mathscr{E}\mathrm{nd}_B(X) \oplus D\mathscr{E}\mathrm{nd}_B(X)[d]$. Then $\mathscr{E}\mathrm{nd}_A(X \otimes_B A)$ is isomorphic to $A' = B' \oplus DB'[d]$. So $\mathsf{per}\, A$ is triangle equivalent to $\mathsf{per}\, A'$. □

In the sequel, we only consider the special case that $\inf(B) = 0$ and B has finite global dimension. In this case, A is a Gorenstein proper dg k-algebra. If A' considered in Proposition 2.6.1 is also Gorenstein proper, then we have $\underline{\mathsf{CM}}A \simeq \underline{\mathsf{CM}}A'$ by the result above. We show $\underline{\mathsf{CM}}A$ is a cluster category in the following sense. For the details of orbit category, we refer to [K2].

Definition 2.6.2. Let B be a finite dimensional hereditary k- algebra. The $(-d)$-*cluster category* $\mathcal{C}_{-d}(B)$ is defined as the orbit category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod} B)/\nu[d]$, where ν is the Nakayama functor.

Keller proved the following result.

Proposition 2.6.3. [K2, Theorem 2][K4] *Let B be a finite-dimensional hereditary k-algebra. Let $A = B \oplus DB[d-1]$ be the trivial extension dg algebra. Then*
(1) *$\mathcal{C}_{-d}(B)$ has a structure of triangulated category;*
(2) *$\mathcal{C}_{-d}(B)$ is triangle equivalent to $\mathsf{D}_{\mathrm{sg}}(A)$.*

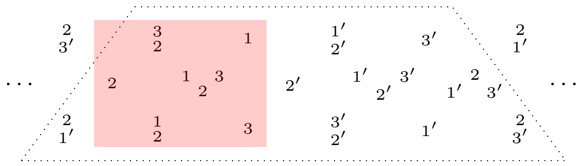
Notice that in this case $\mathsf{D}_{\mathrm{sg}}(A) = \mathsf{thick}_A(B)/\mathsf{per}\, A$ holds by the fact $\mathsf{thick}_A(B) = \mathsf{D}^{\mathrm{b}}(A)$ (see Lemma 0.3.5). By using this proposition, we have a useful observation, where we denote by $\mathfrak{A}(\underline{\mathsf{CM}}A)$ the AR quiver of $\underline{\mathsf{CM}}A$.

Corollary 2.6.4. *Let B be a finite-dimensional hereditary k-algebra and let $A = B \oplus DB[d-1]$ for $d \geq 1$. Then*

(1) *The stable category $\underline{\mathsf{CM}}A$ is triangle equivalent to $\mathcal{C}_{-d}(B)$;*
(2) *We have $\mathfrak{A}(\underline{\mathsf{CM}}A) = \mathfrak{A}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, B)/\nu[d]) = \mathbb{Z}\Delta/\nu[d]$. In particular, $\mathfrak{A}(\mathsf{mod}\, B)$ is a full subquiver of $\mathfrak{A}(\underline{\mathsf{CM}}A)$.*

We end this section with a concrete example.

Example 2.6.5. Let B be the k-algebra given by the quiver $1 \to 2 \leftarrow 3$. Let $A = B \oplus DB[1]$ be the trivial extension dg k-algebra. Then we may regard A as the dg k-algebra given by $1 \underset{\alpha_1}{\overset{\alpha_2}{\rightleftarrows}} 2 \underset{\beta_1}{\overset{\beta_2}{\rightleftarrows}} 3$, with relations $\{\alpha_1\alpha_2\alpha_1, \beta_1\beta_2\beta_1, \alpha_1\beta_2, \beta_1\alpha_2, \alpha_2\alpha_1 - \beta_2\beta_1\}$ and 0 differential. Further, the degrees of A are induced by $\deg \alpha_1 = 0 = \deg \beta_1$, $\deg \alpha_2 = -1 = \deg \beta_2$. By Corollary 2.6.4, we know the $\underline{\mathsf{CM}}A$ is triangle equivalent to $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, B)/\nu[2]$. We describe $\mathfrak{A}(\underline{\mathsf{CM}}A)$ as follows.



where the arrows are omitted and a fundamental domain is outlined in dotted line. Notice that the shaded part is exactly the AR quiver of $\mathsf{mod}\, B$.

2.7. CY configurations and symmetric dg algebras

The aim of this section is to show the following theorem, which is the converse of Theorem 2.5.5.

Theorem 2.7.1. *Let Δ be a Dynkin diagram and $d \geq 1$. Let C be a subset of vertices of $\mathbb{Z}\Delta/\mathbb{S}[d]$. The following are equivalent.*

(1) *C is a $(-d)$-CY configuration;*
(2) *There exists a d-symmetric dg k-algebra A with the AR quiver of $\mathsf{CM}A$ is $(\mathbb{Z}\Delta)_C/\mathbb{S}[d]$.*

To prove this, we need some preparations. We first study the connection between simple-minded collections (SMCs) in $\mathsf{D}^{\mathrm{b}}(A)$ and $(-d)$-CY configurations in $\underline{\mathsf{CM}}A$.

2.7.1. $(-d)$**-CY configurations are given by SMCs.** In this subsection, our aim is to show the following result, which plays a key role in the proof of Theorem 2.7.1.

Theorem 2.7.2. *Let Δ be a Dynkin diagram. Let $A = k\Delta \oplus D(k\Delta)[d-1]$ be the trivial extension dg k-algebra. Then the quotient functor $\mathsf{D}^{\mathrm{b}}(A) \to \mathsf{D}_{\mathrm{sg}}(A) \cong \underline{\mathsf{CM}}A$ induces a surjective map*

$$\{\textit{SMCs in } \mathsf{D}^{\mathrm{b}}(A)\} \longrightarrow \{(-d)\textit{-CY configurations in } \underline{\mathsf{CM}}A\}.$$

Notice that in this setting, the notion 'd-SMS' coincides with '$(-d)$-CY configuration' by [CSP, Proposition 2.13], since $\mathsf{CM}\, A$ has only finitely many indecomposable objects. So we may regard $(-d)$-CY configuration and d-SMS as the same notion later in this section.

Let R be a pre-SMC of $\mathsf{D}^{\mathrm{b}}(A)$. Recall the *SMC reduction* $\mathcal{U}$ *of* $\mathsf{D}^{\mathrm{b}}(A)$ *with respect to* R is defined as the Verdier quotient (see Section 1.2.1)

$$\mathcal{U} := \mathsf{D}^{\mathrm{b}}(A)/\mathsf{thick}(R).$$

Let E be a pre-d-SMS of $\mathsf{D}_{\mathrm{sg}}(A)$. The *SMS reduction* $(\mathsf{D}_{\mathrm{sg}}(A))_E$ of $\mathsf{D}_{\mathrm{sg}}(A)$ with respect to E is defined as the subcategory (see [CSP, Section 6])

$$(\mathsf{D}_{\mathrm{sg}}(A))_E := \{X \in \mathsf{D}_{\mathrm{sg}} \mid \mathrm{Hom}_{\mathsf{D}_{\mathrm{sg}}(A)}(X, Y[-i]) = 0 \text{ for any } i = 0, 1, \cdots, d \text{ and } Y \in E\}.$$

Let us recall some important properties of SMC and SMS reduction for later use.

Proposition 2.7.3. (1) *(Theorem 1.2.1) Let R be a pre-SMC of $\mathsf{D}^{\mathrm{b}}(A)$. Assume $\mathsf{Filt}(R)$ is functorially finite in $\mathsf{D}^{\mathrm{b}}(A)$. Then the natural functor $\mathsf{D}^{\mathrm{b}}(A) \to \mathcal{U}$ induces a bijection*

$$\{\textit{SMCs in } \mathsf{D}^{\mathrm{b}}(A) \textit{ containing } R\} \longleftrightarrow \{\textit{SMCs in } \mathcal{U}\};$$

(2) [CSP, Theorem 6.6] *Let E be a pre-d-SMS of $\mathsf{D}_{\mathrm{sg}}(A)$. Assume $\mathsf{Filt}(E)$ is functorially finite in $\mathsf{D}_{\mathrm{sg}}(A)$. Then the SMS reduction $(\mathsf{D}_{\mathrm{sg}}(A))_E$ has a structure of triangulated category and there is a bijection*

$$\{d\textit{-SMSs in } \mathsf{D}_{\mathrm{sg}}(A) \textit{ containing } E\} \longleftrightarrow \{d\textit{-SMSs in } (\mathsf{D}_{\mathrm{sg}}(A))_E\};$$

(3) *(Corollary 1.3.15 and Proposition 1.4.1) Let A be a representation finite d-self-injective dg algebra. Then the quotient functor $\mathsf{D}^{\mathrm{b}}(A) \to \mathsf{D}_{\mathrm{sg}}(A) \cong \underline{\mathsf{CM}}A$ induces a well-defined map*

$$\{\textit{SMCs in } \mathsf{D}^{\mathrm{b}}(A)\} \longrightarrow \{d\textit{-SMSs in } \underline{\mathsf{CM}}A\}.$$

Now we are ready to show Theorem 2.7.2.

Proof of Theorem 2.7.2. We first mention that $\underline{\mathsf{CM}}A$ is triangle equivalent to the orbit category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, k\Delta)/\nu[d]$ and the AR quiver of $\underline{\mathsf{CM}}A$ is $\mathbb{Z}\Delta/\mathbb{S}[d]$ by Corollary 2.6.4. Moreover, we may assume Δ has an alternating orientation (that is, each vertex in Δ is either sink or source). We apply induction on the number n of vertices of Δ. If $n = 1$, then $\Delta = A_1$. In this case, any indecomposable object in $\underline{\mathsf{CM}}A$ is a shift of the simple object and the argument is clearly true.

For the general case, let C be a $(-d)$-CY configuration in $\underline{\mathsf{CM}}A$. Since the quiver Δ has an alternating orientation, each τ-orbit in the AR quiver of $\mathsf{mod}\, k\Delta$ contains a simple $k\Delta$-module. Then by Corollary 2.6.4 (2), there exists a simple dg A-module S and an integer s, such that $\tau^s S \in C$ in $\underline{\mathsf{CM}}A$. Notice that $\tau = \mathbb{S}[-1]$ and $\mathbb{S} = [-d]$, so we have $S[t] \in C$, where $t = -sd$. Without loss of generality, in the following we may assume $S \in C$.

It is clear that $\{S\}$ is a pre-SMC in $\mathsf{D}^{\mathrm{b}}(A)$, and the SMC reduction $\mathsf{D}^{\mathrm{b}}(A)/\mathsf{thick}(S)$ is triangle equivalent to $\mathsf{D}^{\mathrm{b}}(eAe)$ by [Ji2, Proposition 3.9], where $e \in k\Delta$ is an idempotent such that $\mathrm{top}(1-e)A = S$. Considering the SMS reduction $(\mathsf{D}_{\mathrm{sg}}(A))_S$ of $\mathsf{D}_{\mathrm{sg}}(A)$ with respect to S, then by [Ji2, Theorem 6.4], we have a triangle equivalence $\mathsf{D}_{\mathrm{sg}}(eAe) \simeq (\mathsf{D}_{\mathrm{sg}}(A))_S$ and the following commutative diagram.

$$\begin{array}{ccc} \{\text{SMCs in } \mathsf{D}^{\mathrm{b}}(A) \text{ containing } S\} & \longrightarrow & \{d\text{-SMSs in } \mathsf{D}_{\mathrm{sg}}(A) \text{ containing } S\} . \\ \downarrow \simeq & & \downarrow \simeq \\ \{\text{SMCs in } \mathsf{D}^{\mathrm{b}}(eAe)\} & \longrightarrow & \{d\text{-SMSs in } \mathsf{D}_{\mathrm{sg}}(eAe)\} \end{array}$$

Notice that $eAe \cong k\Delta' \oplus D(k\Delta')[d-1]$, where Δ' is obtained from Δ by deleting the vertex i, which corresponds to S, and the arrows connected to i. Then Δ' also has an alternating orientation and has $n-1$ vertices. By induction, we may assume the lower map is surjective. Since C is a $(-d)$-CY configuration in $\mathsf{D}_{\mathrm{sg}}(A)$ containing S, then $C\backslash\{S\}$ is a $(-d)$-CY configuration in $(\mathsf{D}_{\mathrm{sg}}(A))_S$ by Proposition 2.7.3 (2). So there is a SMC H of $\mathsf{D}^{\mathrm{b}}(eAe)$ such that $H \cong C\backslash\{S\}$ in $\mathsf{D}_{\mathrm{sg}}(eAe)$. We may regard H as a subset of $\mathsf{D}^{\mathrm{b}}(A)/\mathsf{thick}(S)$ through $\mathsf{D}^{\mathrm{b}}(eAe) \simeq \mathsf{D}^{\mathrm{b}}(A)/\mathsf{thick}(S)$. Then $H \cup \{S\}$ is a SMC of $\mathsf{D}^{\mathrm{b}}(A)$ by Proposition 2.7.3 (1). By the diagram above, we have that $H \cup \{S\}$ sends to C in $\mathsf{D}_{\mathrm{sg}}(A)$. By induction, the assertion is true. □

2.7.2. **Proof of Theorem 2.7.1.** To prove our main theorem, we need the following generalization of SMC-silting bijection [KoY, Theorem 6.1] due to [SY].

Proposition 2.7.4. [SY, Theorem 1.1] *Let A be a non-positive proper dg k-algebra. Then*

(1) *There is a bijection*

$$\{\textit{SMCs of } \mathsf{D}^{\mathrm{b}}(A)\} \longleftrightarrow \{\textit{silting objects of } \mathsf{per}\, A\};$$

(2) *Let* $\{X_1, \cdots, X_n\}$ *be a SMC of* $\mathsf{D}^{\mathrm{b}}(A)$ *which corresponds to a silting* $P \in \mathsf{per}\, A$. *Let* $B := \mathscr{E}\mathrm{nd}_A(P)$ *be the endmoprhism dg* k*-algebra. Then the natural equivalence* $? \otimes^{\mathbf{L}}_B P : \mathsf{D}^{\mathrm{b}}(B) \to \mathsf{D}^{\mathrm{b}}(A)$ *sends simple* B*-modules to* $\{X_1, \cdots, X_n\}$.

We also need the following observation.

Lemma 2.7.5. *Let* A *be a proper dg* k*-algebra and let* P *be a silting dg* A*-modules. Let* $B := \mathscr{E}\mathrm{nd}_A(P)$. *Then if* A *is* d*-symmetric, so is* B.

Proof. Let A be a d-symmetric dg k-algebra, that is $\mathsf{add}\, A[-d+1] = \mathsf{add}\, DA$ in $\mathsf{D}(A^{\mathrm{op}} \otimes_k A)$. We may assume $\mathsf{D}A \cong A[-d+1]$ in $\mathsf{D}(A^{\mathrm{op}} \otimes_k A)$. We may also assume DA and A have property (P) as $(A^{\mathrm{op}} \otimes_k A)$-modules and P has property (P) as $(B^{\mathrm{op}} \otimes_k A)$-module (see [K1, Section 3.1]). Then there is a quasi-isomorphim $f : A \to DA[d-1]$ of $(A^{\mathrm{op}} \otimes_k A)$-modules, which induces a quasi-isomorphism $1 \otimes f : P \to P \otimes_A DA[d-1]$ of $(B^{\mathrm{op}} \otimes_k A)$-modules.

Applying the functor $\mathscr{H}\mathrm{om}_A(P, ?)$, we have a quasi-isomorphism $B \to \mathscr{H}\mathrm{om}_A(P, P \otimes_A DA[d-1])$ of $B^{\mathrm{op}} \otimes_k B$-modules. Notice that we have an isomorphism

$$\mathscr{H}\mathrm{om}_A(P, P \otimes_A DA[d-1]) \cong D\mathscr{H}\mathrm{om}_A(P, P)[d-1] = DB[d-1]$$

of $(B^{\mathrm{op}} \otimes_k B)$-modules, then $B \cong DB[d-1]$ in $\mathsf{D}(B)$. Thus B is also d-symmetric. □

Now we are ready to prove Theorem 2.7.1.

Proof of Theorem 2.7.1. Let C be a $(-d)$-CY configuration of $\mathbb{Z}\Delta/\mathbb{S}[d]$. Let $A = k\Delta \oplus D(k\Delta)[d-1]$ be the trivial extension dg algebra. We may regard C as a $(-d)$-CY configuration in $\underline{\mathsf{CM}}A$. There is a SMC $\{X_1, \cdots, X_n\}$ in $\mathsf{D}^{\mathrm{b}}(A)$ which is sent to C in $\underline{\mathsf{CM}}A$ by Theorem 2.7.2 and there exists a silting object P in $\mathsf{per}\, A$ corresponds to $\{X_1, \cdots, X_n\}$ by Proposition 2.7.4. Let $B = \mathscr{E}\mathrm{nd}_A(P)$. Then B is a d-symmetric dg k-algebra by Lemma 2.7.5 and moreover, $\underline{\mathsf{CM}}B$ is triangle equivalent to $\underline{\mathsf{CM}}A$. By Proposition 2.7.4, the simple modules of B corresponds to $\{X_1, \cdots, X_n\}$. Then the AR quiver of $\underline{\mathsf{CM}}B$ is isomorphic to $(\mathbb{Z}\Delta)_C/\mathbb{S}[d]$. □

2.8. Maximal d-Brauer relations and Brauer tree dg algebras

In the section, we give a combinatorial proof of Theorem 2.7.1 for the case $\Delta = A_n$. We will see in A_n case, there is a very nice description of $(-d)$-CY configurations by *maximal* d*-Brauer relations*. We develop some technical concepts and results on them. Then we introduce *Brauer tree dg algebras* from maximal d-Brauer relations and we show the simples of such dg algebras correspond to the given CY configurations.

2.8.1. **Maximal d-Brauer relations.** We start with the following definition.

Definition 2.8.1. Let $d \geq 1$ and $n > 0$ be two integers and let $N := (d+1)n + d - 1$. Let Π be an N-gon with vertices numbered clockwise from 1 to N.

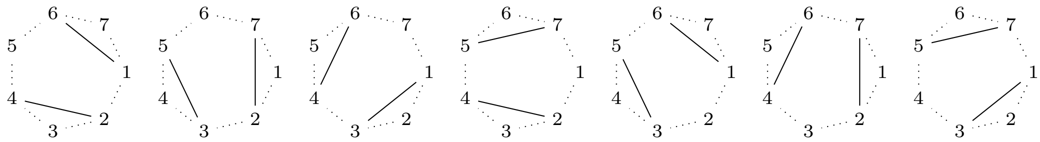
(1) A *diagonal* in Π is a straight line segment that joins two of the vertices and goes through the interior of Π. The diagonal which joins two vertices i and j is denoted by $(i, j) = (j, i)$.
(2) A d*-diagonal* in Π is a diagonal of the form $(i, i + d + j(d+1))$, where $0 \leq j \leq n-1$.

The definition of maximal d-Brauer relation is as follows. It is some special kind of 2-Brauer relation in the sense of [L, Definition 6.1].

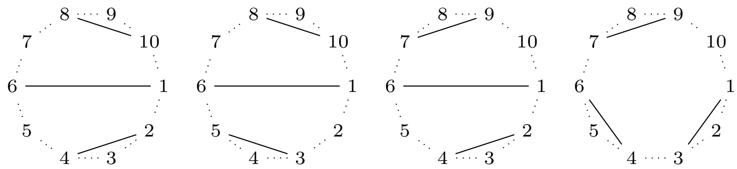
Definition 2.8.2. Let B be a set of d-diagonals in Π. We call B a d*-Brauer relation of* Π if any two d-diagonals in B are disjoint. We call a d-Brauer relation B *maximal*, if it is maximal with respect to inclusions.

We denote by $\mathbf{B}$ the set of maximal d-Brauer relations on Π. Let θ be the clockwise rotation by $2\pi/N$. If $I = (i_1, i_2)$ is a diagonal, then $\theta^t(I) = (i_1 + t, i_2 + t)$ gives us a new diagonal. For any $B, B' \in \mathbf{B}$, if there exists $n \in \mathbb{Z}$ such that $B = \theta^n(B')$, we say B and B' are *equivalent up to rotation*, denoting by $B \sim B'$. It gives rise to an equivalence relation on $\mathbf{B}$. We denote by $\underline{\mathbf{B}}$ the set of equivalence classes of $\mathbf{B}$. We give two simple examples to show what the maximal d-Brauer relations look like.

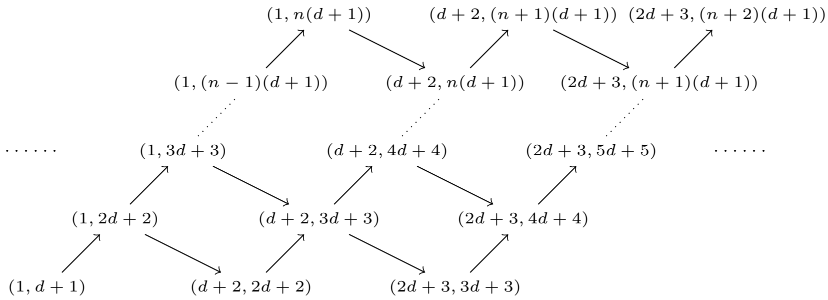
Example 2.8.3. Let $d = 2$ and $n = 2$. Then $N = 7$ and $\mathbf{B}$ consists of the following and $\#\underline{\mathbf{B}} = 1$.



Example 2.8.4. Let $d = 2$ and $n = 3$. Then $N = 10$, $\#\mathbf{B} = 30$ and $\underline{\mathbf{B}}$ consists of the following.

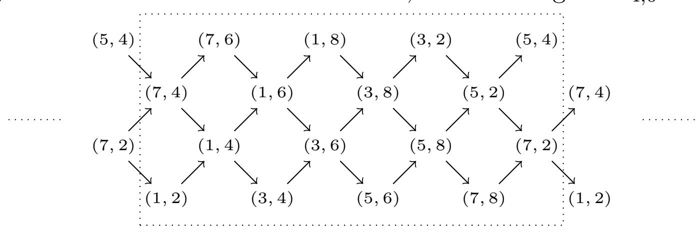


Now we give a description of $(-d)$-Calabi-Yau configurations of type A_n by using maximal d-Brauer relations. To each vertex in $\mathbb{Z}A_n$, we associate a label in $\mathbb{Z} \times \mathbb{Z}$ as follows.

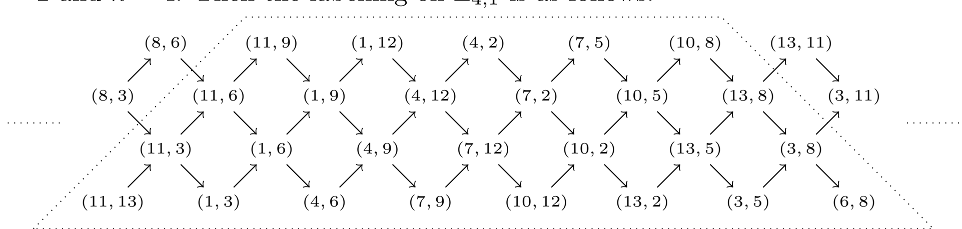


Let $\mathbb{Z}A_{n,d}$ be the stable translation quiver $\mathbb{Z}A_n/\mathbb{S}[d]$. Since by the labelling above, $\mathbb{S}[d]$ sends (i,j) to $(i+N, j+N)$ if d is even, and to $(j+N, i+N)$ if d is odd, then we may label $\mathbb{Z}A_{n,d}$ by taking the labelling in $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, where we identify (i,j) and (j,i). Let us see some examples.

Example 2.8.5. (1) Let $d = 1$ and $n = 4$. In this case, the labelling on $\mathbb{Z}_{4,0}$ is as follows.



(2) Let $d = 2$ and $n = 4$. Then the labelling on $\mathbb{Z}_{4,1}$ is as follows.



By the labelling above, we have the following theorem. This result has been show in [CS2], we put a new proof in Appendix by using concepts developed here. Let $\mathbf{C}$ be the set of $(-d)$-CY configurations in $\mathbb{Z}A_{n,d}$.

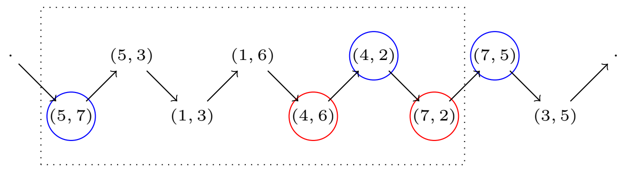
Theorem 2.8.6. [CS2, Theorem 6.5]

(1) *There is a bijection between the vertices of $\mathbb{Z}A_{n,d}$ and the d-diagonals in Π sending the vertex (i,j) of $\mathbb{Z}A_{n,d}$ to the diagonal (i,j) of Π.*

(2) *The bijection in* (1) *gives a bijection between* $\mathbf{C}$ *and* $\mathbf{B}$*;*
(3) *Any* $(-d)$*-CY configuration in* $\mathbb{Z}A_{n,d}$ *contains exactly* n *elements.*

We give an example to show how the bijection works.

Example 2.8.7. Let $n = 2$ and $d = 2$. We associate to each vertex of $\mathbb{Z}A_{2,1}$ a label in $\mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$ as following:



It is easy to check the set $\{(4,6),(7,2)\}$ is a (-2)-CY configuration of $\mathbb{Z}A_{2,1}$ and it gives rise to a maximal d-Brauer relation of 7-gon as follows (left part):

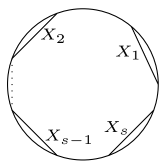
On the other hand, any maximal d-Brauer relation, for example $\{(2,4),(7,5)\}$, gives us a (-2)-CY configuration in $\mathbb{Z}A_{2,1}$.

The following lemma is immediately from the definition.

Lemma 2.8.8. *Let* $X \in \mathbb{Z}A_{n,d}$ *with labelling* (x_1, x_2)*. Then* $X[1] = (x_1 + 1, x_2 + 1)$ *and* $X[1] = \theta(X)$ *as* d*-diagonals.*

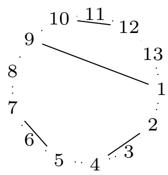
In the rest of this subsection, we introduce some technical concepts and results. They give us a better understanding of maximal d-Brauer relations and in particular, Proposition 2.8.16 will play a crucial role in the proof of Theorem 2.8.32.

Definition 2.8.9. (1) Let C be a set of diagonals in Π. We call C a *cycle* if C is contained in the closure of some connect component (denoted by Π_C) of the subset $\Pi \backslash \bigcup_{X \in C} X$ of Π. In this case, elements in C has a anti-clockwise ordering $C = \{X_1, \cdots, X_s\}$ given as follows.



(2) Let $B \in \mathbf{B}$ and $C \subset B$. We call C a B*-cycle* if C is a cycle and $\{X \in \mathbf{B} \mid X \in \overline{\Pi_C}\} = C$.

Example 2.8.10. Let $d = 2$ and $n = 4$. Let B be the following maximal d-Brauer relation.



By the definition above, $C = \{(2,4),(5,7),(10,12)\}$ is a cycle but not a B-cycle and $C' = \{(2,4),(5,7),(1,9)\}$ is a B-cycle.

Here are some elementary properties of these concepts. The proof is left to the reader.

Proposition 2.8.11. *Let* $B \in \mathbf{B}$*, then*
(1) B *is the union of* B*-cycles;*
(2) *Any two* B*-cycles have at most one common diagonal;*

(3) *Let* $C := \{X_1, \dots, X_s\}$ *be a set of diagonals in* Π. *Let* $X_{s+1} := X_1$. *Then* C *is a cycle if and only if for any* i, $2 \leq i \leq s$, X_{i-1} *and* X_{i+1} *are in the same connect component of* $\Pi \backslash X_i$
(4) *Let* $X, Y \in B$. *Then* X *and* Y *are in the same* B*-cycle if and only if for any* $Z \neq X, Y$ *in* B, X *and* Y *are in the same connected component part of* $\Pi \backslash Z$
(5) *Let* $X, Y, Z \in B$. X *and* Y *are in the same connected component of* $\Pi \backslash Z$ *if and only if there is a sequence* $X = X_1, X_2, \cdots, X_t = Y$ *of* B, *such that* $Y \neq X_i$, $1 \leq i \leq t$ *and* X_j, X_{j+1} *are in the same* B*-cycle for* $1 \leq j \leq t-1$.

We give an easy observation.

Lemma 2.8.12. *Let* $B \in \mathbf{B}$ *and* $X \in B$. *Let* Π_1 *and* Π_2 *be two connect components of* $\Pi \backslash X$. *Then*
(1) $B \cap \Pi_i := \{Y \in B \mid Y \subset \Pi_i\}$ *is a maximal* d*-Brauer relation of* Π_i *for* $i = 1, 2$;
(2) *If* X *has the form* $(i, i+d+1+(d+2)j)$, *then* $\{\#B \cap \Pi_1, \#B \cap \Pi_2\} = \{j, n-j-1\}$.

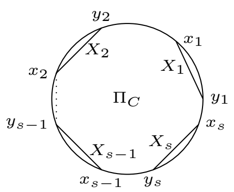
Let X and Y be two disjoint d-diagonals. We denote by $\delta(X, Y)$ the smallest positive integer m such that $\theta^{-m}(X) \cap Y \neq \emptyset$.

Remark 2.8.13. Let $X, Y \in \mathbb{Z}A_{n,d}$. If X and Y are disjoint as d-diagonals, then $\delta(X, Y) = \min\{i > 0 \mid \bar{h}_X(Y[i]) \neq 0\}$ by Lemma 2.8.8.

We will give a description of B-cycles by δ. Before this, we show a lemma. Let $\mathfrak{S}_s$ be the permutation group.

Lemma 2.8.14. *Let* $B \in \mathbf{B}$ *and let* $C \subset B$ *be a cycle with anti-clockwise ordering* $\{X_1, \cdots, X_s\}$. *Let* Π_C *be the connect component of* $\Pi \backslash C$ *given in Definition 2.8.9. Let* $X_{s+1} = X_1$. *Then the following statement holds.*
(1) *Let* $m = \#(B \cap \Pi_C)$, *then* $\sum_{l=1}^{s} \delta(X_l, X_{l+1}) = d + s + (d+1)m - 1$;
(2) *For any* $\tau \in \mathfrak{S}_s$, *we have* $\sum_{l=1}^{s} \delta(X_{\tau(l)}, X_{\tau(l+1)}) \geq d + s + (d+1)m - 1$. *Moreover, the equality holds if and only if* $\tau(l+1) = \tau(l) + 1$ *for all* $1 \leq l \leq s$;
(3) C *is a* B*-cycle if and only if* $\sum_{l=1}^{s} \delta(X_l, X_{l+1}) = d + s - 1$.

Proof. (1) Assume X_i has the form (x_i, y_i) as follows, where $y_i = x_i + d + (d+1)j_i$ with $0 \leq j_i \leq n-1$.



Since by definition, $\delta(X_i, X_{i+1}) = 1+$ the number of vertices between X_i and X_{i+1} (anti-clockwise), then $\sum_{l=1}^{s} \delta(X_l, X_{l+1}) = s + \#\Pi_C$. We count $\Pi \backslash \Pi_C$ first. By our labelling, it is easy to see the number of vertices in $\Pi \backslash \Pi_C$ is $\sum_{i=1}^{s}(d+1)(j_i+1)$. Then $\sum_{l=1}^{s} \delta(X_l, X_{l+1}) = s + d + (d+1)(n - \sum_{i}^{s}(j_i+1)) - 1$. By Lemma 2.8.12 (2), $\#B \cap (\Pi \backslash \Pi_C) = \sum_{i=1}^{s}(j_i+1)$. Then by Theorem 2.8.6 (3), $m = \#B \cap \Pi_C = n - \sum_{i}^{s}(j_i+1)$. Thus $\sum_{l=1}^{s} \delta(X_l, X_{l+1}) = d + s + (d+1)m - 1$.

(2) For any $\tau \in \mathfrak{S}_n$, we have $\delta(X_{\tau(l)}, X_{\tau(l+1)}) \geq \delta(X_{\tau(l)}, X_{\tau(l)+1})$ and the equality holds if and only if $\tau(l+1) = \tau(l) + 1$. Then by (1),

$$\sum_{l=1}^{s} \delta(X_{\tau(l)}, X_{\tau(l+1)}) \geq \sum_{l=1}^{s} \delta(X_{\tau(l)}, X_{\tau(l)+1}) = d + s + (d+1)m - 1.$$

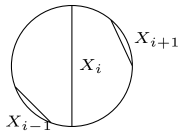
(3) By Definition 2.8.9, C is a B-cycle if and only if $m = 0$, then by (1), it holds if and only if $\sum_{l=1}^{s} \delta(X_l, X_{l+1}) = d + s - 1$. □

The following proposition gives us a useful criterion for being B-cycle.

Proposition 2.8.15. *Let $B \in \mathbf{B}$ and let C be a subset of B. Then C is a B-cycle if and only if there is a numbering $C = \{X_1, \dots, X_s\}$ such that $\sum_{l=1}^{s} \delta(X_l, X_{l+1}) = d+s-1$, where $X_{s+1} = X_1$. In this case, $\{X_1, \dots, X_s\}$ is an anti-clockwise ordering or C.*

Proof. The "only if" part. Assume C is a B-cycle with anti-clockwise ordering $\{X_1, \cdots, X_l\}$, then by Lemma 2.8.14 (1), $\sum_{l=1}^{s} \delta(X_l, X_{l+1}) = d+s-1$.

The "if" part. To prove C is a B-cycle, it suffices to show C is a cycle by Lemma 2.8.14 (2) and (3). If it is not true, then by Proposition 2.8.11 (3), there exists some i, $2 \le i \le s$, such that X_{i-1} and X_{i+1} are in different connect components of $\Pi \backslash X_i$ as follows.



In this case, we have

$$\delta(X_{i-1}, X_i) + \delta(X_i, X_{i+1}) = \delta(X_{i-1}, X_{i+1}). \tag{2.8.1}$$

Now consider the new set $C' := C \backslash X_i$. If it is a cycle, then it is clear that $X_i \in B \cap \Pi_{C'}$, where $\Pi_{C'}$ is the connected component given in Definition 2.8.9 (1). Then $\# B \cap \Pi_{C'} \ge 1$ and by Lemma 2.8.14 (2), the following inequality holds.

$$\sum_{l=1}^{i-2} \delta(X_l, X_{l+1}) + \delta(X_{i-1}, X_{i+1}) + \sum_{l=i+1}^{s} \delta(X_l, X_{l+1}) \ge 2d + s - 1. \tag{2.8.2}$$

Notice that by equation (2.8.1), the left hand of (2.8.2) equals $d+s-1$. Then $d+s-1 \ge 2d+s-1$, a contradiction. If C' is not a cycle, we do the same thing on C' as on C, and after finite steps, we get a contradiction. Thus C is a cycle, therefore a B-cycle. □

Let $B \in \mathbf{B}$. Then B is determined by B-cycles in the following sense.

Proposition 2.8.16. *Let $B, B' \in \mathbf{B}$ and let $\phi : B \to B'$ be a bijective map. If for any B-cycle C with anti-clockwise ordering $C = \{X_1, \cdots, X_s\}$, we have $\delta(X_i, X_{i+1}) = \delta(\phi(X_i), \phi(X_{i+1}))$. Then ϕ is the restriction of θ^n for some integer n, that is, B is isomorphic to B' up to rotation.*

To prove this proposition, we need prepare several lemmas.

Lemma 2.8.17. *Let B, B' and ϕ as above. Let $C = \{X_1, \cdots, X_s\}$ be a subset of B. The following are equiavlent.*

(1) *$C = \{X_1, \cdots, X_s\}$ is a B-cycle with anti-clockwise ordering;*
(2) *$\phi(C) = \{\phi(X_1), \cdots, \phi(X_s)\}$ is a B'-cycle with anti-clockwise ordering.*

Proof. (1) to (2). If $C = \{X_1, \cdots, X_s\}$ is a B-cycle with anti-clockwise ordering, then

$$\sum_{i=1}^{s} \delta(\phi(X_i), \phi(X_{i+1})) = \sum_{i=1}^{s} \delta(X_i, X_{i+1}) = d + s - 1.$$

Then by Proposition 2.8.15, $\phi(C) = \{\phi(X_1), \cdots, \phi(X_s)\}$ is a B'-cycle with anti-clockwise ordering.

(2) to (1). It is suffices to show if $\phi(X_i)$ and $\phi(X_j)$ are in the same B'-cycle, then so are X_i and X_j. If it is not true, then by Proposition 2.8.11, there exists $Y \in B$, such that X_i and X_j are in different connected component of $\Pi \backslash Y$, which is contradict to Lemma 2.8.18 below. □

Lemma 2.8.18. *Let $X, Y, Z \in B$. Then X, Y are in the same connected component of $\Pi \backslash Z$ if and only if $\phi(X)$ and $\phi(Y)$ are in the same connected component of $\Pi \backslash \phi(Z)$.*

Proof. It is immediately from Proposition 2.8.11 (5) and Lemma 2.8.17 (1) to (2) part. □

Proof of Proposition 2.8.16. We first show for any X in B, X and $\phi(X)$ have the same length. Let $X = (x, x + d + (d + 1)j) \in B$, $1 \leq j \leq n - 1$. Let Π_1 and Π_2 be the connected components of $\Pi \backslash X$. By Lemma 2.8.12, j is determined by the set $\{\#B \cap \Pi_1, \#B \cap \Pi_2\}$. Since by Lemma 2.8.18, $\{\#B \cap \Pi_1, \#B \cap \Pi_2\} = \{\#B' \cap \Pi'_1, \#B' \cap \Pi'_2\}$, then $\phi(X)$ has the form $(x', x' + d + (d + 1)j)$, where Π'_1 and Π'_2 are the connected components of $B' \backslash \phi(X)$. So there is an integer n, such that $\phi(X) = \theta^n(X)$.

We claim $\theta^n(B) = B'$. Let $C = \{X = X_1, \cdots, X_s\}$ be a B-cycle. Since $\delta(X_i, X_{i+1}) = \delta(\phi(X_i), \phi(X_{i+1}))$, then $\phi(C) = \theta^n(C)$. For any $Y \in B$, Y and X are connected by a series of B-cycles, thus $\theta^n(B) = \phi(B)$ holds. □

2.8.2. **The cardinality of maximal d-Brauer relations.** In this section, we compute the cardinality of maximal Brauer relations. Let $d \geq 1$ and $n > 0$ be two integers. Let Π be a $((d+1)n+d-1)$-gon. Recall we denote by $\mathbf{B}$ the set of maximal d-Brauer relations on Π. We have the following theorem.

Theorem 2.8.19. $\#\mathbf{B} = \frac{1}{n+1}\binom{(d+1)n+d-1}{n}$.

Corollary 2.8.20. *There are* $\frac{1}{n+1}\binom{(d+1)n+d-1}{n}$ *different* $(-d)$*-CY configurations in* $\mathbb{Z}A_{n,d}$.

Remark 2.8.21. For the classical case $d = 1$, the cardinality of Riedtmann's configurations in $\mathbb{Z}A_{n,d}$ is given by the positive Fuss-Catalan number (see [CS1, Corollary 5.8]).

Let $\mathbf{V} := \{$ subset V of vertices of Π such that $\#V = n$ $\}$. Then the cardinality of $\mathbf{V}$ is $\binom{(d+1)n+d-1}{n}$. The main idea of the proof of Theorem 2.8.19 is to construct a surjective map from $\mathbf{V}$ to $\mathbf{B}$. For any $V \in \mathbf{V}$, to construct a maximal d-Brauer relation corresponds to V, we need the following observation.

Lemma 2.8.22. *Let* $V = \{v_1, \ldots, v_n\} \in \mathbf{V}$. *Then for any* $v_i \in V$, *there exists a* d*-diagonal with the form* $(v_i, v_i + d + (d + 1)a_i)$, $0 \leq a_i \leq n - 1$, *such that*

$$\#\{v \in V \mid v_i < v < v_i + d + (d + 1)a_i\} = a_i$$

and $v_i + d + (d + 1)a_i \notin V$.

Proof. Let $b_i \in \{0, 1, 2, \ldots, n - 1\}$ be the biggest number such that $v_i + d + (d + 1)b_i \notin V$. Since $\#V = n$, then

$$\#\{v \in V \mid v_i < v < v_i + d + (d + 1)b_i\} \leq b_i.$$

On the other hand, we have

$$\#\{v \in V \mid v_i < v < v_i + d\} \geq 0.$$

So there exists $0 \leq a_i \leq b_i$ satisfies our conditions. □

For any $v_i \in V$, let $J_{v_i} = (v_i, w_i := v_i + d + (d + 1)a_i)$ be the d-diagonal such that a_i is the smallest number satisfies the conditions in Lemma 2.8.22. We have the following result.

Proposition 2.8.23. *Let* $V = \{v_1, \ldots, v_n\} \in \mathbf{V}$. *Then* $\{J_{v_1}, \ldots, J_{v_n}\}$ *defined above is a maximal* d*-Brauer relation on* Π.

Before prove this proposition, we give some basic properties of J_{v_i} first.

Lemma 2.8.24. *Let* $J_{v_i} = (v_i, w_i = v_i + d + (d + 1)a_i)$ *defined as above, then*

(1) *For any* $0 \leq c_i < a_i$, *we have*

$$\#\{v \in V \mid v_i < v \leq v_i + d + (d + 1)c_i\} > c_i.$$

(2)

$$\#\{v \in V \mid v_i + d + (d + 1)(a_i - 1) < v < v_i + d + (d + 1)a_i\} = 0.$$

Proof. (1) If $\#\{v \in V \mid v_i < v \leq v_i + d + (d+1)c_i\} \leq c_i$, then we can find $0 \leq d_i \leq c_i$, such that d_i satisfies the conditions in Lemma 2.8.22, it contradicts the minimality of a_i. Then the assertion is true.

(2) By (1), we have $\#\{v \in V \mid v_i < v \leq v_i + d + (d+1)(a_i - 1)\} > a_i - 1$. On the other hand, $\#\{v \in V \mid v_i < v < v_i + d + (d+1)a_i\} = a_i$, then the statement holds clearly. □

Proof of Proposition 2.8.23. By Theorem 2.8.6, we only need to show that any two diagonals in $\{J_{v_1}, \dots, J_{v_n}\}$ are disjoint. Let $v_i, v_j \in V$. If neither $v_j < v_i < w_j$ nor $v_i < v_j < w_i$ holds, then it is clear J_{v_i} and J_{v_j} are disjoint. Otherwise, we may assume $v_j < v_i < w_j$. It suffices to show $v_j < u_i < w_j$. We consider the following two cases.

If $v_j + d + (d+1)b_j < v_i \leq v_j + d + (d+1)(b_j+1)$, for some $0 \leq b_j < a_j$. By Lemma 2.8.24 (2), we know that $b_j + 1 \leq a_j - 1$. Consider the diagonal $(v_i, v_i + d + (d+1)(a_j - b_j - 2))$, we claim that

$$\#\{v \in V \mid v_i < v \leq v_i + d + (d+1)(a_j - b_j - 2)\} \leq a_j - b_j - 2.$$

Indeed by Lemma 2.8.24 (1),

$$\#\{v \in V \mid v_j < v \leq v_j + d + (d+1)b_j\} > b_j,$$

and by the definition of w_j,

$$\#\{v \in V \mid v_j < v < v_j + d + (d+1)a_i\} = a_i.$$

Then $\#\{v \in V \mid v \neq v_i \text{ and } v_j + d + (d+1)b_j < v < w_j\} \leq a_j - b_j - 2$. So the claim is true and $a_i \leq a_j - b_j - 2 < a_j$. Then $w_i < w_j$ and J_{v_i} and J_{v_j} are disjoint.

If $v_j < v_i \leq v_j + d$. Consider the diagonal $(v_i, v_i + d + (d+1)(a_j - 1))$. It is clear that $\#\{v \in V \mid v_i < v \leq v_i + d + (d+1)(a_j - 1)\} \leq a_j - 1$. Then $a_i \leq a_j - 1 < a_j$. So $w_i < w_j$. Moreover, Y_{v_i} and Y_{v_j} are disjoint.

Thus $\{J_{v_1}, \dots, J_{v_n}\}$ is a maximal d-Brauer relation. □

Now we can construct a map $\Theta : \mathbf{V} \longrightarrow \mathbf{B}$ by sending $V \in \mathbf{V}$ to $\Theta(V) := \{J_v \mid v \in V\}$. By Proposition 2.8.23, it is well defined. Next for $B \in \mathbf{B}$, we need to determine the preimage of B.

Lemma 2.8.25. *Let Θ be defined as above. Then Θ is surjective. More precisely, for any $B \in \mathbf{B}$, we have $\#\{V \in \mathbf{V} \mid \Theta(V) = B\} = n+1$.*

Proof. Let $B = \{X_1, \dots, X_n\}$ be a maximal d-Brauer relation. Assume X_t has the form (x_t, y_t) for $1 \leq t \leq n$. Given any x_t, we construct a set $V_{x_t} \in \mathbf{V}$ as follows.

(1) For any $1 \leq s \leq n$, one of x_s and y_s belongs to V_{x_t};
(2) $i_s \in V_{i_t}$ if and only if $i_t \leq i_s < j_s$ by clockwise ordering.

We construct V_{y_t} in a similar way. It is easy to show $\Theta(V_{x_t}) = \Theta(V_{y_t}) = B$. Then Θ is surjective.

We claim that $\#\{V_{x_t}, V_{y_t} \mid 1 \leq t \leq n\} = n+1$. We show this by induction. If $n = 1$, it is clear. Assume the claim holds for $n \leq m-1$. For the case $n = m$. Let Π_1 and Π_2 be the two connect components of $\Pi \backslash X_i$. Assume $y_i = x_i + d + (d+1)j$, where $0 \leq j \leq n-1$. By Lemma 2.8.12, $B \cap \Pi_l$ is a maximal d-Brauer relation on Π_l, $1 \leq l \leq 2$ and moreover $\{\#B \cap \Pi_1, \#B \cap \Pi_2\} = \{m-j-1, j\}$. Then by induction, $\#\{V_{x_t}, V_{y_t} \mid 1 \leq t \leq n\} = (m-j-1+1) + (j+1) = m+1$. So the claim is true.

For $V \in \Theta^{-1}(B)$, by our construction of $\{J_v \mid v \in V\}$, one can show that V is given by some V_{x_t} or V_{y_t}. Thus by the claim above $\#\{V \in \mathbf{V} \mid \Theta(V) = B\} = n+1$. □

Theorem 2.8.19 is deduced by the Lemma 2.8.25 directly.

Proof of Theorem 2.8.19. By Lemma 2.8.25, we know $\#\mathbf{B} = \frac{1}{n+1}\#\mathbf{V} = \frac{1}{n+1}\binom{(d+1)n+d-1}{n}$. □

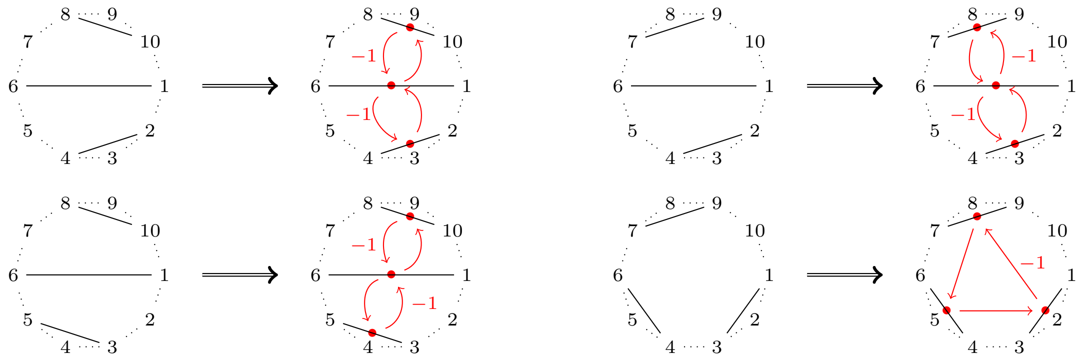
2.8.3. **Brauer tree dg algebras.** We first introduce a graded quiver from given maximal d-Brauer relation in the following way.

Definition 2.8.26. Let $B \in \mathbf{B}$. The graded quiver Q_B associated to B is defined as follows.
(1) The vertices of Q_B are given by the d-diagonals in B;
(2) For any B-cycle C with anti-clockwise ordering $\{X_1, \cdots, X_s\}$, we draw arrows $X_i \to X_{i+1}$ with degree $1 - \delta(X_i, X_{i+1})$, where $1 \le i \le s$ and $X_{s+1} = X_1$.

We say a cycle in Q_B is *minimal*, if it is given by some B-cycle.

In Example 2.8.4, we give the maximal d-Brauer relations for the case $d = 2$ and $n = 3$. Now we draw the d-Brauer quivers associate to them.

Example 2.8.27. Let $d = 2$ and $n = 3$. The graded quivers associate to the maximal d-Brauer relations are as follows.



where the quivers are drawn by red lines and the numbers with red color are degrees correspond to the arrows near them.

We give some basic properties on Q_B, which are induced by Proposition 2.8.11 and Lemma 2.8.14 (2).

Proposition 2.8.28. *Let $B \in \mathbf{B}$. Then Q_B satisfies the following*
(1) *Every vertex of Q belongs to one or two minimal cycles;*
(2) *Any two minimal cycles meet in one vertex at most;*
(3) *There are no loops in Q;*
(4) *Every arrow is equipped with a non-positive degree and the sum of degrees of each minimal cycle is $-d+1$.*

Remark 2.8.29. The above properties (1), (2), (3) imply that Q_B a Brauer quiver in the sense of Gabriel and Riedtmann (see [GR]).

Now we introduce the following main object in this section.

Definition 2.8.30. Let $B \in \mathbf{B}$. The *Brauer tree dg algebra* A_{Q_B} is defined as kQ_B/I_B with zero differential and grading given by that of Q_B, where the admissible ideal I_B is generated by the following relations.
(1) For any minimal cycle
$$X_1 \xrightarrow{\alpha_1} X_2 \to \cdots \to X_{m-1} \xrightarrow{\alpha_{m-1}} X_m \xrightarrow{\alpha_m} X_1,$$
$\alpha_i\alpha_{i+1}\cdots\alpha_m\alpha_1\cdots\alpha_i \in I$ for each $1 \le i \le m$;
(2) If X is the common d-diagonal of two B-cycles
$$X = X_1 \xrightarrow{\alpha_1} X_2 \to \cdots \to X_{m-1} \xrightarrow{\alpha_{m-1}} X_m \xrightarrow{\alpha_m} X_1$$
$$X = Y_1 \xrightarrow{\beta_1} Y_2 \to \cdots \to Y_{s-1} \xrightarrow{\beta_{m-1}} Y_s \xrightarrow{\beta_s} Y_1,$$
then $\beta_s\alpha_1 \in I$ and $\alpha_m\beta_1 \in I$ and $\alpha_1\alpha_2\cdots\alpha_m - \beta_1\beta_2\cdots\beta_s \in I$.

The following proposition is an easy generalization of well-known result for ungraded case.

Proposition 2.8.31. *The dg algebra A_{Q_B} is d-symmetric.*

Now we are ready to state the following main result, which implies Theorem 2.7.1 for the case $\Delta = A_n$. Recall from Definition 0.5.1 the definition of $(\mathbb{Z}A_{n,d})_C$.

Theorem 2.8.32. *Let B be a maximal d-Brauer relation on $((d+1)n+d-1)$-gon and let C be the $(-d)$-CY configuration in $\mathbb{Z}A_{n,d}$ corresponding to B. Then for the Brauer tree dg algebra A_{Q_B}, the AR quiver of $\mathsf{CM}\, A_{Q_B}$ is isomorphic to $(\mathbb{Z}A_{n,d})_C$.*

The outline of our proof is the following. Consider the $(-d)$-CY configuration C_A given by the simples of A_{Q_B}. Then the AR quiver of $\mathsf{CM}\, A_{Q_B}$ is isomorphic to $(\mathbb{Z}A_{n,d})_{C_A}$. So we only need to show $C = C_A$. To show this, let B_A be the maximal d-Brauer relation corresponds to C_A.

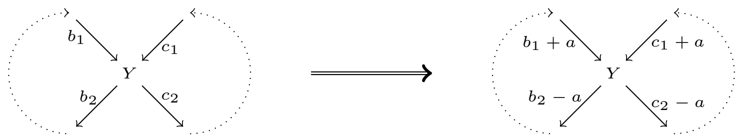
$$C \longleftrightarrow B \longrightarrow A_{Q_B} \xrightarrow{\text{simples}} C_A \longleftrightarrow B_A$$

Then it suffices to prove B is isomorphic to B_A up to rotation.

We first describe the AR quiver of the stable category $\underline{\mathsf{CM}} A_{Q_B}$.

Proposition 2.8.33. *The AR quiver of $\underline{\mathsf{CM}} A_{Q_B}$ is $\mathbb{Z}A_{n,d}$.*

To prove this proposition, we need some observations. Let $Y \in (Q_B)_0$ be a vertex and $a \in \mathbb{Z}$. We construct a new graded quiver $Q_{Y,a}$. It is isomorphic to Q_B as ungraded quiver. The degrees of arrows ending at Y and starting at Y are changed as follows.



Q_B $\qquad\qquad$ $Q_{Y,a}$

And other degrees of arrows are the same as in Q_B. Let $T_{Y,a} := P_Y[a] \bigoplus (\bigoplus_{Y' \in B, Y' \neq Y} P_{Y'})$ be a dg A_{Q_B}-module, where P_Y is the indecomposable projective module corresponds to the vertex Y. Consider the Brauer tree dg algebra $A_{Q_{Y,a}}$. Then one can show that $A_{Q_{Y,a}}$ is isomorphic to the endmorphism dg algebra $\mathscr{E}\mathrm{nd}(T_{Y,a})$. Immediately, we have

Lemma 2.8.34. *The functor $\mathbf{R}\mathscr{H}\mathrm{om}(T_{Y,a}, ?)$ induces a triangle equivalence $\mathsf{D}^{\mathrm{b}}(A_{Q_B})/\operatorname{\mathsf{per}} A_{Q_B} \to \mathsf{D}^{\mathrm{b}}(A_{Q_{Y,a}})/\operatorname{\mathsf{per}} A_{Q_{Y,a}}$.*

Proof. It is clear that $T_{Y,a}$ is a compact generator of $\operatorname{\mathsf{per}} A_{Q_B}$. Then $\mathbf{R}\mathscr{H}\mathrm{om}(T_{Y,a}, ?) : \operatorname{\mathsf{per}} A_{Q_B} \to \operatorname{\mathsf{per}} A_{Q_{Y,a}}$ is an equivalence, which induces a triangle equivalence $\mathsf{D}^{\mathrm{b}}(A_{Q_B}) \to \mathsf{D}^{\mathrm{b}}(A_{Q_{Y,a}})$. Thus the assertion is true. □

Now we prove Proposition 2.8.33 by adjusting degrees of Q_B to some special case.

Proof of Proposition 2.8.33. Let $B \in \mathbf{B}$. We say Q_B is *admissible* if each minimal cycle in Q_B has an arrow with degree $-d+1$ and other arrows with degree 0. We consider the following two cases.

(1) If Q_B is admissible. Let D be the set of arrows in Q_B with degree $-d+1$. It is an admissible cutting set in the sense of [FP, Schr]. Therefore A_{Q_B} is isomorphic to the trivial extension $\Lambda \oplus D\Lambda[d-1]$ by [Schr, Theorem 1.3], where Λ is the factor algebra $A_{Q_B}/(D)$. By Corollary 2.6.4, $\underline{\mathsf{CM}}(A_{Q_B})$ is triangle equivalent to $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)/\nu[d]$. By [H1, Theorem 6.7], Λ is an iterated titled algebra of type A_n. So the AR-quiver of $\underline{\mathsf{CM}}(A_{Q_B})$ is given by $\mathbb{Z}A_{n,d}$.

(2) For general Q_B. We claim there exists $B' \in \mathbf{B}$, such that $Q_{B'}$ is admissible and there is a triangle equivalence $\underline{\mathsf{CM}}(A_{Q_B}) \xrightarrow{\sim} \underline{\mathsf{CM}}(A_{Q_{B'}})$. In fact, we can start from any minimal cycle. Under a suitable ordering, we may change the degrees of Q_B to obtain an admissible quiver $Q_{B'}$ step by step by our discussion above. Then by Theorem 2.2.4 (3) and by Lemma 2.8.34, $\underline{\mathsf{CM}} A_{Q_B} = \mathsf{D}^{\mathrm{b}}(A_{Q_B})/\operatorname{\mathsf{per}} A_{Q_B}$ is triangle equivalence to $\underline{\mathsf{CM}} A_{Q_{B'}} = \mathsf{D}^{\mathrm{b}}(A_{Q_{B'}})/\operatorname{\mathsf{per}} A_{Q_{B'}}$. Then by (1), the AR quiver of $\underline{\mathsf{CM}}(A_{Q_B})$ is $\mathbb{Z}A_{n,d}$. □

Let $B = \{Y_1, \cdots, Y_n\}$ be a maximal d-Brauer relation on $((d+1)n+d-1)$-gon. Recall that the vertices of Q_B are given by $\{Y_1, \cdots, Y_n\}$. By Theorem 2.5.5, the set $C_A := \{S_1, \cdots, S_n\}$ of simple dg A_{Q_B}-modules is a $(-d)$-CY configuration, where S_i is the simple module corresponds to vertex Y_i. And by Proposition 2.8.33, the AR quiver of $\underline{\mathsf{CM}}A_{Q_B}$ is $\mathbb{Z}A_{n,d}$. Thus we can also regard C_A as the subset of $\mathbb{Z}A_{n,d}$. Let B_A be the maximal d-Brauer relation corresponds to C_A. By abuse of notation, the d-diagonals in B_A are also denoted by $\{S_1, \cdots, S_n\}$.

Let $\{Y_{j_1}, Y_{j_2}, \cdots, Y_{j_s}\}$ be a B-cycle with anti-clockwise ordering. Then it gives a minimal cycle in Q_B.

$$Y_{j_1} \xrightarrow{\alpha_1} Y_{j_2} \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{s-1}} Y_{j_s} \xrightarrow{\alpha_s} Y_{j_1}$$

where $\deg \alpha_i = 1 - \delta(Y_{j_i}, Y_{j_{i+1}})$ by Definition 2.8.26. The following proposition gives us some information which determines B uniquely.

Proposition 2.8.35. *Assume $\alpha_i : Y_{j_i} \to Y_{j_{i+1}}$ is an arrow in Q_B. Then $\delta(S_{j_i}, S_{j_{i+1}}) = \delta(Y_{j_i}, Y_{j_{i+1}})$, where we regard S_{j_i} and Y_{j_i} as d-diagonals in B_A and B respectively.*

Proof. By Remark 2.8.13, $\delta(S_{j_i}, S_{j_{i+1}}) = \min\{t > 0 \mid \bar{h}_{S_{j_i}}(S_{j_{i+1}}[t]) \neq 0\}$ and by Proposition 2.5.12, we have $\bar{h}_{S_{j_i}}(S_{j_{i+1}}[t]) = \mathrm{Hom}_{\underline{\mathsf{CM}}A_{Q_B}}(S_{j_i}, S_{j_{i+1}}[t])$. Thus

$$\begin{aligned} \delta(S_{j_i}, S_{j_{i+1}}) &= \min\{t > 0 \mid \mathrm{Hom}_{\underline{\mathsf{CM}}A_{Q_B}}(S_{j_i}, S_{j_{i+1}}[t]) \neq 0\} \\ &= \min\{t > 0 \mid \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A_{Q_B})}(S_{j_i}, S_{j_{i+1}}[t]) \neq 0\} \end{aligned}$$

where the second equality holds by the fact that $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A_{Q_B})}(S_{j_i}, A) = \mathrm{H}^{-d+1}(S_{j_i}) = 0$. Let $l = -\deg \alpha_i$. By our construction of Q_B, it is clear that every path from Y_{j_i} to $Y_{j_{i+1}}$ has degree no more than $-l$. Then $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A_{Q_B})}(S_{j_i}, S_{j_{i+1}}[t]) = 0$ for any $0 \leq t \leq l$ and $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(A_{Q_B})}(S_{j_i}, S_{j_{i+1}}[l+1]) \neq 0$ by Proposition 0.3.8. Thus $\delta(S_{j_i}, S_{j_{i+1}}) = l + 1 = 1 - \deg \alpha_i = \delta(Y_{j_i}, Y_{j_{i+1}})$. □

Now we are ready to prove Theorem 2.8.32.

Proof of Theorem 2.8.32. Consider the map $\phi : B \to B_A$ sending Y_j to S_j. It is clearly bijective and for any B cycle C with anti-clockwise ordering $\{Y_{j_1}, Y_{j_2}, \cdots, Y_{j_m}\}$, we have $\delta(Y_{j_i}, Y_{j_{i+1}}) = \delta(S_{j_i}, S_{j_{i+1}})$ by Proposition 2.8.35. Then by Proposition 2.8.16, B is isomorphic to B_A up to rotation. Then the AR quiver of $\mathsf{CM}\, A_{Q_B}$ is isomorphic to $(\mathbb{Z}A_{n,d})_C$. □

Appendix B. A new proof of Theorem 2.8.6

In this part, we give a new proof of Theorem 2.8.6 by using the results developed in Section 2.8.1. We first point out the following property.

Proposition 2.8.36. *For any $B \in \mathbf{B}$, we have $\#B = n$.*

Proof. Let $B \in \mathbf{B}$. We apply the induction on n.

If $n = 1$, then Π is a $(2d+2)$-gon and every d-diagonal has the form $(i, i+d+1)$. In this case, any two d-diagonals intersect, which implies that B contains only one d-diagonal.

Assume our argument is true for $n \leq m$, where $m \geq 1$. Now consider the case $n = m+1$. Assume $I \in B$ has the form $(i_1, i_1 + d + 1 + (d+2)j)$. Then $\Pi \backslash I$ has two connect components Π_1 and Π_2, where Π_1 is a $((d+2)j+d)$-gon and Π_2 is a $((d+2)(n-j-1)+d)$-gon. By Lemma 2.8.12, $B \cap \Pi_1$ (resp. $B \cap \Pi_2$) is a maximal d-Brauer relation of Π_1 (resp. Π_2). By induction, $\#(B \cap \Pi_1) = j$ and $\#(B \cap \Pi_2) = n-j-1$. Then $\#B = \#(B \cap \Pi_1) + \#(B \cap \Pi_2) + 1 = n$. Therefore the statement holds for any $n \geq 1$. □

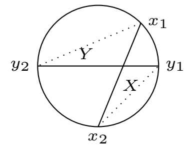
The following lemma is immediately from our labelling on $\mathbb{Z}A_n$.

Lemma 2.8.37. *Let $X, Y \in \mathbb{Z}(A_n)_0$, where $X = (x, x+d+1+(d+2)m)$, $0 \leq m \leq n-1$. Then $h_X(Y) \neq 0$ if and only if $Y = (x+(d+2)i, x+d+1+(d+2)j)$, where $0 \leq i \leq m \leq j \leq n-1$.*

The following lemma gives us a way to read $\bar{h}_X(Y)$ from the relative position of X and Y in Π.

Lemma 2.8.38. *Let $X, Y \in \mathbb{Z}A_{n,d}$. We also regard them as d-diagonals in Π. Then*

(1) *If X and Y are disjoint, then $\bar{h}_X(Y) = 0$;*
(2) *If X and Y are joint, then $\bar{h}_X(Y) \neq 0$ if and only if X and Y are connected by d-diagonals as follows*



that is, if and only if (y_1, x_2) (or equivalently, (y_2, x_1)) is a d-diagonal.

Then by the description above, we have the following result.

Proposition 2.8.39. *Let $X, Y \in \mathbb{Z}A_{n,d}$. Then the following are equivalent*

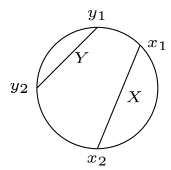
(1) *$\bar{h}_X(Y[-s]) = 0$ and $\bar{h}_Y(X[-s]) = 0$ for $0 \leq s \leq d$;*
(2) *X and Y are disjoint as d-diagonals.*

Proof. From (1) to (2). If $X \cap Y \neq \emptyset$. We may assume $X = (x_1, x_2 = x_1 + d + 1 + (d+2)i)$ and $x_1 \leq y_1 < x_2 \leq y_2$, where $0 \leq i \leq n-1$. We consider the following cases.

- If $x_2 \leq y_2 \leq x_2 + d$ and $y_1 - x_1 > y_2 - x_2$, then $\bar{h}_X(Y[x_2 - y_2]) \neq 0$;
- If $x_2 \leq y_2 \leq x_2 + d$ and $y_1 - x_1 \leq y_2 - x_2$, then $\bar{h}_X(Y[x_1 - y_1]) \neq 0$;
- If $x_2 + d < y_2$ and $y_1 = x_1 + d + 1 + (d+2)i'$, $0 \leq i' \leq i$, then (x_1, y_1) is a d-diagonal. Then by our discussion above, $\bar{h}_Y(X) \neq 0$;
- If $x_2 + d < y_2$ and $y_1 \neq x_1 + d + 1 + (d+2)i'$ for any $0 \leq i' \leq i$. Then there exist $0 \leq t \leq d$, such that $Y[-t]$ has the form $(x_1 + (d+2)j, y_2 - t)$ for some $0 \leq j < i$. In this case, $\bar{h}_X(Y[-t]) \neq 0$.

All the cases above are contradictory to the condition (1). So we know X and Y are disjoint.

From (2) to (1). Assume X and Y are disjoint as follows



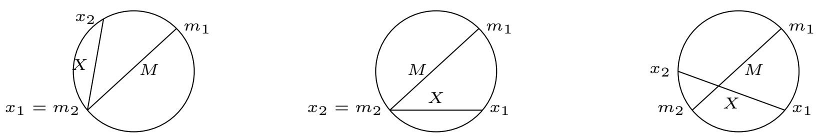
For $0 \leq s \leq d$, $Y[-s] = (y_1 - s, y_2 - s)$. If $X \cap Y[-s] = \emptyset$, it is clear $\bar{h}_X(Y[-s]) = 0$. If $X \cap Y[-s] \neq \emptyset$, *i.e.* $y_2 - s \leq x_2$. Then $(y_2 - s, x_2)$ can not be a d-diagonal (it is possible only when $s > d + 1$). So we still have $\bar{h}_X(Y[-s]) = 0$. Similarly, $\bar{h}_Y X[-s] = 0$ for $0 \leq s \leq d$. □

Remark 2.8.40. Let $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ be two d-diagonals. Assume $x_1 < y_1 < x_2 < y_2$. Then by the proof of Proposition 2.8.39, if $y_1 \neq x_1 + d + 1 + (d+2)i'$ for any $0 \leq i' \leq i$, in other words, if (x_1, y_1) is not a d-diagonal, then there exists $0 \leq s \leq d$ such that $\bar{h}_X(Y[-s]) \neq 0$.

To prove Theorem 2.8.6, we need another lemma.

Lemma 2.8.41. *Let $B \in \mathbf{B}$ and let M be a d-diagonal. Then there exists $X \in B$ and $0 \leq i \leq d$ such that $\bar{h}_X(M[-i]) \neq 0$.*

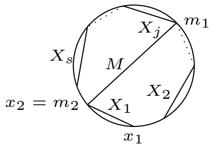
Proof. Since B is maximal, there exists $X \in B$ such that $X \cap M \neq \emptyset$. Up to rotation, there are three types of positional relationships between M and X as follows.



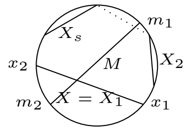
type 1 type 2 type 3

We show the statement case by case. For type 1, it is clear $\bar{h}_X(M) \neq 0$ by Lemma 2.8.38. For type 2, if there is $m_1 < t < x_1$, such that $T = (m_1, t)$ is a d-diagonal in B, then $\bar{h}_T(M) \neq 0$. If there is no such a T, we claim that $\exists\, Y \in B$ such that Y and M are of type 3.

To prove this claim, let us consider the B-cycle B_X containing X such that B_X and M are on the same side of X. If the claim is not true, then for any $X' \in B_X$, M and X' are disjoint or of type 2 (Notice that by our assumption, type 1 never happens). Labelling B_X anti-clockwise starting from X. Let $X_{s+1} = X = X_1$ (see figure (a) below). We may write $x_1 = x_2 + d + 1 + (d+2)i'$ and $m_1 = m_2 + d + 1 + (d+2)i''$, where $0 \leq i'' < i' \leq n-1$, then $x_1 - m_1 = (d+2)(i' - i'')$ and the number of vertices between m_1 and x_1 is $(d+2)(i'-i'')-1 = d+1+(d+2)(i'-i''-1)$. Let j be the smallest number such that X_j and X_1 are on the different sides of M. Then the sum of number of vertices between X_i and X_{i+1} for $1 \leq i \leq j-1$ is at least $d+1$. Then $\sum_{i=1}^{s} \delta(X_i, X_{i+1}) \geq d+s+1$. It is contradictory to Proposition 2.8.15, which says $\sum_{i=1}^{s} \delta(X_i, X_{i+1}) = d+s$. So the claim holds. Then we only need to consider type 3.



(a)



(b)

Assume X and M are of type 3. We may assume there is no $X' \in B$, such that X' and the vertex m_1 are on the same side of X, and X', M are of type 3 (if such X' exists, replace X by X'). Now we show (x_2, m_1) is not a d-diagonal. If (x_2, m_1) is a d-diagonal, consider the B-cycle B_X containing X, B_X and the vertex m_1 are on the same side of X. Labelling B_X anti-clockwise (see figure (b) above). Since (x_2, m_1) is a d-diagonal, then $(d+2)|(x_1 - m_1)$. Similar to our discussion for type 2, we have $\sum_{i=1}^{s} \delta(X_i, X_{i+1}) \geq d+s+1$, which is contradictory to Proposition 2.8.15. So we know (x_2, m_1) is not a d-diagonal. Then by Remark 2.8.40, there exists $0 \leq i \leq d$ such that $\bar{h}_X(M[-i]) \neq 0$. Therefore the assertion is true. □

We are ready to prove Theorem 2.8.6 now.

The proof of Theorem 2.8.6. Given a $(-d-1)$-CY configuration C in $\mathbb{Z}A_{n,d}$. By Definition 2.5.9, for any two different objects X and Y in C, we have $\bar{h}_X(Y[-s]) = 0$ and $\bar{h}_Y(X[-s]) = 0$ for $0 \leq s \leq d$. Then by Proposition 2.8.39, X and Y are disjoint. So the set $\{X | X \in C\}$ gives rise to a d-Brauer relation B. We claim B is maximal. If not, there exists a d-diagonal M such that for any $X \in C$, X and M are disjoint. Then by Proposition 2.8.39, $\bar{h}_X(M[-s]) = 0$ for $0 \leq s \leq d$. It is contradictory to that C is a $(-d-1)$-CY configuration (see Definition 2.5.9).

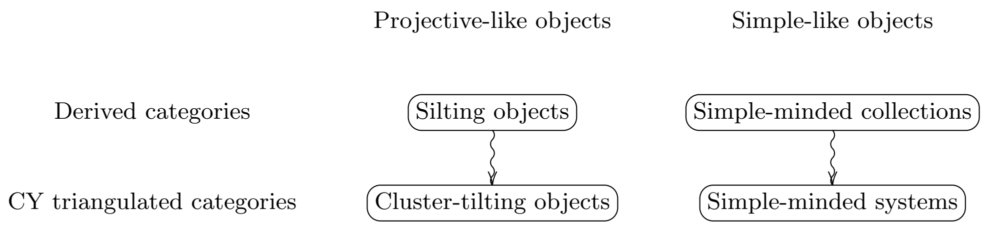
On the other hand, given a maximal d-Brauer relation B. Let C be the set of vertices of $\mathbb{Z}A_{n,d}$ corresponds to the d-diagonals in B. By Proposition 2.8.39, for any two different objects X and Y in C, we have $\bar{h}_X(Y[-j]) = 0$, where $0 \leq j \leq d$. Let M be any vertex in $\mathbb{Z}A_{n,d}$. Since B is maximal, then by Lemma 2.8.41, there exists $X \in \mathcal{C}$ and $0 \leq i \leq d$ such that $\bar{h}_X(M[-i]) \neq 0$. So C is a $(-d-1)$-CY configuration. □

Part 3. Positive Fuss-Catalan numbers and Simple-minded systems in negative Calabi-Yau categories

3.1. Introduction

Fomin and Zelevinsky [FZ1] showed that cluster algebras of finite type correspond bijectively with finite root systems Φ. As a generalization of their combinatorial structure, Fomin and Reading [FR] introduced generalized cluster complex $\Delta^d(\Phi)$ for each positive integer d. It is a simplicial complex whose ground set is the disjoint union of d copies of the set Φ^+ of positive roots and the set of negative simple roots, and studied actively in combinatorics [Ar, STW]. It is known that $\Delta^d(\Phi)$ is categorified by $(d+1)$-Calabi-Yau ($(d+1)$-CY) cluster categories $\mathcal{C}_{d+1}(kQ)$ [2] for the corresponding Dynkin quiver Q [K2, T]. The category $\mathcal{C}_{d+1}(kQ)$ has special objects called cluster tilting objects, which correspond bijectively with maximal simplices in $\Delta^d(\Phi)$ [Z] and with silting objects contained in some subcategory of $\mathsf{D}^{\mathrm{b}}(kQ)$ [BRT1]. Culster tilting objects also play a key role in Cohen-Macaulay representations [I2].

Recently there is increasing interest in negative CY triangulated categories (see [CS1, CS2, CS3, CSP, HJY, Ji1, Ji2, Jo1, KYZ]), including $(-d)$-CY cluster categories $\mathcal{C}_{-d}(kQ)$. These categories often contain special objects called d-simple-minded systems (or d-SMS) [CS2] (see Definition 0.2.7). It plays a key role in the study of Cohen-Macaulay dg modules [Ji1], see Part 2.



In some important cases, cluster tilting objects and d-SMSs are shadows of more fundamental objects, namely, silting objects and simple-minded collections (SMCs) respectively [KN, IYa1, Ji2].

The aim of this chapter is to show that there is a bijection between d-SMSs and maximal simplices in $\Delta^d(\Phi)$ without negative simple roots. In particular, the number of d-SMSs in $\mathcal{C}_{-d}(kQ)$ is precisely the positive Fuss-Catalan number. Our method is based on a refined version of silting-t-structure correspondence.

3.1.1. **Counting d-simple-minded systems.** Let Φ be a simply-laced finite root system, and W the corresponding Weyl group. The *Fuss-Catalan number* is given by the formula

$$C_d(W) := \prod_{i=1}^{n} \frac{dh + e_i + 1}{e_i + 1}$$

where n is the rank of W, h is its Coxeter number, and $e_1, \ldots, e_n$ are its exponents (see Figure 1). It is well-known that $C_d(W)$ equals the number of maximal simplices in $\Delta^d(\Phi)$ and also equals the number of d-noncrossing partitions for W (see [Ar, FR]). There is a variant of $C_d(W)$, called the *positive Fuss-Catalan number*, denoted by $C_d^+(W)$ and given by the formula

$$C_d^+(W) := \prod_{i=1}^{n} \frac{dh + e_i - 1}{e_i + 1},$$

see Figure 1 for the explicit value.

Let k be a field and Q a Dynkin quiver. The $(-d)$-*CY cluster category* $\mathcal{C}_{-d}(kQ)$ is defined as the orbit category $\mathcal{C}_{-d}(kQ) := \mathcal{D}^{\mathrm{b}}(kQ)/\nu[d]$, where ν is the Nakayama functor of $\mathsf{D}^{\mathrm{b}}(kQ)$. This is a triangulated category by [K2] and has AR quiver $\mathbb{Z}Q/\nu[d]$. We denote by d-$\mathsf{SMS}\,\mathcal{C}_{-d}(kQ)$ the set of d-SMSs in $\mathcal{C}_{-d}(kQ)$, and by $\mathsf{max\text{-}sim}\,\Delta^d(\Phi)$ (resp. $\mathsf{max\text{-}sim}^+\,\Delta^d(\Phi)$) the set of maximal

[2] $(d+1)$-CY cluster categories are usually called d-cluster categories in hereditary setting, and $(d+1)$-cluster categories in non-hereditary setting.

Q	h	$e_1, \dots, e_n$	$C_d^+(W) = \#\, d\text{-}\mathsf{SMS}\, \mathcal{C}_{-d}(kQ)$
A_n	$n+1$	$1, 2, \dots, n$	$\frac{1}{n+1}\binom{(d+1)n+d-1}{n}$
D_n	$2(n-1)$	$1, 3, \dots, 2n-3, n-1$	$\frac{(2d+1)n-2d-2}{n}\binom{(n-1)(d+1)-1}{n-1}$
E_6	12	$1, 4, 5, 7, 8, 11$	$\frac{d(2d+1)(3d+1)(4d+1)(6d+5)(12d+7)}{30}$
E_7	18	$1, 5, 7, 9, 11, 13, 17$	$\frac{d(3d+1)(3d+2)(9d+2)(9d+4)(9d+5)(9d+8)}{280}$
E_8	30	$1, 7, 11, 13, 17, 19, 23, 29$	$\frac{d(3d+1)(5d+1)(5d+2)(5d+3)(15d+8)(15d+11)(15d+14)}{1344}$

FIGURE 1. Positive Fuss-Catalan numbers

simplices (resp. maximal simplices without negative simple roots) in $\Delta^d(\Phi)$. We will prove the following result.

Theorem 3.1.1. *Let Q be a Dynkin quiver and let $d \geq 1$.*

(1) *There is a bijection*

$$d\text{-}\mathsf{SMS}\, \mathcal{C}_{-d}(kQ) \overset{1:1}{\longleftrightarrow} \mathsf{max\text{-}sim}^+\, \Delta^d(\Phi).$$

(2) *We have $\#\, d\text{-}\mathsf{SMS}\, \mathcal{C}_{-d}(kQ) = C_d^+(W)$, where W is the Weyl group of kQ.*

The result (2) is known for the case $d = 1$ by [CS1] and for the case $Q = A_n$ by [Ji1]. Figure 1 gives us concrete formulas for type A, D and E.

To prove Theorem 3.1.1, we need to introduce some categorical notions. We have a standard t-structure $(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0})$ on $\mathsf{D}^{\mathrm{b}}(kQ)$, where $\mathsf{D}^{\leq 0} := \{X \in \mathsf{D}^{\mathrm{b}}(kQ) \mid \mathrm{H}^{>0}(X) = 0\}$ and $\mathsf{D}^{\geq 0} := \{X \in \mathsf{D}^{\mathrm{b}}(kQ) \mid \mathrm{H}^{<0}(X) = 0\}$. Notice that

$$\mathsf{D}^{\leq -1} \subset \nu \mathsf{D}^{\leq 0} \subset \mathsf{D}^{\leq 0} \text{ and } \mathsf{D}^{\geq 1} \subset \nu^{-1}\mathsf{D}^{\geq 0} \subset \mathsf{D}^{\geq 0}. \tag{3.1.1}$$

For $m \leq n$, we consider three subcategories

$$\mathsf{D}_-^{[m,n]} := \mathsf{D}^{\leq n} \cap \nu\mathsf{D}^{\geq m+1} \subset \mathsf{D}^{[m,n]} := \mathsf{D}^{\leq n} \cap \mathsf{D}^{\geq m} \subset \mathsf{D}_+^{[m,n]} := \mathsf{D}^{\leq n} \cap \nu^{-1}\mathsf{D}^{\geq m-1}.$$

We denote by $\mathsf{silt}\, \mathsf{D}^{\mathrm{b}}(kQ)$ the set of silting objects in $\mathsf{D}^{\mathrm{b}}(kQ)$ and by $\mathsf{SMC}\, \mathsf{D}^{\mathrm{b}}(kQ)$ the set of SMCs in $\mathsf{D}^{\mathrm{b}}(kQ)$ (see Definition 0.2.4). The following is a main result of Part 3.

Theorem 3.1.2. *Let Q be a Dynkin quiver and $d \geq 1$. Then there are bijections*

$$(\mathsf{silt}\, \mathcal{D}^{\mathrm{b}}(kQ)) \cap \mathcal{D}^{[1-d,0]} \overset{1:1}{\longleftrightarrow} (\mathsf{SMC}\, \mathsf{D}^{\mathrm{b}}(kQ)) \cap \mathsf{D}_-^{[-d,0]}, \tag{3.1.2}$$

$$\overset{1:1}{\longleftrightarrow} d\text{-}\mathsf{SMS}\, \mathcal{C}_{-d}(kQ), \tag{3.1.3}$$

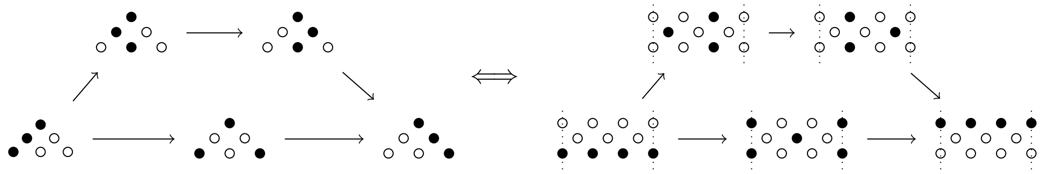
where the map (1.3) is induced by the natural functor $\mathcal{D}^{\mathrm{b}}(kQ) \xrightarrow{\pi} \mathcal{D}^{\mathrm{b}}(kQ)/\nu[d]$.

The bijection (1.2) holds for any finite-dimensional Iwanaga-Gorenstein algebras (see Corollary 3.1.10). The bijection (1.3) holds for acyclic quiver Q, see the recent paper [CSPP].

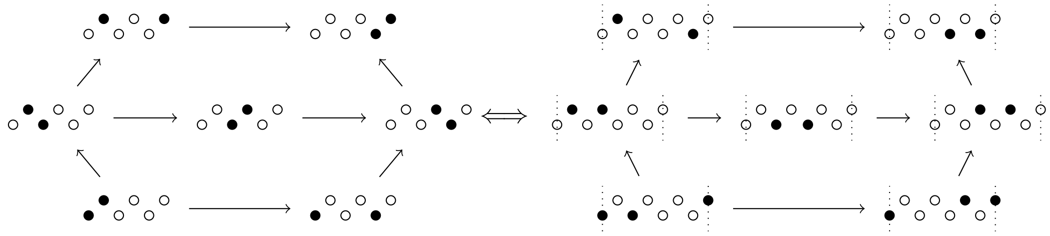
Our theorems and the results in [BRT1, Z] mentioned above are summarized as follows, where we denote by $\mathsf{ctilt}\, \mathcal{C}_{d+1}(kQ)$ the set of cluster tilting objects in $\mathcal{C}_{d+1}(kQ)$.

$$\begin{array}{ccccc}
d\text{-}\mathsf{SMS}\, \mathcal{C}_{-d}(kQ) & & \xleftrightarrow[\sim]{\text{Thm.3.1.2}} & & (\mathsf{SMC}\, \mathsf{D}^{\mathrm{b}}(kQ)) \cap \mathsf{D}_-^{[-d,0]} \\
\text{Thm.3.1.1} \updownarrow \wr & & & & \text{Thm.3.1.2} \updownarrow \wr \\
\mathsf{max\text{-}sim}^+\, \Delta^d(\Phi) & & \xleftrightarrow[\sim]{} & & (\mathsf{silt}\, \mathsf{D}^{\mathrm{b}}(kQ)) \cap \mathsf{D}^{[1-d,0]} \\
\big\downarrow & & & & \big\downarrow \\
\mathsf{max\text{-}sim}\, \Delta^d(\Phi) & \xleftrightarrow[\sim]{[\mathrm{Z}]} & \mathsf{ctilt}\, \mathcal{C}_{d+1}(kQ) & \xleftrightarrow[\sim]{[\mathrm{BRT1}]} & (\mathsf{silt}\, \mathsf{D}^{\mathrm{b}}(kQ)) \cap \mathsf{D}_+^{[1-d,0]}.
\end{array}$$

Example 3.1.3. (1) Let $Q = A_3$ and $d = 1$, then the bijection between silting objects of $\mathsf{D}^{\mathrm{b}}(kA_3)$ contained in $\mathsf{mod}\, kA_3$ and (-1)-SMS of $\mathcal{C}_{-1}(kA_3)$ is as follows.



(2) Let $Q = A_2$ and $d = 2$. Then the bijection is as follows.



3.1.2. **Silting-t-structure correspondence.** The bijection (1.2) above is similar to the bijection between silting objects and d-$\mathrm{Hom}_{\leq 0}$-configurations in [BRT2]. Our approach here is more direct and based on silting-t-structure correspondence. Let us recall it first.

Theorem 3.1.4. [KaY1] *Let A be a finite-dimensional k-algebra. Then there are bijections*

$$\mathsf{silt}\,\mathsf{per}\,A \overset{1:1}{\longleftrightarrow} \{\textit{bounded } t\textit{-structures of } \mathsf{D}^{\mathrm{b}}(A) \textit{ with length hearts}\} \overset{1:1}{\longleftrightarrow} \mathsf{SMC}\,\mathsf{D}^{\mathrm{b}}(A).$$

In this subsection, we give two refined versions of silting-t-structure correspondence in triangulated categories, both of which imply the bijection (1.2) above. Our common assumption is the following, which is satisfied for $\mathcal{T} = \mathsf{D}^{\mathrm{b}}(A)$ and the perfect derived category $\mathcal{U} = \mathsf{per}\,A$ for a finite-dimensional k-algebra A.

Assumption 3.1.5. Let $\mathcal{T}$ be a triangulated category and $\mathcal{U}$ a thick subcategories of $\mathcal{T}$. Assume that for any $P \in \mathsf{silt}\,\mathcal{U}$, we have a bounded t-structure

$$\mathcal{T} = \mathcal{T}_P^{\leq 0} \perp \mathcal{T}_P^{>0}, \text{ where } \mathcal{T}_P^{\leq 0} := P[<0]^{\perp} \text{ and } \mathcal{T}_P^{>0} := P[\geq 0]^{\perp}. \tag{3.1.4}$$

See Section 0.1 for the definition of $(\,)^{\perp}$.

We call (3.1.4) the *silting t-structure* associated with P, and call its heart $\mathcal{H}_P := \mathcal{T}_P^{\leq 0} \cap \mathcal{T}_P^{\geq 0}$ the *silting heart*. Then P can be recovered from the subcategory $\mathcal{H}_P$ (see Lemma 3.2.1). Denote by $\mathsf{silt\text{-}heart}\,\mathcal{T}$ the set of silting hearts of $\mathcal{T}$. Notice that $\mathsf{silt}\,\mathcal{U}$ and $\mathsf{silt\text{-}heart}\,\mathcal{T}$ have canonical partial orders (see Section 0.2).

Theorem 3.1.6. *Under Assumption 3.1.5, let $\mathcal{T} = \mathcal{X} \perp \mathcal{X}^{\perp} = \mathcal{Y} \perp \mathcal{Y}^{\perp}$ be two silting t-structures. Then there is a poset isomorphism*

$$\{P \in \mathsf{silt}\,\mathcal{U} \mid P \in \mathcal{X} \cap {}^{\perp}\mathcal{Y}\} \overset{\cong}{\longleftrightarrow} \{\mathcal{H} \in \mathsf{silt\text{-}heart}\,\mathcal{T} \mid \mathcal{H} \subset \mathcal{X} \cap \mathcal{Y}^{\perp}\}.$$

Let us see an example, which is well-known for the case $d = 2$ [BY].

Corollary 3.1.7. *Let A be a finite-dimensional k-algebra and $d \geq 1$. Assume that $P \in \mathsf{silt}\,\mathsf{per}\,A$ is d-*term silting*, that is,* $\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(A)}(A[<0], P) = 0 = \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(A)}(P, A[\geq d])$. *We have a poset isomorphism.*

$$\{d\textit{-term silting objects in } \mathsf{per}\,A\} \overset{\cong}{\longleftrightarrow} (\mathsf{SMC}\,\mathsf{D}^{\mathrm{b}}(A)) \cap \mathsf{D}^{[1-d,0]}.$$

When $\mathcal{T}$ has relative Serre functor in the following sense, we can improve Theorem 3.1.6 by dropping the assumption that two t-structures are silting.

Assumption 3.1.8. Keep Assumption 3.1.5. Assume that $\mathcal{T}$ is a k-linear triangulated category and we have a *relative Serre functor* ν. That is, there is an auto-equivalence $\nu : \mathcal{T} \simeq \mathcal{T}$ which restricts to an auto-equivalence $\nu : \mathcal{U} \simeq \mathcal{U}$, such that there exists a functorial isomorphism

$$D\mathrm{Hom}_{\mathcal{T}}(X,Y) \cong \mathrm{Hom}_{\mathcal{T}}(Y,\nu X),$$

for any $X \in \mathcal{U}$ and $Y \in \mathcal{T}$, where D is the k dual.

Then there is a poset isomorphism as follows.

Theorem 3.1.9. *Under Assumption 3.1.8, let $\mathcal{T} = \mathcal{X} \perp \mathcal{X}^{\perp} = {}^{\perp}\mathcal{Z} \perp \mathcal{Z}$ be any two t-structures. There is a poset isomorphism*

$$\{P \in \mathsf{silt}\,\mathcal{U} \mid P \in \mathcal{X} \cap \mathcal{Z}\} \overset{\cong}{\longleftrightarrow} \{\mathcal{H} \in \mathsf{silt\text{-}heart}\,\mathcal{T} \mid \mathcal{H} \subset \mathcal{X} \cap \nu\mathcal{Z}\}.$$

The following result plays a key role to prove Theorem 3.1.2.

Corollary 3.1.10. *Let A be a finite-dimensional k-algebra and $d \geq 1$. If A is Iwanaga-Gorenstein (that is, the A-module A has finite injective dimension both sides), then there is a poset isomorphism*

$$(\mathsf{silt}\,\mathsf{per}\,A) \cap \mathsf{D}^{[1-d,0]} \overset{\cong}{\longleftrightarrow} (\mathsf{SMC}\,\mathsf{D}^{\mathrm{b}}(A)) \cap \mathsf{D}_{-}^{[-d,0]}.$$

3.2. Proof of main Theorems

3.2.1. **Silting-t-structure correspondence.** We first show Theorem 3.1.6. The following observation is useful.

Lemma 3.2.1. *Under Assumption 3.1.5, there is a poset isomorphism* $\mathsf{silt}\,\mathcal{U} \overset{\cong}{\longleftrightarrow} \mathsf{silt\text{-}heart}\,\mathcal{T}$.

Proof. The map $\mathsf{silt}\,\mathcal{U} \to \mathsf{silt\text{-}heart}\,\mathcal{T}$ is clearly surjective. For $P, Q \in \mathsf{silt}\,\mathcal{U}$, we have

$$Q \geq P \overset{(0.2.3)}{\iff} P[\geq 0] \in Q[<0]^{\perp} \overset{(3.1.4)}{\iff} P[\geq 0] \in \mathcal{T}_Q^{\leq 0} \overset{(3.1.4)}{\iff} \mathcal{T}_Q^{>0} \supset \mathcal{T}_P^{>0} \overset{(0.2.1)}{\iff} \mathcal{H}_Q \geq \mathcal{H}_P.$$

Thus the map is a poset isomorphism. □

Proposition 3.2.2. *Under Assumption 3.1.5, let $Q, R \in \mathsf{silt}\,\mathcal{U}$. Then there is a poset isomorphism*

$$(\mathsf{silt}\,\mathcal{U}) \cap \mathcal{U}^{Q}_{\leq 0} \cap \mathcal{U}^{R}_{\geq 0} \overset{\cong}{\longleftrightarrow} \{\mathcal{H} \in \mathsf{silt\text{-}heart}\,\mathcal{T} \mid \mathcal{H} \subset \mathcal{T}_Q^{\leq 0} \cap \mathcal{T}_R^{\geq 0}\}.$$

Proof. Let $P \in \mathsf{silt}\,\mathcal{U}$. Then

$$P \in \mathcal{U}^{Q}_{\leq 0} \cap \mathcal{U}^{R}_{\geq 0} \overset{(0.2.3)}{\iff} Q \geq P \geq R \overset{\text{Lem. 3.2.1}}{\iff} \mathcal{H}_Q \geq \mathcal{H}_P \geq \mathcal{H}_R \overset{(0.2.1)}{\iff} \mathcal{H}_P \subset \mathcal{T}_Q^{\leq 0} \cap \mathcal{T}_R^{\geq 0}.$$

Thus the assertion holds. □

Now we are ready to prove Theorem 3.1.6.

Proof of Theorem 3.1.6. There exists $Q, R \in \mathsf{silt}\,\mathcal{U}$ such that $\mathcal{X} = Q[<0]^{\perp}$ and $\mathcal{Y} = R[\leq 0]^{\perp}$. Since $\mathcal{X} \cap \mathcal{Y}^{\perp} = \mathcal{T}_Q^{\leq 0} \cap \mathcal{T}_R^{\geq 0}$ and $\mathcal{X} \cap {}^{\perp}\mathcal{Y} \cap \mathcal{U} = \mathcal{U}^{Q}_{\leq 0} \cap \mathcal{U}^{R}_{\geq 0}$ hold, the assertion follows from Proposition 3.2.2. □

Proof of Corollary 3.1.7. Let $\mathcal{T} = \mathsf{D}^{\mathrm{b}}(A)$ and $\mathcal{U} = \mathsf{per}\,A$. Let $\mathcal{X} = \mathsf{D}^{\leq 0}$ and $\mathcal{Y} = \mathsf{D}^{\leq -d}$. Then $\mathcal{Y}^{\perp} = \mathsf{D}^{\geq 1-d}$ and $\mathsf{per}\,A \cap {}^{\perp}\mathcal{Y} = \mathsf{Filt}(A[<d])$. By Theorem 3.1.6, we have a poset isomorphism $\{P \in \mathsf{per}\,A \mid P \in \mathsf{D}^{\leq 0} \cap \mathsf{Filt}A[<d]\} \overset{\cong}{\longleftrightarrow} \{\mathcal{H} \in \mathsf{silt\text{-}heart}\,\mathsf{D}^{\mathrm{b}}(A) \mid \mathcal{H} \subset \mathsf{D}^{[1-d,0]}\}$. Using the bijection $\mathsf{silt\text{-}heart}\,\mathsf{D}^{\mathrm{b}}(A) \overset{\cong}{\longleftrightarrow} \mathsf{SMC}\,\mathsf{D}^{\mathrm{b}}(A)$ in Theorem 3.1.4, we obtain the assertion. □

Next we prove Theorem 3.1.9.

Proof of Theorem 3.1.9. Let $P \in \mathsf{silt}\,\mathcal{U}$. Thanks to Lemma 3.2.1, it suffices to show that $P \in \mathcal{X}$ if and only if $\mathcal{H}_P \subset \mathcal{X}$, and $P \in \mathcal{Z}$ if and only if $\mathcal{H}_P \subset \nu\mathcal{Z}$.
(a) By (3.1.4) and (0.2.2), we have ${}^{\perp}(\mathcal{U}^P_{\leq 0}{}^{\perp}) = {}^{\perp}\mathcal{T}_P^{>0} = \mathcal{T}_P^{\leq 0}$. Thus

$$P \in \mathcal{X} \Longleftrightarrow \mathcal{U}^P_{\leq 0} \subset \mathcal{X} \overset{{}^{\perp}(\mathcal{X}^{\perp})=\mathcal{X}}{\Longleftrightarrow} {}^{\perp}(\mathcal{U}^P_{\leq 0}{}^{\perp}) \subset \mathcal{X} \Longleftrightarrow \mathcal{T}_P^{\leq 0} \subset \mathcal{X} \Longleftrightarrow \mathcal{H}_P \subset \mathcal{X}.$$

(b) By (3.1.4) and (0.2.2), we have $({}^{\perp}\nu\mathcal{U}^P_{\geq 0})^{\perp} = (\mathcal{U}^P_{\geq 0}{}^{\perp})^{\perp} = \mathcal{T}_P^{<0\perp} = \mathcal{T}_P^{\geq 0}$. Thus

$$P \in \mathcal{Z} \Longleftrightarrow \mathcal{U}^P_{\geq 0} \subset \mathcal{Z} \Longleftrightarrow \nu\mathcal{U}^P_{\geq 0} \subset \nu\mathcal{Z} \overset{({}^{\perp}\nu\mathcal{Z})^{\perp}=\nu\mathcal{Z}}{\Longleftrightarrow} ({}^{\perp}\nu\mathcal{U}^P_{\geq 0})^{\perp} \subset \nu\mathcal{Z} \Longleftrightarrow \mathcal{T}_P^{\geq 0} \subset \nu\mathcal{Z} \Longleftrightarrow \mathcal{H}_P \subset \nu\mathcal{Z}.$$

So the assertion is true. □

Proof of Corollary 3.1.10. Let $\mathcal{T} = \mathsf{D}^{\mathrm{b}}(A)$ and $\mathcal{U} = \mathsf{per}\,A$. Let $\mathcal{X} = \mathsf{D}^{\leq 0}$ and $\mathcal{Z} = \mathsf{D}^{\geq 1-d}$. Since A is Iwanaga-Gorenstein, then the Nakayama functor ν is the relative Serre functor. By Theorem 3.1.9, we have a poset isomorphism

$$(\mathsf{silt}\,\mathsf{per}\,A) \cap \mathsf{D}^{[1-d,0]} \overset{\cong}{\longleftrightarrow} \{\mathcal{H} \in \mathsf{silt\text{-}heart}\,\mathsf{D}^{\mathrm{b}}(A) \mid \mathcal{H} \subset \mathsf{D}_{-}^{[-d,0]}\}.$$

By Theorem 3.1.4, we obtain the assertion. □

3.2.2. **Proof of Theorems 3.1.1 and 3.1.2.** In this subsection, we will write $A = kQ$, D (resp. $\mathcal{C}$) for $\mathsf{D}^{\mathrm{b}}(A)$ (resp. $\mathcal{C}_{-d}(A)$) for simplicity, and $\mathcal{H} = \mathsf{mod}\,A$. Let S be an SMC of D. Then $\mathcal{H}_S := \mathsf{Filt}S$ is the heart of a t-structure $(\mathsf{D}_S^{\leq 0}, \mathsf{D}_S^{\geq 0})$ given by

$$\mathsf{D}_S^{\leq 0} := \mathsf{Filt}(S[\geq 0]) \text{ and } \mathsf{D}_S^{\geq 0} := \mathsf{Filt}(S[\leq 0]).$$

We need the following observation.

Lemma 3.2.3. *Let S, T be two SMCs of D. Then the following are equivalent.*
(1) $\mathcal{H}_S \subset \mathsf{D}_T^{\leq 0} \cap \nu\mathsf{D}_T^{\geq 1-d}$;
(2) $\mathcal{H}_T \subset \nu^{-1}\mathsf{D}_S^{\leq d-1} \cap \mathsf{D}_S^{\geq 0}$;
(3) $\mathsf{D}_S^{\leq 0} \subset \mathsf{D}_T^{\leq 0}$ *and* $\mathsf{D}_S^{\geq 0} \subset \nu\mathsf{D}_T^{\geq 1-d}$;
(4) $\mathsf{D}_T^{\leq 0} \subset \nu^{-1}\mathsf{D}_S^{\leq d-1}$ *and* $\mathsf{D}_T^{\geq 0} \subset \mathsf{D}_S^{\geq 0}$.

Proof. One can check (1) ⇔ (3) ⇔ (4) ⇔ (2) easily. □

Recall that $\pi : \mathsf{D} \to \mathcal{C}$ is the natural functor. Then π gives a bijection

$$\mathsf{ind}\,\mathsf{D}_{-}^{[-d,0]} \xrightarrow{\simeq} \mathsf{ind}\,\mathcal{C}. \tag{3.2.1}$$

In the rest, we write $\pi(X)$ as X for any $X \in \mathsf{D}$. We give a lemma which plays an important role in the sequel.

Lemma 3.2.4. *Let $X, Y \in \mathsf{D}_{-}^{[-d,0]}$ and $0 \leq i \leq d$. Then we have*

$$\mathrm{Hom}_{\mathcal{C}}(X, Y[-i]) = \mathrm{Hom}_{\mathsf{D}}(X, Y[-i]) \oplus D\mathrm{Hom}_{\mathsf{D}}(Y, X[i-d]).$$

Proof. By the definition of $\mathcal{C}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(X, Y[-i]) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{D}}(X, \nu^n Y[nd-i]).$$

If $n < 0$, then $nd - i \leq -d$ and $\nu^n Y[nd-i] \in \nu^{n+1}\mathsf{D}^{\geq 1} \subset \mathsf{D}^{\geq 1}$ by (3.1.1). So $\mathrm{Hom}_{\mathsf{D}}(X, \nu^n Y[nd - i]) = 0$. If $n > 1$, then $m := 1 - n < 0$ and

$$\mathrm{Hom}_{\mathsf{D}}(X, \nu^n Y[nd-i]) = D\mathrm{Hom}_{\mathsf{D}}(\nu^{n-1} Y[nd-i], X) = D\mathrm{Hom}_{\mathsf{D}}(Y, \nu^m X[md-(d-i)]) = 0,$$

by the first case. Thus the assertion follows. □

For $m \geq n$, we denote by $\mathsf{D}_S^{[m,n]}$ the intersection $\mathsf{D}_S^{\leq n} \cap \mathsf{D}_S^{\geq m}$. The following lemma is useful.

Lemma 3.2.5. *Let Q be an acyclic quiver, $S \subset \mathsf{SMC}\,\mathsf{D} \cap \mathsf{D}^{\leq 0} \cap \nu\mathsf{D}^{\geq 1-d}$ and $N \in \mathsf{D}^{\leq 0} \cap \nu\mathsf{D}^{\geq 1-d}$. For $1 \leq i \leq d$, if $\mathrm{Hom}_{\mathsf{D}}(S[i], N) = 0 = \mathrm{Hom}_{\mathsf{D}}(N, S[i-d])$, then $N = 0$.*

Proof. By (3.1.1) and Lemma 3.2.3, we have

$$\mathcal{H} = \mathsf{mod}\, A \subset \nu\mathsf{D}^{\leq 1} \cap \mathsf{D}^{\geq 0} \subset \mathsf{D}_S^{\leq d} \cap \mathsf{D}_S^{\geq 0}. \tag{3.2.2}$$

Since $DA \in \nu\mathsf{D}^{\leq 0}$ and $S[-d] \in \nu\mathsf{D}^{\geq 1}$, then $\mathrm{Hom}_{\mathsf{D}}(DA, S[-d]) = 0$ and $DA \in \mathsf{D}_S^{[0,d-1]}$ by Lemma 3.2.3 and (3.2.2). Thus $\mathrm{Hom}_{\mathsf{D}}(N, DA) = 0$ (that is $\mathrm{H}^0(N) = 0$) and moreover, $N \in \mathsf{D}^{\leq -1} \cap \nu\mathsf{D}^{\geq 1-d} \subset \mathsf{D}^{[-d,-1]} \subset \mathsf{D}_S^{[-d,d-1]}$ by (3.2.2). Thus $N = 0$. □

We denote by $(-d)$- $\mathsf{CY}\text{-}\mathsf{conf}\,\mathcal{C}$ the set of $(-d)$-Calabi-Yau configurations of $\mathcal{C}$.

Proposition 3.2.6. *Let Q be an acyclic quiver. Then the map*

$$(\mathsf{SMC}\,\mathsf{D}) \cap \mathsf{D}_-^{[-d,0]} \xrightarrow{\pi} (-d)\text{-}\,\mathsf{CY}\text{-}\mathsf{conf}\,\mathcal{C}. \tag{3.2.3}$$

is well-defined.

Proof. Let S be an SMC contained in $\mathsf{D}^{[-d,0]}$. We show S is a $(-d)$-CY configuration in $\mathcal{C}$. Let $X, Y \in S$ and $0 \leq i < d$. By Lemma 3.2.4, we have

$$\mathrm{Hom}_{\mathcal{C}}(X, Y[-i]) = \mathrm{Hom}_{\mathsf{D}}(X, Y[-i]) \oplus D\mathrm{Hom}_{\mathsf{D}}(Y, X[i-d]) = \mathrm{Hom}_{\mathsf{D}}(X, Y[-i]).$$

Immediately S satisfies the conditions (1) and (2) in Definition 2.5.1.

It remains to check that $\bigcap_{j=0}^{d-1} {}^{\perp}S[-j] = 0$. Let $M \in \mathcal{C}$ be an indecomposable object satisfying $\mathrm{Hom}_{\mathcal{C}}(M, S[-j]) = 0$ for any $0 \leq j \leq d-1$. By (3.2.1), there exists an indecomposable object $N \in \mathsf{D}_-^{[-d,0]}$, such that $\pi(N) = M$. By Lemma 3.2.4, we have $\mathrm{Hom}_{\mathsf{D}}(N, S[i-d]) = 0$ and $\mathrm{Hom}_{\mathsf{D}}(S[i], N) = 0$ for any $1 \leq i \leq d$. Then $N = 0$ by Lemma 3.2.5. □

We are ready to prove Theorem 3.1.2 now.

Proof of Theorem 3.1.2. The bijection (1.2) follows directly from Corollary 3.1.10. The map (1.3) is well-defined by Proposition 3.2.6. Since this is injective by (3.2.1), it suffices to show that (1.3) is surjective.

Let S be any d-SMS of $\mathcal{C}$. We also denote by S the preimage $\pi^{-1}(S)$ of S via the bijection (3.2.1). We claim S is an SMC of D. Let $X, Y \in S$. We know

$$\dim_k \mathrm{Hom}_{\mathcal{C}}(X, Y[-i]) \geq \dim_k \mathrm{Hom}_{\mathsf{D}}(X, Y[-i]),$$

for any $i \in \mathbb{Z}$. For $0 \leq i \leq d-1$, the left hand side is $\delta_{XY}\delta_{0i}$ for the Kronecker delta, so is the right hand side. We show $\mathrm{Hom}_{\mathsf{D}}(X, Y[-d]) = 0$. This is clear if $X = Y$. If $X \neq Y$, then

$$0 = \mathrm{Hom}_{\mathcal{C}}(Y, X) = \mathrm{Hom}_{\mathsf{D}}(Y, X) \oplus D\mathrm{Hom}_{\mathsf{D}}(X, Y[-d]),$$

and hence $\mathrm{Hom}_{\mathsf{D}}(X, Y[-d]) = 0$. For $i > d$, we have

$$Y[-i] \in \nu\mathsf{D}^{\geq 1-d+i} \subset \nu\mathsf{D}^{>1} \subset \mathsf{D}^{\geq 1}$$

by (3.1.1). Thus $\mathrm{Hom}_{\mathsf{D}}(X, Y[<-d]) = 0$.

It remains to show $\mathsf{D} = \mathsf{thick}\, S$. Since D is locally finite, $\mathcal{S} = \mathsf{thick}\, S$ is functorially finite in D. Thus we have a torsion pair $\mathsf{D} = {}^{\perp}\mathcal{S} \perp \mathcal{S}$ by [IYo, Proposition 2.3]. Thus it suffices to show ${}^{\perp}\mathcal{S} = 0$. Let $X \in {}^{\perp}\mathcal{S}$ be an indecomposable object. Since kQ is hereditary, $\mathsf{D} = \mathsf{add}(\mathcal{H}[i] \mid i \in \mathbb{Z})$. We may assume $X \in \mathcal{H}$. Then $\mathrm{Hom}_{\mathsf{D}}(X, S[-i]) = 0$ for all $i \in \mathbb{Z}$. Moreover, for any $0 \leq i < d$, we have $X[i-d] \in \mathsf{D}^{\geq 1}$, and hence $\mathrm{Hom}_{\mathsf{D}}(S, X[i-d]) = 0$. Since $X, S \in \mathsf{D}_-^{[-d,0]}$, we have $\mathrm{Hom}_{\mathcal{C}}(X, S[-i]) = 0$ by Lemma 3.2.4. Since S is a d-SMS, $X = 0$. Thus ${}^{\perp}\mathcal{S} = 0$ as desired. □

Theorem 3.1.1 is clear now.

Proof of Theorem 3.1.1. (1) follows form Theorem 3.1.2, [BRT1, Proposition 2.4] and [Z, Theorem 5.7]. (2) follows from (1) and [FR, Proposition 12.4]. □

References

[ABu] Maurice Auslander, Ragnar-Olaf Buchweitz, *The homological theory of maximal Cohen-Macaulay approximations*, Colloque en l'honneur de Pierre Samuel (Orsay, 1987). Mém. Soc. Math. France (N.S.) No. 38 (1989), 5–37.

[ABr] Maurice Auslander, Mark Bridger, *Stable module theory*, Memoirs of the American Mathematical Society, No. 94 American Mathematical Society, Providence, R.I. 1969.

[AI] Takuma Aihara, Osamu Iyama, *Silting mutation in triangulated categories*, J. Lond. Math. Soc. (2) 85 (2012), no. 3, 633–668.

[AKLY] Lidia Angeleri Hügel, Steffen Koenig, Qunhua Liu, Dong Yang, *Ladders and simplicity of derived module categories*, J. Algebra 472 (2017), 15–66.

[Al] Salah Al-Nofayee, *Simple objects in the heart of a t-structure*, J. Pure Appl. Algebra 213 (2009), no. 1, 54–59.

[Am] Claire Amiot, *Cluster categories for algebras of global dimension 2 and quivers with potential*, Annales de l'institut Fourier. 2009, 59(6): 2525–2590.

[AMY] Takahide Adachi, Yuya Mizuno, Dong Yang, *Discreteness of silting objects and t-structures in triangulated categories*, Proc. Lond. Math. Soc. (3) 118 (2019), no. 1, 1–42.

[An] Marcus Andrei, *Tilting complexes for group graded algebras*, J. Group Theory 6 (2003), no. 2, 175–193.

[Ar] Drew Armstrong, *Generalized noncrossing partitions and combinatorics of Coxeter groups*, Mem. Amer. Math. Soc. 202 (2009), no. 949.

[ARS] Maurice Auslander, Idun Reiten, Sverre O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, 36. Cambridge University Press, Cambridge, 1995. xiv+423 pp. ISBN: 0-521-41134-3.

[B] Ragnar-Olaf Buchweitz, *Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings*, unpublished manuscript.

[BBD] Aleksandr A. Beilinson, Joseph N. Bernstein, Pierre Deligne, *Faisceaux pervers* (French) [Perverse sheaves] Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque, 100, Soc. Math. France, Paris, 1982.

[BGS] Alexander Beilinson, Victor Ginzburg, Wolfgang Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. 9 (1996), no. 2, 473–527.

[BK] Alexey Bondal, Mikhail Kapranov, *Enhanced triangulated categories*, Mat. Sb. **181** (1990), no. 5, 669–683; translation in Math. USSR-Sb. **70** (1991), no.1, 93–107.

[BH] Winfried Bruns, Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, 39. Cambridge University Press, Cambridge, 1993.

[BMRRT] Aslak Bakke Buan, Robert Marsh, Markus Reineke, Idun Reiten, Gordana Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. 204 (2006), no. 2, 572–618.

[Bo] Mikhail V. Bondarko, *Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general)*, J. K-Theory 6 (2010), no. 3, 387–504.

[BRT1] Aslak Bakke Buan, Idun Reiten, Hugh Thomas, *Three kinds of mutation*, J. Algebra 339 (2011), 97–113.

[BRT2] Aslak Bakke Buan, Idun Reiten, Hugh Thomas, *From m-clusters to m-noncrossing partitions via exceptional sequences*, Math. Z. 271 (2012), no. 3-4, 1117–1139.

[BY] Thomas Brüstle, Dong Yang, *Ordered exchange graphs*, Advances in representation theory of algebras, 135–193, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2013.

[C1] Xiao-Wu Chen, *On extensions of covariantly finite subcategories*, arXiv:0809.3387.

[C2] Xiao-Wu Chen, *Singularity categories, Schur functors and triangular matrix rings*, Algebr. Represent. Theory 12 (2009), no. 2-5, 181–191.

[C3] Xiao-Wu Chen, *Gorenstein Homological Algebra of Artin Algebras*, arXiv:1712.04587.

[CR] Charles W. Curtis, Irving Reiner, *Methods of representation theory. Vol. I. With applications to finite groups and orders*, Pure and Applied Mathematics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1981.

[CS1] Raquel Coelho Simões, *Hom-configurations and noncrossing partitions*, J. Algebraic Combin. 35 (2012), no. 2, 313–343.

[CS2] Raquel Coelho Simões, *Hom-configurations in triangulated categories generated by spherical objects*, J. Pure Appl. Algebra 219 (2015), no. 8, 3322–3336.

[CS3] Raquel Coelho Simões, *Mutations of simple-minded systems in Calabi-Yau categories generated by a spherical object*, Forum Math. 29 (2017), no. 5, 1065–1081.

[CSP] Raquel Coelho Simões, David Pauksztello, *Simple-minded systems and reduction for negative Calabi-Yau triangulated categories*, Trans. Amer. Math. Soc. 373 (2020), no. 4, 2463–2498.

[CSPP] Raquel Coelho Simões, David Pauksztello, David Ploog, *Functorially finite hearts, simple-minded systems in negative cluster categories, and noncrossing partitions*, arXiv:2004.00604.

[D] Alex Dugas, *Torsion pairs and simple-minded systems in triangulated categories*, Appl. Categ. Structures 23 (2015), no. 3, 507–526.

[EJ] Edgar E. Enochs, Overtoun M. G. Jenda, *Relative homological algebra*, Volume 1. Second revised and extended edition. De Gruyter Expositions in Mathematics, 30. Walter de Gruyter GmbH & Co. KG, Berlin, 2011.

[FP] Elsa A. Fernández, María Inés Platzeck, *Isomorphic trivial extensions of finite dimensional algebras*, J. Pure Appl. Algebra 204 (2006), no. 1, 9–20.

[FR] Sergey Fomin, Nathan Reading, *Generalized cluster complexes and Coxeter combinatorics*, Int. Math. Res. Not. 2005, no. 44, 2709–2757.

[FZ1] Sergey Fomin, Andrei Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. 15 (2002), no. 2, 497–529.

[FZ2] Sergey Fomin, Andrei Zelevinsky, *Cluster algebras. II. Finite type classification*, Invent. Math. 154 (2003), no. 1, 63–121.

[Ga] Peter Gabriel, *Auslander-Reiten sequences and representation-finite algebras*, Representation theory, I (Proc. Workshop, Carleton Univ., Ottawa, Ont., 1979), pp. 1–71, Lecture Notes in Math., 831, Springer, Berlin, 1980.

[Gu] Lingyan Guo, *Cluster tilting objects in generalized higher cluster categories*, J. Pure Appl. Algebra 215 (2011), no. 9, 2055–2071.

[GR] Peter Gabriel, Christine Riedtmann, *Group representations without groups*, Comment. Math. Helv. 54 (1979), no. 2, 240–287.

[H1] Dieter Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, 119. Cambridge University Press, Cambridge, 1988.

[H2] Dieter Happel, *On Gorenstein algebras*, Representation theory of finite groups and finite-dimensional algebras (Bielefeld, 1991), 389–404, Progr. Math., 95, Birkhäuser, Basel, 1991.

[HJY] Thorsten Holm, Peter Jørgensen, Dong Yang, *Sparseness of t-structures and negative Calabi-Yau dimension in triangulated categories generated by a spherical object*, Bull. Lond. Math. Soc. 45 (2013), no. 1, 120–130.

[I1] Osamu Iyama, *τ-categories I: Ladders*, Algebr. Represent. Theory 8 (2005), no. 3, 297–321.

[I2] Osamu Iyama, *Tilting Cohen-Macaulay representations*, Proceedings of the International Congress of Mathematicians–Rio de Janeiro 2018. Vol. II. Invited lectures, 125–162, World Sci. Publ., Hackensack, NJ, 2018.

[INP] Osamu Iyama, Hiroyuki Nakaoka, Yann Palu, *Auslander–Reiten theory in extriangulated categories*, arXiv:1805.03776.

[IW] Osamu Iyama, Michael Wemyss, *Maximal modifications and Auslander-Reiten duality for non-isolated singularities*, Invent. Math. 197 (2014), no. 3, 521–586.

[IYa1] Osamu Iyama, Dong Yang, *Silting reduction and Calabi-Yau reduction of triangulated categories*, Trans. Amer. Math. Soc. 370 (2018), no. 11, 7861–7898.

[IYa2] Osamu Iyama, Dong Yang, *Quotients of triangulated categories and Equivalences of Buchweitz, Orlov and Amiot–Guo–Keller*, to appear in Amer. J. Math. also arXiv:1702.04475.

[IYo] Osamu Iyama, Yuji Yoshino, *Mutation in triangulated categories and rigid Cohen-Macaulay modules*, Invent. Math. 172 (2008), no. 1, 117–168.

[Ji1] Haibo Jin, *Cohen-Macaulay differential graded modules and negative Calabi-Yau configurations*, arXiv:1812.03737.

[Ji2] Haibo Jin, *Reductions of triangulated categories and simple-minded collections*, arXiv:1907.05114.

[Jo1] Peter Jørgensen, *Auslander-Reiten theory over topological spaces*, Comment. Math. Helv. 79 (2004), no. 1, 160–182.

[Jo2] Peter Jørgensen, *The Auslander-Reiten quiver of a Poincaré duality space*, Algebr. Represent. Theory 9 (2006), no. 4, 323–336.

[K1] Bernhard Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) 27 (1994), no. 1, 63–102.

[K2] Bernhard Keller, *On triangulated orbit categories*. Doc. Math. 10 (2005), 551–581.

[K3] Bernhard Keller, *On differential graded categories*, International Congress of Mathematicians. Vol. II, 151–190, Eur. Math. Soc., Zürich, 2006.

[K4] Bernhard Keller, *Corrections to 'On triangulated orbit categories'*, https://webusers.imj-prg.fr/~bernhard.keller/publ/corrTriaOrbit.pdf.

[KaY1] Martin Kalck, Dong Yang, *Relative singularity categories I: Auslander resolutions*, Adv. Math. 301 (2016), 973–1021.

[KaY2] Martin Kalck, Dong Yang, *Derived categories of graded gentle one-cycle algebras*, J. Pure Appl. Algebra 222 (2018), no. 10, 3005–3035.

[KL] Steffen Koenig, Yuming Liu, *Simple-minded systems in stable module categories*, Q. J. Math. 63 (2012), no. 3, 653–674.

[KMV] Bernhard Keller, Daniel Murfet, Michel Van den Bergh, *On two examples by Iyama and Yoshino*, Compos. Math. 147 (2011), no. 2, 591–612.

[KN] Bernhard Keller, Pedro Nicolás, *Weight structures and simple dg modules for positive dg algebras*, Int. Math. Res. Not. IMRN 2013, no. 5, 1028-1078.

[KoY] Steffen Koenig, Dong Yang, *Silting objects, simple-minded collections, t-structures and co-t-structures for finite-dimensional algebras*, Doc. Math. 19 (2014), 403–438.

[KR] Bernhard Keller, Idun Reiten, *Acyclic Calabi-Yau categories*, Compos. Math. 144 (2008), no. 5, 1332–1348.

[KYZ] Bernhard Keller, Dong Yang, Guodong Zhou, *The Hall algebra of a spherical object*, J. Lond. Math. Soc. (2) 80 (2009), no. 3, 771–784.

[L] Xueyu Luo, 0-*Calabi-Yau configurations and finite Auslander-Reiten quivers of Gorenstein orders*, J. Pure Appl. Algebra 219 (2015), no. 12, 5590–5630.

[LW] Graham J. Leuschke, Roger Wiegand, *Cohen-Macaulay representations*, Mathematical Surveys and Monographs, 181. American Mathematical Society, Providence, RI, 2012.

[Ma] Hideyuki Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, Cambridge,1989.

[Min] Hiroyuki Minamoto, *Resolutions of DG-modules and their applications to commutative DG-algebras*, arXiv:1802.01994.

[Miy] Jun-ichi Miyachi, *Localization of triangulated categories and derived categories*, J. Algebra 141 (1991), no. 2, 463–483.

[Ne] Amnon Neeman, *Triangulated categories*, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001.

[NP] Hiroyuki Nakaoka, Yann Palu, *Extriangulated categories, Hovey twin cotorsion pairs and model structures*, Cah. Topol. Géom. Différ. Catég. 60 (2019), no. 2, 117–193.

[O] Dmitri Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, Proc. Steklov Inst. Math. 2004, no. 3(246), 227–248.

[P] David Pauksztello, *Compact corigid objects in triangulated categories and co-t-structures*, Cent. Eur. J. Math. 6 (2008), no. 1, 25–42.

[PV] Chrysostomos Psaroudakis, Jorge Vitória, *Recollements of module categories* Appl. Categ. Structures 22 (2014), no. 4, 579–593.

[Ric1] Jeremy Rickard, *Derived categories and stable equivalence*, J. Pure Appl. Algebra 61 (1989), no. 3, 303–317.

[Ric2] Jeremy Rickard, *Equivalences of derived categories for symmetric algebras*, J. Algebra 257 (2002), no. 2, 460–481.

[Rie1] Christine Riedtmann, *Algebren, Darstellungsköcher, überlagerungen und zurück*, Comment. Math. Helv. 55 (1980), no. 2, 199–224.

[Rie2] Christine Riedtmann, *Representation-finite self-injective algebras of class* A_n, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), pp. 449–520, Lecture Notes in Math., 832, Springer, Berlin, 1980.

[Rie3] Christine Riedtmann, *Representation-finite self-injective algebras of class* D_n. Compositio Math. 49 (1983), no. 2, 231–282.

[Rie4] Christine Riedtmann, *Configurations of* $\mathbb{Z}D_n$, J. Algebra 82 (1983), no. 2, 309–327.

[RV] Idun Reiten, Michel Van den Bergh, *Noetherian hereditary abelian categories satisfying Serre duality*, J. Amer. Math. Soc. 15 (2002), no. 2, 295–366.

[Schm] Karsten Schmidt, *Families of Auslander-Reiten components for simply connected differential graded algebras*, Math. Z. 264 (2010), no. 1, 43–62.

[Schr] Sibylle Schroll, *Trivial extensions of gentle algebras and Brauer graph algebras*, J. Algebra 444 (2015), 183–200.

[Si] Daniel Simson, *Linear representations of partially ordered sets and vector space categories*, Algebra, Logic and Applications, 4. Gordon and Breach Science Publishers, Montreux, 1992.

[STW] Christian Stump, Hugh Thomas, Nathan Williams, *Cataland: Why the Fuss?*, arXiv:1503.00710.

[SY] Hao Su, Dong Yang, *From simple-minded collections to silting objects via Koszul duality*, Algebr. Represent. Theory 22 (2019), no. 1, 219–238.

[T] Bertrand Toën, *The homotopy theory of dg-categories and derived Morita theory.* Invent. Math. 167 (2007), no. 3, 615–667.

[W] Alfred Wiedemann, *Die Auslander-Reiten Köcher der gitterendlichen Gorensteinordnungen*, Bayreuth. Math. Schr. Volume 23, Pages 1–134.

[Ya] Kunio Yamagata, *On Artinian rings of finite representation type*, J. Algebra 50 (1978), no. 2, 276–283.

[Ye1] Amnon Yekutieli, *Derived Categories*, arXiv:1610.09640.

[Ye2] Amnon Yekutieli, *Duality and Tilting for Commutative DG Rings*, arXiv:1312.6411.

[Yo] Yuji Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, London Mathematical Society Lecture Note Series, 146, Cambridge University Press, Cambridge, 1990.

[Z] Bin, Zhu, *Generalized cluster complexes via quiver representations*, J. Algebraic Combin. 27 (2008), no. 1, 35–54.

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