Categorical Properties and Classifications of Several Subcategories of Module Categories

（加群圏の種々の部分圏の圏論的性質と分類について）

Haruhisa Enomoto

Student ID: 521801034

Graduate School of Mathematics
Nagoya University

Contents

Abstract

In this thesis, we study various subcategories of module categories in the representation theory of algebras. There are several classes of important subcategories of module categories which naturally arise in the representation theory and in connection with other areas of mathematics. Almost all of such subcategories are *extension-closed*, thus can be regarded as Quillen's *exact categories*. In this thesis, we study categorical properties of subcategories and give classifications of several kinds of subcategories, using exact categories.

This thesis consists of six chapters.

In Chapter 1, the introductory chapter of this thesis, we explain general ideas behind this thesis, including several subcategories and exact categories.

In Chapter 2, we consider the Grothendieck group of an exact category, and consider the criterion of the additive finiteness of an exact category via the Grothendieck group. This chapter is based on the author's published paper [Eno3].

In Chapter 3, we establish the theory of *simple objects* and the *Jordan-Hölder property* of exact categories. This chapter is based on the author's paper [Eno4].

In Chapter 4, motivated by the previous chapter, we give a classification of simple objects in torsion-free classes over *preprojective algebras* and *path algebras* of Dynkin type. This chapter is based on the author's paper [Eno5].

In Chapter 5, we introduce and investigate the notion of a *monobrick*, which enables us to study various subcategories in a uniform way by using simple objects. This chapter is based on the author's paper [Eno6].

In Chapter 6, we classify *ICE-closed subcategories* (subcategories closed under Images, Cokernels and Extensions) over path algebras by *rigid* modules. This chapter is based on the author's paper [Eno7].

Acknowledgement

First and foremost, I would like to thank my advisor Osamu Iyama, for his support and many helpful comments over the past six years. I would also like to thank Hiroyuki Nakaoka for valuable discussions.

I would like to express my gratitude to all the administrative staffs of Graduate School of Mathematics for the financial support and the resources.

In addition, I thank some of my colleagues and friends. I'm grateful to H. Inoue for providing me with a lot of funny and relaxing time over the past nine(!) years since I entered this university. I also thank Y. Gyoda and A. Sakai for valuable discussions on my research and pointing out some typos and mistakes. There are more colleagues who support me, but I omit it for want of space. Talking and discussion with them has certainly been stimulating and encouraging my interest in mathematics, and I owe it to them that I was able to study math (almost) continuously.

Last but not least, I thank my parents for their love and support. Also I thank Virtual Cast, Inc. This company provides a VR live communication service *Virtual Cast*. I gave some mathematical (and non-mathematical) talks in a VR space using Virtual cast, and uploaded recordings of them. Talking in a VR space largely has been encouraging my passion in mathematics and my life.

This work is supported by JSPS KAKENHI Grant Number JP18J21556.

CHAPTER 1

Introduction

Representation theory studies how an algebraic object acts on vector spaces. It dates back to Frobenius, who studied finite groups by using their representations. Since then, finite groups, Lie groups and Lie algebras have played leading roles in representation theory. It was not until around 1970 that the representation theory of *algebras*, the main topic of this thesis, had attracted an attention.

Let Λ be a unital associative ring. The purpose of the representation theory of Λ is to understand the structure of the category of Λ-modules. To get interesting results, it is common to impose some restriction on rings and modules we consider, such as noetherian and finitely generated. Then for a noetherian ring Λ, the category $\mathsf{mod}\,\Lambda$ of finitely generated right modules is a target of our investigation, which has a nice property such as being an abelian category.

1.1. Aim of this thesis

In the representation theory of Λ, we often consider full *subcategories* of $\mathsf{mod}\,\Lambda$, instead of $\mathsf{mod}\,\Lambda$ itself. For example, subcategories of $\mathsf{mod}\,\Lambda$ called *torsion classes*, *torsion-free classes* and *wide subcategories* of $\mathsf{mod}\,\Lambda$ have been main objects in the representation theory of finite dimensional algebra. As we shall see later, these subcategories have interesting relation to the algebraic Lie theory via the root system, and actually have applications in various areas of mathematics, such as Lusztig's crystal basis of quantum groups, Schubert varieties or unipotent cells in Kac-Moody groups, resolutions of Kleinian singularities, and so on.

There are two main themes of this thesis. Most of the past studies on these subcategories of $\mathsf{mod}\,\Lambda$ aimed at *classifying nice subcategories* (such as torsion classes) by some methods. However, *categorical structures of a given fixed subcategory* have not been investigated in the literature. This is the first theme of this thesis.

Along with the study of categorical structures, the author noticed that *there are more other kinds of subcategories to study which have not been considered before.* Most of the past studies concerned with torsion classes, torsion-free classes and wide subcategories of module categories. However, there are more subcategories having interesting properties. The second theme of this thesis is to introduce new kinds of subcategories and give classifications of them.

Let us give more details. Most of the subcategories $\mathcal{E}$ of $\mathsf{mod}\,\Lambda$ which naturally arise in the representation theory are *closed under extensions*: that is, for every short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

of Λ-modules, we have that $L, N \in \mathcal{E}$ implies $M \in \mathcal{E}$.

Every extension-closed subcategory of $\mathsf{mod}\,\Lambda$ has an additional structure called an *exact category* in the sense of Quillen. An exact category is a generalization of an abelian category. More precisely, it is an additive category *together with* a class of "short exact sequences" in that category, hence it has more information than its additive structure. We can endow every extension-closed subcategory $\mathcal{E}$ of $\mathsf{mod}\,\Lambda$ with the natural structure of an exact category ("short exact sequences" are just usual short exact sequences in $\mathsf{mod}\,\Lambda$ with all terms in $\mathcal{E}$). We can (and should) regard $\mathcal{E}$ as exact categories, not just as additive ones.

Using the exact structure, we can consider the several *invariants and properties of* $\mathcal{E}$, which we will investigate, and with which we will classify several subcategories, in this thesis:

- The Grothendieck group $\mathsf{K}_0(\mathcal{E})$ of $\mathcal{E}$.
- The Grothendieck monoid $\mathsf{M}(\mathcal{E})$ of $\mathcal{E}$.

- Simple objects in $\mathcal{E}$.
- The validity of the Jordan-Hölder type theorem in $\mathcal{E}$.
- Projective objects in $\mathcal{E}$.

We will explain these concepts in the next section.

1.2. Summary of results

In this section, we summarize main results in each chapter.

1.2.1. Chapter 2: Grothendieck group and additive finiteness. We say that an additive category $\mathcal{E}$ is *additively finite* if there are only finitely many indecomposable objects in $\mathcal{E}$ up to isomorphism. From the beginning of the representation theory of algebras, one of the important questions is: When is $\mathsf{mod}\,\Lambda$ additively finite? Such an algebra Λ is called a *representation-finite algebra*, and there are lots of studies on it.

As for this, the following characterization using the *Grothendieck group* was shown by Butler and Auslander. The Grothendieck group $\mathsf{K}_0(\mathsf{mod}\,\Lambda)$ of $\mathsf{mod}\,\Lambda$ is an abelian group generated by objects in $\mathsf{mod}\,\Lambda$ modulo relations $M = L + N$ whenever we have a short exact sequence $0 \to L \to M \to N \to 0$ in $\mathsf{mod}\,\Lambda$.

THEOREM 1.2.1 ([But, Aus3]). *Let Λ be a finite-dimensional algebra. Then $\mathsf{mod}\,\Lambda$ is additively finite if and only if the relation of the Grothendieck group $\mathsf{K}_0(\mathsf{mod}\,\Lambda)$ is generated by so-called* almost split sequences.

Here almost split sequences are special kinds of short exact sequences which are *minimal* in some sense.

Since we are interested in an extension-closed subcategory $\mathcal{E}$ of $\mathsf{mod}\,\Lambda$ instead of $\mathsf{mod}\,\Lambda$ itself, we want to study when $\mathcal{E}$ is additively finite. We can define the Grothendieck group $\mathsf{K}_0(\mathcal{E})$ of $\mathcal{E}$ in a similar way, and the main result in Chapter 2 is the following, which extends Butler-Auslander's result to a wide class of subcategories of $\mathsf{mod}\,\Lambda$.

THEOREM 1.2.2 (= Theorem A). *Let Λ be a finite-dimensional algebra and $\mathcal{E}$ an extension-closed subcategory of $\mathsf{mod}\,\Lambda$ which is covariantly finite and resolving (see Definitions 2.4.5 and 2.4.6). Then $\mathcal{E}$ is additively finite if and only if the relation of the Grothendieck group $\mathsf{K}_0(\mathcal{E})$ is generated by almost split sequences in $\mathcal{E}$.*

This extends and unifies several known results in the literature such as [MMP, PR].

1.2.2. Chapter 3: Characterization of the Jordan-Hölder property. As everyone majoring in algebra knows, every *composition series* of a given module with finite length is essentially unique up to a permutation of composition factors. This *Jordan-Hölder theorem* is very classical and fundamental result in algebra. Since a composition series of a module M can be regarded as a way to build up M from simple modules by iterated extensions, this theorem says that such ways are essentially unique.

As we have said, we are interested in an extension-closed subcategory $\mathcal{E}$ of a module category, instead of the whole module category. Since $\mathcal{E}$ has a structure of an exact category, we have a notion of short exact sequence, thus we can define a *simple object in $\mathcal{E}$* as follows:

DEFINITION 1.2.3. Let $\mathcal{E}$ be an exact category and M a non-zero object of $\mathcal{E}$. Then M is called *simple in $\mathcal{E}$* if there are no short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

in $\mathcal{E}$ such that $L, N \neq 0$.

Roughly speaking, simple objects in $\mathcal{E}$ are precisely objects which cannot be decomposed into smaller pieces with respect to short exact sequences in $\mathcal{E}$. As in the classical case, we can define composition series in $\mathcal{E}$. We say that $\mathcal{E}$ satisfies the *Jordan-Hölder property*, often abbreviated by *(JHP)*, if the precise analogue of the Jordan-Hölder theorem holds *in $\mathcal{E}$*. It turns out that lots of subcategories of $\mathsf{mod}\,\Lambda$ do not satisfy (JHP) for a finite-dimensional algebra Λ, and the main result in Chapter 3 is the following characterization of (JHP).

THEOREM 1.2.4 (= Theorems F, 3.5.10). *Let $\mathcal{E}$ be an exact category. Then the following are equivalent.*

(1) *$\mathcal{E}$ satisfies (JHP).*
(2) *The Grothendieck monoid $\mathsf{M}(\mathcal{E})$ of $\mathcal{E}$ is a free commutative monoid.*

If $\mathcal{E}$ is a subcategory of $\mathsf{mod}\,\Lambda$ for a finite-dimensional algebra Λ of the form ${}^{\perp}U$ for a module U with finite injective dimension (see Assumption 3.5.6), then the following is also equivalent:

(3) *The number of indecomposable projective objects in $\mathcal{E}$ is equal to the number of simple objects in $\mathcal{E}$.*

Here the *Grothendieck monoid* $\mathsf{M}(\mathcal{E})$ of an exact category $\mathcal{E}$ is a natural monoid version of the Grothendieck group. The assumption on $\mathcal{E}$ above are satisfied for various classes of subcategories such as functorially finite torsion(-free) classes and the category of Gorenstein projective modules, see Example 3.5.13 for the detail.

1.2.3. Chapter 4: Torsion-free classes over preprojective algebras of Dynkin type. The previous result suggests that *it should be interesting to determine simple objects in a given extension-closed subcategory.* In Chapter 4, we do this for a subcategory called a *torsion-free class* of a module category of a preprojective algebra of Dynkin type.

Let Λ be a finite-dimensional algebra. Then a subcategory $\mathcal{F}$ of $\mathsf{mod}\,\Lambda$ is called a *torsion-free class* if it is closed under extensions and submodules. Dually, a subcategory $\mathcal{T}$ of $\mathsf{mod}\,\Lambda$ is called a *torsion class* if it is closed under extensions and quotients. These two classes of subcategories have been playing central roles in the representation theory of algebras, and related to various areas of mathematics. However, properties of a fixed torsion(-free) class have not been investigated in detail, which we will study in Chapter 4 for the preprojective algebra case.

First, we recall about quiver representations and some results. Let k be a field and Q a quiver, that is, a directed graph. Then a *representation M of Q* consists of the following data:

- To each vertex $v \in Q$, we associate a finite-dimensional k-vector space M_v.
- To each arrow $a\colon v \to v' \in Q$, we associate a k-linear map $M_a\colon M_v \to M_{v'}$.

Then we can consider the category of representations of Q. It is known that for a finite acyclic quiver Q, this category coincides with $\mathsf{mod}\,kQ$ for a so-called *path algebra kQ* of Q.

As for this, the *Gabriel's theorem* established a connection of the representation theory of algebras and the Lie theory. To state this, we use the following notation. For a Dynkin quiver Q, we denote by Φ, Φ^+ and α_v for a vertex $v \in Q$, the root system of type Q, the set of positive roots in Φ and the simple root corresponding to the Dynkin vertex v respectively. For a representation M of Q, we put

$$\underline{\dim}\, M := \sum_{v \in Q} (\dim M_v)\alpha_v,$$

and call it the *dimension vector of M*. Then we have the following classification.

THEOREM 1.2.5 (Gabriel's theorem). *Let Q be a Dynkin quiver. Then the assignment $M \mapsto \underline{\dim}\, M$ gives a bijection between the set of isomorphism classes of indecomposable representations of Q and the set of positive roots Φ^+.*

Roughly speaking, it says that *representations of a Dynkin quiver give a categorification of the root system.* By this, to an element w of the Weyl group W of Φ, we can associate the subcategory $\mathcal{F}_Q(w)$ of $\mathsf{mod}\,kQ$ as follows:

$$\mathcal{F}_Q(w) := \{M \in \mathsf{mod}\,kQ \mid \underline{\dim}\, M \in \mathsf{inv}(w)\}.$$

Here $\mathsf{inv}(w)$ is called an *inversion set of w*, the set of positive roots in Φ which are sent to negative roots by w^{-1}. This set is a very fundamental object in the Lie theory and root systems, for example, the cardinality of $\mathsf{inv}(w)$ is known to be equal to the length of w.

To a Dynkin quiver Q, we can associate another algebra Π_Q called a *preprojective algebra of Q*. This algebra has some common futures with the path algebra kQ, but has more symmetries than kQ. More precisely, the representation theory of Q heavily depends on the choice of the *orientations* of the underlying Dynkin graph (for example, the categories of representations of

$1 \leftarrow 2 \leftarrow 3$ and $1 \to 2 \leftarrow 3$ are not equivalent). However, Π_Q does not depend on the orientations. Moreover, Π_Q is something like an *enhancement of* kQ, which means that the representation theory of kQ are included in Π_Q. Thus the representation theory of Π_Q uniformly treats the representation theory of Dynkin quivers with all orientations at the same time.

Now we have the following Gabriel type theorem for preprojective algebras.

PROPOSITION 1.2.6. *Let Q be a Dynkin quiver and Π_Q a preprojective algebra. Then the dimension vector of every* brick *is a positive root of the corresponding root system.*

Here a *brick* is a module whose endomorphism ring is a division ring. For example, every indecomposable kQ-module is a brick. Although there exist infinitely many indecomposable Π_Q-modules for almost all Dynkin quivers Q, there are only finitely many bricks, thus bricks in $\mathsf{mod}\,\Pi_Q$ are more appropriate analogues of indecomposable kQ-modules.

Then [BIRS] defined a subcategory $\mathcal{F}(w)$ of $\mathsf{mod}\,\Pi_Q$, which is an analogue of $\mathcal{F}_Q(w) \subset \mathsf{mod}\,kQ$ above. In fact, $\underline{\dim}\,B \in \mathsf{inv}(w)$ holds for every brick B contained in $\mathcal{F}(w)$. Moreover, this category also plays an important role in the theory of cluster algebras, since it categorifies the cluster structure of the coordinate ring of a unipotent cell in the algebraic group by [GLS].

The following result of Mizuno is the starting point of Chapter 4.

THEOREM 1.2.7 ([Miz, Theorem 2.30]). *Let Q be a Dynkin quiver. Then the assignment $w \mapsto \mathcal{F}(w)$ gives a bijection between the following two sets:*

(1) *The Weyl group W of the associated root system.*
(2) *The set of torsion-free classes in $\mathsf{mod}\,\Pi_Q$.*

Thus torsion-free classes in $\mathsf{mod}\,\Pi_Q$ are parametrized by elements of the Weyl group. Then it is natural to expect that there is a relation between (exact-)categorical properties of $\mathcal{F}(w)$ and Lie-theoretical property of w. In this direction, we obtain the following main result of Chapter 4.

THEOREM 1.2.8 (= Theorems J, L). *Let Q be a Dynkin quiver and w an element of the corresponding Weyl group. Then by taking dimension vectors, we have a bijection between the following two sets.*

(1) *The set of isomorphism classes of simple objects in $\mathcal{F}(w)$.*
(2) *The set of* Bruhat inversions *of w.*

Moreover, $\mathcal{F}(w)$ satisfies (JHP) if and only if the number of supports of w is equal to the number of Bruhat inversions of w. In addition, the same statement holds for $\mathcal{F}_Q(w) \subset \mathsf{mod}\,kQ$ instead of $\mathcal{F}(w)$.

Here *Bruhat inversions* are inversions of w which cannot be written as a sum of smaller inversions of w. This gives a complete classification of simple objects and the validity of (JHP) in $\mathcal{F}(w)$ purely by using Lie theoretic language.

1.2.4. Chapter 5: Monobricks and new class of subcategories. As we mentioned in the previous subsection, torsion classes and torsion-free classes have been central objects in the history of the representation theory of algebras. Recently, Adachi-Iyama-Reiten gave a classification of torsion-free classes using injective objects of them. More precisely, they showed that *functorially finite* torsion-free classes in $\mathsf{mod}\,\Lambda$ for a finite-dimensional algebra Λ are in bijection with *support τ^--tilting Λ-modules.* Here the functorial finiteness is some restriction on a torsion-free class $\mathcal{F}$ which ensures that $\mathcal{F}$ has enough injective objects.

As we have seen, the set of *simple objects* of $\mathcal{F}$, instead of projective or injective objects, seems to be a new interesting object to study. Motivated by this idea, we succeed in classifying *all* torsion-free classes in *any length abelian category*, which is one of the main results in Chapter 5:

THEOREM 1.2.9 (= Theorem N). *Let $\mathcal{A}$ be a length abelian category. Then taking simple objects yields a bijection between the following two sets.*

(1) *The set of all torsion-free classes in $\mathcal{A}$.*
(2) *The set of cofinally closed monobricks in $\mathcal{A}$.*

Here *monobricks* is a set $\mathcal{M}$ of bricks in $\mathcal{A}$ such that every non-zero morphism between objects in $\mathcal{M}$ is an injection, and the cofinal closedness is a kind of poset-theoretical closedness condition, see Definition 5.3.2.

Actually, we establish a bijection which includes the above bijection. A special well-known class of monobricks is a class of *semibricks*, a set of bricks such that every morphism between them is a zero morphism. It is a classical result that semibricks are in bijection with *wide subcategories* of $\mathcal{A}$, exact abelian subcategories closed under extensions.

We introduce the concept of *left Schur subcategory* of $\mathcal{A}$, which unifies both torsion-free classes and wide subcategories. Then we show the following bijection.

THEOREM 1.2.10 (= Theorem O). *Let $\mathcal{A}$ be a length abelian category. Then taking simple objects yields a bijection between the following two sets:*

(1) *The set of left Schur subcategories of $\mathcal{A}$.*
(2) *The set of monobricks in $\mathcal{A}$.*

Moreover, this bijection restricts to the following bijections between

- *the set of wide subcategories of $\mathcal{A}$ and the set of semibricks in $\mathcal{A}$, and*
- *the set of torsion-free classes in $\mathcal{A}$ and the set of cofinally closed monobricks in $\mathcal{A}$.*

One of the benefits of this result is that the relation between torsion-free classes and wide subcategories becomes more transparent by using monobricks. Each monobrick can be regarded as a poset in a natural way, and this poset structure provides us a powerful tool to study these two kinds of subcategories.

For example, it was shown in [MS] that if there are only finitely many torsion-free classes, we have a bijection between the sets of torsion-free classes and wide subcategories. The proof in [MS] heavily depends on [AIR], and does not work in an length abelian category in general. In contrast, we can quickly deduce it by using only some poset theoretical argument on monobricks (Theorem 5.5.4).

1.2.5. Chapter 6: ICE-closed subcategories in quiver representations. In the previous chapter, we introduced left Schur subcategories of an abelian category $\mathcal{A}$. A typical example is a subcategory closed under images, kernels and extensions. Motivated by this, in Chapter 6, we give a classification of such subcategories in the category of representations of a Dynkin quiver Q, or equivalently, $\mathsf{mod}\, kQ$.

For the compatibility of [AIR] and [IT], we work with the dual notion *ICE-closed subcategories* of $\mathsf{mod}\, kQ$, which are subcategories of $\mathsf{mod}\, kQ$ closed under taking Images, Cokernels and Extensions. In particular, torsion classes are ICE-closed, and Ingalls and Thomas classified them by support tilting modules in [IT] using projective objects in torsion classes. In Chapter 6, we extend their classification to include all ICE-closed subcategories of $\mathsf{mod}\, kQ$ as follows:

THEOREM 1.2.11 (= Theorem R). *Let Q be a Dynkin quiver. Then taking projective objects yields a bijection between the following two sets:*

(1) *The set of ICE-closed subcategories of* $\mathsf{mod}\, kQ$.
(2) *The set of isomorphism classes of rigid kQ-modules, that is, a kQ-module M satisfying $\operatorname{Ext}^1_{kQ}(M,M) = 0$.*

This bijection extends a bijection between the set of torsion classes in $\mathsf{mod}\, kQ$ *and the set of support tilting kQ-modules established by* [IT].

Moreover, we show that the number of ICE-closed subcategories of $\mathsf{mod}\, kQ$ depends only on the underlying Dynkin graph of Q, and does not depend on the choice of orientations of arrows. We give an explicit formula of this number for each Dynkin type. In particular, for type A case, this number coincides with the combinatorial number known as the *large Schröder number*.

The representation theory of Dynkin quiver is the most basic, classical and fundamental object which motivated research at the down of the representation theory of algebras, and rigid modules are also quite fundamental and have appeared in many contexts. Therefore, it is a bit surprising to the author that this result has not been observed in the literature.

CHAPTER 2

Relations for Grothendieck groups and representation-finiteness

This chapter is based on the published paper [Eno3].

For an exact category $\mathcal{E}$, we study the Butler's condition "AR=Ex": the relation of the Grothendieck group of $\mathcal{E}$ is generated by Auslander-Reiten conflations. Under some assumptions, we show that AR=Ex is equivalent to that $\mathcal{E}$ has finitely many indecomposables. This can be applied to functorially finite torsion(free) classes and contravariantly finite resolving subcategories of the module category of an artin algebra, and the category of Cohen-Macaulay modules over an order which is Gorenstein or has finite global dimension. Also we showed that under some weaker assumption, AR=Ex implies that the category of syzygies in $\mathcal{E}$ has finitely many indecomposables.

2.1. Introduction

Let Λ be a finite-dimensional k-algebra over a field k. To the abelian category $\mathsf{mod}\,\Lambda$ of finitely generated Λ-modules, we can associate an abelian group $\mathsf{K}_0(\mathsf{mod}\,\Lambda)$ called the *Grothendieck group*. This is the quotient group $\mathsf{K}_0(\mathsf{mod}\,\Lambda, 0)/\,\mathsf{Ex}(\mathsf{mod}\,\Lambda)$, where $\mathsf{K}_0(\mathsf{mod}\,\Lambda, 0)$ denotes the free abelian group with the basis the set of isomorphism classes $[X]$ of indecomposable objects $X \in \mathsf{mod}\,\Lambda$, and $\mathsf{Ex}(\mathsf{mod}\,\Lambda)$ the subgroup generated by $[X]-[Y]+[Z]$ for every exact sequence $0 \to X \to Y \to Z \to 0$.

Auslander-Reiten sequences, or *AR sequences* for short, are special kind of short exact sequences in $\mathsf{mod}\,\Lambda$, which are "minimal" in some sense. We denote by $\mathsf{AR}(\mathsf{mod}\,\Lambda)$ the subgroup of $\mathsf{Ex}(\mathsf{mod}\,\Lambda)$ generated by AR sequences. Then Butler and Auslander proved the following result.

THEOREM 2.1.1 ([But, Aus3]). *Let Λ be a finite-dimensional algebra. Then $\mathsf{AR}(\mathsf{mod}\,\Lambda) = \mathsf{Ex}(\mathsf{mod}\,\Lambda)$ holds if and only if there are only finitely many indecomposable objects in $\mathsf{mod}\,\Lambda$ up to isomorphism.*

Auslander conjectured that similar results hold for a more general class of categories other than $\mathsf{mod}\,\Lambda$. In this paper, we study this conjecture in the context of Quillen's *exact categories.* For a Krull-Schmidt exact category $\mathcal{E}$, the Grothendieck group $\mathsf{K}_0(\mathcal{E}) = \mathsf{K}_0(\mathcal{E}, 0)/\,\mathsf{Ex}(\mathcal{E})$ of $\mathcal{E}$ is defined in the same way. Also the notion of AR sequences is defined over $\mathcal{E}$, which we call *AR conflations.* Thus we have the subgroup $\mathsf{AR}(\mathcal{E})$ of $\mathsf{Ex}(\mathcal{E})$, and the aim of this paper is to study the following question.

QUESTION 2.1.2. *For a Krull-Schmidt exact category $\mathcal{E}$, when are the following equivalent?*

(1) *$\mathcal{E}$ is of finite type, that is, there exist only finitely many indecomposable objects in $\mathcal{E}$ up to isomorphism.*
(2) *$\mathsf{AR}(\mathcal{E}) = \mathsf{Ex}(\mathcal{E})$ holds.*

Let us mention some known results on this question. For the implication (1) $\Rightarrow$ (2), we have the following quite general result by the author:

THEOREM 2.1.3 ([Eno2, Theorem 3.18], Corollary 2.4.4). *Let R be a complete noetherian local ring and $\mathcal{E}$ a Krull-Schmidt exact R-category such that $\mathcal{E}(X, Y)$ is a finitely generated R-module for each $X, Y \in \mathcal{E}$. If $\mathcal{E}$ has enough projectives and is of finite type, then $\mathsf{AR}(\mathcal{E}) = \mathsf{Ex}(\mathcal{E})$ holds.*

This can be applied to various situations in the representation theory of noetherian algebras. We provide a proof of this in Corollary 2.4.4.

On the other hand, the implication (2) $\Rightarrow$ (1) is more subtle. Indeed, some counter-examples are known, e.g. [MMP, Section 3]. However, there are affirmative results for this implication in some concrete situations: $\mathcal{E} = \mathsf{CM}\,\Lambda$ for an R-order Λ with finite global dimension [AR1, Proposition 2.3], and the category of modules with good filtrations over standardly stratified algebras [MMP, PR]. Also some partial results are known ([Hir, Kob]).

In this paper, we first give a sufficient condition for (1) and (2) in Question 2.1.2 to be equivalent, by using a functorial method. For a Krull-Schmidt exact category $\mathcal{E}$, we consider the category $\mathsf{Mod}\,\mathcal{E}$ of functors $\mathcal{E}^{\mathrm{op}} \to \mathcal{A}b$. There is an important subcategory $\mathsf{eff}\,\mathcal{E}$ of $\mathsf{Mod}\,\mathcal{E}$, the category of *effaceable* functors (Definition 2.2.1), which nicely reflects the exact structure of $\mathcal{E}$ and is closely related to the group $\mathsf{Ex}(\mathcal{E})$. Then let us consider the following conditions:

(a) $\mathcal{E}$ is of finite type.
(b) $\mathcal{E}$ is *admissible* [Eno2], that is, every object in $\mathsf{eff}\,\mathcal{E}$ has finite length in $\mathsf{Mod}\,\mathcal{E}$.
(c) $\mathsf{AR}(\mathcal{E}) = \mathsf{Ex}(\mathcal{E})$ holds.

Simple modules contained in $\mathsf{eff}\,\mathcal{E}$ bijectively correspond to AR conflations in $\mathcal{E}$ (Proposition 2.2.3). Thus the condition (b) can be regarded as a functorial analogue of (c), hence it is somewhat closer to (c) than (a) is. The relation of these conditions are summarized in Figure 1.

Secondly, we apply this condition to concrete situations and obtain results on Question 2.1.2. For an artin algebra, we show the following result, which extends the results [MMP, Theorem 3] and [PR, Theorem 3.6].

Theorem A (= Theorem 2.4.11). *Let Λ be an artin algebra and $\mathcal{E}$ a contravariantly finite resolving subcategory of $\mathsf{mod}\,\Lambda$. Then the following are equivalent.*

(1) *$\mathcal{E}$ is of finite type.*
(2) *$\mathsf{AR}(\mathcal{E}) = \mathsf{Ex}(\mathcal{E})$ holds.*

When R is not artinian, we prove the following result. This extends [Hir, Kob] to the non-commutative case, and [AR1, Proposition 2.3] to Gorenstein orders. Also our method provides another proof of [AR1, Proposition 2.3].

Theorem B (= Theorem 2.4.15, Corollary 2.5.3). *Let R be a complete Cohen-Macaulay local ring and Λ an R-order with at most an isolated singularity. Suppose that $\mathsf{Ex}(\mathsf{CM}\,\Lambda) = \mathsf{AR}(\mathsf{CM}\,\Lambda)$ holds. Then the following hold.*

(1) *$\Omega\mathsf{CM}\,\Lambda$, the category of syzygies of Cohen-Macaulay Λ-modules, is of finite type.*
(2) *If Λ has finite global dimension or Λ is a Gorenstein order, then $\mathsf{CM}\,\Lambda$ is of finite type.*

Actually, most of the results above are deduced from the following relations between (a), (b) and (c):

Theorem C (Proposition 2.3.3, Theorem 2.3.7). *Let $\mathcal{E}$ be a Krull-Schmidt exact R-category with a projective generator over a commutative noetherian ring R. Then the implications in Figure 1 hold.*

$$\begin{tikzcd}[column sep=huge]
\text{(a) } \mathcal{E} \text{ is of finite type} \arrow[r, Rightarrow, shift left, "\underline{\mathcal{E}} \text{ is Hom-finite}"] & \text{(b) } \mathcal{E} \text{ is admissible} \arrow[l, Rightarrow, shift left, "\underline{\mathcal{E}} \text{ has a weak cogenerator}"] \arrow[r, Rightarrow, shift left] & \text{(c) } \mathsf{AR}(\mathcal{E}) = \mathsf{Ex}(\mathcal{E}) \arrow[l, Rightarrow, shift left, "\mathcal{E} \text{ satisfies (CF)}"]
\end{tikzcd}$$

Figure 1. Summary of Section 2.3

Here we introduce the condition *(CF)* on exact categories concerning the Grothendieck groups (see Definition 2.3.5).

The paper is organized as follows. In Section 2.2, we introduce basic definitions and collect some results about Grothendieck group and effaceable functors. In Section 2.3, we introduce some conditions on $\mathcal{E}$ and prove the relations indicated in Figure 1. In Section 2.4, we verify that the conditions are satisfied in some concrete cases. In Section 2.5, we study the finiteness of the category of syzygies. In Appendix A, we provide some interesting properties of the category of effaceable functors.

2.1.1. Notation and conventions. Throughout this paper, we assume that *all categories and functors are additive* and *all subcategories are additive, full and closed under isomorphisms.* For an object G of an additive category $\mathcal{E}$, we denote by $\mathsf{add}\, G$ the category consisting of all direct summands of finite direct sums of G. We say that an additive category is *of finite type* if there exists an object $G \in \mathcal{E}$ such that $\mathcal{E} = \mathsf{add}\, G$ holds.

For a Krull-Schmidt category $\mathcal{E}$, we denote by $\mathsf{ind}\, \mathcal{E}$ the set of isomorphism classes of indecomposable objects in $\mathcal{E}$. We refer the reader to [Krau] for the basics of Krull-Schmidt categories. We say that an additive category $\mathcal{E}$ is an *idempotent complete* if every idempotent morphism splits. It is well-known that a Krull-Schmidt category is idempotent complete.

As for exact categories, we use the terminologies *inflations*, *deflations* and *conflations.* We refer the reader to [Büh] for the basics of exact categories. We say that an exact category $\mathcal{E}$ has a *projective generator* P if $\mathcal{E}$ has enough projectives $\mathsf{add}\, P$.

2.2. Preliminaries

First we introduce the basic definition about Auslander-Reiten theory in a Krull-Schmidt category. Let $\mathcal{E}$ be a Krull-Schmidt category and $\mathcal{J}$ its Jacobson radical. A morphism $g : Y \to Z$ in $\mathcal{E}$ is called *right almost split* if Z is indecomposable, g is in $\mathcal{J}$ and any morphism $h : W \to Z$ in $\mathcal{J}$ factors through g. Dually we define *left almost split.*

For a Krull-Schmidt exact category $\mathcal{E}$, we say that a conflation $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{E}$ is an *AR conflation* if f is left almost split and g is right almost split. We say that an indecomposable object Z in $\mathcal{E}$ *admits a right AR conflation* if there exists an AR conflation ending at Z. We say that $\mathcal{E}$ *has right AR conflations* if every indecomposable non-projective object admits a right AR conflation. Left AR conflations are defined dually, and we say that $\mathcal{E}$ *has AR conflations* if it has both right and left AR conflations.

Next we introduce some notation concerning the Grothendieck group. For a Krull-Schmidt exact category $\mathcal{E}$, let $\mathsf{K}_0(\mathcal{E}, 0)$ be a free abelian group $\bigoplus_{[X]\in \mathsf{ind}\, \mathcal{E}} \mathbb{Z} \cdot [X]$ generated by the set $\mathsf{ind}\, \mathcal{E}$ of isomorphism classes of indecomposable objects in $\mathcal{E}$, which we identify with the Grothendieck group of $\mathcal{E}$ with the split exact structure, that is, the exact structure where all conflations are split exact. We denote by $\mathsf{Ex}(\mathcal{E})$ the subgroup of $\mathsf{K}_0(\mathcal{E}, 0)$ generated by

$$\{[X] - [Y] + [Z] \mid \text{there exists a conflation } X \rightarrowtail Y \twoheadrightarrow Z \text{ in } \mathcal{E}\}.$$

We call the quotient group $\mathsf{K}_0(\mathcal{E}) := \mathsf{K}_0(\mathcal{E}, 0)/\mathsf{Ex}(\mathcal{E})$ the *Grothendieck group* of $\mathcal{E}$. We denote by $\mathsf{AR}(\mathcal{E})$ the subgroup of $\mathsf{Ex}(\mathcal{E})$ generated by

$$\{[X] - [Y] + [Z] \mid \text{there exists an AR conflation } X \rightarrowtail Y \twoheadrightarrow Z \text{ in } \mathcal{E}\}.$$

Throughout this paper, we make use of functor categorical arguments. Let us recall the related concepts. For an additive category $\mathcal{E}$, a *right $\mathcal{E}$-module* M is a contravariant additive functor $M : \mathcal{E}^{\mathrm{op}} \to \mathcal{A}b$ from $\mathcal{E}$ to the category of abelian groups $\mathcal{A}b$. We denote by $\mathsf{Mod}\, \mathcal{E}$ the category of right $\mathcal{E}$-modules, and morphisms are natural transformations between them. Then the category $\mathsf{Mod}\, \mathcal{E}$ is a Grothendieck abelian category with enough projectives, and projective objects are precisely direct summands of (possibly infinite) direct sums of representable functors.

We denote by $\mathsf{mod}\, \mathcal{E}$ the category of finitely presented $\mathcal{E}$-modules, that is, the modules M such that there exists an exact sequence $\mathcal{E}(-, X) \to \mathcal{E}(-, Y) \to M \to 0$ for some X, Y in $\mathcal{E}$. This category $\mathsf{mod}\, \mathcal{E}$ is not necessarily an abelian category, but $\mathsf{mod}\, \mathcal{E}$ is closed under extensions in $\mathsf{Mod}\, \mathcal{E}$ by the horseshoe lemma. Thereby we always regard $\mathsf{mod}\, \mathcal{E}$ as an exact category, which has enough projectives in the sense of exact category. Moreover, if $\mathcal{E}$ is idempotent complete, then projectives in $\mathsf{mod}\, \mathcal{E}$ are precisely representable functors.

Now let us introduce the notion of *effaceability* of $\mathcal{E}$-modules, which plays an essential role throughout this paper. This notion was originally introduced by Grothendieck [Gro, p. 141].

DEFINITION 2.2.1. Let $\mathcal{E}$ be an exact category. A right $\mathcal{E}$-module M is called *effaceable* if there exists a deflation $f : Y \twoheadrightarrow Z$ in $\mathcal{E}$ such that $\mathcal{E}(-, Y) \xrightarrow{\mathcal{E}(-,f)} \mathcal{E}(-, Z) \to M \to 0$ is exact. We denote by $\mathsf{eff}\,\mathcal{E}$ the subcategory of $\mathsf{mod}\,\mathcal{E}$ consisting of effaceable functors.

The category $\mathsf{eff}\,\mathcal{E}$ plays a quite essential role to study exact categories in a functorial method. This definition for exact categories is due to [Fio], and the category $\mathsf{eff}\,\mathcal{E}$ plays an important role in [Eno2] to classify exact structures on a given additive category. For the compatibility of Grothendieck's effaceability, see Proposition 2.A.1.

In the rest of this section, we collect basic properties about functor categories and effaceable functors which we need in the sequel.

PROPOSITION 2.2.2. *Let $\mathcal{E}$ be a Krull-Schmidt category and $\mathcal{J}$ its Jacobson radical. Put $S_Z := \mathcal{E}(-, Z)/\mathcal{J}(-, Z)$ for $Z \in \mathcal{E}$. Then the following hold.*

1. *The map $Z \mapsto S_Z$ gives a bijection between $\mathsf{ind}\,\mathcal{E}$ and the set of isomorphism classes of simple $\mathcal{E}$-modules.*
2. *A morphism $g : Y \to Z$ in $\mathcal{E}$ is right almost split if and only if Z is indecomposable and $\mathcal{E}(-, Y) \xrightarrow{\mathcal{E}(-,g)} \mathcal{E}(-, Z) \to S_Z \to 0$ is exact.*

PROOF. It is well-known, see [Aus2] for example. □

In particular, simple $\mathcal{E}$-modules contained in $\mathsf{mod}\,\mathcal{E}$ bijectively correspond to indecomposable objects in $\mathcal{E}$ which admit right almost split maps. Moreover if $\mathcal{E}$ is an exact category, then one can show more on $\mathsf{eff}\,\mathcal{E}$ as follows.

PROPOSITION 2.2.3. *Let $\mathcal{E}$ be a Krull-Schmidt exact category and Z an indecomposable object in $\mathcal{E}$. Then the following are equivalent.*

1. *Z admits a right AR conflation.*
2. *S_Z belongs to $\mathsf{eff}\,\mathcal{E}$.*
3. *S_Z is finitely presented and Z is non-projective.*

In particular, simple $\mathcal{E}$-modules contained in $\mathsf{eff}\,\mathcal{E}$ bijectively correspond to indecomposables in $\mathcal{E}$ which admit right AR conflations.

PROOF. (1) $\Rightarrow$ (2): We have a deflation $g : Y \twoheadrightarrow Z$ which is right almost split. Thus $S_Z \cong \operatorname{Coker} \mathcal{E}(-, g)$ belongs to $\mathsf{eff}\,\mathcal{E}$.

(2) $\Rightarrow$ (3): Since S_Z is in $\mathsf{eff}\,\mathcal{E}$, it is finitely presented. By [Eno2, Propositions 2.11], we have that Z is non-projective since $S_Z(Z) \neq 0$ holds.

(3) $\Rightarrow$ (1): This is precisely [Eno2, Proposition A.3]. □

The following says that the category $\mathsf{eff}\,\mathcal{E}$ has nice properties and controls the exact structure of $\mathcal{E}$. Here we say that an $\mathcal{E}$-module M is *finitely generated* if there exists a surjection from a representable functor onto M.

PROPOSITION 2.2.4. *Let $\mathcal{E}$ be an idempotent complete exact category. Then the following holds.*

1. *Suppose that there exists an exact sequence $0 \to M_1 \to M \to M_2 \to 0$ in $\mathsf{Mod}\,\mathcal{E}$ and that M_1 is finitely generated. Then M is in $\mathsf{eff}\,\mathcal{E}$ if and only if both M_1 and M_2 are in $\mathsf{eff}\,\mathcal{E}$.*
2. *A complex $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{E}$ is a conflation if and only if there exists an exact sequence*
$$0 \to \mathcal{E}(-, X) \xrightarrow{\mathcal{E}(-,f)} \mathcal{E}(-, Y) \xrightarrow{\mathcal{E}(-,g)} \mathcal{E}(-, Z) \to M \to 0$$
in $\mathsf{Mod}\,\mathcal{E}$ with M in $\mathsf{eff}\,\mathcal{E}$.

PROOF. We refer the reader to [Eno2, Proposition 2.10] for the proof of (1), and [Eno2, Theorem 2.7] or the proof of Lemma 2.A.1 for (2). □

The category $\mathsf{eff}\,\mathcal{E}$ enjoys further nice properties, e.g. being abelian, and we refer the interested reader to Appendix A.

If $\mathcal{E}$ has enough projectives, then one has a simpler description of the category $\mathsf{eff}\,\mathcal{E}$. Let $\mathcal{E}$ be an exact category with enough projectives and denote by $\mathcal{P}$ the subcategory of $\mathcal{E}$ consisting of projective objects. We define the *stable category* $\underline{\mathcal{E}}$ by the ideal quotient $\mathcal{E}/[\mathcal{P}]$. Then $\mathsf{eff}\,\mathcal{E}$ can be naturally identified with the module category of $\underline{\mathcal{E}}$, which we omit the proof.

LEMMA 2.2.5 ([Eno2, Lemma 2.13]). *Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$. Then the natural restriction functor $\mathsf{Mod}\,\underline{\mathcal{E}} \to \mathsf{Mod}\,\mathcal{E}$ induces an equivalence $\mathsf{mod}\,\underline{\mathcal{E}} \simeq \mathsf{eff}\,\mathcal{E}$.*

2.3. General conditions for (a), (b) and (c) to be equivalent

In this section, we will provide some general results on the three conditions: (a) $\mathcal{E}$ is of finite type, (b) $\mathcal{E}$ is admissible and (c) $\mathsf{AR}(\mathcal{E}) = \mathsf{Ex}(\mathcal{E})$. Here we say that $\mathcal{E}$ is *admissible* if every object in $\mathsf{eff}\,\mathcal{E}$ has finite length in $\mathsf{Mod}\,\mathcal{E}$ ([Eno2]). See Proposition 2.A.3 for the equivalent condition of the admissibility. Then our results in this section can be summarized in Figure 1 in the introduction.

To begin with, let us introduce some terminologies on R-categories needed in this paper. Let R be a commutative noetherian ring and $\mathcal{E}$ an R-category. We say that an R-category $\mathcal{E}$ is *Hom-noetherian* (resp. *Hom-finite*) if $\mathcal{E}(X,Y)$ is finitely generated (resp. has finite length) as an R-module for each X, Y in $\mathcal{E}$.

To deal with admissibility, it is convenient to introduce the following criterion for deciding whether the functor has finite length or not. Let $\mathcal{E}$ be a Krull-Schmidt category and $M \in \mathsf{Mod}\,\mathcal{E}$ a right $\mathcal{E}$-module. Then the support $\mathsf{supp}\,M$ of M is defined by

$$\mathsf{supp}\,M := \{X \in \mathsf{ind}\,\mathcal{E} \mid M(X) \neq 0\}.$$

The Hom-finiteness ensures that finiteness of modules can be checked only by considering their support:

LEMMA 2.3.1. *Let $\mathcal{E}$ be a Krull-Schmidt category and $M \in \mathsf{mod}\,\mathcal{E}$ an $\mathcal{E}$-module. If M has finite length in $\mathsf{Mod}\,\mathcal{E}$, then $\mathsf{supp}\,M$ is a finite set. The converse holds if $\mathcal{E}$ is Hom-finite R-category.*

PROOF. See, for example, [Aus2, Theorem 2.12]. □

2.3.1. (a) Representation-finiteness and (b) admissibility. First, we consider the relation between: (a) $\mathcal{E}$ is of finite type, and (b) $\mathcal{E}$ is admissible.

DEFINITION 2.3.2. Let $\mathcal{C}$ be an additive category. We say that an object $C \in \mathcal{C}$ is a *weak cogenerator* if $\mathcal{C}(X,C) = 0$ implies that X is a zero object for every $X \in \mathcal{C}$.

If we assume the existence of a weak cogenerator of the stable category, then (a) is almost equivalent to (b).

PROPOSITION 2.3.3. *Let $\mathcal{E}$ be a Krull-Schmidt exact category with a projective generator. Consider the following conditions:*

(1) *$\mathcal{E}$ is admissible and $\underline{\mathcal{E}}$ has a weak cogenerator.*
(2) *$\mathcal{E}$ is of finite type.*

Then (1) ⇒ (2) holds. The converse holds if $\underline{\mathcal{E}}$ is a Hom-finite R-category for a commutative noetherian ring R.

PROOF. (1) ⇒ (2): Since $\mathcal{E}$ has enough projectives, we may identify $\mathsf{eff}\,\mathcal{E}$ with $\mathsf{mod}\,\underline{\mathcal{E}}$ by Lemma 2.2.5. Let $\underline{X}$ be a weak cogenerator of $\underline{\mathcal{E}}$. By the assumption, we have that $\underline{\mathcal{E}}(-,\underline{X})$ has finite length. This implies that $\mathsf{supp}\,\underline{\mathcal{E}}(-,\underline{X})$ is finite by Lemma 2.3.1. Since every indecomposable object $\underline{W}$ in $\underline{\mathcal{E}}$ should satisfy $\underline{\mathcal{E}}(\underline{W},\underline{X}) \neq 0$, it follows that $\underline{\mathcal{E}}$ is of finite type. Now $\mathcal{E}$ itself is of finite type because we have a natural identification $\mathsf{ind}\,\mathcal{E} = \mathsf{ind}\,\underline{\mathcal{E}} \sqcup \mathsf{ind}(\mathsf{add}\,P)$.

(2) ⇒ (1): Suppose that $\mathcal{E}$ is of finite type and $\underline{\mathcal{E}}$ is Hom-finite over R. Then obviously the direct sum of all indecomposables in $\underline{\mathcal{E}}$ is a weak cogenerator of $\underline{\mathcal{E}}$. Moreover, since $\underline{\mathcal{E}}$ is Hom-finite and $\mathsf{ind}\,\underline{\mathcal{E}}$ is finite, every object in $\mathsf{eff}\,\mathcal{E} = \mathsf{mod}\,\underline{\mathcal{E}}$ has finite length by Lemma 2.3.1. □

2.3.2. (b) Admissibility and (c) AR=Ex. Now let us investigate the relation between the condition (b) and (c). First we show that if (b) $\mathcal{E}$ is admissible, then (c) $\mathsf{AR}(\mathcal{E}) = \mathsf{Ex}(\mathcal{E})$ holds. Next we will introduce the *condition (CF)* and show that the converse implication holds under it.

PROPOSITION 2.3.4. *Let $\mathcal{E}$ be a Krull-Schmidt exact category. Suppose that $\mathcal{E}$ is admissible. Then $\mathsf{Ex}(\mathcal{E}) = \mathsf{AR}(\mathcal{E})$ holds.*

PROOF. This is proved in the similar way as [Eno2, Theorem 3.17]. We provide a proof here for the convenience of the reader.

Let $0 \to X_1 \xrightarrow{f_1} Y_1 \xrightarrow{g_1} Z_1 \to 0$ be a conflation in $\mathcal{E}$ and we have to show $[X_1] - [Y_1] + [Z_1] \in \mathsf{AR}(\mathcal{E})$. Put $M := \operatorname{Coker} \mathcal{E}(-, g_1)$ in $\mathsf{Mod}\,\mathcal{E}$, then M is in $\mathsf{eff}\,\mathcal{E}$ by the definition. Suppose that there exists another conflation $0 \to X_2 \xrightarrow{f_2} Y_2 \xrightarrow{g_2} Z_2 \to 0$ in $\mathcal{E}$ such that $M \cong \operatorname{Coker} \mathcal{E}(-, g_2)$. Then we have exact sequences

$$0 \to \mathcal{E}(-, X_i) \xrightarrow{\mathcal{E}(-, f_i)} \mathcal{E}(-, Y_i) \xrightarrow{\mathcal{E}(-, g_i)} \mathcal{E}(-, Z_i) \to M \to 0$$

in $\mathsf{Mod}\,\mathcal{E}$ for $i = 1, 2$. Thus Schanuel's lemma shows that $\mathcal{E}(-, X_1) \oplus \mathcal{E}(-, Y_2) \oplus \mathcal{E}(-, Z_1) \cong \mathcal{E}(-, X_2) \oplus \mathcal{E}(-, Y_1) \oplus \mathcal{E}(-, Z_2)$ in $\mathsf{Mod}\,\mathcal{E}$, hence $X_1 \oplus Y_2 \oplus Z_1 \cong X_2 \oplus Y_1 \oplus Z_2$ in $\mathcal{E}$ by the Yoneda lemma. This implies that $[X_1] - [Y_1] + [Z_1] = [X_2] - [Y_2] + [Z_2]$ in $\mathsf{K}_0(\mathcal{E}, 0)$. Thus it suffices to show the following claim.

Claim: For any $M \in \mathsf{eff}\,\mathcal{E}$, there exists at least one exact sequence in $\mathsf{Mod}\,\mathcal{E}$

$$0 \to \mathcal{E}(-, X) \xrightarrow{\mathcal{E}(-, f)} \mathcal{E}(-, Y) \xrightarrow{\mathcal{E}(-, g)} \mathcal{E}(-, Z) \to M \to 0$$

with $[X] - [Y] + [Z] \in \mathsf{AR}(\mathcal{E})$. Note that such a complex $0 \to X \to Y \to Z \to 0$ is automatically a conflation by Proposition 2.2.4(2).

We will show this claim by induction on $l(M)$, the length of M as an $\mathcal{E}$-module. Suppose that $l(M) = 1$, that is, M is a simple $\mathcal{E}$-module. We may assume that $M = S_Z$ for an indecomposable object $Z \in \mathcal{E}$ by Proposition 2.2.2, and there exists an AR conflation $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{E}$ by Proposition 2.2.3. This gives the desired projective resolution $0 \to \mathcal{E}(-, X) \xrightarrow{\mathcal{E}(-, f)} \mathcal{E}(-, Y) \xrightarrow{\mathcal{E}(-, g)} \mathcal{E}(-, Z) \to S_Z \to 0$ of S_Z, and $[X] - [Y] + [Z] \in \mathsf{AR}(\mathcal{E})$ holds.

Now suppose that $l(M) > 1$. Take a simple $\mathcal{E}$-submodule M_1 of M. Then we have an exact sequence $0 \to M_1 \to M \to M_2 \to 0$ in $\mathsf{Mod}\,\mathcal{E}$. Since M_1 is finitely generated, Proposition 2.2.4(1) shows that M_1 and M_2 are both in $\mathsf{eff}\,\mathcal{E}$. By induction hypothesis, we have an exact sequence $0 \to \mathcal{E}(-, X_i) \xrightarrow{\mathcal{E}(-, f_i)} \mathcal{E}(-, Y_i) \xrightarrow{\mathcal{E}(-, g_i)} \mathcal{E}(-, Z_i) \to M_i \to 0$ with $[X_i] - [Y_i] + [Z_i] \in \mathsf{AR}(\mathcal{E})$ for each $i = 1, 2$. By the horseshoe lemma, we obtain a projective resolution

$$0 \to \mathcal{E}(-, X_1 \oplus X_2) \xrightarrow{\mathcal{E}(-, f)} \mathcal{E}(-, Y_1 \oplus Y_2) \xrightarrow{\mathcal{E}(-, g)} \mathcal{E}(-, Z_1 \oplus Z_2) \to M \to 0.$$

Then we have

$$[X_1 \oplus X_2] - [Y_1 \oplus Y_2] + [Z_1 \oplus Z_2] = \sum_{i=1,2} ([X_i] - [Y_i] + [Z_i]) \in \mathsf{AR}(\mathcal{E}).$$

Thus the claim follows. □

Next let us investigate when the converse of Proposition 2.3.4 holds. The author does not know whether this always holds, but we show that this is the case for particular cases. To this purpose, let us introduce the condition *(CF)*, which stands for the *Conservation of Finiteness.*

DEFINITION 2.3.5. For an exact category $\mathcal{E}$, we say that $\mathcal{E}$ satisfies *(CF)* if it satisfies the following:

(CF) Let $0 \to X \xrightarrow{f_i} Y \xrightarrow{g_i} Z \to 0$ be two conflations in $\mathcal{E}$ for $i = 1, 2$ and put $M_i := \operatorname{Coker} \mathcal{E}(-, g_i) \in \mathsf{eff}\,\mathcal{E}$. If M_1 has finite length in $\mathsf{Mod}\,\mathcal{E}$, then so does M_2.

For later purposes, the following equivalent condition is more useful.

LEMMA 2.3.6. *For an exact category $\mathcal{E}$, the condition* (CF) *is equivalent to the following condition.*

(CF′) *Let* $0 \to X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i \to 0$ *be conflations in* $\mathcal{E}$ *for* $i = 1, 2$ *and put* $M_i := \operatorname{Coker} \mathcal{E}(-, g_i)$ *in* $\mathsf{eff}\,\mathcal{E}$. *Suppose that* $[X_1] - [Y_1] + [Z_1] = [X_2] - [Y_2] + [Z_2]$ *holds in* $\mathsf{K}_0(\mathcal{E}, 0)$. *If* M_1 *has finite length in* $\mathsf{Mod}\,\mathcal{E}$, *then so does* M_2.

Proof. Obviously it suffices to show that (CF) implies (CF′). Since $[X_1] - [Y_1] + [Z_1] = [X_2] - [Y_2] + [Z_2]$ holds in $\mathsf{K}_0(\mathcal{E}, 0)$, we have that $X_2 \oplus Y_1 \oplus Z_2 \cong X_1 \oplus Y_2 \oplus Z_1$ holds in $\mathcal{E}$ by the Krull-Schmidtness of $\mathcal{E}$. Now consider the following two complexes:

$$0 \to X_1 \oplus X_2 \xrightarrow{\begin{bmatrix} 0 & 1 \\ f_1 & 0 \\ 0 & 0 \end{bmatrix}} X_2 \oplus Y_1 \oplus Z_2 \xrightarrow{\begin{bmatrix} 0 & g_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} Z_1 \oplus Z_2 \to 0,$$

$$0 \to X_1 \oplus X_2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & f_2 \\ 0 & 0 \end{bmatrix}} X_1 \oplus Y_2 \oplus Z_1 \xrightarrow{\begin{bmatrix} 0 & 0 & 1 \\ 0 & g_2 & 0 \end{bmatrix}} Z_1 \oplus Z_2 \to 0.$$

These are conflations since they are direct sums of conflations and split exact sequences. Applying (CF) to them, it is easy to check that (CF′) holds. □

Now we can state the following main result in this section.

Theorem 2.3.7. *Let* $\mathcal{E}$ *be a Krull-Schmidt exact category. Then the following are equivalent.*

(1) $\mathcal{E}$ *is admissible.*
(2) $\mathsf{Ex}(\mathcal{E}) = \mathsf{AR}(\mathcal{E})$ *holds and* $\mathcal{E}$ *satisfies* (CF).
(3) $\mathsf{Ex}(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathsf{AR}(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q}$ *holds and* $\mathcal{E}$ *satisfies* (CF).

Proof. (1) $\Rightarrow$ (2): If every object in $\mathsf{eff}\,\mathcal{E}$ has finite length, then we have $\mathsf{Ex}(\mathcal{E}) = \mathsf{AR}(\mathcal{E})$ by Proposition 2.3.4, and the condition (CF) is trivially satisfied.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (1): Let M be in $\mathsf{eff}\,\mathcal{E}$ and take a conflation $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{E}$ such that $0 \to \mathcal{E}(-, X) \xrightarrow{\mathcal{E}(-,f)} \mathcal{E}(-, Y) \xrightarrow{\mathcal{E}(-,g)} \mathcal{E}(-, Z) \to M \to 0$ is exact. By $\mathsf{AR}(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathsf{Ex}(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q}$, there exist a positive integer a, integer b_i and an AR conflation $0 \to X_i \to Y_i \xrightarrow{g_i} Z_i \to 0$ for each i such that we have an equality in $\mathsf{K}_0(\mathcal{E}, 0)$:

$$a([X] - [Y] + [Z]) = \sum_{i=1}^{n} b_i([X_i] - [Y_i] + [Z_i]).$$

We may assume that $b_i > 0$ for $i \leq m$ and $-c_i := b_i < 0$ for $i > m$. Thus we have an equality

$$a([X] - [Y] + [Z]) + \sum_{i=m+1}^{n} c_i([X_i] - [Y_i] + [Z_i]) = \sum_{i=1}^{m} b_i([X_i] - [Y_i] + [Z_i]) \tag{2.3.1}$$

in $\mathsf{K}_0(\mathcal{E}, 0)$ such that all the coefficients are positive integers. Thus each side of the equation (2.3.1) comes from the single conflation in $\mathcal{E}$. Put $S_i := \operatorname{Coker} \mathcal{E}(-, g_i) \in \mathsf{mod}\,\mathcal{E}$ for each i, which is simple $\mathcal{E}$-module contained in $\mathsf{eff}\,\mathcal{E}$. Then the left hand side in (2.3.1) corresponds to $M^a \oplus \bigoplus_{i=m+1}^{n} S_i^{c_i}$ in $\mathsf{eff}\,\mathcal{E}$ and the right hand side corresponds to $\bigoplus_{i=1}^{m} S_i^{b_i}$ in $\mathsf{eff}\,\mathcal{E}$. Since the latter has finite length, the condition (CF′) implies that so does the former. Thus M has finite length. □

2.4. Applications

In Section 2.3, we have shown the equivalence of (a), (b) and (c) under three technical assumptions: Hom-finiteness of $\underline{\mathcal{E}}$, the existence of a weak cogenerator of $\underline{\mathcal{E}}$, and the condition (CF) (see Figure 1 for a summary of these results). In this section, we show that these assumptions are satisfied for some concrete cases.

Main applications we have in mind are the representation theory of noetherian algebras, where the base ring is not necessarily artinian. For the convenience of the reader, we recall some related definitions.

Definition 2.4.1. Let R be a noetherian local ring and Λ an R-algebra.

(1) Λ is called a *noetherian* R*-algebra* if Λ is finitely generated as an R-module. If in addition R is artinian, then we say that Λ is an *artin* R*-algebra*.

(2) Suppose that R is Cohen-Macaulay. For a noetherian R-algebra, we denote by $\mathsf{CM}\,\Lambda$ the category of finitely generated Λ-modules which are maximal Cohen-Macaulay as R-modules.
(3) A noetherian R-algebra Λ is called an *R-order* if $\Lambda \in \mathsf{CM}\,\Lambda$ holds.
(4) We say that an R-order Λ *has at most an isolated singularity* if $\operatorname{gl.dim}\Lambda_p = \operatorname{ht} p$ holds for every non-maximal prime $p \in \operatorname{Spec} R$.
(5) An R-order is called a *Gorenstein* order if $\mathsf{CM}\,\Lambda$ is a Frobenius exact category.

Let R be a complete Cohen-Macaulay local ring and Λ an R-order. Then $\mathsf{CM}\,\Lambda$ is closed under extensions in $\mathsf{mod}\,\Lambda$, thus is an exact category with a projective generator Λ. Moreover, $\mathsf{CM}\,\Lambda$ has AR conflations if and only if Λ has at most an isolated singularity [Aus4].

2.4.1. Hom-finiteness of the stable category. First we consider the Hom-finiteness of $\underline{\mathcal{E}}$. Recall that (a) $\Rightarrow$ (b) holds if $\underline{\mathcal{E}}$ is Hom-finite (Proposition 2.3.3). This condition is trivial if R is artinian, hence we obtain the following.

COROLLARY 2.4.2. *Let $\mathcal{E}$ be a Hom-finite idempotent complete exact R-category over a commutative artinian ring R. Suppose that $\mathcal{E}$ is of finite type. Then $\mathcal{E}$ is admissible and $\mathsf{AR}(\mathcal{E}) = \mathsf{Ex}(\mathcal{E})$ holds.*

For an artin algebra Λ, we can apply this result to any subcategory $\mathcal{E}$ of $\mathsf{mod}\,\Lambda$ which is closed under extensions and direct summands.

If R is not artinian, we can use the following fact on Hom-finiteness.

LEMMA 2.4.3 ([Eno2, Proposition A.5]). *Let R be a complete noetherian local ring and $\mathcal{E}$ a Hom-noetherian idempotent complete exact R-category. Suppose that $\mathcal{E}$ has enough projectives and consider the following conditions:*
(1) *$\mathcal{E}$ is of finite type.*
(2) *$\mathcal{E}$ has AR conflations.*
(3) *$\underline{\mathcal{E}}$ is Hom-finite over R.*

Then the implication (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *holds.*

Thus we immediately obtain the following.

COROLLARY 2.4.4 ([Eno2, Corollary 3.18]). *Let R be a complete noetherian local ring and $\mathcal{E}$ a Hom-noetherian idempotent complete exact R-category. Suppose that $\mathcal{E}$ has enough projectives and $\mathcal{E}$ is of finite type. Then $\mathcal{E}$ is admissible and $\mathsf{AR}(\mathcal{E}) = \mathsf{Ex}(\mathcal{E})$ holds.*

Note that the assumption on $\mathcal{E}$ here is rather mild. In particular, this corollary can be applied to $\mathcal{E} := \mathsf{CM}\,\Lambda$ for an R-order Λ which is *CM-finite*, that is, $\mathsf{CM}\,\Lambda$ is of finite type. Thus this provides a further generalization of [Aus3, Proposition 2.2].

2.4.2. Existence of a weak cogenerator of the stable category. For the implication (b) $\Rightarrow$ (a), we need a weak cogenerator of $\underline{\mathcal{E}}$ (Proposition 2.3.3). To this purpose, let us recall the notions of functorial finiteness and resolving subcategories.

DEFINITION 2.4.5. Let $\mathcal{A}$ be an additive category and $\mathcal{E}$ an additive subcategory of $\mathcal{A}$.
(1) A morphism $f : E_X \to X$ in $\mathcal{A}$ is said to be a *right $\mathcal{E}$-approximation* if E_X is in $\mathcal{E}$ and every morphism $E \to X$ with $E \in \mathcal{D}$ factors through f.
(2) $\mathcal{E}$ is said to be *contravariantly finite* if every object in $\mathcal{A}$ has a right $\mathcal{E}$-approximation.

A *left $\mathcal{E}$-approximation* and *covariantly finiteness* are defined dually. We say that $\mathcal{E}$ is *functorially finite* if it is both contravariantly and covariantly finite.

DEFINITION 2.4.6. Let Λ be a noetherian ring and $\mathcal{E}$ a subcategory of $\mathsf{mod}\,\Lambda$. We say that $\mathcal{E}$ is a *resolving* subcategory of $\mathsf{mod}\,\Lambda$ if it satisfies the following conditions:
(1) $\mathcal{E}$ contains Λ.
(2) $\mathcal{E}$ is closed under extensions, that is, for each exact sequence $0 \to X \to Y \to Z \to 0$, if both X and Z are in $\mathcal{E}$, then so is Y.

(3) $\mathcal{E}$ is closed under kernels of surjections, that is, for each exact sequence $0 \to X \to Y \to Z \to 0$, if both Y and Z are in $\mathcal{E}$, then so is X.
(4) $\mathcal{E}$ is closed under summands.

Since each resolving subcategory is an extension-closed subcategory of an abelian category, it has the natural exact structure. Thereby we always regard it as an exact category.

The following result provides a rich source of exact categories such that its stable category has a weak cogenerator.

PROPOSITION 2.4.7. *Let Λ be a semiperfect noetherian ring and $\mathcal{E}$ a contravariantly finite resolving subcategory of $\mathsf{mod}\,\Lambda$. Then $\underline{\mathcal{E}}$ has a weak cogenerator, which can be chosen as a right $\mathcal{E}$-approximation of $\Lambda/\operatorname{rad}\Lambda$.*

PROOF. Since $\mathcal{E}$ is resolving in $\mathsf{mod}\,\Lambda$, it is straightforward to see that $\mathcal{E}$ has a projective generator Λ. Let us construct a weak cogenerator $\underline{X}$ of $\underline{\mathcal{E}}$ (cf. [AR1, Lemma 2.4]). Since Λ is semiperfect, $\Lambda/\operatorname{rad}\Lambda$ is a direct sum of all simple Λ-modules up to multiplicity. Take a right $\mathcal{E}$-approximation $f : X \to \Lambda/\operatorname{rad}\Lambda$ of $\Lambda/\operatorname{rad}\Lambda$, and we claim that $\underline{X}$ is a weak cogenerator of $\underline{\mathcal{E}}$.

Suppose that $\underline{\mathcal{E}}(\underline{W}, \underline{X}) = 0$. Then it follows from this that every morphism from W to finitely generated semisimple modules factors through some projective module. We will show that W is projective, that is, $\underline{W} = 0$ in $\underline{\mathcal{E}}$.

For a Jacobson radical $\operatorname{rad} W$ of W, let $\pi : W \twoheadrightarrow W/\operatorname{rad} W$ be a natural projection and take a projective cover $p : P \twoheadrightarrow W/\operatorname{rad} W$ (this is possible since Λ is semiperfect). It follows that there exists a projective cover $\varphi : P \twoheadrightarrow W$ such that the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, two heads, "\varphi"] \arrow[dr, two heads, "p"'] & W \arrow[r, dashed, "\psi"] \arrow[d, two heads, "\pi"] & P \arrow[dl, two heads, "p"] \\ & W/\operatorname{rad} W & \end{tikzcd}$$

Since $W/\operatorname{rad} W$ is semisimple, π must factor through some projective module. Thus there exists $\psi : W \to P$ which makes the above diagram commute. On the other hand, since p is right minimal, $\psi \circ \varphi$ must be an isomorphism. It follows that φ is an isomorphism, thus W is projective. $\square$

Now one immediately obtains the following result about the equivalence of (a) and (b) for a contravariantly finite resolving subcategories over noetherian algebras. We will treat the case of artin algebras later (see Theorem 2.4.11) since one can prove more.

COROLLARY 2.4.8. *Let R be a complete noetherian local ring, Λ a noetherian R-algebra and $\mathcal{E}$ a contravariantly finite resolving subcategory of $\mathsf{mod}\,\Lambda$. Then $\mathcal{E}$ is of finite type if and only if $\mathcal{E}$ is admissible.*

PROOF. There exists a weak cogenerator of $\underline{\mathcal{E}}$ by Proposition 2.4.7 since Λ is semiperfect. Thus $\mathcal{E}$ is of finite type if $\mathcal{E}$ is admissible by Proposition 2.3.3. The other implication follows from Corollary 2.4.4. $\square$

EXAMPLE 2.4.9. Let R be a complete Cohen-Macaulay local ring and Λ an R-order. Then it is well-known that $\mathsf{CM}\,\Lambda$ is a contravariantly finite resolving subcategory of $\mathsf{mod}\,\Lambda$ (e.g. [AuBu]). In particular, $\underline{\mathsf{CM}}\,\Lambda$ has a weak cogenerator by Proposition 2.4.7 and Corollary 2.4.8 applies to $\mathsf{CM}\,\Lambda$. Thus Corollary 2.4.8 generalizes the result [Aus3, Lemma 2.4 (b)] on the category $\mathsf{CM}\,\Lambda$.

2.4.3. On the condition (CF). Recall that in Section 2.3.2, we have investigated the relation between (b) admissibility and (c) AR=Ex, and Theorem 2.3.7 shows that the condition (CF) ensures that (b) and (c) are equivalent. Now let us give some actual examples where $\mathcal{E}$ satisfies (CF). *In this subsection, we fix a commutative noetherian ring R.*

First we deal with exact categories which are Hom-finite.

PROPOSITION 2.4.10. *Let $\mathcal{E}$ be a Hom-finite exact R-category. Then $\mathcal{E}$ satisfies* (CF).

Proof. Let $0 \to X \xrightarrow{f_i} Y \xrightarrow{g_i} Z \to 0$ be conflations in $\mathcal{E}$ for $i = 1, 2$, and put $M_i := \operatorname{Coker} \mathcal{E}(-, g_i) \in \mathsf{eff}\,\mathcal{E}$. Then we have that $[M_1] = [\mathcal{E}(-, X)] - [\mathcal{E}(-, Y)] + [\mathcal{E}(-, Z)] = [M_2]$ holds in $\mathsf{K}_0(\mathsf{mod}\,\mathcal{E})$. We will show that if M_1 has finite length, then so does M_2. It suffices to show that $\mathsf{supp}\, M_2$ is finite by Lemma 2.3.1.

Let X be an indecomposable object in $\mathcal{E}$. For each F in $\mathsf{mod}\,\mathcal{E}$, the assignment $F \mapsto \operatorname{length}_R F(X)$ makes sense since $\mathcal{E}$ is Hom-finite. It clearly extends to the group homomorphism $\chi_X : \mathsf{K}_0(\mathsf{mod}\,\mathcal{E}) \to \mathbb{Z}$. Now we have that $X \in \mathsf{supp}\, M_1 \Leftrightarrow \chi_X[M_1] \neq 0 \Leftrightarrow \chi_X[M_2] \neq 0 \Leftrightarrow X \in \mathsf{supp}\, M_2$ by $[M_1] = [M_2]$. Thus $\mathsf{supp}\, M_1 = \mathsf{supp}\, M_2$ holds, and this is finite since M_1 has finite length. □

Now we immediately obtain the following general result on artin algebras.

Theorem 2.4.11. *Let Λ be an artin algebra and $\mathcal{E}$ a contravariantly finite resolving subcategory of $\mathsf{mod}\,\Lambda$. Then the following are equivalent.*

(1) *$\mathcal{E}$ is of finite type.*
(2) *$\mathcal{E}$ is admissible.*
(3) *$\mathsf{AR}(\mathcal{E}) = \mathsf{Ex}(\mathcal{E})$ holds.*
(4) *$\mathsf{AR}(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathsf{Ex}(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q}$ holds.*

Proof. Since Λ is artin algebra, Λ is semiperfect and $\underline{\mathcal{E}}$ is Hom-finite. Thus Propositions 2.3.3 and 2.4.7 imply that (1) and (2) are equivalent. Furthermore, since $\mathcal{E}$ is Hom-finite, $\mathcal{E}$ satisfies (CF) by Proposition 2.4.10. Thus the other conditions are also equivalent by Theorem 2.3.7. □

Contravariantly finite resolving subcategories are closely related to so-called *cotilting modules* by the famous result of [AR2]. For the convenience of the reader, we explain their result here.

Definition 2.4.12. Let Λ be an artin algebra and U a Λ-module in $\mathsf{mod}\,\Lambda$. Then U is called a *cotilting Λ-module* if it satisfies the following conditions:

(1) the injective dimension of U is finite.
(2) $\operatorname{Ext}^{>0}_{\Lambda}(U, U) = 0$ holds.
(3) There exists an exact sequence $0 \to U_n \to \cdots \to U_0 \to D\Lambda \to 0$ of Λ-modules with $U_i \in \mathsf{add}\, U$ for each i, where D denotes the standard duality $D : \mathsf{mod}\,\Lambda^{\mathrm{op}} \to \mathsf{mod}\,\Lambda$.

For a module U in $\mathsf{mod}\,\Lambda$, we denote by ${}^{\perp}U$ the full subcategory of $\mathsf{mod}\,\Lambda$ consisting of modules X satisfying $\operatorname{Ext}^{>0}_{\Lambda}(X, U) = 0$.

Proposition 2.4.13 ([AR2, Theorem 5.5]). *Let Λ be an artin algebra and U a cotilting Λ-module. Then ${}^{\perp}U$ is a contravariantly finite resolving subcategory of $\mathsf{mod}\,\Lambda$.*

Thus we can apply Theorem 2.4.11 to $\mathcal{E} := {}^{\perp}U$ for a cotilting module U over an artin algebra. This provides a rich source of examples:

- Let $\mathcal{E}$ be a functorially finite torsionfree class of $\mathsf{mod}\,\Lambda$ for an artin algebra Λ (see [ASS]). Then by factoring out the annihilator of $\mathcal{E}$, we may assume that $\mathcal{E}$ is a faithful functorially finite torsionfree class. It is well-known that such $\mathcal{E}$ is of the form ${}^{\perp}U$ for a cotilting Λ-module with $\mathsf{id}\, U \leq 1$ (see e.g. [Sma, Theorem]), so Theorem 2.4.11 applies to $\mathcal{E}$. By duality, every functorially finite torsion class over artin algebras is also an example.
- An artin algebra Λ is called *Iwanaga-Gorenstein* if both $\mathsf{id}(\Lambda_\Lambda)$ and $\mathsf{id}({}_\Lambda\Lambda)$ are finite. In this case, we write $\mathsf{GP}\,\Lambda := {}^{\perp}\Lambda$ and call it the category of *Gorenstein-projective* modules. It is immediate that Λ itself is a cotilting Λ-module, so we can apply Theorem 2.4.11 to $\mathcal{E} := \mathsf{GP}\,\Lambda$.

Next we will consider the case $\dim R > 0$. Although $\mathcal{E}$ is rarely Hom-finite, we can conclude (CF) by using the Hom-finiteness of the stable category $\underline{\mathcal{E}}$ in some cases. Let us introduce some terminologies to state our result.

Let $\mathcal{E}$ be an exact category with enough projectives. In this case, we have the syzygy functor $\Omega : \underline{\mathcal{E}} \to \underline{\mathcal{E}}$. We denote by $\underline{X}$ in $\underline{\mathcal{E}}$ for the image of X in $\mathcal{E}$ under the natural projection $\mathcal{E} \twoheadrightarrow \underline{\mathcal{E}}$. For an object X in $\mathcal{E}$, we say that X has *finite projective dimension* if there exists some integer $n \geq 0$ such that $\Omega^n \underline{X} = 0$ holds in $\underline{\mathcal{E}}$.

Recall that an exact category $\mathcal{E}$ is called *Frobenius* if $\mathcal{E}$ has enough projectives and enough injectives, and the classes of projectives and injectives coincide.

PROPOSITION 2.4.14. *Let R be a complete noetherian local ring and $\mathcal{E}$ a Hom-noetherian idempotent complete exact R-category with AR conflations. Suppose that $\mathcal{E}$ satisfies either one of the following conditions:*

(1) *$\mathcal{E}$ has enough projectives and every object in $\mathcal{E}$ has finite projective dimension, or*
(2) *$\mathcal{E}$ is Frobenius.*

Then $\mathcal{E}$ satisfies (CF).

PROOF. Let $0 \to X \xrightarrow{f_i} Y \xrightarrow{g_i} Z \to 0$ be conflations in $\mathcal{E}$ for $i = 1, 2$. Put $M_i := \operatorname{Coker} \mathcal{E}(-, g_i) \in \mathsf{eff}\, \mathcal{E}$ for each i and suppose that M_1 has finite length. We will show that so does M_2. First note that Lemma 2.2.5 applies to this situation since $\mathcal{E}$ has enough projectives, so we identify $\mathsf{eff}\, \mathcal{E}$ with $\mathsf{mod}\, \underline{\mathcal{E}}$ naturally. Then M_i has finite length as an $\mathcal{E}$-module if and only if it does so as an $\underline{\mathcal{E}}$-module. Moreover, since $\underline{\mathcal{E}}$ is Hom-finite by Lemma 2.4.3, this occurs if and only if $\mathsf{supp}\, M_i$ is finite, where the support is considered inside $\mathsf{ind}\, \underline{\mathcal{E}}$.

(1) First we prove that $[M_1] = [M_2]$ holds in $\mathsf{K}_0(\mathsf{mod}\, \underline{\mathcal{E}})$. Choose an integer $n \geq 0$ such that $\Omega^n \underline{Z} = 0$ holds in $\underline{\mathcal{E}}$. Then it is classical that we have an exact sequence in $\mathsf{mod}\, \underline{\mathcal{E}}$ for each i:

$$\begin{aligned} 0 &\to \underline{\mathcal{E}}(-, \Omega^{n-1}\underline{X}) \to \underline{\mathcal{E}}(-, \Omega^{n-1}\underline{Y}) \to \underline{\mathcal{E}}(-, \Omega^{n-1}\underline{Z}) \to \cdots \\ &\to \underline{\mathcal{E}}(-, \Omega\underline{X}) \to \underline{\mathcal{E}}(-, \Omega\underline{Y}) \to \underline{\mathcal{E}}(-, \Omega\underline{Z}) \to \underline{\mathcal{E}}(-, \underline{X}) \to \underline{\mathcal{E}}(-, \underline{Y}) \to \underline{\mathcal{E}}(-, \underline{Z}) \to M_i \to 0. \end{aligned}$$

Therefore it immediately follows that $[M_1] = [M_2]$ holds in $\mathsf{K}_0(\mathsf{mod}\, \underline{\mathcal{E}})$.

Let $\underline{X}$ be an indecomposable object of $\underline{\mathcal{E}}$. Since $\underline{\mathcal{E}}$ is a Hom-finite R-category by Lemma 2.4.3, we have the characteristic map $\chi_{\underline{X}} : \mathsf{K}_0(\mathsf{mod}\, \underline{\mathcal{E}}) \to \mathbb{Z}$ which sends F to $\operatorname{length}_R F(\underline{X})$. Then we have $\underline{X} \in \mathsf{supp}\, M_1 \Leftrightarrow \chi_{\underline{X}}[M_1] \neq 0 \Leftrightarrow \chi_{\underline{X}}[M_2] \neq 0 \Leftrightarrow \underline{X} \in \mathsf{supp}\, M_2$, where the support is considered inside $\mathsf{ind}\, \underline{\mathcal{E}}$. Thus Proposition 2.3.1 implies that M_2 has finite length.

(2) Put $N_i := \operatorname{Ker} \underline{\mathcal{E}}(-, \underline{f_i})$ for each $i = 1, 2$. Then we have exact sequences in $\mathsf{mod}\, \underline{\mathcal{E}}$ as in (1):

$$\begin{aligned} 0 \to N_i \to \underline{\mathcal{E}}(-, \underline{X}) \xrightarrow{\underline{\mathcal{E}}(-, \underline{f_i})} \underline{\mathcal{E}}(-, \underline{Y}) &\xrightarrow{\underline{\mathcal{E}}(-, \underline{g_i})} \underline{\mathcal{E}}(-, \underline{Z}) \to M_i \to 0, \\ \underline{\mathcal{E}}(-, \Omega\underline{Y}) &\xrightarrow{\underline{\mathcal{E}}(-, \Omega\underline{g_i})} \underline{\mathcal{E}}(-, \Omega\underline{Z}) \to N_i \to 0. \end{aligned}$$

It follows from the first exact sequence that $[M_1 \oplus N_1] = [M_1] + [N_1] = [M_2] + [N_2] = [M_2 \oplus N_2]$ holds in $\mathsf{K}_0(\mathsf{mod}\, \underline{\mathcal{E}})$. We will show that N_1 has finite length. By the same argument as in (1), we have that $\mathsf{supp}(M_1 \oplus N_1) = \mathsf{supp}(M_2 \oplus N_2)$ holds in $\mathsf{ind}\, \underline{\mathcal{E}}$. Now $\Omega : \underline{\mathcal{E}} \to \underline{\mathcal{E}}$ is an equivalence since $\mathcal{E}$ is Frobenius, and let us denote by $\Omega^- : \underline{\mathcal{E}} \to \underline{\mathcal{E}}$ the quasi-inverse of Ω. Then it is easily checked that $\underline{X} \in \mathsf{supp}\, N_1 \Leftrightarrow \Omega^- \underline{X} \in \mathsf{supp}\, M_1 \Leftrightarrow \underline{X} \in \Omega(\mathsf{supp}\, M_1)$ holds for an indecomposable object $\underline{X} \in \underline{\mathcal{E}}$, hence $\mathsf{supp}\, N_1 = \Omega(\mathsf{supp}\, M_1)$. Therefore $\mathsf{supp}\, N_1$ is finite, and so is $\mathsf{supp}(M_1 \oplus N_1) = \mathsf{supp}(M_2 \oplus N_2)$. Hence $\mathsf{supp}\, M_2$ is finite, which implies that M_2 has finite length. $\square$

For an R-order Λ with at most an isolated singularity, Proposition 2.4.14 shows that $\mathsf{CM}\, \Lambda$ satisfies (CF) if either Λ has finite global dimension or Λ is a Gorenstein order. Thus we obtain the following result on the category $\mathsf{CM}\, \Lambda$.

THEOREM 2.4.15. *Let R be a complete Cohen-Macaulay local ring and Λ an R-order with at most an isolated singularity. Put $\mathcal{E} := \mathsf{CM}\, \Lambda$. Then the following are equivalent.*

(1) *$\mathcal{E}$ is of finite type.*
(2) *$\mathcal{E}$ is admissible.*

Assume in addition that either Λ has finite global dimension or Λ is a Gorenstein order. Then the following are also equivalent.

(3) *$\mathsf{AR}(\mathcal{E}) = \mathsf{Ex}(\mathcal{E})$ holds.*
(4) *$\mathsf{AR}(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathsf{Ex}(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q}$ holds.*

PROOF. This follows from Example 2.4.9 and Proposition 2.4.14. $\square$

REMARK 2.4.16. Theorem 2.4.15 for the case Λ has finite global dimension was shown in [AR1, Proposition 2.3], but their proof relies on higher algebraic K-theory. Also Theorem 2.4.15 generalizes [Hir] where Λ is assumed to be commutative and Gorenstein. The author does not know whether all the conditions above are equivalent without any assumption on an R-order Λ, even if Λ is commutative.

2.5. Finiteness of syzygies

Let R be a complete Cohen-Macaulay local ring and suppose that $\mathsf{AR}(\mathsf{CM}\, R) = \mathsf{Ex}(\mathsf{CM}\, R)$ holds. Although we do not know whether $\mathsf{CM}\, R$ is of finite type, it is shown in [Kob] that $\Omega\mathsf{CM}\, R$, the category of syzygies of $\mathsf{CM}\, R$ is so. In this section, we will extend this result to a non-commutative order.

Let $\mathcal{E}$ be an exact category with enough projectives. In this section, we denote by $\Omega\mathcal{E}$ the subcategory of $\mathcal{E}$ consisting of objects X such that there exists an inflation $X \rightarrowtail P$ in $\mathcal{E}$ for some projective object P. If $\mathcal{E}$ is Krull-Schmidt, then so is $\Omega\mathcal{E}$ since $\Omega\mathcal{E}$ is closed under direct sums and summands. The essential image of $\Omega\mathcal{E}$ under the natural projection $\mathcal{E} \twoheadrightarrow \underline{\mathcal{E}}$ coincides with $\Omega\underline{\mathcal{E}}$, the essential image of the syzygy functor $\Omega : \underline{\mathcal{E}} \to \underline{\mathcal{E}}$. We begin with the following property about $\Omega\mathcal{E}$, which is of interest in itself.

LEMMA 2.5.1. *Let R be a commutative noetherian ring, and let $\mathcal{E}$ be a Hom-noetherian idempotent complete exact R-category with a projective generator P. Then the syzygy functor $\Omega : \underline{\mathcal{E}} \to \Omega\underline{\mathcal{E}}$ has a fully faithful left adjoint $\Omega^- : \Omega\underline{\mathcal{E}} \to \underline{\mathcal{E}}$.*

PROOF. For each object $W \in \Omega\mathcal{E}$, take a left $(\mathsf{add}\, P)$-approximation $f : W \to P^W$ (this is possible since $\mathcal{E}$ is Hom-noetherian). Since W is in $\Omega\mathcal{E}$, there exists an inflation $f' : W \rightarrowtail P'$ with $P' \in \mathsf{add}\, P$. Thus f' factors through f, which implies that f is an inflation by the idempotent completeness of $\mathcal{E}$ ([Büh, Proposition 7.6]). Now we define $\Omega^- W \in \mathcal{E}$ by the following conflation:

$$0 \to W \xrightarrow{f} P^W \xrightarrow{g} \Omega^- W \to 0.$$

Next suppose that we have a map $\varphi : W_1 \to W_2$ in $\Omega\underline{\mathcal{E}}$. This induces the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & W_1 \arrow[r, "f_1"] \arrow[d, "\varphi"] & P^{W_1} \arrow[r, "g_1"] \arrow[d, "\psi"] & \Omega^- W_1 \arrow[r] \arrow[d, "\Omega^-\varphi"] & 0 \\ 0 \arrow[r] & W_2 \arrow[r, "f_2"] & P^{W_2} \arrow[r, "g_2"] & \Omega^- W_2 \arrow[r] & 0 \end{tikzcd}$$

in $\mathcal{E}$ where both rows are conflations, since $f_2 \circ \varphi$ must factor through f_1. A simple diagram chase shows that this defines a well-defined isomorphism $\Omega^- : \underline{\mathcal{E}}(W_1, W_2) \to \underline{\mathcal{E}}(\Omega^- W_1, \Omega^- W_2)$, with its inverse induced by the syzygy functor $\Omega : \underline{\mathcal{E}} \to \Omega\underline{\mathcal{E}}$. Thus we obtain a functor $\Omega^- : \Omega\underline{\mathcal{E}} \to \underline{\mathcal{E}}$, which is fully faithful and satisfies $\Omega\Omega^- \cong \mathsf{id}_{\Omega\underline{\mathcal{E}}}$.

Finally we show that $\Omega^- : \Omega\underline{\mathcal{E}} \to \underline{\mathcal{E}}$ is a left adjoint of $\Omega : \underline{\mathcal{E}} \to \Omega\underline{\mathcal{E}}$. Let W be in $\Omega\mathcal{E}$ and X in $\mathcal{E}$. Then the syzygy functor induces a morphism $\underline{\mathcal{E}}(\Omega^- W, X) \to \underline{\mathcal{E}}(W, \Omega X)$ by $\Omega\Omega^- W \cong W$ in $\underline{\mathcal{E}}$. It is easy to show that this map is bijective, by considering the following diagram

$$\begin{tikzcd} 0 \arrow[r] & W \arrow[r, "f"] \arrow[d] & P^W \arrow[r, "g"] \arrow[d] & \Omega^- W \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Omega X \arrow[r] & P_X \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

where P_X is projective and f is a left $\mathsf{add}\, P$-approximation. The details are left to the reader. □

THEOREM 2.5.2. *Let R be a complete noetherian local ring and $\mathcal{E}$ a Hom-noetherian idempotent complete exact R-category with a projective generator P and AR conflations. Suppose that $\mathsf{AR}(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathsf{Ex}(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q}$ holds. Then the following holds.*

(1) *For every object M in $\mathsf{eff}\, \mathcal{E}$, we have that $\mathsf{supp}\, M \cap \mathsf{ind}(\Omega\underline{\mathcal{E}})$ is finite, where the support is considered inside $\mathsf{ind}\, \underline{\mathcal{E}}$.*
(2) *Assume in addition that $\underline{\mathcal{E}}$ has a weak cogenerator. Then $\Omega\mathcal{E}$ is of finite type.*

Proof. Throughout this proof, supports of functors are always considered inside $\mathsf{ind}\,\underline{\mathcal{E}}$.

(1) First recall that the stable category $\underline{\mathcal{E}}$ is Hom-finite over R by Lemma 2.4.3. By the similar argument as in Proposition 2.3.7, it suffices to prove the following weaker version of (CF).

(CF)$_\Omega$ Let $0 \to X \xrightarrow{f_i} Y \xrightarrow{g_i} Z \to 0$ be conflations in $\mathcal{E}$ for $i = 1, 2$ and put $M_i := \operatorname{Coker} \mathcal{E}(-, g_i) \in \mathsf{eff}\,\mathcal{E}$. If $\mathsf{supp}\, M_1$ is finite, then $\mathsf{supp}\, M_2 \cap \mathsf{ind}(\Omega\underline{\mathcal{E}})$ is finite.

Assume the above situation and put $N_i := \operatorname{Ker} \underline{\mathcal{E}}(-, \underline{f_i})$ for each $i = 1, 2$. Then we have exact sequences in $\mathsf{mod}\,\underline{\mathcal{E}}$ for $i = 1, 2$, as in the proof of Proposition 2.4.14(2):

$$0 \to N_i \to \underline{\mathcal{E}}(-, \underline{X}) \xrightarrow{\underline{\mathcal{E}}(-,\underline{f_i})} \underline{\mathcal{E}}(-, \underline{Y}) \xrightarrow{\underline{\mathcal{E}}(-,\underline{g_i})} \underline{\mathcal{E}}(-, \underline{Z}) \to M_i \to 0,$$

$$\underline{\mathcal{E}}(-, \Omega\underline{Y}) \xrightarrow{\underline{\mathcal{E}}(-,\Omega\underline{g_i})} \underline{\mathcal{E}}(-, \Omega\underline{Z}) \to N_i \to 0.$$

As in the proof of Proposition 2.4.14(2), it is enough to show that $\mathsf{supp}\, N_1 \cap \mathsf{ind}(\Omega\underline{\mathcal{E}})$ is finite. Actually we will show that every element of $\mathsf{supp}\, N_1 \cap \mathsf{ind}(\Omega\underline{\mathcal{E}})$ is isomorphic to $\Omega\underline{A}$ for some $\underline{A} \in \mathsf{supp}\, M_1$, which obviously implies the desired claim.

Let $\underline{W}$ be an element of $\mathsf{supp}\, N_1 \cap \mathsf{ind}(\Omega\underline{\mathcal{E}})$, so $\underline{\mathcal{E}}(\underline{W}, \Omega\underline{g_1}) : \underline{\mathcal{E}}(\underline{W}, \Omega\underline{Y_1}) \to \underline{\mathcal{E}}(\underline{W}, \Omega\underline{Z_1})$ is not surjective. By Lemma 2.5.1, this is equivalent to that $\underline{\mathcal{E}}(\Omega^-\underline{W}, \underline{g_1}) : \underline{\mathcal{E}}(\Omega^-\underline{W}, \underline{Y_1}) \to \underline{\mathcal{E}}(\Omega^-\underline{W}, \underline{Z_1})$ is not surjective. On the other hand, since $\Omega^- : \Omega\underline{\mathcal{E}} \to \underline{\mathcal{E}}$ is fully faithful by Lemma 2.5.1, we have that $\Omega^-\underline{W}$ is indecomposable. Thus $\Omega^-\underline{W}$ belongs to $\mathsf{supp}\, M_1$. Now $\underline{W} \cong \Omega\Omega^-\underline{W}$ in $\underline{\mathcal{E}}$ holds, which completes the proof.

(2) Let $\underline{X}$ be a weak cogenerator of $\underline{\mathcal{E}}$. By (1), we have that that $\mathsf{ind}(\Omega\underline{\mathcal{E}}) \cap \mathsf{supp}\, \underline{\mathcal{E}}(-, \underline{X})$ is finite. Since every indecomposable object $\underline{W}$ in $\underline{\mathcal{E}}$ should satisfy $\underline{\mathcal{E}}(\underline{W}, \underline{X}) \neq 0$, it follows that $\Omega\underline{\mathcal{E}}$ is of finite type. Now $\Omega\mathcal{E}$ itself is of finite type because we have a natural identification $\mathsf{ind}(\Omega\mathcal{E}) = \mathsf{ind}(\Omega\underline{\mathcal{E}}) \sqcup \mathsf{ind}(\mathsf{add}\, P)$. $\square$

Now we apply this theorem to the category of Cohen-Macaulay modules. The obtained result extends [Hir, Kob], where Λ was assumed to be commutative.

Corollary 2.5.3. *Let R be a complete Cohen-Macaulay local ring and Λ an R-order with at most an isolated singularity. If $\mathsf{AR}(\mathsf{CM}\,\Lambda) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathsf{Ex}(\mathsf{CM}\,\Lambda) \otimes_{\mathbb{Z}} \mathbb{Q}$ holds, then $\Omega\mathsf{CM}\,\Lambda$ is of finite type.*

Proof. We can apply Theorem 2.5.2 to $\mathcal{E} := \mathsf{CM}\,\Lambda$ since $\underline{\mathcal{E}}$ has a weak cogenerator by Example 2.4.9. $\square$

2.A. Some properties of the category of effaceable functors

First we show that our notion of effaceability is equivalent to that of Grothendieck (see [Gro, p. 148] or [Wei, Exercise 2.4.5] for the standard definition).

Proposition 2.A.1. *Let $\mathcal{E}$ be an idempotent complete exact category and $M \in \mathsf{Mod}\,\mathcal{E}$ a right $\mathcal{E}$-module. Then the following are equivalent.*

1. *M is in $\mathsf{eff}\,\mathcal{E}$.*
2. *M is finitely presented, and for every $W \in \mathcal{E}$ and $w \in M(W)$, there exists a deflation $\psi : E \twoheadrightarrow W$ in $\mathcal{E}$ such that $(M\psi)(w) = 0$.*

Proof. (1) $\Rightarrow$ (2): Suppose that M is in $\mathsf{eff}\,\mathcal{E}$ and take a conflation $0 \to X \to Y \xrightarrow{g} Z \to 0$ in $\mathcal{E}$ such that $M \cong \operatorname{Coker} \mathcal{E}(-, g)$. Denote by $\pi : \mathcal{E}(-, Z) \twoheadrightarrow M$ the natural projection. Clearly M is finitely presented. Let $W \in \mathcal{E}$ and $w \in M(W)$. By the Yoneda lemma, we have a morphism $w_* : \mathcal{E}(-, W) \to M$. By the projectivity of $\mathcal{E}(-, W)$ and the Yoneda lemma, we have a map $\varphi : W \to Z$ such that $\pi \circ \mathcal{E}(-, \varphi) = w_*$. Now let us take the pullback of g along φ:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & E \arrow[r, "\psi"] \arrow[d] & W \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & X \arrow[r] & Y \arrow[r, "g"] & Z \arrow[r] & 0
\end{tikzcd}$$

Note that ψ is a deflation. This yields the following commutative diagram in $\mathsf{Mod}\,\mathcal{E}$:

$$\begin{tikzcd}
 & & \mathcal{E}(-,E) \arrow[r, "\mathcal{E}(-,\psi)"] \arrow[d] & \mathcal{E}(-,W) \arrow[d, "\mathcal{E}(-,\varphi)"'] \arrow[dr, "w_*"] & & \\
0 \arrow[r] & \mathcal{E}(-,X) \arrow[r] & \mathcal{E}(-,Y) \arrow[r, "\mathcal{E}(-,g)"'] & \mathcal{E}(-,Z) \arrow[r, "\pi"'] & M \arrow[r] & 0
\end{tikzcd}$$

By applying these functors to E, it is straightforward to check that $(M\psi)(w) = 0$.

(2) $\Rightarrow$ (1): Suppose that M satisfies the condition of (2). Since M is finitely presented, we have a morphism $g : Y \to Z$ in $\mathcal{E}$ such that $\mathcal{E}(-,Y) \to \mathcal{E}(-,Z) \to M \to 0$ is exact. We have an element $z \in M(Z)$ which corresponds to $\mathcal{E}(-,Z) \to M$ by the Yoneda lemma. By the assumption, there exists a deflation $\psi : E \twoheadrightarrow Z$ such that $(M\psi)(z) = 0$. The Yoneda lemma implies that the composition $\mathcal{E}(-,E) \to \mathcal{E}(-,Z) \to M$ is a zero map, hence the map $\mathcal{E}(-,\psi)$ lifts to $\mathcal{E}(-,Y)$. Thus we have a map $h : E \to Y$ such that $g \circ h = \psi$. Since ψ is a deflation, we can conclude that so is g by the idempotent completeness of $\mathcal{E}$. Therefore M is in $\mathsf{eff}\,\mathcal{E}$. $\square$

Next we prove that $\mathsf{eff}\,\mathcal{E}$ is an abelian subcategory of $\mathsf{Mod}\,\mathcal{E}$.

THEOREM 2.A.2. *Let $\mathcal{E}$ be an exact category. Then $\mathsf{eff}\,\mathcal{E}$ is closed under kernels and cokernels in $\mathsf{Mod}\,\mathcal{E}$. In particular, $\mathsf{eff}\,\mathcal{E}$ is an abelian category such that the inclusion $\mathsf{eff}\,\mathcal{E} \to \mathsf{Mod}\,\mathcal{E}$ is exact. Moreover, $\mathsf{eff}\,\mathcal{E}$ is closed under extensions in $\mathsf{Mod}\,\mathcal{E}$.*

PROOF. Let $\varphi : M_1 \to M_2$ be a morphism in $\mathsf{eff}\,\mathcal{E}$. For each $i = 1, 2$, we have the corresponding conflations $0 \to X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i \to 0$ in $\mathcal{E}$ satisfying $M_i = \operatorname{Coker} \mathcal{E}(-, g_i)$. We will show that the image, kernel and cokernel of φ in $\mathsf{Mod}\,\mathcal{E}$ are contained in $\mathsf{eff}\,\mathcal{E}$. Sine φ induces a morphism between the projective resolutions of M_1 and M_2, we have the following commutative diagram in $\mathcal{E}$ by the Yoneda lemma:

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r, "f_1"] \arrow[d, "a"] & Y_1 \arrow[r, "g_1"] \arrow[d, "b"] & Z_1 \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & X_2 \arrow[r, "f_2"] & Y_2 \arrow[r, "g_2"] & Z_2 \arrow[r] & 0
\end{tikzcd}$$

Take the pushout E of the top conflation along $a : X_1 \to X_2$. By the universal property, we obtain the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r, "{{}^t[-a,f_1]}"] \arrow[d, equal] & X_2 \oplus Y_1 \arrow[r, "{[f,b_1]}"] \arrow[d, "{[0,1]}"] & E \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & X_1 \arrow[r, "f_1"] \arrow[d, "a"] & Y_1 \arrow[r, "g_1"] \arrow[d, "b_1"] & Z_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X_2 \arrow[r, "f"] \arrow[d, equal] & E \arrow[r, "g"] \arrow[d, "b_2"] & Z_1 \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & X_2 \arrow[r, "f_2"] \arrow[d, "f"] & Y_2 \arrow[r, "g_2"] \arrow[d, "{{}^t[1,0]}"] & Z_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E \arrow[r, "{{}^t[b_2,g]}"] & Y_2 \oplus Z_1 \arrow[r, "{[g_2,-c]}"] & Z_2 \arrow[r] & 0
\end{tikzcd} \tag{2.A.1}$$

in $\mathcal{E}$ such that $b_2 \circ b_1 = b$ holds. It is straightforward to see that diagram (2.A.1) commutes. Moreover, all the rows are conflations, and the top-right square and the bottom-left square are pullback-pushout squares, see [Büh, Proposition 2.12]

By the Yoneda embedding, we obtain the following commutative diagram in $\mathsf{Mod}\,\mathcal{E}$ corresponding to (2.A.1), where all the rows are exact.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}(-,X_1) \arrow[r] \arrow[d, equal] & \mathcal{E}(-,X_2\oplus Y_1) \arrow[r] \arrow[d] & \mathcal{E}(-,E) \arrow[r] \arrow[d] & K \arrow[r] \arrow[d, "\iota"] & 0 \\
0 \arrow[r] & \mathcal{E}(-,X_1) \arrow[r] \arrow[d] & \mathcal{E}(-,Y_1) \arrow[r] \arrow[d] & \mathcal{E}(-,Z_1) \arrow[r] \arrow[d, equal] & M_1 \arrow[r] \arrow[d, "\varphi_1"] & 0 \\
0 \arrow[r] & \mathcal{E}(-,X_2) \arrow[r] \arrow[d, equal] & \mathcal{E}(-,E) \arrow[r] \arrow[d] & \mathcal{E}(-,Z_1) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\varphi_2"] & 0 \\
0 \arrow[r] & \mathcal{E}(-,X_2) \arrow[r] \arrow[d] & \mathcal{E}(-,Y_2) \arrow[r] \arrow[d] & \mathcal{E}(-,Z_2) \arrow[r] \arrow[d, equal] & M_2 \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & \mathcal{E}(-,E) \arrow[r] & \mathcal{E}(-,Y_2\oplus Z_1) \arrow[r] & \mathcal{E}(-,Z_2) \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

We have that K, M and N belong to $\mathsf{eff}\,\mathcal{E}$. We can check that φ_1 is surjective and φ_2 is injective, hence M is a image of φ. Moreover, one can show that ι is a kernel of φ_1 (or equivalently, φ) and π is a cokernel of φ_2 (or equivalently φ). We leave the details to the reader.

For the extension-closedness of $\mathsf{eff}\,\mathcal{E}$, we refer the reader to [Eno2, Proposition 2.10]. □

In this paper, we often deal with the finiteness of length of effaceable $\mathcal{E}$-modules. Since $\mathsf{eff}\,\mathcal{E}$ and $\mathsf{Mod}\,\mathcal{E}$ are both abelian, the length of effaceable modules seems to depend on the ambient category we adopt. This is not the case by the following.

PROPOSITION 2.A.3. *Let $\mathcal{E}$ be an exact category and M an object in $\mathsf{eff}\,\mathcal{E}$.*

(1) *M has finite length in $\mathsf{eff}\,\mathcal{E}$ if and only if it does so in $\mathsf{Mod}\,\mathcal{E}$.*
(2) *Suppose that $\mathsf{mod}\,\mathcal{E}$ is abelian. Then M has finite length in $\mathsf{Mod}\,\mathcal{E}$ if and only if it does so in $\mathsf{mod}\,\mathcal{E}$.*

Moreover, the length and the composition factors of M in $\mathsf{Mod}\,\mathcal{E}$ coincide with those in $\mathsf{eff}\,\mathcal{E}$ and in $\mathsf{mod}\,\mathcal{E}$.

This follows from the following elementary observation.

LEMMA 2.A.4. *Let $\mathcal{E}$ be an additive category and $\mathcal{A}$ an abelian subcategory of $\mathsf{Mod}\,\mathcal{E}$ such that the inclusion $\mathcal{A} \to \mathsf{Mod}\,\mathcal{E}$ is exact. Suppose that $\mathcal{A}$ is closed under finitely generated submodules. Then the following are equivalent for every object M in $\mathcal{A}$.*

(1) *M has finite length in $\mathsf{Mod}\,\mathcal{E}$.*
(2) *M has finite length in $\mathcal{A}$.*

Moreover, the length and the composition factors of M in $\mathsf{Mod}\,\mathcal{E}$ coincides with those in $\mathcal{A}$.

PROOF. (1) $\Rightarrow$ (2): This is clear since the lattice of subobjects of M in $\mathcal{A}$ is a subposet of that in $\mathsf{Mod}\,\mathcal{E}$.

(2) $\Rightarrow$ (1): It suffices to show that every simple object in $\mathcal{A}$ is also simple in $\mathsf{Mod}\,\mathcal{E}$. Let M be a simple object in $\mathcal{A}$ and N a non-zero submodule of M in $\mathsf{Mod}\,\mathcal{E}$. By using the Yoneda lemma, it is easily checked that there is a non-zero finitely generated submodule N' of N in $\mathsf{Mod}\,\mathcal{E}$. Since N' is a finitely generated submodule of M, we have that N' belongs to $\mathcal{A}$ by the assumption. Thus $N' = M$ holds by $N' \leq M$ in $\mathcal{A}$ and $N' \neq 0$.

The remaining assertions are clear from the above proof and the Jordan-Hölder theorem. □

PROOF OF PROPOSITION 2.A.3. (1) The category $\mathsf{eff}\,\mathcal{E}$ satisfies the conditions in Lemma 2.A.4 by Propositions 2.A.2 and 2.2.4(1).

(2) Suppose that $\mathsf{mod}\,\mathcal{E}$ is abelian. Then it is well-known that the embedding $\mathsf{mod}\,\mathcal{E} \to \mathsf{Mod}\,\mathcal{E}$ is exact and that $\mathsf{mod}\,\mathcal{E}$ is closed under finitely generated submodules, see [Aus1]. Thus Lemma 2.A.4 applies. □

CHAPTER 3

The Jordan-Hölder property and Grothendieck monoids of exact categories

This chapter is based on [Eno4].

We investigate the Jordan-Hölder property (JHP) in exact categories. First we introduce Grothendieck monoids of exact categories, and show that (JHP) holds if and only if the Grothendieck monoid is free. Moreover, we give a criterion for this which only uses the Grothendieck group and the number of simple objects. Next we apply these results to the representation theory of artin algebras. For a large class of exact categories including functorially finite torsion(-free) classes, (JHP) holds precisely when the number of indecomposable projectives is equal to that of simples. We study torsion-free classes in a quiver of type A in detail using the combinatorics of symmetric groups. In particular, we show that simples correspond to Bruhat inversions of a c-sortable element, and give the combinatorial criterion for (JHP).

3.1. Introduction

To begin with, let us recall the *Jordan-Hölder theorem* for modules. This classical theorem says that the ways in which M can be built up from simple modules are essentially unique.

THEOREM (the Jordan-Hölder theorem). *Let Λ be a ring and M a Λ-module of finite length. Then any composition series of M are equivalent. In particular, composition factors of M together with their multiplicities are uniquely determined by M.*

Let us express this situation as follows: the category of Λ-modules of finite length satisfies *the Jordan-Hölder property*, abbreviated by *(JHP)*. The aim of this paper is to investigate to what extent this property is valid for various settings, especially those arising in the representation theory of algebras. It is known that an abelian category satisfies (JHP) if every object has finite length (e.g. [Ste, p.92]), so the abelian case is rather trivial. In this paper, we study (JHP) in the context of Quillen's *exact categories*, which generalize abelian categories and serve as a useful categorical framework for studying various subcategories of module categories.

As in the case of module categories, we can define a *poset of admissible subobjects* of an object of an exact category, and classical notions like simple objects, composition series and (JHP) make sense in this setting (see Section 2). Typical examples of exact categories are extension-closed subcategories of $\mathsf{mod}\,\Lambda$ for an artin algebra Λ, and in this case, all objects have *at least one* composition series. However, it turns out that there exist many categories which do not satisfy (JHP), as well as those which do. Here are some examples (we refer to Section 3.8.2 for more examples).

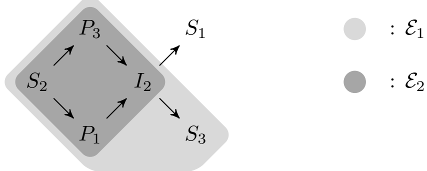
EXAMPLE D (c.f. [BHLR, Example 6.9]). Let k be a field and define $\mathcal{E}$ as the category of finite-dimensional k-vector spaces whose dimensions are not equal to 1. Then obviously $\mathcal{E}$ is closed under extensions in $\mathsf{mod}\,k$, thus an exact category, and k^2 and k^3 are *simple* objects in $\mathcal{E}$. However, we have the following two different decompositions of k^6 into simples:

$$k^3 \oplus k^3 = k^6 = k^2 \oplus k^2 \oplus k^2.$$

Thus composition series (and lengths) are not unique.

The next examples are *torsion-free classes* of a module category, which have been studied in the representation theory of algebras. See Section 3.6 for the detailed explanation of this example.

Example E (c.f. Example 3.6.17 (1)). Let Q be a quiver $1 \to 2 \leftarrow 3$ and consider the path algebra kQ. Then the Auslander-Reiten quiver of $\mathsf{mod}\, kQ$ is given as follows:



Consider the subcategory $\mathcal{E}_1 := \mathsf{add}\{S_2, P_1, P_3, I_2, S_3\}$ of $\mathsf{mod}\,\Lambda$. Then it is checked that S_2, P_1, S_3 are simples of $\mathcal{E}_1$, and $\mathcal{E}_1$ satisfies (JHP). For example, P_3 decomposes into S_2 and S_3, and I_2 decomposes into P_1 and S_3.

On the other hand, consider $\mathcal{E}_2 := \mathsf{add}\{S_2, P_1, P_3, I_2\}$. Then (JHP) fails in $\mathcal{E}_2$: in fact, all four indecomposables are simple in $\mathcal{E}_2$, but we have two exact sequences

$$\begin{aligned} & 0 \to P_1 \to P_1 \oplus P_3 \to P_3 \to 0 \quad \text{(a split sequence)}, \\ \text{and} \quad & 0 \to S_2 \to P_1 \oplus P_3 \to I_2 \to 0 \quad \text{(an almost split sequence)}, \end{aligned}$$

which implies that $P_1 \oplus P_3$ has two different decomposition: one into P_1 and P_3 and the other into S_2 and I_2. Thus *composition factors* of $P_1 \oplus P_3$ are not unique.

To explorer (JHP), we make use of a lesser-known invariant of exact categories, *Grothendieck monoids*. First let us make some observation on the Grothendieck *group* $\mathsf{K}_0(\mathcal{E})$ of $\mathcal{E}$, which is more famous. If $\mathcal{E}$ satisfies (JHP), then we can easily check that $\mathsf{K}_0(\mathcal{E})$ is a free abelian group, since simple objects form a free basis of $\mathsf{K}_0(\mathcal{E})$ by (JHP). Thus (JHP) implies the freeness of $\mathsf{K}_0(\mathcal{E})$. However, the converse is not true: $\mathsf{K}_0(\mathcal{E}_i) \cong \mathbb{Z}^3$ holds for $i = 1, 2$ in Example E but (JHP) fails in $\mathcal{E}_2$. Therefore, we need a *more sophisticated invariant* of exact categories than Grothendieck groups. To this purpose, we propose to study a *Grothendieck monoid* $\mathcal{M}(\mathcal{E})$ of an exact category $\mathcal{E}$, which is a monoid defined by the same universal property as the Grothendieck group. In the representation theory of algebras, this monoid is closely related to the monoid of dimension vectors of modules (see Corollary 3.5.9).

By using $\mathcal{M}(\mathcal{E})$, we obtain the following simple characterization of (JHP):

Theorem F (= Theorem 3.4.12). *Let $\mathcal{E}$ be a skeletally small exact category. Then the following are equivalent:*

(1) *$\mathcal{E}$ satisfies (JHP).*
(2) *$\mathcal{M}(\mathcal{E})$ is a free monoid.*
(3) *$\mathsf{K}_0(\mathcal{E})$ is a free abelian group, and the images of non-isomorphic simple objects in $\mathsf{K}_0(\mathcal{E})$ form a basis of $\mathsf{K}_0(\mathcal{E})$.*

By using this theorem, we obtain the following easier criterion for (JHP) which can be useful in actual situations.

Theorem G (= Theorem 3.4.13). *Let $\mathcal{E}$ be a skeletally small exact category. Suppose that $\mathsf{K}_0(\mathcal{E})$ is finitely generated. Then the following are equivalent:*

(1) *$\mathcal{E}$ satisfies (JHP).*
(2) *$\mathsf{K}_0(\mathcal{E})$ is a free abelian group, and* $\operatorname{rank} \mathsf{K}_0(\mathcal{E})$ *is equal to the number of non-isomorphic simple objects in $\mathcal{E}$.*

We apply these results to the context of the representation theory of artin algebras. In many cases, the Grothendieck group turns out to be free of finite rank, whose rank is equal to the number of non-isomorphic indecomposable projective objects (Proposition 3.5.8). Thus Theorem G tells us that all we have to do to check (JHP) is to count the number of simple objects in $\mathcal{E}$, and then to compare it to that of projectives. In particular, we can give a criterion whether a functorially finite torsion(-free) class satisfies (JHP) using the language of τ-tilting theory (Corollary 3.5.15). As an easy application of this theorem, we show the following result on Nakayama algebras.

Corollary H (= Corollary 3.5.19). *Let Λ be a Nakayama algebra. Then every torsion-free class of* $\mathsf{mod}\,\Lambda$ *satisfies (JHP).*

Finally, we discuss simple objects in torsion-free classes of the category $\mathsf{mod}\, kQ$ for a type A_n quiver Q. Torsion-free classes of $\mathsf{mod}\, kQ$ are classified in [IT, AIRT, Tho]: they are in bijection with c_Q-*sortable elements* in the symmetric group S_{n+1}. Let $\mathcal{F}(w)$ be the torsion-free class of $\mathsf{mod}\, kQ$ corresponding to a c_Q-sortable element w. Then it is natural to expect that we can describe simples in $\mathcal{F}(w)$ and the validity of (JHP) by using the combinatorics of S_{n+1}. We obtain the following result along this line.

THEOREM I (= Theorem 3.6.14, Corollary 3.6.15). *Let Q be a quiver of type A_n and $w \in S_{n+1}$ a c_Q-sortable element. Consider the corresponding torsion-free class $\mathcal{F}(w)$ of $\mathsf{mod}\, kQ$. Then we have the following:*

(1) *Simple objects in $\mathcal{F}(w)$ are in bijection with Bruhat inversions of w.*
(2) *$\mathcal{F}(w)$ satisfies (JHP) if and only if the number of supports of w is equal to that of Bruhat inversions of w.*

Here *Bruhat inversions* are inversions of w which give rise to covering relations in the Bruhat order of S_{n+1} (see Definition 3.6.2). In the paper [Eno5], it is shown that this holds for other Dynkin types (see Remark 3.6.19).

3.1.1. Organization. The content of each section is as follows. In Section 2, we study posets of admissible subobjects of objects in exact categories, and give several basic definitions which we use throughout this paper, such as (JHP). In the latter part of Section 2, we give some categorical conditions which ensure that subobject posets are (modular) lattices. In Section 3, we define the Grothendieck monoid of an exact category, and study its basic properties as a monoid. In Section 4, we give characterizations of (JHP) in terms of Grothendieck monoids or groups (Theorems F and G).

In Section 5, we apply the general results to the representation theory of artin algebras. We mainly consider *good* extension-closed subcategories of module categories (see Assumption 3.5.6), and study (JHP) for this class of categories. In Section 6, we consider torsion-free classes of $\mathsf{mod}\, kQ$ where Q is a quiver of type A and prove Theorem I. In Section 7, we give examples of computations of the Grothendieck monoids. In Section 8, we collect counter-examples on various conditions which we have investigated. In Section 9, some open problems are discussed. In Appendix A, we collect basic properties on monoids which we need in the body part.

3.1.2. Conventions and notation. Throughout this paper, *we assume that all categories are skeletally small*, that is, the isomorphism classes of objects form a set. In addition, *all subcategories are assumed to be full and closed under isomorphisms.* For a category $\mathcal{E}$, we denote by $\mathsf{Iso}\,\mathcal{E}$ the set of all isomorphism classes of objects in $\mathcal{E}$. For an object X in $\mathcal{E}$, the isomorphism class of X is denoted by $[X] \in \mathsf{Iso}\,\mathcal{E}$.

For a set of object $\mathcal{C}$ of an additive category $\mathcal{E}$, we denote by $\mathsf{add}\,\mathcal{C}$ the subcategory of $\mathcal{E}$ consisting of direct summands of finite direct sums of objects in $\mathcal{C}$.

For a Krull-Schmidt category $\mathcal{E}$, we denote by $\mathsf{ind}\,\mathcal{E}$ the set of isomorphism classes of indecomposable objects in $\mathcal{E}$. We denote by $|X|$ the number of non-isomorphic indecomposable direct summands of X.

By a *module* we always mean right modules unless otherwise stated. For a noetherian ring Λ, we denote by $\mathsf{mod}\,\Lambda$ (resp. by $\mathsf{proj}\,\Lambda$) the category of finitely generated right Λ-modules (resp. finitely generated projective right Λ-modules).

As for exact categories, we use the terminologies *inflations*, *deflations* and *conflations.* We refer the reader to [Büh] for the basics of exact categories. For an inflation $A \rightarrowtail X$ in an exact category, we denote by $X \twoheadrightarrow X/A$ the cokernel of $A \rightarrowtail X$. We say that an exact category $\mathcal{E}$ *has a progenerator P* (resp. *an injective cogenerator I*) if P is projective (resp. I is injective) and every object in $\mathcal{E}$ admits a deflation from a finite direct sum of P (resp. an inflation into a finite direct sum of I). For an exact category $\mathcal{E}$, we denote by $\mathcal{P}(\mathcal{E})$ (resp. $\mathcal{I}(\mathcal{E})$) the category consisting of all projective (resp. injective) objects in $\mathcal{E}$.

Let $\mathcal{A}$ be an abelian category and $\mathcal{E}$ a subcategory of $\mathcal{A}$. We say that $\mathcal{E}$ is *extension-closed* if for every short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, we have that Y belongs to $\mathcal{E}$

whenever both X and Z belong to $\mathcal{E}$. In such a case, unless otherwise stated, we regard $\mathcal{E}$ as an exact category with the natural exact structure; conflations are precisely short exact sequences of $\mathcal{A}$ whose terms all belong to $\mathcal{E}$.

For a poset P and two elements $a, b \in P$ with $a \leq b$, we denote by $[a, b]$ the *interval poset* $[a, b] := \{x \in P \mid a \leq x \leq b\}$ with the obvious partial order.

By a *monoid* M we mean a *commutative* semigroup with a unit, and we always use an additive notation: the operation is denoted by $+$, and the unit of addition is always denoted by 0. A *homomorphism* between monoids is a map which preserves the addition and 0.

We denote by $\mathbb{N}$ the monoid of non-negative integers: $\mathbb{N} = \{0, 1, 2, \cdots\}$ with the addition $+$.

For a set A, we denote by $\#A$ the cardinality of A.

3.2. Posets of admissible subobjects

To study the Jordan-Hölder property (JHP) on exact categories, one has to define what (JHP) exactly means, which we will do in this section. The contents of this section are natural generalizations of the corresponding ones in module categories or abelian categories.

3.2.1. Basic properties. First we collect some basic definitions on the poset of admissible subobjects, which we use throughout this paper.

DEFINITION 3.2.1. Let $\mathcal{E}$ be a skeletally small exact category and X an object of $\mathcal{E}$.

- We call an inflation $A \rightarrowtail X$ an *admissible subobject of* X (c.f. [BHLR, Definition 3.1]). We often call A an admissible subobject of X in this case.
- Two subobjects A, B of X are called *equivalent* if there exists an isomorphism $A \xrightarrow{\sim} B$ which makes the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "\wr"] & X \\ B \arrow[ur, tail] & \end{tikzcd}$$

  We denote by $\mathsf{P}(X)$ the equivalence class of admissible subobjects of X. When we want to emphasize the ambient category $\mathcal{E}$, we write $\mathsf{P}_{\mathcal{E}}(X)$.
- We define a partial order on $\mathsf{P}(X)$ as follows: We write $A \leq B$ for $A, B \in \mathsf{P}(X)$ if there exists an *inflation* $A \rightarrowtail B$ which makes the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] & X \\ B \arrow[ur, tail] & \end{tikzcd}$$

  It is easily checked that this relation actually gives a partial order on $\mathsf{P}(X)$.

We remark that $\mathsf{P}(X)$ always has the greatest element X and the smallest element 0.

EXAMPLE 3.2.2. Let $\mathcal{A}$ be an abelian category and $\mathcal{E}$ an extension-closed subcategory of $\mathcal{A}$. Then for an object $X \in \mathcal{E}$, we have that

$$\mathsf{P}_{\mathcal{E}}(X) = \{A \mid A \text{ is a subobject of } X \text{ in } \mathcal{A} \text{ which satisfies } A, X/A \in \mathcal{E}\}.$$

Although we assume that there exists an *inflation* $A \rightarrowtail B$ in order to have $A \leq B$, if $\mathcal{E}$ is *weakly idempotent complete*, then this condition is not necessary. We refer to [Büh, Section 7] for weakly idempotent completeness.

LEMMA 3.2.3. *Let $\mathcal{E}$ be a weakly idempotent complete exact category. Then for an object X and $A, B \in \mathsf{P}(X)$ of X, the following are equivalent:*

(1) *$A \leq B$ holds in $\mathsf{P}(X)$.*

(2) *There exists a morphism $\varphi\colon A \to B$ which makes the following diagram commutes:*

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "\varphi"'] & X \\ B \arrow[ur, tail, "\iota"'] & \end{tikzcd}$$

PROOF. (1) $\Rightarrow$ (2): Obvious.

(2) $\Rightarrow$ (1): It is enough to check that φ is an inflation. This follows from that $\iota\varphi$ is an inflation and $\mathcal{E}$ is weakly idempotent complete, see [Büh, Proposition 7.6] for example. $\square$

EXAMPLE 3.2.4. Let $\mathcal{A}$ be an abelian category and $\mathcal{E}$ an extension-closed subcategory of $\mathcal{A}$. Suppose that $\mathcal{E}$ is *closed under direct summands*, that is, if $A \oplus B$ belongs to $\mathcal{E}$ then so do A and B. Then $\mathcal{E}$ is idempotent complete, thus weakly idempotent complete. Many important exact categories investigated in the representation theory of algebras arise in this way.

The next proposition ensures that for any interval $[A, B]$ in $\mathsf{P}(X)$, we have an isomorphism of posets $[A, B] \cong \mathsf{P}(B/A)$, as in the case of abelian categories.

PROPOSITION 3.2.5. *Let $\mathcal{E}$ be a skeletally small exact category and X an object of $\mathcal{E}$. Then the following hold, where all the intervals are considered in $\mathsf{P}(X)$.*

(1) *For $A \in \mathsf{P}(X)$, we have an isomorphism of posets $[0, A] \cong \mathsf{P}(A)$.*
(2) *For $A \in \mathsf{P}(X)$, we have an isomorphism of posets $(-)/A\colon [A, X] \cong \mathsf{P}(X/A)$. Moreover, $X/B \cong (X/A)/(B/A)$ holds for any B in $[A, X]$.*
(3) *For $A, B \in \mathsf{P}(X)$ with $A \leq B$, we have an isomorphism of posets $(-)/A\colon [A, B] \cong \mathsf{P}(B/A)$. Moreover, for any X_1, X_2 in $\mathsf{P}(X)$ with $A \leq X_1 \leq X_2 \leq B$, we have that $X_2/X_1 \cong (X_2/A)/(X_1/A)$.*

PROOF. (1) For an element $B \in [0, A]$, there exists an inflation $B \rightarrowtail A$ since $B \leq A$ holds. Thus $B \in \mathsf{P}(A)$ holds. On the other hand, let $B \in \mathsf{P}(A)$. Then we have an inflation $B \rightarrowtail A$. Since inflations are closed under compositions, it follows that the composition $B \rightarrowtail A \rightarrowtail X$ is an inflation, hence B is an admissible subobject of X. Consequently we have $B \in \mathsf{P}(X)$, and clearly $B \in [0, A]$ holds. These assignments are easily shown to be mutually inverse isomorphisms of posets between $[0, A]$ and $\mathsf{P}(A)$.

(2) For an element $B \in [A, X]$, we have inflations $A \rightarrowtail B \rightarrowtail X$. Then we obtain the following commutative diagram

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, tail] & B/A \arrow[d, tail] \\ A \arrow[r, tail] & X \arrow[r, two heads] \arrow[d, two heads] & X/A \arrow[d, two heads] \\ & X/B \arrow[r, equal] & X/B \end{tikzcd}$$

in $\mathcal{E}$ (see [Büh, Lemma 3.5]). Thus the assignment $B \mapsto B/A$ gives a morphism of poset $(-)/A\colon [A, X] \to \mathsf{P}(X/A)$. Moreover, by the above sequence, $X/B \cong (X/A)/(B/A)$ holds.

Conversely, let $M \rightarrowtail X/A$ be an admissible subobject of X/A. By taking the pullback of $X \twoheadrightarrow X/A$ along $M \rightarrowtail X/A$, we obtain the following diagram

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, tail] & M \arrow[d, tail] \\ A \arrow[r, tail] & X \arrow[r, two heads] \arrow[d, two heads] & X/A \arrow[d, two heads] \\ & X/B \arrow[r, equal] & X/B \end{tikzcd}$$

in $\mathcal{E}$ where all rows and columns are conflations (see [Büh, Proposition 2.15]). Since $A \leq B$ holds in $\mathsf{P}(X)$, the assignment $M \mapsto B$ gives a morphism of poset $\mathsf{P}(X/A) \to [A, X]$. These maps are easily seen to be mutually inverse to each other.

(3) This follows from (1) and (2), since we have the following chain of isomorphisms of posets

$$\begin{tikzcd} & & \mathsf{P}(A) \\ & \mathsf{P}(B) \arrow[r, "\cong"] & {[0,B]} \arrow[u, hook] \\ \mathsf{P}(B/A) \arrow[r, "\cong"] & {[A,B]} \arrow[u, hook] \arrow[r, "\cong"] & {[A,B]} \arrow[u, hook] \end{tikzcd}$$

where the middle interval is considered in $\mathsf{P}(B)$ and the two right intervals are considered in $\mathsf{P}(A)$.

The isomorphism $X_2/X_1 \cong (X_1/A)/(X_2/A)$ is obtained by applying (2) to $A \in \mathsf{P}(X_2)$. $\square$

Now we introduce *simple* objects in exact categories, that is, objects which cannot be decomposed into smaller pieces with respect to conflations. This objects plays a central role throughout this paper, and to determine simple objects in a given exact category is an essential (and difficult) task when we check (JHP).

Definition 3.2.6. Let $\mathcal{E}$ be an exact category and X an object of X. We say that X is *simple* if the poset $\mathsf{P}(X)$ of admissible subobjects of X has exactly two distinct elements 0 and X. This is equivalent to that X is not zero and there exists no conflation of the form $L \rightarrowtail X \twoheadrightarrow N$ in $\mathcal{E}$ with $L, N \neq 0$. We denote by $\mathsf{sim}\,\mathcal{E}$ the set of isomorphism classes of simple objects in $\mathcal{E}$.

We remark that the notion of simple objects in exact categories was also defined in [BHLR, Definition 3.2] and [BeGr, Definition 5.2].

3.2.2. Basic definitions on inflation series, composition series and (JHP). In module categories, *submodule series* (a chain of submodules) serves as a basic tool when we consider composition series and the Jordan-Hölder theorem. It is natural to introduce the corresponding notion, *inflation series*, in the context of exact categories. Here are the definitions of it and related notions, including (JHP). Note that some of the definitions were also given in [BeGr, BHLR, HR].

First recall some notions from poset theory. A *chain* of a poset P is a totally ordered subset of P. A chain T of P is called *maximal* if there exists no chain of P which properly contains T. A chain T is called *finite* if T is a finite set. Such a chain is of the form $x_0 < x_1 < \cdots < x_n$, and we say that this chain *has length* n in this case.

Let $\mathcal{E}$ be a skeletally small exact category.

- For X in $\mathcal{E}$, an *inflation series* of X is a finite sequence of inflations $0 = X_0 \rightarrowtail X_1 \rightarrowtail \cdots \rightarrowtail X_n = X$. We often identify it with a weakly increasing sequence $0 = X_0 \leq X_1 \leq \cdots \leq X_n = X$ in $\mathsf{P}(X)$.
- We say that an inflation series $0 = X_0 \rightarrowtail X_1 \rightarrowtail \cdots \rightarrowtail X_n = X$ of X is a *proper inflation series* if none of $X_i \rightarrowtail X_{i+1}$ is an isomorphism, or equivalently, $X_0 < X_1 < \cdots < X_n$ holds in $\mathsf{P}(X)$. In this case, we say that this inflation series has *length* n. We often identify proper inflation series with finite chains of $\mathsf{P}(X)$ which contain 0 and X.
- Let X and Y be objects in $\mathcal{E}$, and let $0 = X_0 \rightarrowtail X_1 \rightarrowtail \cdots \rightarrowtail X_n = X$ and $0 = Y_0 \rightarrowtail Y_1 \rightarrowtail \cdots \rightarrowtail Y_m = Y$ be two inflation series of X and Y respectively. Then these inflation series are called *isomorphic* if $m = n$ and there exists a permutation σ of the set $\{1, 2, \cdots n\}$ such that X_i/X_{i-1} and $Y_{\sigma(i)}/Y_{\sigma(i)-1}$ are isomorphic in $\mathcal{E}$ for all i. In this case, we say that X and Y *have isomorphic inflation series*.
- An inflation series $0 = X_0 \rightarrowtail X_1 \rightarrowtail \cdots \rightarrowtail X_n = X$ of X is called a *composition series* if $X_0 < X_1 < \cdots < X_n$ is a maximal chain in $\mathsf{P}(X)$, or equivalently, each quotient X_i/X_{i-1} is a simple object for all i (this follows from Proposition 3.2.5). We often identify composition series with finite maximal chains of $\mathsf{P}(X)$.

- $\mathcal{E}$ is called a *length exact category* if every object X in $\mathcal{E}$, the set of lengths of proper inflation series of X has an upper bound, or equivalently, the set of lengths of finite chains of $\mathsf{P}(X)$ has an upper bound.
- A length exact category $\mathcal{E}$ satisfies the *unique length property* if any composition series of X have the same length for every object X in $\mathcal{E}$.
- A length exact category $\mathcal{E}$ satisfies the *Jordan-Hölder property*, abbreviated by *(JHP)*, if any composition series of X are isomorphic to each other for every object X in $\mathcal{E}$.

First we prove some properties of length exact categories which easily follow from definitions. Actually the proof only uses the general theory of posets.

PROPOSITION 3.2.7. *For a length exact category $\mathcal{E}$ and an object X in $\mathcal{E}$, the following holds:*

(1) *Every chain of $\mathsf{P}(X)$ is finite.*
(2) *Composition series of X are precisely maximal chains of $\mathsf{P}(X)$.*
(3) *Every proper inflation series of X can be refined to a composition series.*
(4) *X has at least one composition series.*

PROOF. (1) Suppose that $\mathsf{P}(X)$ has a chain T consisting of infinitely many elements. Then by choosing a finite subset of T, we can obtain a finite chain of $\mathsf{P}(X)$ with an arbitrary large length, which is a contradiction.

(2) This follows from (1) and the definition of compositions series.

(3) Suppose that a proper chain $0 = X_0 < X_1 < \cdots < X_n = X$ of $\mathsf{P}(X)$ is given. If this chain is maximal, then we have nothing to do. Suppose that this is not the case. If $[X_{i-1}, X_i] = \{X_{i-1}, X_i\}$ for every i, then this chain is clearly maximal. Thus there exist some i and $Y \in \mathsf{P}(X)$ with $X_{i-1} < Y < X_i$. Consequently, we obtain a chain $X_0 < X_1 < \cdots < X_{i-1} < Y < X_i < \cdots < X_n$ of $\mathsf{P}(X)$ with length $n+1$, which is a refinement of the original chain. By iterating this process, we will eventually obtain a finite maximal chain, since lengths of finite chains of $\mathsf{P}(X)$ have an upper bound.

(4) We may assume that $X \neq 0$. Then (4) follows by applying (3) to a chain $0 < X$ of $\mathsf{P}(X)$. □

REMARK 3.2.8. Let $\mathcal{E}$ be an exact category and suppose that for every $X \in \mathcal{E}$ there exists at least one composition series of X. Even so, $\mathcal{E}$ may not be a length exact category, since lengths of finite chains of $\mathsf{P}(X)$ may be arbitrary large (unfortunately the author does not know such an example). However, if $\mathcal{E}$ is an abelian category (or more generally, an exact category such that every subobject poset is a modular lattice), then $\mathcal{E}$ is a length exact category. This follows from Theorem 3.2.18.

REMARK 3.2.9. In the recent paper [HR], it was shown that *nice* exact categories, which are pre-abelian categories satisfying the *nice axiom*, satisfy (JHP).

By definition, (JHP) implies the unique length property for length exact categories. For concrete (counter-)examples concerning (JHP) and the unique length property, we refer the reader to Section 3.8.2.

In the remaining part of this section, we give a sufficient condition for the unique length property by using the theory of modular lattices.

3.2.3. Quasi-abelian implies lattice. Recall that a *lattice* L is a poset such that for every two elements $a, b \in L$, there exist the *meet* $a \wedge b$, the greatest lower bound, and the *join* $a \vee b$, the least upper bound. In abelian categories, it is classical that the subobject poset $\mathsf{P}(X)$ is always a lattice by considering sum and intersection of subobjects. However, this does not hold in general, *even if we consider pre-abelian categories* (see Examples 3.8.1 and 3.8.2). In this subsection, we will show that $\mathsf{P}(X)$ is a lattice when we consider torsion(-free) classes of abelian categories, or more generally, *quasi-abelian* categories.

First we recall the definition of quasi-abelian categories.

DEFINITION 3.2.10. Let $\mathcal{E}$ be an additive category.

(1) $\mathcal{E}$ is *pre-abelian* if every morphism in $\mathcal{E}$ has a kernel and a cokernel. It is well-known that pre-abelian category has all pullbacks and pushouts.
(2) A pre-abelian category $\mathcal{E}$ is *quasi-abelian* if the class of kernel morphisms is closed under pushouts and the class of cokernel morphisms is closed under pullbacks.

Quasi-abelian categories in the above sense are sometimes called *almost abelian* [Rum1, Rum2], or *semi-abelian* [Raĭ]. See Historical remark in [Rum3, Section 2] for the somewhat involved history of quasi-abelian categories.

Every quasi-abelian category has the natural greatest exact structure, and we always consider this greatest exact structure in what follows. More precisely, the following proposition holds. We refer to [Büh, Proposition 4.4] for the proof.

PROPOSITION 3.2.11. *Let $\mathcal{E}$ be a quasi-abelian category. Then $\mathcal{E}$ has the greatest exact structure, in which conflations, inflations and deflations are precisely kernel-cokernel pairs, kernel morphisms and cokernel morphisms respectively.*

By this proposition, for an object X in a quasi-abelian category $\mathcal{E}$, an inflation $A \rightarrowtail X$ is nothing but a kernel morphism in $\mathcal{E}$. Moreover, for two admissible subobjects $\iota_A : A \rightarrowtail X$ and $\iota_B : B \rightarrowtail X$, we have that $A \leq B$ holds in $\mathsf{P}(X)$ if and only if ι_A factors through ι_B. This follows from Proposition 3.2.3 since every pre-abelian category is idempotent complete.

A typical example of quasi-abelian categories are torsion(-free) classes of abelian categories.

DEFINITION 3.2.12. Let $\mathcal{A}$ be an abelian category.

(1) Let $\mathcal{T}$ and $\mathcal{F}$ be subcategories of $\mathcal{A}$. We say that a pair $(\mathcal{T}, \mathcal{F})$ is a *torsion pair* in $\mathcal{A}$ if it satisfies the following conditions:
   (a) $\mathcal{A}(\mathcal{T}, \mathcal{F}) = 0$ holds, that is, $\mathcal{A}(T, F) = 0$ holds for every $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
   (b) For every object $X \in \mathcal{A}$, there exists a short exact sequences
$$0 \to TX \to X \to FX \to 0$$
   in $\mathcal{A}$ with $TX \in \mathcal{T}$ and $FX \in \mathcal{F}$.

   In this case, we say that $\mathcal{T}$ is a torsion class and $\mathcal{F}$ is a torsion-free class.
(2) We say that a torsion pair $(\mathcal{T}, \mathcal{F})$ is *hereditary* if $\mathcal{T}$ is closed under subobjects in $\mathcal{A}$. In this case, we say that $\mathcal{F}$ is a *hereditary torsion-free class.*

For a torsion pair $(\mathcal{T}, \mathcal{F})$ on an abelian category, it is easily checked that $\mathcal{T}$ is closed under quotients and extensions, and that $\mathcal{F}$ is closed under subobjects and extensions. Hence we can regard $\mathcal{T}$ and $\mathcal{F}$ as exact categories.

The next proposition is classical, e.g. [Rum2, Theorem 2]. For the convenience of the reader, we give a simple proof which makes use of the exact structure.

PROPOSITION 3.2.13. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in an abelian category $\mathcal{A}$. Then $\mathcal{T}$ and $\mathcal{F}$ are both quasi-abelian.*

PROOF. We only show that $\mathcal{F}$ is quasi-abelian. Since $\mathcal{F}$ is closed under extensions in $\mathcal{A}$, we have that $\mathcal{F}$ has the natural exact structure, where conflations are short exact sequences with all terms in $\mathcal{F}$. By the axiom of exact categories, inflations are closed under pushouts and deflations are closed under pullbacks. Thus it suffices to show that $\mathcal{E}$ is pre-abelian, that every kernel morphism in $\mathcal{F}$ is an inflation and that every cokernel morphism in $\mathcal{F}$ is a deflation.

First we show that every morphism has a kernel, and every kernel morphism is an inflation. Let $f : X \to Y$ be an arbitrary morphism in $\mathcal{F}$. Then we have the following commutative diagram in $\mathcal{A}$ with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\iota"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, equal] & Y & \\ 0 \arrow[r] & K \arrow[r] & X \arrow[r] & \operatorname{Im} f \arrow[r] \arrow[u, hook] & 0. \end{tikzcd}$$

Since $\mathcal{F}$ is closed under subobjects in $\mathcal{A}$, we have $K, \operatorname{Im} f \in \mathcal{F}$. Therefore $\iota : K \to X$ is a kernel of f in $\mathcal{F}$, and this is actually an inflation in $\mathcal{F}$ since the bottom row is a conflation in $\mathcal{F}$.

Next we show that every morphism in $\mathcal{F}$ has a cokernel and every cokernel morphism is a deflation in $\mathcal{F}$. Let $f\colon X \to Y$ be an arbitrary morphism in $\mathcal{F}$ and denote by C the cokernel of f in $\mathcal{A}$. Then we have the following commutative diagram in $\mathcal{A}$ with exact rows and columns

$$\begin{tikzcd}
 & & X \arrow[d, "f"] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & Y \arrow[r, "\pi"] \arrow[d] & FC \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & TC \arrow[r] & C \arrow[r] \arrow[d] & FC \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

where $TC \in \mathcal{T}$ and $FC \in \mathcal{F}$. Then $K \in \mathcal{F}$ holds since $\mathcal{F}$ is closed under subobjects, and it is straightforward to see that π is a cokernel of f in $\mathcal{F}$. It follows that the middle row is a conflation in $\mathcal{F}$ and that π is actually a deflation in $\mathcal{F}$. $\square$

We omit the following lemma, which can be checked straightforwardly.

LEMMA 3.2.14. *Let $\mathcal{E}$ be an additive category and suppose that we have a commutative diagram*

$$\begin{tikzcd}
 & E \arrow[r, "i'"] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & W \arrow[r, "fg"] \arrow[d, "g"] & Y \arrow[d, equal] \\
0 \arrow[r] & K \arrow[r, "i"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

such that the left square is pullback and i is a kernel of f. Then i' is a kernel of fg. In particular, kernel morphisms are closed under pullbacks if exists.

Now we can prove that *quasi-abelian implies lattice property.*

PROPOSITION 3.2.15. *Let $\mathcal{E}$ be a quasi-abelian category with the greatest exact structure and X an object of $\mathcal{E}$. Then $\mathsf{P}(X)$ is a lattice.*

PROOF. Let $A, B \in \mathsf{P}(X)$ be two admissible subobjects of X.

We first construct the meet $A \cap B$. Note that since $\mathcal{E}$ is pre-abelian, $\mathcal{E}$ has pullbacks. Now we take the following pullback:

$$\begin{tikzcd}
A \cap B \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & A \arrow[d, tail] \\
B \arrow[r, tail] & X
\end{tikzcd}$$

Then by Lemma 3.2.14, the above morphisms $A \cap B \to A$ and $A \cap B \to B$ are kernel morphisms, hence inflations. Thus the composition $A \cap B \rightarrowtail A \rightarrowtail X$ is an inflation, so it is an admissible subobject of X. Hence we have $A \cap B \in \mathsf{P}(X)$ and that $A \cap B \leq A, B$. Suppose that $W \in \mathsf{P}(X)$ satisfies $W \leq A$ and $W \leq B$. Then by the universal property of the pullback, we have a morphism $W \to A \cap B$ such that the composition $W \to A \cap B \rightarrowtail X$ is an inflation. Note that $\mathcal{E}$ is idempotent complete because $\mathcal{E}$ is pre-abelian. Then Lemma 3.2.3 shows that $W \leq A \cap B$ holds in $\mathsf{P}(X)$. Therefore $A \cap B$ is the meet of A and B in the poset $\mathsf{P}(X)$.

Next we will show the existence of the join $A + B \in \mathsf{P}(X)$. Here we shall give two constructions.

(First Construction): This construction is in fact the dual of the previous one, but we include it here for the completeness. Since $\mathcal{E}$ is pre-abelian, it has pushouts. Now we take the following pushout:

$$\begin{tikzcd}
X \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\text{p.o.}"] & X/A \arrow[d] \\
X/B \arrow[r] & C
\end{tikzcd}$$

Then by the dual of Lemma 3.2.14, we have that the above morphism $X/A \to C$ is a cokernel morphism, hence a deflation. It follows that the composition $X \twoheadrightarrow X/A \twoheadrightarrow C$ is a deflation, so C can be written as $C = X/(A+B)$ for some $A+B \in \mathsf{P}(X)$. Now we leave it the reader to verify that $A+B$ is actually a join of A and B in $\mathsf{P}(X)$.

(Second Construction): The second construction is similar to the usual construction of the join in abelian categories: the join $A+B$ is obtained by taking the image of the map $A \oplus B \to X$. Moreover, we will use this construction later to show the modular property of the lattice. To state the construction, we use the following basic property on quasi-abelian categories.

Lemma 3.2.16 ([Büh, Proposition 4.8]). *Every morphism f in a quasi-abelian category admits a factorization $f = \iota p$ such that ι is a kernel morphism and p is an epimorphism.*

Now for admissible subobjects $\iota_A \colon A \rightarrowtail X$ and $\iota_B \colon B \rightarrowtail Y$, consider the induced morphism $[\iota_A, \iota_B] \colon A \oplus B \to X$. By the above lemma, there exist an object $A+B$ and morphisms $A \oplus B \xrightarrow{p} A+B \xrightarrow{\iota} X$ such that $[\iota_A, \iota_B] = \iota p$ holds, p is an epimorphism and ι is a kernel morphism. We claim that $\iota \colon A+B \rightarrowtail X$ is a join of A and B.

Clearly ι_A and ι_B factors through ι, so $A, B \le A+B$ holds by Proposition 3.2.3. On the other hand, suppose that $\iota_C \colon C \rightarrowtail X$ satisfies $A, B \le C$. Then since ι_A, ι_B factors through ι_C, the universal property of $A \oplus B$ yield a morphism $\varphi \colon A \oplus B \to C$ which makes the following diagram commutes, where the bottom row is a conflation.

$$\begin{tikzcd}
 & A \oplus B \arrow[r, "p"] \arrow[d, "\varphi"] & A+B \arrow[d, tail, "\iota"] & & \\
0 \arrow[r] & C \arrow[r, tail, "\iota_C"] & X \arrow[r, two heads, "\pi_C"] & X/C \arrow[r] & 0
\end{tikzcd}$$

It follows that $\pi_C \iota p = 0$, and since p is an epimorphism, we have that $\pi_C \iota = 0$. Thus ι factors through ι_C, so $A+B \le C$ holds in $\mathsf{P}(X)$ by Proposition 3.2.3. $\square$

Remark 3.2.17. Proposition 3.2.15 can be proved directly in the case of a torsion-free class, as follows. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in an abelian category $\mathcal{A}$ and X an object of $\mathcal{F}$. We give a construction of a meet and a join of $A, B \in \mathsf{P}_{\mathcal{F}}(X)$.

- A meet $A \cap B$ is the usual meet of subobjects in an abelian category $\mathcal{A}$. It suffices to observe that $A \cap B$ is actually *admissible*, that is, $X/(A \cap B)$ belongs to $\mathcal{F}$. We have the following exact sequence in $\mathcal{A}$:

$$0 \to X/(A \cap B) \to X/A \oplus X/B$$

  Since $\mathcal{F}$ is closed under direct sums and subobjects, it follows that $X/(A \cap B)$ belongs to $\mathcal{F}$, that is, $A \cap B$ is an admissible subobject of X.
- A join of A and B in $\mathsf{P}_{\mathcal{F}}(X)$ is in general *different from* the usual join $A+B$ of subobjects in an abelian category $\mathcal{A}$. The problem is that $A+B$ is not necessarily admissible, that is, $X/(A+B)$ may not belong to $\mathcal{F}$. However, we have the following exact sequence

$$0 \to T\left(X/(A+B)\right) \to X/(A+B) \to F\left(X/(A+B)\right) \to 0.$$

  with the left term in $\mathcal{T}$ and the right term in $\mathcal{F}$, and let us define $\overline{A+B} \in \mathsf{P}_{\mathcal{F}}(X)$ by the isomorphism $X/(\overline{A+B}) \cong F\left(X/(A+B)\right)$. We leave it to the reader to check that $\overline{A+B}$ is indeed a join of A and B in $\mathsf{P}_{\mathcal{F}}(X)$.

3.2.4. Integral quasi-abelian implies modularity and the unique length property. It is known that a submodule lattice, or more generally, a subobject lattice in an abelian category, is a *modular lattice* (see e.g. [Ste, Proposition IV.5.3]). First, let us see what modularity of subobject lattices implies in our context.

Let us recall the definition of modularity. A lattice L is called *modular* if for every a, b, x in L with $a \le b$, we have $(x \vee a) \wedge b = (x \wedge b) \vee a$. Modularity is an useful tool to study composition series, since we have the following *Jordan-Hölder theorem* for modular lattices. For the proof, we refer the reader to [Ste, Corollary 3.2, Proposition 3.3].

THEOREM 3.2.18 (The Jordan-Hölder theorem for modular lattices). *Let L be a modular lattice with the least element 0 and the greatest element 1. Suppose that L has at least one finite maximal chain. Then the following holds:*

(1) *Every chain of L must be finite.*
(2) *Every maximal chain of L has the same length.*
(3) *Every chain can be refined to some maximal chain, thus the set of lengths of chains of L has an upper bound.*

Moreover, we have some kind of *uniqueness of composition series*: for two maximal chains $0 = x_0 < x_1 < \cdots < x_l = 1$ and $0 = y_0 < y_1 < \cdots < y_l = 1$, each intervals $[x_{i-1}, x_i]$ and $[y_{j-1}, y_j]$ are *"isomorphic"* up to permutations (*isomorphic* here means *projective* in lattice theory, see e.g. [Ste, III. Section 2] for the detail). Actually, the Jordan-Hölder theorem for abelian categories can be proved by using the above lattice-theoretic one (see [Ste, p.92] for example).

For exact categories, modularity of subobject lattices does not imply (JHP) in general because *"isomorphic"* above may not induce an isomorphism. However, modularity clearly *does* imply the unique length property, so the following is an immediate corollary of Theorem 3.2.18.

COROLLARY 3.2.19. *Let $\mathcal{E}$ be an exact category. Suppose that for every object $X \in \mathcal{E}$, the poset $\mathsf{P}(X)$ is a modular lattice and that at least one composition series of X exists. Then $\mathcal{E}$ is a length exact category and satisfies the unique length property.*

In the rest of this section, we will show that *integral quasi-abelian categories* are exact categories in which this situation occurs.

DEFINITION 3.2.20. A quasi-abelian category $\mathcal{E}$ is called *integral quasi-abelian* if the class of epimorphisms is closed under pullbacks and the class of monomorphisms is closed under pushouts.

The typical example is a *hereditary torsion-free class* of abelian categories (Definition 3.2.12), see e.g. [Rum1, Lemma 6] for the proof.

To show modularity, we need the following lemma of integral quasi-abelian categories, which asserts that a regular morphism is an *essential epimorphism*:

LEMMA 3.2.21. *Let $\mathcal{E}$ be an integral quasi-abelian category. Suppose that we have morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{E}$ which satisfy the following conditions:*

(1) *g is a monomorphism and an epimorphism (that is, g is regular).*
(2) *gf is an epimorphism.*

Then f is an epimorphism.

PROOF. Consider the following commutative diagram in $\mathcal{E}$:

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, two heads, "\pi"] \arrow[d, "g"'] \arrow[dr, phantom, "\text{p.o.}"] & \operatorname{Coker} f \arrow[r] \arrow[d, "g'"] & 0. \\ & C \arrow[r, "\pi'"'] & W & \end{tikzcd}$$

Then we have $\pi' g f = g' \pi f = 0$, and since gf is an epimorphism, $\pi' = 0$. Thus $g' \pi = 0$ holds. On the other hand, since monomorphisms are stable under pushouts and g is a monomorphism, so is g'. Therefore $\pi = 0$ holds, so f is an epimorphism. □

To prove modularity, the following criterion is quite useful.

LEMMA 3.2.22 ([Ste, Proposition III.2.3]). *Let L be a lattice. Then the following are equivalent:*

(1) *L is a modular lattice.*
(2) *Take any $a, b \in L$ with $a \leq b$, and take $x, c, c' \in [a, b]$. Suppose that $x \vee c = b = x \vee c'$ and $x \wedge c = a = x \wedge c'$ hold (that is, c and c' are complements of x in $[a, b]$), and that $c \leq c'$ holds. Then we have $c = c'$.*

PROPOSITION 3.2.23. *Let $\mathcal{E}$ be an integral quasi-abelian category with the greatest exact structure and X an object of $\mathcal{E}$. Then $\mathsf{P}(X)$ is a modular lattice.*

PROOF. For admissible subobjects A and B of X, we denote by $A \cap B$ (resp. $A+B$) the meet (resp. join) of them constructed in Proposition 3.2.15.

We make use of Lemma 3.2.22. By Proposition 3.2.5, every interval of $\mathsf{P}(X)$ is isomorphic to $\mathsf{P}(Y)$ for some object Y of $\mathcal{E}$. Thus it suffices to show the following claim:

(Claim): *Let A, B_1, B_2 be admissible subobjects of X. Suppose that $A \cap B_1 = A \cap B_2 = 0$, $A + B_1 = A + B_2 = X$ and $B_1 \leq B_2$ hold in $\mathsf{P}(X)$. Then $B_1 = B_2$ holds.*

(Proof of Claim). Let ι_A, ι_{B_1} and ι_{B_2} be the inflations corresponding to A, B_1 and B_2 respectively. By the construction of the meet given by Proposition 3.2.15, we have the following pullback diagram for each $i = 1, 2$.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & B_i \arrow[d, tail, "\iota_{B_i}"] \\ A \arrow[r, tail, "\iota_A"] & X \end{tikzcd}$$

It follows that $0 \to A \oplus B_i$ is a kernel of $[\iota_A, \iota_{B_i}] \colon A \oplus B_i \to X$, so $[\iota_A, \iota_{B_i}]$ is a monomorphism for each $i = 1, 2$.

On the other hand, by the second construction of the join given in Proposition 3.2.15, we have the following factorization on the join $A + B_i$ for each i:

$$[\iota_A, \iota_{B_i}] \colon A \oplus B_i \xrightarrow{r_i} A + B_i \rightarrowtail X$$

such that r_i is an epimorphism. Now since $A + B_i = X$, we have $r_i = [\iota_A, \iota_{B_i}] \colon A \oplus B_i \to X$.

We have shown that r_i is both a monomorphism and an epimorphism for each i. Let $\iota \colon B_1 \rightarrowtail B_2$ be the inflation corresponding to $B_1 \leq B_2$. Then we have the following commutative diagram:

$$\begin{tikzcd} A \oplus B_1 \arrow[r, "r_1"] \arrow[d, tail, "{\left[\begin{smallmatrix}1 & 0\\ 0 & \iota\end{smallmatrix}\right]}"'] & X \arrow[d, equal] \\ A \oplus B_2 \arrow[r, "r_2"'] & X \end{tikzcd}$$

Since r_2 is a monomorphism and an epimorphism and r_1 is an epimorphism, it follows that $\left[\begin{smallmatrix}1 & 0\\ 0 & \iota\end{smallmatrix}\right] : A \oplus B_1 \to A \oplus B_2$ is an epimorphism by Lemma 3.2.21. However, $\left[\begin{smallmatrix}1 & 0\\ 0 & \iota\end{smallmatrix}\right]$ is a kernel morphism, so it must be an isomorphism. Thus ι also is an isomorphism by the standard matrix calculation. $\square$

COROLLARY 3.2.24. *Let $\mathcal{E}$ be an integral quasi-abelian category which is length as an exact category (for example, a hereditary torsion-free class of length abelian category). Then $\mathcal{E}$ satisfies the unique length property.*

PROOF. This immediately follows from Proposition 3.2.23 and Corollary 3.2.19. $\square$

We refer the reader to Example 3.8.5 for examples which satisfy the unique length property.

3.3. Grothendieck monoid of an exact category

To an exact category $\mathcal{E}$, we can associate the *Grothendieck group* $\mathsf{K}_0(\mathcal{E})$, which is the abelian group defined by the universal property with respect to conflations. The investigation and calculation of this group is a very classical topic in the various area of mathematics. However, much less attention has been paid to the *monoid version* of the Grothendieck group, the *Grothendieck monoid* $\mathcal{M}(\mathcal{E})$, which we introduce in this section.

REMARK 3.3.1. Let us mention some related works on Grothendieck monoids. The notion of Grothendieck monoids was originally introduced and investigated in [BeGr] to study Hall algebras of exact categories. Actually there is some overlap between this paper and [BeGr], and the author is grateful to J. Greenstein for pointing out it.

Also Brookfield studied the structure of the Grothendieck monoid of the category of finitely generated modules over noetherian rings in [Bro1, Bro2, Bro3].

3.3.1. Construction and basic properties of the Grothendieck monoid. Recall that *monoids are assumed to be commutative* in our convention. First we give the definition of Grothendieck monoids and study their basic properties as monoids. We refer to Appendix A for some unexplained notions on monoids.

DEFINITION 3.3.2. Let $\mathcal{E}$ be a skeletally small exact category.

(1) A map $f\colon \mathsf{Iso}\,\mathcal{E} \to M$ to a monoid M is said to *respect conflations* if it satisfies the following conditions:
   (a) $f[0] = 0$ holds.
   (b) For every conflation
$$0 \to X \to Y \to Z \to 0$$
   in $\mathcal{E}$, we have that $f[Y] = f[X] + f[Z]$ holds in M.

(2) A *Grothendieck monoid* $\mathcal{M}(\mathcal{E})$ is a monoid $\mathcal{M}(\mathcal{E})$ together with a map $\pi\colon \mathsf{Iso}\,\mathcal{E} \to \mathcal{M}(\mathcal{E})$ which satisfies the following universal property:
   (a) π respects conflations in $\mathcal{E}$.
   (b) Every map $f\colon \mathsf{Iso}\,\mathcal{E} \to M$ to an monoid M which respects conflations in $\mathcal{E}$ uniquely factors through π, that is, there exists a unique monoid homomorphism $\overline{f}\colon \mathcal{M}(\mathcal{E}) \to M$ which satisfies $f = \overline{f}\pi$.

$$\begin{tikzcd} \mathsf{Iso}\,\mathcal{E} \arrow[r, "f"] \arrow[d, "\pi"'] & M \\ \mathcal{M}(\mathcal{E}) \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

By abuse of notation, we often write $\pi[X] = [X]$ to represent an element in $\mathcal{M}(\mathcal{E})$.

First of all, we must show that the Grothendieck monoid *actually exists*.

PROPOSITION 3.3.3. *For a skeletally small exact category $\mathcal{E}$, its Grothendieck monoid $\mathcal{M}(\mathcal{E})$ exists.*

PROOF. Define the operation $+$ on $\mathsf{Iso}\,\mathcal{E}$ by $[A]+[B] := [A \oplus B]$, then clearly $\mathsf{Iso}\,\mathcal{E}$ is a monoid with an additive unit $[0]$.

Now let us define a monoid congruence $\sim$ on $\mathsf{Iso}\,\mathcal{E}$ which is generated by the following relations (see Proposition 3.A.6): *For every conflation $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$, we impose* $[Y] \sim [X]+[Z]$. Using this, we obtain a monoid $\mathcal{M}(\mathcal{E}) := \mathsf{Iso}\,\mathcal{E}/\sim$. Now it is clear from the construction that $\mathcal{M}(\mathcal{E})$ enjoys the required universal property of the Grothendieck monoid. □

This construction was used to define Grothendieck monoids in [BeGr, Definition 2.6]. The following is a more direct characterization whether $[X] \sim [Y]$ holds in $\mathcal{M}(\mathcal{E})$. Note that for an inflation series $0 = X_0 \rightarrowtail X_1 \rightarrowtail \cdots \rightarrowtail X_n = X$ of X, we can easily show inductively that $[X] = [X_1] + [X_2/X_1] + \cdots + [X_n/X_{n-1}]$ holds in $\mathcal{M}(\mathcal{E})$. Thus if X and Y have isomorphic inflation series, then $[X] = [Y]$ holds in $\mathcal{M}(\mathcal{E})$.

PROPOSITION 3.3.4. *Let $\mathcal{E}$ be a skeletally small exact category and X, Y two objects of $\mathcal{E}$. Then $[X] = [Y]$ holds in $\mathcal{M}(\mathcal{E})$ if and only if there exist a sequence of objects $X = X_0, X_1, \cdots, X_m = Y$ in $\mathcal{E}$ such that X_i and X_{i-1} have isomorphic inflation series for each i.*

PROOF. We freely use the construction of $\mathcal{M}(\mathcal{E})$ in Proposition 3.3.3. For two elements $[X], [Y]$ in $\mathsf{Iso}\,\mathcal{E}$, we write $[X] \approx [Y]$ if there exists objects $X = X_0, X_1, \cdots, X_m = Y$ such that X_i and X_{i-1} have isomorphic inflation series for each i. It is clear that $\approx$ is an equivalence relation on $\mathsf{Iso}\,\mathcal{E}$, and it suffices to show that $\approx$ and $\sim$ coincides.

First we show that $\approx$ is a monoid congruence on $\mathsf{Iso}\,\mathcal{E}$. It is enough to show that for every object A in $\mathcal{E}$, if X and Y have isomorphic inflation series, then so do $A \oplus X$ and $A \oplus Y$. Let $0 = X_0 \rightarrowtail X_1 \rightarrowtail \cdots \rightarrowtail X_n = X$ and $0 = Y_0 \rightarrowtail Y_1 \rightarrowtail \cdots \rightarrowtail Y_n = Y$ be two isomorphic inflation series. Then it is obvious that two inflation series $X_0 \rightarrowtail \cdots \rightarrowtail X_n \rightarrowtail X_n \oplus A$ and $Y_0 \rightarrowtail Y_1 \rightarrowtail \cdots \rightarrowtail Y_n \rightarrowtail Y_n \oplus A$ are isomorphic, where the last inflations are the inclusion into direct summands. Therefore $\approx$ is a monoid congruence on $\mathsf{Iso}\,\mathcal{E}$.

Next we will prove that $\approx$ coincides with $\sim$. From the argument above this proposition, we have that $[X] \approx [Y]$ implies $[X] \sim [Y]$ since $[X] = [X_0] \sim [X_1] \sim \cdots \sim [X_m] = [Y]$. To show the converse implication, it suffices to show that for every conflation $X \rightarrowtail Y \twoheadrightarrow Z$, we have that $[Y] \approx [X] + [Z] = [X \oplus Z]$. This is clear since we have inflation series $0 \rightarrowtail X \rightarrowtail Y$ of Y and $0 \rightarrowtail X \rightarrowtail X \oplus Z$ of $X \oplus Z$ which are isomorphic since $Y/X \cong Z$. $\square$

Using this description of the Grothendieck monoid, we can prove some properties of it.

Proposition 3.3.5 (c.f. [BeGr, Lemma 2.9]). *Let $\mathcal{E}$ be a skeletally small exact category. Then the following hold.*

(1) *For an object X in $\mathcal{E}$, we have that $[X] = 0$ in $\mathcal{M}(\mathcal{E})$ if and only if $X \cong 0$.*
(2) *$\mathcal{M}(\mathcal{E})$ is reduced.*

Proof. (1) This follows from Proposition 3.3.4, since if 0 and X has isomorphic inflation series, then clearly $X \cong 0$ holds.

(2) Suppose that $[X] + [Y] = 0$ holds in $\mathcal{M}(\mathcal{E})$. Then $[X \oplus Y] = 0$ holds, hence $X \oplus Y \cong 0$ by (1). Therefore both X and Y must be isomorphic to 0 in $\mathcal{E}$. Thus $\mathcal{M}(\mathcal{E})$ is reduced. $\square$

We can prove that non-isomorphic simples in $\mathcal{E}$ are distinct in $\mathcal{M}(\mathcal{E})$, and in fact they are in bijection with *atoms* of $\mathcal{M}(\mathcal{E})$. This property is remarkable compared to the Grothendieck group, since non-isomorphic simples may represent the same element in the Grothendieck group $\mathsf{K}_0(\mathcal{E})$ (see Section 3.8.3.1). We refer the reader to Definition 3.A.8 for the notion of atoms in monoids.

Proposition 3.3.6 (c.f. [BeGr, Lemma 5.3]). *Let $\mathcal{E}$ be a skeletally small exact category. Then the following hold.*

(1) *Let S and X be two objects in $\mathcal{E}$. Suppose that $[S] = [X]$ holds in $\mathcal{M}(\mathcal{E})$ and that S is simple. Then $S \cong X$ holds in $\mathcal{E}$.*
(2) *Let S_1 and S_2 be two simple objects in $\mathcal{E}$. Then $[S_1] = [S_2]$ holds in $\mathcal{M}(\mathcal{E})$ if and only if $S_1 \cong S_2$ holds in $\mathcal{E}$.*
(3) *The assignment $S \mapsto [S]$ from $\mathsf{Ob}\,\mathcal{E}$ to $\mathcal{M}(\mathcal{E})$ induces a bijection $\mathsf{sim}\,\mathcal{E} \xrightarrow{\simeq} \mathsf{Atom}\,\mathcal{M}(\mathcal{E})$.*

Proof. (1) Suppose that S and X have isomorphic inflation series for $X \in \mathcal{E}$ and a simple object S in $\mathcal{E}$. Then it is clear that $S \cong X$ holds, since possible inflation series of S is of the form $0 \rightarrowtail \cdots \rightarrowtail 0 \rightarrowtail S = \cdots = S$. Now the assertion of (1) inductively follows from Proposition 3.3.4.

(2) This is a special case of (1).

(3) We first show that $[S]$ is an atom in the monoid $\mathcal{M}(\mathcal{E})$. Suppose that $[S] = [X] + [Y]$. Then we have that $[S] = [X \oplus Y]$, so (1) implies that $S \cong X \oplus Y$. Since simple objects are indecomposable, it follows that $X \cong 0$ or $Y \cong 0$ holds, that is, $[X] = 0$ or $[Y] = 0$. Therefore $[S]$ is an atom.

Conversely, suppose that X is not simple. We may assume that $X \neq 0$, so we have a non-trivial conflation $X_1 \rightarrowtail X \twoheadrightarrow X_2$ with $X_1, X_2 \neq 0$. Then $[X] = [X_1] + [X_2]$ holds in $\mathcal{M}(\mathcal{E})$ and we have $[X_1], [X_2] \neq 0$ by Proposition 3.3.5. Thus $[X]$ is not an atom.

Now we have proved that atoms in $\mathcal{M}(\mathcal{E})$ are precisely elements which come from simple objects. Then the remaining claim follows immediately from (2). $\square$

For later use, we show another necessary condition for $[X] = [Y]$ in $\mathcal{M}(\mathcal{E})$. Recall that a subcategory $\mathcal{D}$ of an exact category $\mathcal{E}$ is called *Serre* if for any conflation $0 \to X \to Y \to Z \to 0$ in $\mathcal{E}$, we have that $Y \in \mathcal{D}$ holds if and only if both $X \in \mathcal{D}$ and $Z \in \mathcal{D}$ hold.

Proposition 3.3.7. *Let $\mathcal{E}$ be a skeletally small exact category and X, Y two objects of $\mathcal{E}$. Suppose that X belongs to a Serre subcategory $\mathcal{D}$ of $\mathcal{E}$ and that $[X] = [Y]$ holds in $\mathcal{M}(\mathcal{E})$. Then Y also belongs to $\mathcal{D}$.*

Proof. By Proposition 3.3.4, it suffices to show the assertion when X and Y have isomorphic inflation series. Take inflation series $0 = X_0 \rightarrowtail X_1 \rightarrowtail \cdots \rightarrowtail X_n = X$ and $0 = Y_0 \rightarrowtail Y_1 \rightarrowtail \cdots \rightarrowtail Y_n = Y$ and a permutation σ of $\{1, \cdots, n\}$ such that $X_i/X_{i-1} \cong Y_{\sigma(i)}/Y_{\sigma(i)-1}$ for each i. Since $X \in \mathcal{D}$ and $\mathcal{D}$ is closed under subquotients, X_i/X_{i-1} belongs to $\mathcal{D}$, thus so does Y_i/Y_{i-1} for each i. Therefore, it inductively follows that $Y \in \mathcal{D}$ holds, since $\mathcal{D}$ is extension-closed and Y can be obtained from Y_i/Y_{i-1}'s by extensions. $\square$

3.3.2. Grothendieck monoid and positive part of Grothendieck group. The *Grothendieck group* of an exact category is a classical invariant, which is defined by the similar universal property as the Grothendieck monoid. In this section, we will recall the Grothendieck group and discuss its relation to the Grothendieck monoid.

DEFINITION 3.3.8. Let $\mathcal{E}$ be a skeletally small exact category.

(1) A map $f\colon \mathsf{Iso}\,\mathcal{E} \to G$ to an abelian group G is said to *respect conflations* if it satisfies $f[Y] = f[X] + f[Z]$ in G for every conflation $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$.
(2) A *Grothendieck group* $\mathsf{K}_0(\mathcal{E})$ of an exact category $\mathcal{E}$ is an abelian group $\mathsf{K}_0(\mathcal{E})$ together with a map $\pi\colon \mathsf{Iso}\,\mathcal{E} \to \mathsf{K}_0(\mathcal{E})$ which satisfies the following universal property:
   (a) π respects conflations in $\mathcal{E}$.
   (b) Every map $f\colon \mathsf{Iso}\,\mathcal{E} \to G$ to an abelian group G which respects conflations in $\mathcal{E}$ uniquely factors through π, that is, there exists a unique group homomorphism $\overline{f}\colon \mathsf{K}_0(\mathcal{E}) \to G$ which satisfies $f = \overline{f}\pi$.

$$\begin{tikzcd} \mathsf{Iso}\,\mathcal{E} \arrow[r, "f"] \arrow[d, "\pi"] & G \\ \mathsf{K}_0(\mathcal{E}) \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

By abuse of notation, we often write $\pi[X] = [X]$ to represent an element of $\mathsf{K}_0(\mathcal{E})$.

By the defining properties of the Grothendieck monoid and the Grothendieck group, it is clear that the latter is obtained from the former by taking the *group completion*, as the following proposition claims. We refer the reader to Appendix A for this notion.

PROPOSITION 3.3.9. *Let $\mathcal{E}$ be a skeletally small exact category. Then the Grothendieck group $\mathsf{K}_0(\mathcal{E})$ exists, and it is realized as the group completion $\mathsf{gp}\,\mathcal{M}(\mathcal{E})$ of the Grothendieck monoid $\mathcal{M}(\mathcal{E})$.*

In what follows, we always identify $\mathsf{gp}\,\mathcal{M}(\mathcal{E})$ with $\mathsf{K}_0(\mathcal{E})$ and we denote by $\iota\colon \mathcal{M}(\mathcal{E}) \to \mathsf{K}_0(\mathcal{E})$ the natural map which satisfies $\iota[X] = [X]$ for every X in $\mathcal{E}$.

DEFINITION 3.3.10. Let $\mathcal{E}$ be a skeletally small exact category. We denote by $\mathsf{K}_0^+(\mathcal{E})$ the image of the natural map $\iota\colon \mathcal{M}(\mathcal{E}) \to \mathsf{K}_0(\mathcal{E})$, that is, $\mathsf{K}_0^+(\mathcal{E}) := \{[X] \mid X \in \mathcal{E}\}$. We call $\mathsf{K}_0^+(\mathcal{E})$ the *positive part* of the Grothendieck group.

It follows from Proposition 3.A.15 that $\iota\colon \mathcal{M}(\mathcal{E}) \to \mathsf{K}_0(\mathcal{E})$ induces an isomorphism of monoids $\mathcal{M}(\mathcal{E})_{\mathsf{can}} \cong \mathsf{K}_0^+(\mathcal{E})$, where $\mathcal{M}(\mathcal{E})_{\mathsf{can}}$ is the largest cancellative quotient of $\mathcal{M}(\mathcal{E})$. Thus the positive part has lost information on *non-cancellative part* compared to the Grothendieck monoid. Even so, $\mathsf{K}_0^+(\mathcal{E})$ is a more sophisticated invariant of $\mathcal{E}$ than $\mathsf{K}_0(\mathcal{E})$. Note that we have many examples in which the Grothendieck monoids are not cancellative, see Section 3.8.3.

We shall see later in Corollary 3.5.9 that for a large class of exact categories $\mathcal{E}$ arising in the representation theory of artin algebras, including functorially finite torsion(-free) classes, we can identify $\mathsf{K}_0^+(\mathcal{E})$ with the monoid of dimension vectors of modules in $\mathcal{E}$.

3.4. Characterization of (JHP)

In this section, We will show the basic relationship between structures of exact categories and combinatorial properties of the Grothendieck monoids.

3.4.1. Length-like functions and length exact categories. First we will introduce an analogue of *dimension* or *length* of modules, which characterizes length exact categories.

DEFINITION 3.4.1. Let $\mathcal{E}$ be a skeletally small exact category. We say that a map $\nu\colon \mathsf{Iso}\,\mathcal{E} \to \mathbb{N}$ is a *weakly length-like function* if it satisfies the following conditions:

(1) For every conflation $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$, we have $\nu[Y] \geq \nu[X] + \nu[Z]$.
(2) $\nu[X] = 0$ implies $X \cong 0$ for every $X \in \mathcal{E}$.

We say that ν is a *length-like* function if it satisfies (2) and the following:

(1′) For every conflation $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$, we have $\nu[Y] = \nu[X] + \nu[Z]$.

Remark 3.4.2. Obviously, a length-like function is weakly length-like. It is clear that we can identify a length-like function with a length-like function on the monoid $\mathcal{M}(\mathcal{E})$ by Proposition 3.3.5 (1), see Definition 3.A.16.

Note that (weakly) length-like functions are far from being unique (for example, any positive multiple of a length-like function is a length-like function).

A weakly length-like function gives an upper bound for all possible lengths of X, as follows.

Lemma 3.4.3. *Let $\mathcal{E}$ be a skeletally small exact category with a weakly length-like function ν. For every $X \in \mathcal{E}$ and proper inflation series $0 = X_0 < X_1 < \cdots < X_n = X$ of X, we have $n \leq \nu[X]$. In particular, $\mathcal{E}$ is a length exact category.*

Proof. Since ν is weakly length-like, we have inductively

$$\begin{aligned}\nu[X] &\geq \nu[X_1] + \nu[X/X_1] \\ &\geq \nu[X_1] + \nu[X_2/X_1] + \nu[X/X_2] \\ &\cdots \\ &\geq \sum_{i=1}^{n} \nu[X_i/X_{i-1}].\end{aligned}$$

Since $X_{i-1} \neq X_i$ for each i, we have $X_i/X_{i-1} \neq 0$. Therefore $\nu[X_i/X_{i-1}] > 0$, hence $\nu[X_i/X_{i-1}] \geq 1$, since ν is weakly length-like. It follows that $\nu[X] \geq n$.

This means that the set of lengths of finite chains of $\mathsf{P}(X)$ has an upper bound $\nu[X]$ for every $X \in \mathcal{E}$, thus $\mathcal{E}$ is a length exact category. □

Actually, the existence of a weakly length-like function is equivalent to the length-ness:

Theorem 3.4.4. *Let $\mathcal{E}$ be a skeletally small exact category. Then the following are equivalent:*

(1) *$\mathcal{E}$ has a weakly length-like function.*
(2) *$\mathcal{E}$ is a length exact category.*

Proof. (1) $\Rightarrow$ (2): This is Lemma 3.4.3.

(2) $\Rightarrow$ (1): For an object $X \in \mathcal{E}$, the lengths of chains of $\mathsf{P}(X)$ has an upper bound, thus we can define $\nu[X]$ by the following:

$$\nu[X] := \max\{\, l \mid \text{there exists a chain of } \mathsf{P}(X) \text{ of length } l \,\}.$$

We will show that $\nu \colon \mathsf{Iso}\,\mathcal{E} \to \mathbb{N}$ is a weakly length-like function. If $\nu[X] = 0$, then $X \cong 0$ holds since otherwise we have a chain $0 < X$ of length one.

Suppose that we have a conflation $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$ and put $n := \nu[X]$ and $m := \nu[Z]$. We have chains $0 = X_0 < X_1 < \cdots < X_n = X$ of $\mathsf{P}(X)$ and $0 = Z_0 < Z_1 < \cdots < Z_m = Z$ of $\mathsf{P}(Z)$. By Proposition 3.2.5, we have an isomorphism of posets $[X, Y] \cong \mathsf{P}(Z)$, so we have a chain $X = \overline{Z_0} < \overline{Z_1} < \cdots < \overline{Z_m} = Y$ of $\mathsf{P}(Y)$ which corresponds to Z_i's. Thus we obtain a chain $0 = X_0 < \cdots < X_n (= X = \overline{Z_0}) < \overline{Z_1} < \cdots < \overline{Z_m} = Y$ of length $m + n$. Therefore $\nu[Y] \geq m + n$ follows by the definition of $\nu[Y]$. □

Remark 3.4.5. The weakly length-like function constructed in the proof of (2) $\Rightarrow$ (1) is the same as the *length* given in [BHLR, Section 6].

Using this theorem, we obtain numerous examples of length exact categories.

Example 3.4.6. The typical example in the representation theory of artin algebra is given as follows. Let Λ be an artin algebra and $\mathcal{E}$ an extension-closed subcategory of $\mathsf{mod}\,\Lambda$. The assignment $X \mapsto l(X)$, where $l(X)$ denotes the (usual) length of X as a Λ-module, gives a length-like function on $\mathcal{E}$. It is clear that the same argument holds for any extension-closed subcategory of a length abelian category.

Remark 3.4.7. It can be showed that the category of maximal Cohen-Macaulay modules $\mathsf{CM}\,R$ over a commutative Cohen-Macaulay ring R (or more generally, $\mathsf{CM}\,\Lambda$ for an R-order Λ) has a length-like function. It is an interesting problem to investigate when these categories satisfy (JHP).

The following gives the criterion for the Grothendieck monoid to be finitely generated.

PROPOSITION 3.4.8. *Let $\mathcal{E}$ be a skeletally small exact category. Suppose that $\mathcal{E}$ is a length exact category. Then $\mathcal{M}(\mathcal{E})$ is atomic, and the following are equivalent:*

(1) $\mathsf{sim}\,\mathcal{E}$ *is a finite set.*
(2) $\mathsf{Atom}\,\mathcal{M}(\mathcal{E})$ *is a finite set.*
(3) $\mathcal{M}(\mathcal{E})$ *is a finitely generated.*

PROOF. Since $\mathcal{E}$ is a length exact category, each object X of $\mathcal{E}$ admits a composition series $0 = X_0 \rightarrowtail X_1 \rightarrowtail \cdots \rightarrowtail X_n$, and X_i/X_{i-1} is simple for each i. It follows that $[X] = [X_1/X_0] + [X_2/X_1] + \cdots + [X_n/X_{n-1}]$ holds in $\mathcal{M}(\mathcal{E})$, and Proposition 3.3.6 implies that each $[X_i/X_{i-1}]$ is an atom in $\mathcal{M}(\mathcal{E})$. Thus $\mathcal{M}(\mathcal{E})$ is atomic.

Proposition 3.3.6 shows that (1) is equivalent to (2). On the other hand, Proposition 3.A.9 (2) shows that (2) is equivalent to (3). ☐

3.4.2. Freeness of monoids and (JHP). Now we can prove the characterization of (JHP) in terms of its Grothendieck monoid.

THEOREM 3.4.9. *Let $\mathcal{E}$ be a skeletally small exact category. Then the following are equivalent:*

(1) $\mathcal{E}$ *is a length exact category and satisfies (JHP).*
(2) $\mathcal{M}(\mathcal{E})$ *is a free monoid, or equivalently, a factorial monoid.*

In this case, $\mathcal{M}(\mathcal{E})$ is a free monoid with basis $\mathsf{Atom}\,\mathcal{M}(\mathcal{E}) = \{[S] \mid S \in \mathsf{sim}\,\mathcal{E}\}$, and $\mathsf{K}_0(\mathcal{E})$ is a free abelian group with basis $\{[S] \mid S \in \mathsf{sim}\,\mathcal{E}\}$.

PROOF. Recall that we have an equality $\mathsf{Atom}\,\mathcal{M}(\mathcal{E}) = \{[S] \mid S \in \mathsf{sim}\,\mathcal{E}\}$ by Proposition 3.3.6. Also note that the conditions in (2) are equivalent by Proposition 3.A.9 (5) and Proposition 3.3.5.

(1) $\Rightarrow$ (2): Let us denote by F the free monoid with basis $\mathsf{Atom}\,\mathcal{M}(\mathcal{E})$. Then we have a natural monoid morphism $\varphi\colon F \to \mathcal{M}(\mathcal{E})$ defined by $\varphi[S] = [S]$ for $[S]$ in $\mathsf{Atom}\,\mathcal{M}(\mathcal{E})$. We shall construct the inverse of this map.

For $[X]$ in $\mathsf{Iso}\,\mathcal{E}$, take a composition series $0 = X_0 < X_1 < \cdots < X_n = X$ of X. Define the map $\mathsf{Iso}\,\mathcal{E} \to F$ by $[X] \mapsto [X_1/X_0] + [X_2/X_1] + \cdots + [X_i/X_{i-1}] \in F$. This map does not depend on the choice of composition series by (JHP). We show that this map respects conflations in $\mathcal{E}$.

Suppose that we have a conflation $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$ and that we have composition series $0 = X_0 < X_1 < \cdots < X_n = X$ in $\mathsf{P}(X)$ and $0 = Z_0 < Z_1 < \cdots < Z_m = Z$ in $\mathsf{P}(Z)$. By Proposition 3.2.5 (2), we have an isomorphism of posets $[X, Y] \cong \mathsf{P}(Z)$, so we have a chain $X = \overline{Z_0} < \overline{Z_1} < \cdots < \overline{Z_m} = Y$ in $\mathsf{P}(Y)$ which corresponds to the chosen composition series of Z. For each i, we have $\overline{Z_i}/\overline{Z_{i-1}} \cong Z_i/Z_{i-1}$ by Proposition 3.2.5 (2), so $\overline{Z_i}/\overline{Z_{i-1}}$ is simple. Thus $0 = X_0 < \cdots < X_n (= X = \overline{Z_0}) < \overline{Z_1} < \cdots < \overline{Z_m} = Y$ is a composition series of Y. Now it is obvious that the map $\mathsf{Iso}\,\mathcal{E} \to F$ constructed above respects conflations. Thus we obtain a monoid homomorphism $\psi\colon \mathcal{M}(\mathcal{E}) \to F$.

We claim that φ and ψ are mutually inverse to each other.

($\psi\varphi = \mathsf{id}_F$): Since F is generated by $[S]$ with $S \in \mathsf{sim}\,\mathcal{E}$, it suffices to show that $\psi\varphi[S] = [S]$ holds. This is obvious since $\varphi[S] = [S]$, and S has the composition series $0 < S$.

($\varphi\psi = \mathsf{id}_{\mathcal{M}(\mathcal{E})}$): Let X be an object in $\mathcal{E}$ with a composition series $0 = X_0 < X_1 < \cdots < X_n = X$. Then $\varphi\psi[X] = [X_1/X_0] + [X_2/X_1] + \cdots + [X_n/X_{n-1}]$. On the other hand, inductively we have $[X] = [X_1/X_0] + [X_2/X_1] + \cdots + [X_n/X_{n-1}]$ in $\mathcal{M}(\mathcal{E})$, thus $\varphi\psi[X] = [X]$ holds.

(2) $\Rightarrow$ (1): First we show that $\mathcal{E}$ is a length exact category. By Lemma 3.A.18, we have that $\mathcal{M}(\mathcal{E})$ has a length-like function. Thus $\mathcal{E}$ has a length-like function, therefore it is a length exact category by Proposition 3.4.4.

Next we will show that $\mathcal{E}$ satisfies (JHP). Let X be an object of $\mathcal{E}$ and let $0 = X_0 < X_1 < \cdots < X_m = X$ and $0 = Y_0 < Y_1 < \cdots < Y_n = X$ be two composition series of X. Then we have

$$[X] = [X_1/X_0] + [X_2/X_1] + \cdots + [X_m/X_{m-1}] = [Y_1/Y_0] + [Y_2/Y_1] + \cdots + [Y_n/Y_{n-1}]$$

in $\mathcal{M}(\mathcal{E})$, where $[X_i/X_{i-1}]$ and $[Y_j/Y_{j-1}]$ belong to $\mathsf{Atom}\,\mathcal{M}(\mathcal{E})$ for each i. Then since $\mathcal{M}(\mathcal{E})$ is free on $\mathsf{Atom}\,\mathcal{M}(\mathcal{E})$ by Proposition 3.A.9 (3), it follows that $m = n$ holds, and that there exists a permutation σ of the set $\{1, 2, \cdots, n\}$ such that $[X_i/X_{i-1}] = [Y_{\sigma(i)}/Y_{\sigma(i)-1}]$ holds in $\mathcal{M}(\mathcal{E})$

for each i. This implies that $X_i/X_{i-1} \cong Y_{\sigma(i)}/Y_{\sigma(i)-1}$ by Proposition 3.3.6 (2). Thus these two composition series are isomorphic.

The remaining assertions follow from Proposition 3.A.9 (3) and Example 3.A.11. $\square$

The most classical example in which (JHP) holds is, as stated in the introduction, a *length abelian category.* Recall that an abelian category is called *length* if every object has a composition series, which implies that it is length as an exact category (see Remark 3.2.8).

COROLLARY 3.4.10. *Let $\mathcal{A}$ be a length abelian category. Then $\mathcal{M}(\mathcal{A})$ is a free monoid with basis $\{[S] \mid S \in \mathsf{sim}\,\mathcal{A}\}$. In particular, $\mathsf{K}_0(\mathcal{A})$ is a free abelian group with the same basis.*

PROOF. This follows from Theorem 3.4.9 and the Jordan-Hölder theorem for length abelian categories (see e.g. [Ste, p.92]). $\square$

3.4.3. Grothendieck groups and (JHP). We have shown that (JHP) can be checked by the freeness of the Grothendieck monoid, but the computation of it is rather hard. In this subsection, we give a criterion for (JHP) in terms of the Grothendieck group, which is easier to compute than the Grothendieck monoid. All contents of this subsection is just a formal consequence of the general theory of monoids, that is, the general criterion for a given monoid to be free (Theorem 3.A.20 and Corollary 3.A.21).

Before we state this, we will make an inequality concerning the Grothendieck groups.

PROPOSITION 3.4.11. *Let $\mathcal{E}$ be a skeletally small length exact category. Suppose that $\mathsf{K}_0(\mathcal{E})$ is a free abelian group. Then the following inequality holds:*

$$\operatorname{rank} \mathsf{K}_0(\mathcal{E}) \leq \# \mathsf{Atom}\, \mathcal{M}(\mathcal{E}) = \# \mathsf{sim}\, \mathcal{E}.$$

PROOF. This follows from the corresponding statement on monoids, Proposition 3.A.19. $\square$

Now let us state our main characterizations of (JHP) in terms of the Grothendieck group.

THEOREM 3.4.12. *Let $\mathcal{E}$ be a skeletally small exact category. Then the following are equivalent:*

(1) *$\mathcal{E}$ is a length exact category and satisfies (JHP).*
(2) *$\mathcal{M}(\mathcal{E})$ is free.*
(3) *$\mathcal{E}$ is a length exact category, and $\mathsf{K}_0(\mathcal{E})$ is a free abelian group of basis $\{[S] \mid S \in \mathsf{sim}\,\mathcal{E}\}$.*
(4) *$\mathcal{E}$ is a length exact category, and all elements $[S] \in \mathsf{K}_0(\mathcal{E})$ with $S \in \mathsf{sim}\,\mathcal{E}$ are linearly independent over $\mathbb{Z}$ in $\mathsf{K}_0(\mathcal{E})$.*

PROOF. This immediately follows from Theorems 3.4.9 and 3.A.20 once we observe the following:

- $\mathsf{Atom}\,\mathcal{M}(\mathcal{E}) = \{[S] \mid S \in \mathsf{sim}\,\mathcal{E}\}$ holds (Proposition 3.3.6).
- $\mathcal{E}$ has a length-like function if and only if $\mathcal{M}(\mathcal{E})$ has a length-like function (Definition 3.4.1).
- $\mathcal{M}(\mathcal{E})$ is reduced (Proposition 3.3.5).
- If $\mathcal{E}$ is a length exact category, then $\mathcal{M}(\mathcal{E})$ is atomic (Proposition 3.4.8).

$\square$

If the Grothendieck group is known to be free of finite rank, then we have a more convenient characterization: we only have to count the number of simples and compare it to the rank.

THEOREM 3.4.13. *Let $\mathcal{E}$ be a skeletally small exact category. Then the following are equivalent:*

(1) *$\mathcal{E}$ is a length exact category satisfying (JHP) and $\#\,\mathsf{sim}\,\mathcal{E}$ is finite.*
(2) *$\mathcal{M}(\mathcal{E})$ is free and $\#\,\mathsf{sim}\,\mathcal{E}$ is finite.*
(3) *$\mathcal{M}(\mathcal{E})$ is a finitely generated free monoid.*
(4) *$\mathcal{E}$ is a length exact category, $\#\,\mathsf{sim}\,\mathcal{E}$ is finite and $\mathsf{K}_0(\mathcal{E})$ is a free abelian group with basis $\{[S] \mid S \in \mathsf{sim}\,\mathcal{E}\}$.*
(5) *The following conditions hold:*
    (a) *$\mathcal{E}$ is a length exact category.*
    (b) *$\mathsf{K}_0(\mathcal{E})$ is a free abelian group of finite rank.*

(c) $\#\,\mathsf{sim}\,\mathcal{E} = \operatorname{rank} \mathsf{K}_0(\mathcal{E})$ *holds.*

Proof. This immediately follows from Theorems 3.4.9 and 3.A.20, as in the proof of Theorem 3.4.12. □

This characterization has lots of applications later, since Grothendieck groups of various exact categories arising in the representation theory of algebras are turned out to be free of finite rank (see Proposition 3.5.12).

3.4.4. Half-factoriality of monoids and the unique length property. In this subsection, we will give a characterization of the *unique length property* in terms of Grothendieck monoids. As we have seen in Theorem 3.4.9, (JHP) corresponds to freeness, or equivalently, factoriality of monoids. We will see that the unique length property corresponds to *half-factoriality.* This is natural since both two properties are about the uniqueness of length of factorizations. We refer the reader to Definition 3.A.8 for the notion of half-factorial monoids.

Theorem 3.4.14. *Let $\mathcal{E}$ be a skeletally small exact category. Then the following are equivalent:*

1. *$\mathcal{E}$ is a length exact category and satisfies the unique length property.*
2. *$\mathcal{E}$ has a length-like function l satisfying $l[S] = 1$ for every simple object S in $\mathcal{E}$.*
3. *$\mathcal{M}(\mathcal{E})$ is half-factorial.*

Proof. The proof is similar to that of Theorem 3.4.9, and actually is easier.

(1) ⇒ (2): We will construct a length-like function $l\colon \mathsf{Iso}\,\mathcal{E} \to \mathbb{N}$. Let X be an object of $\mathcal{E}$. Since $\mathcal{E}$ satisfies the unique length property, X has at least one composition series, and every composition series of X has the same length n. We define $l[X] := n$. Then l is a length-like function by the similar argument to the proof of Theorem 3.4.4. Moreover, clearly $l[S] = 1$ holds for every simple object S in $\mathcal{E}$.

(2) ⇔ (3): By Remark 3.4.2, we can identify a length-like function on $\mathcal{E}$ with that of $\mathcal{M}(\mathcal{E})$. Moreover, we have an equality $\mathsf{Atom}\,\mathcal{M}(\mathcal{E}) = \{[S] \mid S \in \mathsf{sim}\,\mathcal{E}\}$ by Proposition 3.3.6. Thus this equivalence follows from Lemma 3.A.18.

(2) ⇒ (1): By Proposition 3.4.4, our category $\mathcal{E}$ is a length exact category. Let X be an object of $\mathcal{E}$. Then it is easily checked that the length of any composition series of X is equal to $l[X]$. Thus $\mathcal{E}$ satisfies the unique length property. □

3.5. (JHP) for extension-closed subcategories of module categories

In this section, we investigate (JHP) for extension-closed subcategories of module categories of artin algebras. *In the rest of this paper, we fix a commutative artinian ring R.* An R-algebra Λ is called an *artin R-algebra* if Λ is finitely generated as an R-module. We often omit the base ring R and call Λ an *artin algebra.* For an artin R-algebra Λ, we denote by $D\colon \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\Lambda^{\mathrm{op}}$ the standard Matlis duality.

3.5.1. Basic properties on Grothendieck monoids. First we collect some basic properties on the Grothendieck monoid, which immediately follows from the general observations we have made. In particular, our exact category always becomes a length exact category.

Proposition 3.5.1. *Let Λ be an artin algebra and $\mathcal{E}$ an extension-closed subcategory of $\mathsf{mod}\,\Lambda$. Then the following hold.*

1. *The assignment $X \mapsto l(X)$, where $l(X)$ denotes the usual length of X as a Λ-module, induces a length-like function on $\mathcal{E}$.*
2. *$\mathcal{E}$ is a length exact category.*
3. *$\mathcal{M}(\mathcal{E})$ is atomic.*
4. *$\mathcal{M}(\mathcal{E})$ is finitely generated if and only if $\mathsf{sim}\,\mathcal{E}$ is a finite set.*

Proof. These follow from Example 3.4.6 (1), Theorem 3.4.4 and Proposition 3.4.8. □

The simplest example is $\mathcal{E} = \mathsf{mod}\,\Lambda$, and we can compute $\mathcal{M}(\mathsf{mod}\,\Lambda)$ and $\mathsf{K}_0(\mathsf{mod}\,\Lambda)$ as follows.

Proposition 3.5.2. *Let Λ be an artin algebra, and let $S_1, \cdots, S_n$ be a complete set of simple right Λ-modules up to isomorphism. Then $\mathcal{M}(\mathsf{mod}\,\Lambda)$ is a free monoid with basis $[S_1], \cdots, [S_n]$. In particular, $\mathsf{K}_0(\mathsf{mod}\,\Lambda)$ is a free abelian group with the same basis. Moreover, $|\Lambda| = \#\,\mathsf{ind}\,\mathcal{P}(\mathsf{mod}\,\Lambda) = \#\,\mathsf{sim}(\mathsf{mod}\,\Lambda) = n$ holds.*

Proof. All the assertions except the last follow form Corollary 3.4.10. The last one holds because there is a bijection between non-isomorphic indecomposable projective Λ-modules and non-isomorphic simple Λ-modules. $\square$

By this, the Grothendieck group $\mathsf{K}_0(\mathsf{mod}\,\Lambda)$ is often identified with $\mathbb{Z}^n$, where n is the number of simple Λ-modules. This is what is usually called the *dimension vectors* of modules.

Definition 3.5.3. Let Λ be an artin algebra, and fix a complete set of non-isomorphic simple Λ-modules $\{[S_1], \cdots, [S_n]\}$. Then we denote by $\underline{\dim}\colon \mathsf{mod}\,\Lambda \to \mathbb{Z}^n$ the assignment which sends X to $(a_1, \cdots, a_n)$, where a_i is the Jordan-Hölder multiplicity of S_i in X. This induces an isomorphism $\mathsf{K}_0(\mathsf{mod}\,\Lambda) \xrightarrow{\sim} \mathbb{Z}^n$. For a Λ-module X, we call $\underline{\dim}\, X$ the *dimension vector* of X.

A typical example of extension-closed subcategories is an *Ext-perpendicular category* with respect to a given module.

Definition 3.5.4. Let Λ be an artin algebra and $U \in \mathsf{mod}\,\Lambda$ a Λ-module. We denote by ${}^{\perp}U$ the subcategory of $\mathsf{mod}\,\Lambda$ which consists of $X \in \mathsf{mod}\,\Lambda$ such that $\mathrm{Ext}_\Lambda^{>0}(X, U) = 0$ holds, that is, $\mathrm{Ext}_\Lambda^i(X, U) = 0$ for all $i > 0$.

Note that ${}^{\perp}U$ is an extension-closed subcategory of $\mathsf{mod}\,\Lambda$, and we always regard it as an exact category from now on. Exact categories arising in this way have nice homological properties as follows. These can be checked directly, so we omit the proofs.

Proposition 3.5.5. *Let Λ be an artin algebra and $U \in \mathsf{mod}\,\Lambda$ a Λ-module, and put $\mathcal{E} := {}^{\perp}U$. Then the following hold.*

(1) *For any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathsf{mod}\,\Lambda$, if Y and Z belong to $\mathcal{E}$, then so does X.*
(2) *$\mathcal{E}$ is closed under direct summands, thus it is idempotent complete.*
(3) *$\mathsf{proj}\,\Lambda \subset \mathcal{E}$ holds.*
(4) *$\mathcal{E}$ is an exact category with a progenerator Λ.*

3.5.2. Ext-perpendicular categories of modules with finite injective dimension. To check whether (JHP) holds or not, it is easier to deal with the case where the Grothendieck group is free of finite rank, since we can use Theorem 3.4.13. Let us introduce an assumption which ensures this.

Assumption 3.5.6. There exists an artin algebra Λ and a Λ-module $U \in \mathsf{mod}\,\Lambda$ with finite injective dimension such that $\mathcal{E}$ is exact equivalent to ${}^{\perp}U$.

We give examples of such $\mathcal{E}$ later in Example 3.5.13.

Lemma 3.5.7. *Let $\mathcal{E}$, Λ and U be as in Assumption 3.5.6. Then the following hold.*

(1) *Put $n := \mathsf{id}(U_\Lambda)$. For every module $X \in \mathsf{mod}\,\Lambda$, there exists an exact sequence*
$$0 \to \Omega^n X \to P_{n-1} \to \cdots \to P_1 \to P_0 \to X \to 0 \tag{3.5.1}$$
in $\mathsf{mod}\,\Lambda$ such that each P_i is finitely generated projective and $\Omega^n X$ belongs to $\mathcal{E}$.
(2) *The natural inclusion $\mathcal{E} \hookrightarrow \mathsf{mod}\,\Lambda$ induces an isomorphism of the Grothendieck groups $\mathsf{K}_0(\mathcal{E}) \xrightarrow{\sim} \mathsf{K}_0(\mathsf{mod}\,\Lambda)$.*

Proof. (1) By taking a projective resolution of X, obviously there exists a short exact sequence of the form (3.5.1) such that each P_i is finitely generated projective. Thus it suffices to show that $\Omega^n X \in {}^{\perp}U$. This is because $\mathrm{Ext}_\Lambda^{>0}(\Omega^n X, U) = \mathrm{Ext}_\Lambda^{>n}(X, U) = 0$ since $\mathsf{id}\, U = n$.

(2) This follows from (1) and Quillen's Resolution Theorem [Qui, §4] on algebraic K-theory. We give an elementary proof here, which is similar as given in [Yos, Lemma 13.2].

We shall construct the inverse of the natural homomorphism $\mathsf{K}_0(\mathcal{E}) \to \mathsf{K}_0(\mathsf{mod}\,\Lambda)$. For a module $X \in \mathsf{mod}\,\Lambda$, take an exact sequence of the form (3.5.1). Then consider the assignment $X \mapsto \sum_{0\le i<n}(-1)^i[P_i] + (-1)^n[\Omega^n X]$. This assignment does not depend on the choice of exact sequences of the form (3.5.1) by the Schanuel lemma, and it respects short exact sequences by the Horseshoe lemma. Thus we obtain the map $\mathsf{K}_0(\mathsf{mod}\,\Lambda) \to \mathsf{K}_0(\mathcal{E})$. This is the desired inverse, and we leave it the reader to check the details. $\square$

To sum up, our exact category has the following nice properties.

PROPOSITION 3.5.8. *Let $\mathcal{E}$ be as in Assumption 3.5.6. Then the following hold.*

(1) *$\mathcal{E}$ is a length exact category with a progenerator.*
(2) *$\mathsf{K}_0(\mathcal{E})$ is free of finite rank.*
(3) $\operatorname{rank} \mathsf{K}_0(\mathcal{E}) = \#\,\mathsf{ind}\,\mathcal{P}(\mathcal{E}) = |P|$ *holds, where P is a progenerator of $\mathcal{E}$.*

PROOF. (1) follows from Propositions 3.5.1 (2) and 3.5.5 (4). (2) and (3) follows directly from Lemma 3.5.7 and Proposition 3.5.2. $\square$

In this situation, the positive part of the Grothendieck group of $\mathcal{E}$ can be identified with the set of dimension vectors of modules which belong to $\mathcal{E}$.

COROLLARY 3.5.9. *Let Λ and $\mathcal{E}$ be as in Assumption 3.5.6, and fix a complete set of simple Λ-modules $\{[S_1], \cdots, [S_n]\}$. Then the natural map $\mathcal{M}(\mathcal{E}) \to \mathsf{K}_0(\mathcal{E}) \to \mathsf{K}_0(\mathsf{mod}\,\Lambda) \cong \mathbb{Z}^n$ induces an isomorphism of monoids between $\mathcal{M}(\mathcal{E})_{\mathsf{can}}$ and the monoid of dimension vectors of modules in $\mathcal{E}$:*

$$\underline{\dim}\,\mathcal{E} = \{\, \underline{\dim}\,X \mid X \in \mathcal{E}\,\} \subset \mathbb{N}^n.$$

PROOF. This is immediate from Proposition 3.A.15, the definition of $\mathsf{K}_0^+(\mathcal{E})$, an isomorphism $\mathsf{K}_0(\mathcal{E}) \xrightarrow{\sim} \mathsf{K}_0(\mathsf{mod}\,\Lambda)$ shown in Proposition 3.5.5 and the definition of the dimension vector. $\square$

Now our previous characterization of (JHP) has rather simple consequence in this situation.

THEOREM 3.5.10. *Let $\mathcal{E}$ be an exact category which satisfies Assumption 3.5.6. Then $\mathcal{E}$ satisfies (JHP) if and only if $\#\,\mathsf{sim}\,\mathcal{E} = \#\,\mathsf{ind}\,\mathcal{P}(\mathcal{E})$ holds, that is, the number of non-isomorphic simples in $\mathcal{E}$ is equal to that of non-isomorphic indecomposable projectives in $\mathcal{E}$.*

PROOF. Immediately follow from Proposition 3.5.8 and Theorem 3.4.13. $\square$

Actually most exact categories which have been investigated in the representation theory of artin algebras satisfy Assumption 3.5.6. Among them, those arising from *cotilting modules* have been attracted an attention.

DEFINITION 3.5.11. Let Λ be an artin algebra. We say that a Λ-module $U \in \mathsf{mod}\,\Lambda$ is *cotilting* if it satisfies the following conditions:

(1) The injective dimension of U is finite.
(2) $\operatorname{Ext}^{>0}_\Lambda(U,U) = 0$ holds.
(3) There exists an exact sequence

$$0 \to U_n \to \cdots \to U_1 \to U_0 \to D\Lambda \to 0$$

in $\mathsf{mod}\,\Lambda$ for some n such that $U_i \in \mathsf{add}\,U$ for each i.

The simplest example of cotilting modules is $D\Lambda$. In this case, the perpendicular category ${}^\perp(D\Lambda)$ coincides with the module category $\mathsf{mod}\,\Lambda$. For a general cotilting module, although ${}^\perp U$ is not abelian, it has nice properties.

PROPOSITION 3.5.12. *Let Λ be an artin algebra and U a cotilting Λ-module. Put $\mathcal{E} := {}^\perp U$. Then the following hold:*

(1) *$\mathcal{E}$ has a progenerator Λ and an injective cogenerator U.*
(2) *$\mathsf{K}_0(\mathcal{E})$ is a free abelian group of finite rank.*
(3) $\operatorname{rank} \mathsf{K}_0(\mathcal{E}) = \#\,\mathsf{ind}\,\mathcal{P}(\mathcal{E}) = \#\,\mathsf{ind}\,\mathcal{I}(\mathcal{E})$ *holds.*

PROOF. (1) By Proposition 3.5.5, the exact category $\mathcal{E}$ has a progenerator Λ. We refer to [AR2, Theorem 5.4] for the proof of the fact that U is an injective cogenerator of $\mathcal{E}$.

(2) This follows from Propositions 3.5.5 and 3.5.2.

(3) By Propositions 3.5.5 and 3.5.2, we only have to show that $\# \operatorname{ind} \mathcal{I}(\mathcal{E}) = |U|$ is equal to $\# \operatorname{ind} \mathcal{P}(\mathcal{E}) = |\Lambda|$. This follows from tilting theory, e.g. [Hap]. $\square$

In particular, it says the number of indecomposable *projectives* and *injectives* in $\mathcal{E}$ coincide. In the case of $\operatorname{mod} \Lambda$, this number is also equal to the number of *simples*. Thus Theorem 3.5.10 says that the violation of this coincidence is nothing but an obstruction for (JHP).

EXAMPLE 3.5.13. The following exact categories $\mathcal{E}$ satisfy Assumption 3.5.6, hence Theorem 3.5.10 holds for them.

(1) $\mathcal{E} := {}^{\perp}U$ for a cotilting Λ-module U over an artin algebra Λ. Note that $\# \operatorname{ind} \mathcal{P}(\mathcal{E}) = \# \operatorname{ind} \mathcal{I}(\mathcal{E})$ holds in this case, by Proposition 3.5.12.
(2) $\mathcal{E} = \operatorname{GP} \Lambda$ for an Iwanaga-Gorenstein artin algebra Λ. Here Λ is called *Iwanaga-Gorenstein* if $\operatorname{id} \Lambda_\Lambda$ and $\operatorname{id} {}_\Lambda\Lambda$ are both finite, and in this case, we denote by $\operatorname{GP} \Lambda$ the category ${}^{\perp}\Lambda$, called the category of *Gorenstein-projective* modules.
(3) Functorially finite torsion-classes and torsion-free classes of $\operatorname{mod} \Lambda$ for an artin algebra Λ. This classes of categories has been attracted an attention, since *τ-tilting theory* gives a powerful combinatorial tool to investigate them, e.g. [AIR].

PROOF. (1) and (2) follow from definition. (3) is well-known to experts (e.g. see [Iya, Proposition 1.2.1]), but we give a sketch here for the convenience of the reader.

We show that a functorially finite torsion-free class $\mathcal{F}$ is a special case of (1). By factoring out the annihilator, we may assume that $\mathcal{F}$ is faithful torsion-free class of Λ, that is, the intersection of annihilators of modules in $\mathcal{F}$ is zero. Then it can be shown that $\Lambda_\Lambda \in \mathcal{F}$ holds. The well-known characterizations on classical (co)-tilting modules tells us that $\mathcal{F} = {}^{\perp}U$ holds for a cotilting module U with $\operatorname{id} U \leq 1$ (see e.g. [ASS, Theorem VI.6.5]).

For a functorially finite torsion class $\mathcal{T}$, the similar argument shows that $\mathcal{T}$ comes from the classical tilting module. Now the Brenner-Butler theorem ([ASS, Theorem VI.3.8]) tells us that $\mathcal{T}$ is exact equivalent to the torsion-free class over another algebra which is induced by some classical cotilting module. Thus $\mathcal{T}$ is also a special case of (1). $\square$

For those familiar with τ-tilting theory, we will state a consequence of our result in τ-tilting-theoretical language. We omit the related definitions and notation, see [AIR] for the detail.

COROLLARY 3.5.14. *Let Λ be an artin algebra and $\mathcal{F} = \operatorname{Sub} U$ a functorially finite torsion-free class of $\operatorname{mod} \Lambda$ where U is a support τ^--tilting module. Then the following are equivalent:*

(1) *$\mathcal{F}$ satisfies (JHP).*
(2) *The number of non-isomorphic simple objects is equal to $|U|$.*

Dually we obtain the τ-tilting and torsion-class version:

COROLLARY 3.5.15. *Let Λ be an artin algebra and $\mathcal{T} = \operatorname{Fac} T$ a functorially finite torsion class of $\operatorname{mod} \Lambda$ where T is a support τ-tilting module. Then the following are equivalent:*

(1) *$\mathcal{F}$ satisfies (JHP).*
(2) *The number of non-isomorphic simple objects is equal to $|T|$.*

In the case of functorially finite torsion-free classes, we obtain the following finiteness result on the positive cone $\mathcal{M}(\mathcal{F})_{\mathsf{can}} \cong \mathsf{K}_0^+(\mathcal{F})$. Note that $\mathcal{M}(\mathcal{F})$ itself is not in general finitely generated (see Section 3.8.3.1 for example).

PROPOSITION 3.5.16. *Let Λ be an artin algebra and $\mathcal{F}$ a torsion-free class of $\operatorname{mod} \Lambda$ such that $\mathcal{F}$ is the smallest torsion-free class which contains some Λ-module U (for example, $\mathcal{F}$ is functorially finite). Then $\mathsf{K}_0^+(\mathcal{F})$ is isomorphic to a finitely generated submonoid of $\mathbb{N}^n$ for some n.*

PROOF. This follows from the following fact: $\mathcal{F}$ is described as the category of modules which have finite filtrations such that each successive quotient is a submodule of U (see [MS, Lemma

3.1]). It follows that $\mathcal{M}(\mathcal{F})$ is generated by $\{[V] \mid V$ is a submodule of $U\}$. By Corollary 3.5.9, we may identify $\mathsf{K}_0^+(\mathcal{F})$ with the monoid of dimension vectors of modules in $\mathcal{F}$, which is generated by the set of dimension vectors of submodules of U. This set is obviously finite, so $\mathsf{K}_0^+(\mathcal{F})$ is finitely generated. $\square$

3.5.3. Torsion-free classes over Nakayama algebras. In this subsection, we investigate torsion-free classes over Nakayama algebras, and show that any such categories satisfy (JHP).

First we recall the notion of Nakayama algebras. For an artin algebra Λ, we say that a Λ-module M is *uniserial* if the set of submodules of M is totally ordered by inclusion. An artin algebra Λ is called *Nakayama* if every indecomposable right and left projective Λ-module is uniserial. We will use the following description of indecomposable modules over Nakayama algebras (see e.g. [ASS, Chapter V] or [ARS, Section VI.2] for the detail).

PROPOSITION 3.5.17. *Let Λ be a Nakayama algebra. Then the following hold.*

(1) *Every indecomposable modules in* $\mathsf{mod}\,\Lambda$ *is uniserial.*
(2) $M \in \mathsf{ind}(\mathsf{mod}\,\Lambda)$ *is uniquely determined by the following data:*
   (a) *The simple module* $S := \operatorname{top} M = M/\operatorname{rad} M$.
   (b) $m := l(M)$, *the length of M as a Λ-module.*

   In particular, Λ is of finite representation type. We denote this module by $S^{(m)}$.
(3) *If two indecomposable modules M and N in* $\mathsf{mod}\,\Lambda$ *satisfy* $\operatorname{top} M \cong \operatorname{top} N$ *and* $l(M) \geq l(N)$, *then there exists a surjection* $M \twoheadrightarrow N$.

We will investigate simple objects in a torsion-free class of $\mathsf{mod}\,\Lambda$ for a Nakayama algebra. Note that since $\mathsf{mod}\,\Lambda$ has finitely many indecomposables, every torsion-free class of $\mathsf{mod}\,\Lambda$ is functorially finite, so it satisfies Assumption 3.5.6 by Example 3.5.13 (3).

THEOREM 3.5.18. *Let Λ be a Nakayama algebra and $\mathcal{F}$ a torsion-free class of* $\mathsf{mod}\,\Lambda$. *Then there exists bijections between the following three sets:*

(1) $\operatorname{top}\mathcal{F}$, *the set of isomorphism classes of simple modules* $\operatorname{top} M$ *for* $M \in \mathsf{ind}\,\mathcal{F}$.
(2) $\mathsf{sim}\,\mathcal{F}$, *the set of isomorphism classes of simple objects in $\mathcal{F}$.*
(3) $\mathsf{ind}\,\mathcal{P}(\mathcal{F})$, *the set of isomorphism classes of indecomposable projective objects in $\mathcal{F}$.*

The maps from (2) *and* (3) *to* (1) *are given by* $M \mapsto \operatorname{top} M$. *On the other hand, for a simple module* $S \in \operatorname{top}\mathcal{F}$ *in* (1), *the corresponding objects are given by* $S^{(m)}$ *in* (2) *and* $S^{(n)}$ *in* (3), *where*

$$m := \min\{i \mid S^{(i)} \in \mathcal{F}\},$$
$$\textit{and} \quad n := \max\{i \mid S^{(i)} \in \mathcal{F}\}.$$

PROOF. Let us denote by $\operatorname{top}\colon \mathsf{ind}\,\mathcal{F} \to \operatorname{top}\mathcal{F}$ the map which sends M to $\operatorname{top} M$. This induces maps $\mathsf{sim}\,\mathcal{F} \to \operatorname{top}\mathcal{F}$ and $\mathsf{ind}\,\mathcal{P}(\mathcal{F}) \to \operatorname{top}\mathcal{F}$. We will show that these two maps are bijections.

($\operatorname{top}\colon \mathsf{sim}\,\mathcal{F} \to \operatorname{top}\mathcal{F}$ is a bijection): First we will show that this map is a surjection. For $S \in \operatorname{top}\mathcal{F}$, put $m := \min\{i \mid S^{(i)} \in \mathcal{F}\}$. We claim that $S^{(m)}$ is a simple object in $\mathcal{F}$. Suppose this is not the case. Then there exists a short exact sequence

$$0 \to K \to S^{(m)} \to M \to 0$$

in $\mathcal{F}$ with $K, M \neq 0$. Since M is a quotient of $S^{(m)}$, it has top S. But this contradicts the minimality of m because $0 < l(M) < l(S^{(m)}) = m$ holds. Thus $S^{(m)}$ belongs to $\mathsf{sim}\,\mathcal{F}$. Hence the map $\mathsf{sim}\,\mathcal{F} \to \operatorname{top}\mathcal{F}$ is a surjection.

Next we will show that this map is an injection. Suppose that this is not the case. Then there exist a simple module S and $0 < i < j$ such that both $S^{(i)}$ and $S^{(j)}$ are simple in $\mathcal{F}$. Then we have a short exact sequence

$$0 \to K \to S^{(j)} \to S^{(i)} \to 0$$

in $\mathsf{mod}\,\Lambda$ by Proposition 3.5.17 (3), and $K \neq 0$ since $i \neq j$. However, K belongs to $\mathcal{F}$ since $\mathcal{F}$ is closed under submodules. Thus the above is a conflation in $\mathcal{F}$, which shows that $S^{(j)}$ is not a simple object in $\mathcal{F}$. This is a contradiction, so $\mathsf{sim}\,\mathcal{F} \to \operatorname{top}\mathcal{F}$ is an injection.

It is clear from the above argument that the inverse of the map $\operatorname{top}\colon \mathsf{sim}\,\mathcal{F} \to \operatorname{top}\mathcal{F}$ is given by $S \mapsto S^{(m)}$ as claimed.

(top: $\mathsf{ind}\,\mathcal{P}(\mathcal{F}) \to \operatorname{top}\mathcal{F}$ is a bijection): First we will show that this map is a surjection. Let M be an indecomposable object in $\mathcal{F}$ and put $S := \operatorname{top} M$. Since $\mathcal{F}$ satisfies Assumption 3.5.6, it has enough projectives by Proposition 3.5.8 [1]. Thus there exists a deflation

$$P_1 \oplus P_2 \oplus \cdots \oplus P_l \twoheadrightarrow M$$

in $\mathcal{F}$, where $P_i \in \mathsf{ind}\,\mathcal{P}(\mathcal{F})$ for each i. This map is a surjection in $\mathsf{mod}\,\Lambda$, so it induces a surjection

$$\operatorname{top} P_1 \oplus \operatorname{top} P_2 \cdots \oplus \operatorname{top} P_l \twoheadrightarrow \operatorname{top} M = S.$$

It follows that $\operatorname{top} P_i = S$ holds for some i. This means that the map $\mathsf{ind}\,\mathcal{P}(\mathcal{F}) \to \operatorname{top}\mathcal{F}$ is surjective.

Next we will show that this map is an injection. Suppose that this is not the case. Then there exists a simple module S and $0 < i < j$ such that both $S^{(i)}$ and $S^{(j)}$ are projective objects in $\mathcal{F}$. Now Proposition 3.5.17 (3) implies that there exists a short exact sequence

$$0 \to K \to S^{(j)} \to S^{(i)} \to 0$$

in $\mathsf{mod}\,\Lambda$. Since $\mathcal{F}$ is closed under submodules, this is a conflation in $\mathcal{F}$, and since $i < j$, we have $K \neq 0$. However, the projectivity of $S^{(i)}$ implies that the above sequence splits, which is a contradiction since $S^{(j)}$ is indecomposable. Thus top: $\mathsf{ind}\,\mathcal{P}(\mathcal{F}) \to \operatorname{top}\mathcal{F}$ is an injection.

Finally, we shall describe the inverse of top: $\mathsf{ind}\,\mathcal{P}(\mathcal{F}) \to \operatorname{top}\mathcal{F}$. For an object $S \in \operatorname{top}\mathcal{F}$, put $n := \max\{i \mid S^{(i)} \in \mathcal{F}\}$, and we claim that the map $S \mapsto S^{(n)}$ is the inverse. Take any $1 \leq i < n$ with $S^{(i)} \in \mathcal{F}$. Then as in the proof of injectivity, we have the following conflation in $\mathcal{F}$

$$0 \to K \to S^{(n)} \to S^{(i)} \to 0$$

with $K \neq 0$, and this sequence does not split since $S^{(n)}$ is indecomposable. Thus $S^{(i)}$ cannot be projective in $\mathcal{F}$. By this fact and the fact that top: $\mathsf{ind}\,\mathcal{P}(\mathcal{F}) \to \operatorname{top}\mathcal{F}$ is surjective, we must have that the inverse of this map is given by $S \mapsto S^{(n)}$. $\square$

As an immediate corollary, we obtain the following result.

COROLLARY 3.5.19. *Let Λ be a Nakayama algebra and $\mathcal{F}$ a torsion-free class of $\mathsf{mod}\,\Lambda$. Then $\mathcal{F}$ satisfies (JHP).*

PROOF. By Theorem 3.5.18, we have an equality $\#\,\mathsf{sim}\,\mathcal{F} = \#\,\mathsf{ind}\,\mathcal{P}(\mathcal{F})$. Thus the assertion follows from Theorem 3.5.10. $\square$

3.6. Torsion-free classes of Type A and Bruhat inversion

In this section, we investigate simple objects in a torsion-free class of the category of representation of a quiver of type A by using the combinatorics of the symmetric group. For a quiver Q, we denote by kQ the path algebra of Q over a field k. As usual, we identify representations of Q with right kQ-modules.

For an acyclic quiver Q, a classification of torsion-free classes of $\mathsf{mod}\,kQ$ with finitely many indecomposables is known: they are in bijection with so-called *c-sortable elements* of the Coxeter group of Q ([IT] for the Dynkin case and [AIRT, Tho] for the general case). For a quiver of type A, the corresponding Coxeter group is just a symmetric group, and we can describe all indecomposable kQ-modules in a quite explicit way. In this section, we freely use these description particular to type A, but in a paper [Eno5], we will show that the results are valid in other Dynkin types, or more generally, preprojective algebras of Dynkin types (see Remark 3.6.19).

[1] Another way to show this is to use [Eno2, Corollary 3.15]: Every Hom-finite Krull-Schmidt exact k-category has enough projectives if it has only finitely many indecomposables.

3.6.1. Bruhat inversions of elements in the symmetric group. First we recall combinatorial notions on the symmetric groups we need later. These notions are well-studied in the context of Coxeter groups, but we will give an explicit description for type A case here for the convenience of the reader. The standard reference is [BB].

We denote by S_{n+1} the symmetric group which acts on the set $\{1, 2, \cdots, n, n+1\}$ from left. We often use the *one-line notation* to represent elements of S_{n+1}, that is, we write $w = w(1)w(2)\cdots w(n+1)$ for $w \in S_{n+1}$. We denote by $(i\ j)$ for $1 \le i, j \le n+1$ the transposition of the letters i and j, and write T for the set of all transpositions in S_{n+1}. Then S_{n+1} is generated by the *simple reflections* $s_i := (i\ i+1)$ for $1 \le i \le n$. We write S for the set of all simple reflections. For example, we have $s_2 s_1 s_3 s_2 = 3412$ in S_4. Note that for a transposition $t = (i\ j)$ and $w \in S_{n+1}$, the element tw is obtained by interchanging two letters i and j in the one-line notation for w, e.g. we have $(3\ 4) \cdot 3412 = 4312$.

Each element $w \in S_{n+1}$ can be written as a product of simple reflections:

$$w = s_{i_1} s_{i_2} \cdots s_{i_l}$$

This expression of w is called *reduced* if l is the minimal among all such expressions. In this case, we call l the *length* of w and write $\ell(w) := l$.

For an element $w \in S_{n+1}$, a transposition $t \in T$ is called an *inversion of* w if $\ell(tw) < \ell(w)$ holds, and we denote by $\mathsf{inv}(w)$ the set of all inversions of w. It is known that a transposition $(i\ j)$ with $i < j$ is an inversion of w if and only if j precedes i in the one-line notation for w, that is, $w^{-1}(i) > w^{-1}(j)$. It is also known that $\ell(w) = \#\,\mathsf{inv}(w)$ holds.

Example 3.6.1. The following are examples of inversions in S_5.

(1) For $w_1 = s_1 s_3 s_2 s_4 s_1 s_3 s_2 s_4 = 45231$ we have:

$$\mathsf{inv}(w_1) = \{(1\ 2),\ (1\ 3),\ (1\ 4),\ (1\ 5),\ (2\ 4),\ (2\ 5),\ (3\ 4),\ (3\ 5)\}.$$

(2) For $w_2 = s_1 s_3 s_2 s_4 s_1 s_3 s_2 s_1 = 54213$ we have:

$$\mathsf{inv}(w_2) = \{(1\ 2),\ (1\ 4),\ (1\ 5),\ (2\ 4),\ (2\ 5),\ (3\ 4),\ (3\ 5),\ (4\ 5)\}$$

The following class of inversions plays an important role in this paper, since we shall see that this corresponds to simple objects in a torsion-free class.

Definition 3.6.2. We say that an inversion t of an element $w \in S_{n+1}$ is a *Bruhat inversion of* w if it satisfies $\ell(tw) = \ell(w) - 1$. We denote by $\mathsf{Binv}(w)$ the set of Bruhat inversions of w.

We can interpret Bruhat inversions in terms of the *cover relation of the Bruhat order*. Recall that the Bruhat order on S_{n+1} is a partial order $\le$ generated by the following relation: *for every $t \in T$ and $w \in S_{n+1}$ with $\ell(tw) < \ell(w)$, we have that $tw < w$ holds.* In what follows, we always denote by $\le$ (and $<$) the Bruhat order on S_{n+1}.

Lemma 3.6.3 ([BB, Lemma 2.1.4]). *For a transposition $t = (i\ j) \in T$ with $i < j$ and an element $w \in S_{n+1}$, the following are equivalent:*

(1) *t is a Bruhat inversion of w.*
(2) *tw is covered by w in the Bruhat order, that is, $tw < w$ holds and there exists no element $u \in S_{n+1}$ satisfying $tw < u < w$.*
(3) *In the one-line notation for w, the letter j precedes i, and there exists no l with $i < l < j$ such that the letter l appears between j and i.*
(4) *$(i\ j) \in \mathsf{inv}(w)$ and there exists no l with $i < l < j$ satisfying $(i\ l), (l\ j) \in \mathsf{inv}(w)$.*

Example 3.6.4. The following are Bruhat inversions of elements in Example 3.6.1.

(1) For $w_1 = 45231$ we have:

$$\mathsf{Binv}(w_1) = \{(1\ 2),\ (1\ 3),\ (2\ 4),\ (2\ 5),\ (3\ 4),\ (3\ 5)\}.$$

(2) For $w_2 = 54213$ we have:

$$\mathsf{Binv}(w_2) = \{(1\ 2),\ (2\ 4),\ (3\ 4),\ (4\ 5)\}$$

Note that $\#\,\mathsf{inv}(w_1) = \#\,\mathsf{inv}(w_2) = 8$ but $\mathsf{Binv}(w_1) = 6 > 4 = \#\,\mathsf{Binv}(w_2)$.

We will use the notion of support of elements in S_{n+1}.

DEFINITION 3.6.5. Let w be an element of S_{n+1}. Then $i \in \{1, \cdots, n\}$ is called a *support* of w if there exists some reduced expression of w which contains s_i. We denote by $\mathsf{supp}(w)$ the set of all supports of w. We say that w *has full support* if $\mathsf{supp}(w) = \{1, 2, \cdots, n\}$.

In fact, if i is in $\mathsf{supp}(w)$, then any reduced expression of w contains s_i ([BB, Corollary 1.4.8]). We will use the following characterization of the support later.

LEMMA 3.6.6. *For $w \in S_{n+1}$ and $i \in \{1, \cdots, n\}$, the following are equivalent:*

(1) $i \in \mathsf{supp}(w)$.
(2) *There exists some j with $1 \le j \le i$ such that $w^{-1}(j) > i$, that is, j does not appear in the initial segment of length i in the one-line notation for w.*
(3) *There exists some j with $1 \le j \le i$ such that $w(j) > i$.*
(4) *There exists some l with $i < l$ such that l appears in the initial segment of length i in the one-line notation for w.*

For example, by Using this criterion, we can easily check that $\mathsf{supp}(45231) = \{1, 2, 3, 4, 5\}$, $\mathsf{supp}(21543) = \{1, 3, 4\}$, and $\mathsf{supp}(12543) = \{3, 4\}$.

3.6.2. Coxeter element, c-sortable elements and torsion-free classes. We will describe the *Ingalls-Thomas bijection* [IT] between sortable elements and torsion-free classes for a quiver of type A, following [Tho]. In what follows, Q is a quiver of type A with n vertices, whose underlying graph is $1 \frac{\quad\quad}{} 2 \frac{\quad\quad}{} \cdots \frac{\quad\quad}{} n$. As usual, we identify right kQ-modules over the path algebra Q and representations of Q. For a kQ-module M and a vertex i of Q, we denote by M_i the vector space attached to i.

A *Coxeter element* of S_{n+1} is an element $c \in S_{n+1}$ which is obtained as the product of all simple reflections $s_1, \cdots, s_n \in S$ in some order, or equivalently, an element with length n which has full support. We say that a Coxeter element c is *associated to* Q if s_i appears before s_j in c whenever there exists an arrow $i \leftarrow j$ in Q. It is known that Coxeter elements are in bijection with orientations of edges in the underlying graph of Q.

EXAMPLE 3.6.7. We give an example of the correspondence between Coxeter elements in S_4 and orientations of A_3 quiver.

Orientations of A_3	Coxeter elements in S_4
$1 \leftarrow 2 \leftarrow 3$	$s_1 s_2 s_3 = 2341$
$1 \leftarrow 2 \rightarrow 3$	$s_1 s_3 s_2 = s_3 s_1 s_2 = 2413$
$1 \rightarrow 2 \leftarrow 3$	$s_2 s_1 s_3 = s_2 s_3 s_1 = 3142$
$1 \rightarrow 2 \rightarrow 3$	$s_3 s_2 s_1 = 4123$

In what follows, we adopt the following convention:

ASSUMPTION 3.6.8. Q is a quiver of type A with n vertices, whose underlying graph is given by

$$1 \frac{\quad\quad}{} 2 \frac{\quad\quad}{} \cdots \frac{\quad\quad}{} n$$

and c is the Coxeter element of S_{n+1} associated with Q.

Torsion-free classes of $\mathsf{mod}\, kQ$ are classified by the combinatorial notion called *c-sortable elements*, which was introduced by Reading [Rea1].

DEFINITION 3.6.9. Let c be a Coxeter element of S_{n+1}. We say that an element w of S_{n+1} is *c-sortable* if there exists a reduced expression of the form $w = c^{(0)} c^{(1)} \cdots c^{(m)}$ such that each $c^{(i)}$ is a subword of c satisfying $\mathsf{supp}(c^{(0)}) \supset \mathsf{supp}(c^{(1)}) \supset \cdots \supset \mathsf{supp}(c^{(m)})$.

Now we can state the correspondence between c-sortable elements and torsion-free classes. A *support* of a module $M \in \mathsf{mod}\, kQ$ is a vertex $i \in Q$ with $M_i \neq 0$, and we denote by $[i, j)$ the set of elements $l \in \{1, 2, \cdots, n\}$ with $i \le l < j$. The following is just a restatement of the well-known classification of indecomposable representations of Q.

PROPOSITION 3.6.10. *Let Q be a quiver as in Assumption 3.6.8. For a transposition $(i\ j) \in T$ with $i < j$, there exists a unique indecomposable module $M := M_{[i,j)}$ in $\mathsf{mod}\, kQ$ whose support is $[i, j)$. This module is a representation of Q defined by the following:*

- *For vertices, $M_l = k$ if $l \in [i, j)$ and $M_l = 0$ otherwise.*
- *For arrows $i \to j$ in Q, we put $\mathsf{id}_k \colon k \to k$ if $M_i = M_j = k$, and 0 otherwise.*

Moreover, this give a bijection $M_{[\ )} \colon T \xrightarrow{\sim} \mathsf{ind}(\mathsf{mod}\, kQ)$ which restricts to $S \xrightarrow{\sim} \mathsf{sim}(\mathsf{mod}\,\Lambda)$.

EXAMPLE 3.6.11. Let Q be the quiver $Q = 1 \to 2 \leftarrow 3$. Then the Auslander-Reiten quiver of $\mathsf{mod}\, kQ$ is as follows, where indecomposables are labelled according to Proposition 3.6.10.

$$\begin{tikzcd}
 & M_{[2,4)} \arrow[dr] & & M_{[1,2)} \\
M_{[2,3)} \arrow[ur] \arrow[dr] & & M_{[1,4)} \arrow[ur] \arrow[dr] & \\
 & M_{[1,3)} \arrow[ur] & & M_{[3,4)}
\end{tikzcd}$$

Now torsion-free classes of $\mathsf{mod}\, kQ$ are classified as follows. Here a *support* of a subcategory $\mathcal{F}$ of $\mathsf{mod}\, kQ$ is the union of supports of modules in $\mathcal{F}$.

THEOREM 3.6.12 ([Tho, Theorem 4.2]). *Let Q and c be as in Assumption 3.6.8. For $w \in S_{n+1}$, define the subcategory $\mathcal{F}(w)$ of $\mathsf{mod}\, kQ$ by*

$$\mathcal{F}(w) := \mathsf{add}\{M_{[i,j)} \mid (i\ j) \in \mathsf{inv}(w) \text{ with } i < j\}.$$

then the following hold.

(1) *If w is c-sortable, then $\mathcal{F}(w)$ is a torsion-free class of $\mathsf{mod}\, kQ$, and the bijection $T \xrightarrow{\sim} \mathsf{ind}(\mathsf{mod}\, kQ)$ given by $(i\ j) \mapsto M_{[i,j)}$ restricts a bijection $\mathsf{inv}(w) \xrightarrow{\sim} \mathsf{ind}\, \mathcal{F}(w)$.*
(2) *The map $w \mapsto \mathcal{F}(w)$ gives a bijection between c-sortable elements of S_{n+1} and torsion-free classes of $\mathsf{mod}\, kQ$.*
(3) *Let w be a c-sortable element. Then $\mathsf{supp}(w)$ coincides with the support of $\mathcal{F}(w)$, and the equality $\#\, \mathsf{supp}(w) = \#\, \mathsf{ind}\, \mathcal{P}(\mathcal{F}(w)) = \#\, \mathsf{ind}\, \mathcal{I}(\mathcal{F}(w))$ holds.*

PROOF. The proof is essentially contained in other references such as [Tho], but we will clarify the relation between our convention and others, since others often use the language of root systems.

We freely use basics of the root system. Let Φ be a root system of type A_n whose Dynkin graph is as in Assumption 3.6.8, and let $e_1, \cdots, e_n$ be the set of simple roots corresponding to vertices $1, \cdots, n$. We identify S_{n+1} with the Weyl group W of Φ as usual. Under this identification, a transposition $(i\ j)$ of S_{n+1} corresponds to a reflection with respect to a positive root $\alpha_{[i,j)} := \sum_{l \in [i,j)} e_l$. By this, the set of transpositions are in bijection with the set of positive roots.

For a kQ-module M, we define the dimension vector $\underline{\dim}\, M$ by $\underline{\dim}\, M := \sum_l \dim_k(M_l) e_l$. Then $M_{[i,j)}$ in Proposition 3.6.10 satisfies $\underline{\dim}\, M_{[i,j)} = \alpha_{[i,j)}$.

The following description of $\mathsf{inv}(w)$ is well-known, see e.g. [BB, Proposition 4.4.6]:

LEMMA 3.6.13. *Let $t \in S_{n+1} = W$ be a transposition, $\alpha \in \Phi$ the positive root corresponding to t and $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ a reduced expression of arbitrary $w \in W$. Then $t \in \mathsf{inv}(w)$ if and only if $\alpha = s_{i_1} s_{i_2} \cdots s_{i_{l-1}}(e_{i_l})$ for some $1 \le l \le r$.*

Now return to our proof. (1) and (2) follow from [Tho, Theorem 4.2] and our above identification. Let us prove (3).

Let w be a c-sortable element. By Example 3.5.13, the equality $\#\, \mathsf{ind}\, \mathcal{P}(\mathcal{F}(w)) = \#\, \mathsf{ind}\, \mathcal{I}(\mathcal{F}(w))$ holds. On the other hand, the number of supports of $\mathcal{F}(w)$ is equal to $\#\, \mathsf{ind}\, \mathcal{I}(\mathcal{F}(w))$ by the general theory of support (τ-)tilting theory, see e.g. [IT, Proposition 2.5(4), Lemma 2.9] or [AIR, Theorem 2.7]. Thus it suffices to show that the number of supports of $\mathcal{F}(w)$ is equal to $\#\, \mathsf{supp}(w)$.

Take a reduced expression $w = c^{(0)} c^{(1)} \cdots c^{(m)}$ such that $c^{(i)}$ is a subword of c and satisfies $\mathsf{supp}(c^{(0)}) \supset \mathsf{supp}(c^{(1)}) \supset \cdots \supset \mathsf{supp}(c^{(m)})$. First suppose that $l \in \mathsf{supp}(w)$, which means that s_l appears in the word $c^{(0)}$. Then we can write the reduced expression of $c^{(0)}$ as $c^{(0)} = c' s_l c''$ such that c' (and c'') does not contain the letter s_l. Let $(i\ j)$ be a transposition which corresponds to a positive root $\alpha := c'(e_l)$, then it belongs to $\mathsf{inv}(w)$ by Lemma 3.6.13. Clearly the e_l-component of α is equal to 1, thus l is a support of $M_{[i,j)}$ since $\underline{\dim}\, M_{[i,j)} = \alpha$.

Conversely, suppose that l is a support of $M := M_{[i,j)}$ for some $(i\ j) \in \mathsf{inv}(w)$. Take a positive root which corresponds to $(i\ j)$. Then we have $\underline{\dim}\, M = \alpha$, thus the e_l-component of α is strictly positive. On the other hand, Lemma 3.6.13 shows that l must be appear in a reduced expression of w, thus $l \in \mathsf{supp}(w)$. □

3.6.3. Bruhat inversions and simples. Our main result in this section is the following, which establishes a bijection between simples in $\mathcal{F}(w)$ and Bruhat inversions of w.

THEOREM 3.6.14. *Let Q and c be as in Assumption 3.6.8 and w a c-sortable element of S_{n+1}. Then the natural bijection $\mathsf{inv}(w) \xrightarrow{\sim} \mathsf{ind}\,\mathcal{F}(w)$ restricts to a bijection $\mathsf{Binv}(w) \to \mathsf{sim}\,\mathcal{F}(w)$. In other words, for $(i\ j) \in \mathsf{inv}(w)$ with $i < j$, the object $M_{[i,j)}$ is simple in $\mathcal{F}(w)$ if and only if $(i\ j)$ is a Bruhat inversion of w.*

Before we give a proof, let us state the immediate consequence of this, which characterizes (JHP) in a purely combinatorial way. See Example 3.6.17 for the actual example of Theorem 3.6.14 and Corollary 3.6.15.

COROLLARY 3.6.15. *Let Q and c be as in Assumption 3.6.8 and w a c-sortable element of S_{n+1}. Then $\mathcal{F}(w)$ satisfies (JHP) if and only if $\#\,\mathsf{supp}(w) = \#\,\mathsf{Binv}(w)$ holds.*

PROOF. Theorem 3.6.14 implies $\#\,\mathsf{sim}\,\mathcal{F}(w) = \#\,\mathsf{Binv}(w)$. On the other hand, Theorem 3.6.12 implies $\#\,\mathsf{ind}\,\mathcal{P}(\mathcal{F}(w)) = \#\,\mathsf{supp}(w)$. Thus our assertion follows from Theorem 3.5.10. □

To prove Theorem 3.6.14, we use the following explicit exact sequences in $\mathsf{mod}\,kQ$.

LEMMA 3.6.16. *Let $(i\ j)$ be a transposition in S_{n+1} with $i < j$ and consider the kQ-module $M_{[i,j)}$ given in Proposition 3.6.10. Then the following hold.*

(1) *For each l with $i < l < j$, one of the following two exact sequences exists:*

$$\text{either} \quad 0 \to M_{[i,l)} \to M_{[i,j)} \to M_{[l,j)} \to 0, \tag{3.6.1}$$
$$\text{or} \quad 0 \to M_{[l,j)} \to M_{[i,j)} \to M_{[i,l)} \to 0. \tag{3.6.2}$$

(2) *Suppose that we have a monomorphism $M_{[l,l')} \hookrightarrow M_{[i,j)}$ for some $l < l'$. Then we have $[l,l') \subset [i,j)$, and there exists an exact sequence*

$$0 \to M_{[l,l')} \to M_{[i,j)} \to M_{[i,l)} \oplus M_{[l',j)} \to 0. \tag{3.6.3}$$

Moreover, there exist the following two exact sequences:

$$0 \to M_{[i,l')} \to M_{[i,j)} \to M_{[l',j)} \to 0, \tag{3.6.4}$$
$$\text{and} \quad 0 \to M_{[l,j)} \to M_{[i,j)} \to M_{[i,l)} \to 0. \tag{3.6.5}$$

Here we put $M_{[a,a)} = 0$ for $a \in \{1, \cdots, n+1\}$.

PROOF. (1) Consider the orientation of the edge between $l-1$ and l in Q. Suppose that we have $l-1 \leftarrow l$ in Q. A subspace $M_{[i,l)}$ of $M_{[i,j)}$ is closed under the action of Q in $M_{[i,j)}$ because there exists no path which starts from one of $[i,l)$ and ends at one of $[l,j)$. Thus $M_{[i,l)}$ is a submodule of $M_{[i,j)}$, and we obtain (3.6.1). By the same reason, if we have $l-1 \to l$ in Q, then we have (3.6.2).

(2) Since $M_{[l,l')}$ must be a subspace of $M_{[i,j)}$, it is clear that $[l,l') \subset [i,j)$ holds. The support of the quotient $M_{[i,j)}/M_{[l,l')}$ is a disjoint union $[i,l) \sqcup [l',j)$, and it is easily checked that the action of kQ on this quotient coincides with that on $M_{[i,l)} \oplus M_{[l',j)}$. Thus we obtain (3.6.3). Moreover, since $M_{[l,l')}$ is closed under actions of kQ in $M_{[i,j)}$, we must have that either $i = l$ or $l-1 \to l$ in Q holds, and that either $l' = j$ or $l'-1 \leftarrow l'$ in Q holds. Thus the existence of two exact sequences (3.6.4) and (3.6.5) follows from the proof of (1). □

PROOF OF THEOREM 3.6.14. Let $(i\ j)$ be an inversion of w with $i < j$. Since any simple object is indecomposable, it suffices to show that $(i\ j)$ is a Bruhat inversion of w if and only if $M_{[i,j)}$ is simple in $\mathcal{F}(w)$.

First, suppose that $(i\ j)$ is *not* a Bruhat inversion of w. Then by Proposition 3.6.3, there exists some l with $i < l < j$ such that both $(i\ l)$ and $(l\ j)$ belong to $\mathsf{inv}(w)$. Thus both $M_{[i,l)}$ and $M_{[l,j)}$ belong to $\mathcal{F}(w)$. Now we have an exact sequence (3.6.1) or (3.6.2) by Lemma 3.6.16. In either case, this gives a conflation in $\mathcal{F}(w)$, and since $M_{[i,l)}$ and $M_{[l,j)}$ are non-zero, $M_{[i,j)}$ is *not* a simple object in $\mathcal{F}(w)$.

Conversely, suppose that $M_{[i,j)}$ is *not* a simple object in $\mathcal{F}(w)$. Then we have a non-isomorphic inflation $N \rightarrowtail M_{[i,j)}$ in $\mathcal{F}(w)$ for some object $0 \neq N$ of $\mathcal{F}(w)$. Take an indecomposable direct summand $M_{[l,l')}$ of N. Then the composition $M_{[l,l')} \hookrightarrow N \rightarrowtail M_{[i,j)}$ is a non-isomorphic inflation in $\mathcal{F}(w)$, because the section $M_{[l,l')} \hookrightarrow N$ is an inflation and inflations are closed under compositions. Now Lemma 3.6.16 (2) tells us that $[l, l') \subset [i, j)$ and that $M_{[i,l)} \oplus M_{[l',j)}$ belongs to $\mathcal{F}(w)$, hence both $M_{[i,l)}$ and $M_{[l',j)}$ belong to $\mathcal{F}(w)$ since $\mathcal{F}(w)$ is closed under direct summands.

Since $M_{[l,l')} \hookrightarrow M_{[i,j)}$ is not an isomorphism, we have $i < l$ or $l' < j$. Suppose that the former holds. Then since $M_{[i,l)} \in \mathcal{F}(w)$, the transposition $(i\ l)$ is an inversion of w. On the other hand, the exact sequence (3.6.5) of Lemma 3.6.16 implies that $M_{[l,j)}$ belongs to $\mathcal{F}(w)$, since $\mathcal{F}(w)$ is closed under submodules. Thus $(l\ j)$ is also an inversion of w. This implies that $(i\ j)$ is *not* a Bruhat inversion by Lemma 3.6.3. The case $i < l' < j$ is completely similar, except we use (3.6.4) instead of (3.6.5). $\square$

EXAMPLE 3.6.17. Let us look at several examples.

(1) Let Q be the quiver $Q = 1 \to 2 \leftarrow 3$. Then the Coxeter element associated with Q is $c = s_2 s_1 s_3 = 3142 \in S_4$. In Table 1, we list all c-sortable elements and the corresponding inversions, Bruhat inversions and torsion-free classes. The black vertices indicate simple objects in $\mathcal{F}(w)$ and the white ones indicate the rest of indecomposables in $\mathcal{F}(w)$. According to this table, we conclude that $\mathcal{F}(w)$ satisfies (JHP) except $w = 3412$.

(2) Let Q be the quiver $Q = 1 \leftarrow 2 \to 3 \leftarrow 4$. The associated Coxeter element is $c = s_1 s_3 s_2 s_4 = 24153 \in S_5$. The Auslander-Reiten quiver of $\mathsf{mod}\, kQ$ is as follows:

$$\begin{tikzcd}[column sep=small, row sep=small]
 & M_{[3,5)} \arrow[dr] & & M_{[1,3)} \arrow[dr] & \\
M_{[3,4)} \arrow[ur] \arrow[dr] & & M_{[1,5)} \arrow[ur] \arrow[dr] & & M_{[2,3)} \\
 & M_{[1,4)} \arrow[ur] \arrow[dr] & & M_{[2,5)} \arrow[ur] \arrow[dr] & \\
M_{[1,2)} \arrow[ur] & & M_{[2,4)} \arrow[ur] & & M_{[4,5)}
\end{tikzcd}$$

In Table 2, we list all faithful torsion-free classes of $\mathsf{mod}\, kQ$, which corresponds to c-sortable elements with full support. Here as in (1), the black vertices indicate simple objects in $\mathcal{F}(w)$ and the white the rest. From this table, for example, we can check that the number of faithful torsion-free classes satisfying (JHP) is 8. By using the computer program, we can calculate that the number of all torsion-free classes is 42, which is the Catalan number, and 34 ones among them satisfy (JHP).

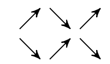
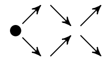
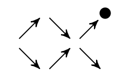
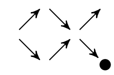
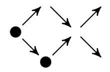
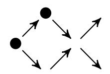
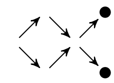
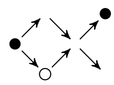
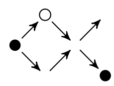
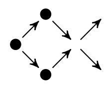
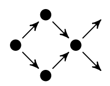
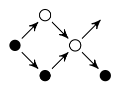
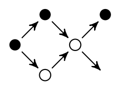
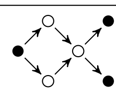
As an application of Corollary 3.6.15, we obtain the following result on the linearly oriented case. The proof is purely combinatorial. Note that this also follows from Corollary 3.5.19, since kQ is a Nakayama algebra in this case.

COROLLARY 3.6.18. *Let Q be a linearly oriented quiver of type A. Then every torsion-free classes $\mathcal{F}$ of* $\mathsf{mod}\, kQ$ *satisfies (JHP).*

PROOF. We may assume that Q is $1 \to 2 \to \cdots \to n$, so $c = s_n \cdots s_2 s_1$. We use the combinatorial characterization of the c-sortable-ness given in [Rea1, Theorem 4.12]: an element $w \in S_{n+1}$ is c-sortable if it is *231-avoiding*, that is, there exists no $i < l < j$ such that l appears before j and j appears before i in the one-line notation for w.

By Theorem 3.6.12, there exists a c-sortable element $w \in S_{n+1}$ satisfying $\mathcal{F} = \mathcal{F}(w)$. Then according to Corollary 3.6.15, the assertion amounts to the following purely combinatorial lemma (or equivalently, this lemma follows from two categorical statements: Corollaries 3.6.15 and 3.5.19).

TABLE 1. Example of Theorem 3.6.14 for $Q = 1 \to 2 \leftarrow 3$

c-sortable elements w	$\mathsf{supp}(w)$	$\mathsf{inv}(w)$	$\mathsf{Binv}(w)$	$\mathcal{F}(w)$
$e = 1234$	$\varnothing$	$\varnothing$	$\varnothing$	
$s_2 = 1324$	2	$(2\ 3)$	$(2\ 3)$	
$s_1 = 2134$	1	$(1\ 2)$	$(1\ 2)$	
$s_3 = 1243$	3	$(3\ 4)$	$(3\ 4)$	
$s_2s_1 = 3124$	$1, 2$	$(1\ 3), (2\ 3)$	$(1\ 3), (2\ 3)$	
$s_2s_3 = 1342$	$2, 3$	$(2\ 3), (2\ 4)$	$(2\ 3), (2\ 4)$	
$s_1s_3 = 2143$	$1, 3$	$(1\ 2), (3\ 4)$	$(1\ 2), (3\ 4)$	
$s_2s_1s_2 = 3214$	$1, 2$	$(1\ 2), (1\ 3), (2\ 3)$	$(1\ 2), (2\ 3)$	
$s_2s_3s_2 = 1432$	$2, 3$	$(2\ 3), (2\ 4), (3\ 4)$	$(2\ 3), (3\ 4)$	
$s_2s_3s_1 = 3142$	$1, 2, 3$	$(1\ 3), (2\ 3), (2\ 4)$	$(1\ 3), (2\ 3), (2\ 4)$	
$s_2s_3s_1s_2 = 3412$	$1, 2, 3$	$(1\ 3), (1\ 4),$ $(2\ 3), (2\ 4)$	$(1\ 3), (1\ 4),$ $(2\ 3), (2\ 4)$	
$s_2s_3s_1s_2s_1 = 4312$	$1, 2, 3$	$(1\ 3), (1\ 4),$ $(2\ 3), (2\ 4), (3\ 4)$	$(1\ 3), (2\ 3), (3\ 4)$	
$s_2s_3s_1s_2s_3 = 3421$	$1, 2, 3$	$(1\ 2), (1\ 3), (1\ 4),$ $(2\ 3), (2\ 4)$	$(1\ 2), (2\ 3), (2\ 4)$	
$s_2s_3s_1s_2s_3s_1 = 4321$	$1, 2, 3$	$(1\ 2), (1\ 3), (1\ 4),$ $(2\ 3), (2\ 4), (3\ 4)$	$(1\ 2), (2\ 3), (3\ 4)$	

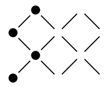
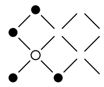
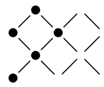
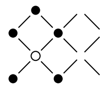
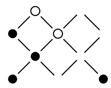
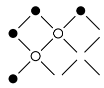
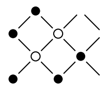
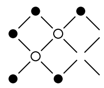
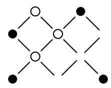
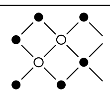
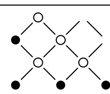
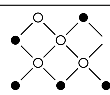
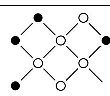
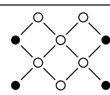
LEMMA. *For every* 231*-avoiding element* w *of* S_{n+1}*, we have* $\#\,\mathsf{Binv}(w) = \#\,\mathsf{supp}(w)$.

To prove this lemma, we will construct an explicit bijection $\mathsf{Binv}(w) \xrightarrow{\sim} \mathsf{supp}(w)$. In what follows, we will work on the one-line notation for w.

Let $(i\ j)$ be a Bruhat inversion of w with $i < j$. We claim that $i \in \mathsf{supp}(w)$ holds. If this is not the case, then the initial segment of length i contains exactly $1, 2, \cdots, i$ by Lemma 3.6.6. This is a contradiction since $j\ (> i)$ appears before i by $(i\ j) \in \mathsf{inv}(w)$. Thus $i \in \mathsf{supp}(w)$ holds, and we obtain a map $\mathsf{Binv}(w) \to \mathsf{supp}(w)$ by $(i\ j) \mapsto i$.

We will show that this map is an injection. Suppose $(i\ j_1)$ and $(i\ j_2)$ are two different Bruhat inversion of w with $i < j_1 < j_2$, so j_1 and j_2 appears before i. Then j_1 must appear before

TABLE 2. Example of Theorem 3.6.14 for faithful torsion-free classes over $Q = 1 \leftarrow 2 \rightarrow 3 \leftarrow 4$.

c-sortable elements w	$\mathsf{inv}(w)$	$\mathsf{Binv}(w)$	$\mathcal{F}(w)$	$\#\,\mathsf{sim}\,\mathcal{F}(w)$
$c = s_1 s_3 s_2 s_4$ $= 24153$	$\mathsf{inv}(c) =$ $(1\ 2), (1\ 4)$ $(3\ 4), (3\ 5)$	$\mathsf{inv}(c)$		4
$cs_1 = 42153$	$\mathsf{inv}(c), (2\ 4)$	$(1\ 2), (2\ 4),$ $(3\ 4), (3\ 5)$		4
$cs_3 = 24513$	$\mathsf{inv}(c), (1\ 5)$	$\mathsf{inv}(c), (1\ 5)$		5
$cs_1 s_3 = 42513$	$\mathsf{inv}(c), (1\ 5), (2\ 4)$	$(1\ 2), (1\ 5),$ $(2\ 4), (3\ 4), (3\ 5)$		5
$cs_3 s_2 = 25413$	$\mathsf{inv}(c), (1\ 5), (4\ 5)$	$(1\ 2), (1\ 4),$ $(3\ 4), (4\ 5)$		4
$cs_3 s_4 = 24531$	$\mathsf{inv}(c), (1\ 3), (1\ 5)$	$(1\ 2), (1\ 3),$ $(3\ 4), (3\ 5)$		4
$cs_1 s_3 s_2 = 45213$	$\mathsf{inv}(c), (1\ 5),$ $(2\ 4), (2\ 5)$	$(1\ 2), (2\ 4), (2\ 5),$ $(3\ 4), (3\ 5)$		5
$cs_1 s_3 s_4 = 42531$	$\mathsf{inv}(c), (1\ 3),$ $(1\ 5), (2\ 4)$	$(1\ 2), (1\ 3),$ $(2\ 4), (3\ 4), (3\ 5)$		5
$cs_3 s_2 s_4 = 25431$	$\mathsf{inv}(c), (1\ 3),$ $(1\ 5), (4\ 5)$	$(1\ 2), (1\ 3),$ $(3\ 4), (4\ 5)$		4
$c^2 = cs_1 s_3 s_2 s_4$ $= 45231$	$\mathsf{inv}(c), (1\ 3), (1\ 5),$ $(2\ 4), (2\ 5)$	$(1\ 2), (1\ 3), (2\ 4),$ $(2\ 5), (3\ 4), (3\ 5)$		6
$cs_1 s_3 s_2 s_1$ $= 54213$	$\mathsf{inv}(c), (1\ 5),$ $(2\ 4), (2\ 5), (4\ 5)$	$(1\ 2), (2\ 4),$ $(3\ 4), (4\ 5)$		4
$c^2 s_1 = cs_1 s_3 s_2 s_4 s_1$ $= 54231$	$\mathsf{inv}(c), (1\ 3), (1\ 5),$ $(2\ 4), (2\ 5), (4\ 5)$	$(1\ 2), (1\ 3), (2\ 4),$ $(3\ 4), (4\ 5)$		5
$c^2 s_3 = cs_1 s_3 s_2 s_4 s_3$ $= 45321$	$\mathsf{inv}(c), (1\ 3), (1\ 5),$ $(2\ 3), (2\ 4), (2\ 5)$	$(1\ 2), (2\ 3),$ $(3\ 4), (3\ 5)$		4
$c^2 s_1 s_3 = cs_1 s_3 s_2 s_4 s_1 s_3$ $= 54321$	$\mathsf{inv}(c), (1\ 3), (1\ 5),$ $(2\ 3), (2\ 4), (2\ 5), (4\ 5)$	$(1\ 2), (2\ 3),$ $(3\ 4), (4\ 5)$		4

j_2, since otherwise $(i\ j_2)$ would not be a Bruhat inversion. Thus w looks like $\cdots j_1 \cdots j_2 \cdots i \cdots$, which contradicts to that w is 231-avoiding. Thus this map is an injection.

Next we will show that this map is a surjection. For $i \in \mathsf{supp}(w)$, we will show the following:

(Claim): *There exists some letter j such that $i < j$ and j appears before i.*

By Lemma 3.6.6, there exists some l with $l > i$ such that l appears in the initial segment of length i. Suppose that any letters left to i are less than i, that is, there is no j as in the above claim. Then in particular i belongs to the initial segment of length i, and is left to l. Also by Lemma 3.6.6 (2), there exists a letter i' with $i' \leq i$ such that i' does not appear in the initial segment of length i. However we must have $i' \neq i$, thus $i' < i$, since i appears in the initial segment of length i. Now we have $i' < i < l$ and w looks like$\cdots i \cdots l \cdots i' \cdots$, which contradicts to that w is 231-avoiding. Thus the claim follows.

Take the rightmost letter j with the claimed property. Then $(i\ j)$ is clearly a Bruhat inversion. Thus the map $\mathsf{Binv}(w) \to \mathsf{supp}(w)$ is surjective. $\square$

REMARK 3.6.19. In a paper [Eno5], we will show that the natural analogue for Theorem 3.6.14 holds for $\mathsf{mod}\, kQ$ with an arbitrary Dynkin quiver Q, or more generally, for $\mathsf{mod}\, \Pi$ for a preprojective algebra Π of an arbitrary Dynkin type.

3.7. Computations of the Grothendieck monoids

So far, we do not know any method to compute the Grothendieck monoid $\mathcal{M}(\mathcal{E})$ for a given length exact category $\mathcal{E}$ in general, except the following information:

- $\mathcal{M}(\mathcal{E})$ is atomic, that is, generated by $\mathsf{Atom}\,\mathcal{M}(\mathcal{E}) = \{[S] \mid S \in \mathsf{sim}\,\mathcal{E}\}$ (Proposition 3.4.8).
- $\mathcal{M}(\mathcal{E})$ is just a free monoid if $\mathcal{E}$ turns out to satisfy (JHP) (Theorem 3.4.9).
- The cancellative quotient $\mathcal{M}(\mathcal{E})_{\mathsf{can}}$, which is isomorphic to the positive part $\mathsf{K}_0^+(\mathcal{E})$ of the Grothendieck group (Proposition 3.A.15), is nothing but the monoid of dimension vectors of modules belonging to $\mathcal{E}$ under mild assumption (Corollary 3.5.9).

Thus the problem is: we do not know how to check whether $\mathcal{M}(\mathcal{E})$ is cancellative or not (if $\mathcal{M}(\mathcal{E})$ is cancellative, then $\mathcal{M}(\mathcal{E})$ can be calculated in principle by the above), and the calculation seems to be rather difficult if $\mathcal{M}(\mathcal{E})$ is *not* cancellative.

In this section, we will show (a bit artificial) examples of calculations of $\mathcal{M}(\mathcal{E})$ such that $\mathcal{M}(\mathcal{E})$ is *not* cancellative.

3.7.1. A category of modules with a designated set of dimension vectors. We will convince the reader that lots of monoids can appear as the Grothendieck monoids of exact categories. First we consider the case of split exact categories.

PROPOSITION 3.7.1. *Let $\mathcal{E}$ be a split exact category, that is, every conflation splits. Then $\mathcal{M}(\mathcal{E})$ is isomorphic to $\mathsf{Iso}\,\mathcal{E}$ (with addition given by $\oplus$) as monoids.*

PROOF. This follows from the construction given in Proposition 3.3.3, or one can show this by checking the universal property directly. The details are left to the reader. $\square$

Using this, we can realize any submonoids of $\mathbb{N}^n$ as Grothendieck monoids.

PROPOSITION 3.7.2. *Let M be a submonoid of $\mathbb{N}^n$ for some $n \geq 0$. Then there exists a split exact category $\mathcal{E}$ whose Grothendieck monoid is isomorphic to M.*

PROOF. Let k be a field and consider a semisimple k-algebra $\Lambda := k^n$. Then Λ has n non-isomorphic simple modules, and we denote by $\underline{\dim}\, X \in \mathbb{N}^n$ for $X \in \mathsf{mod}\,\Lambda$ the dimension vector of X, see Definition 3.5.3. Define the full subcategory $\mathcal{E}$ of $\mathsf{mod}\,\Lambda$ by the following:

$$\mathcal{E} := \{X \in \mathsf{mod}\,\Lambda \mid \underline{\dim}\, X \in M\}.$$

This is an extension-closed subcategory of $\mathsf{mod}\,\Lambda$, and we claim that $\mathcal{M}(\mathcal{E}) \cong M$ holds. First observe that every short exact sequence in $\mathsf{mod}\,\Lambda$ splits, so $\mathcal{E}$ is a split exact category. Thus it suffices to show $\mathsf{Iso}\,\mathcal{E} \cong M$ by Proposition 3.7.1. The map $X \mapsto \underline{\dim}\, X$ induces an isomorphism $\mathsf{Iso}(\mathsf{mod}\,\Lambda) \cong \mathbb{N}^n$, and by construction, this map restricts an isomorphism $\mathsf{Iso}\,\mathcal{E} \cong M$. $\square$

If $\mathcal{E}$ is a full subcategory of a module category, then by regarding $\mathcal{E}$ as an exact category with a split exact structure, Proposition 3.7.1 shows that our monoid $\mathcal{M}(\mathcal{E})$ only concerns with direct sum decompositions of modules. In this case, $\mathcal{M}(\mathcal{E})$ is free if and only if the uniqueness of direct sum decompositions holds (e.g. Krull-Schmidt categories), thus the combinatorial property of the monoid $\mathcal{M}(\mathcal{E})$ encodes information on the non-unique direct sum factorizations of modules. This has been studied by several authors (e.g. [Fac1, Fac2, BaGe]), and we refer the reader to the recent article [BaGe] and references therein for more information on this direction.

In a similar way to the construction of $\mathcal{E}$ above, we can attach extension-closed subcategories of module categories to any submonoids of $\mathsf{K}_0(\mathsf{mod}\,\Lambda)$ for an artin algebra Λ.

Definition 3.7.3. Let Λ be an artin algebra with n non-isomorphic simple modules. For a submonoid M of $\mathbb{N}^n$, we define the subcategory $\mathcal{E}_M$ of $\mathsf{mod}\,\Lambda$ by the following:

$$\mathcal{E}_M := \{X \in \mathsf{mod}\,\Lambda \mid \underline{\dim}\, X \in M\}.$$

Then it is an extension-closed subcategory of $\mathsf{mod}\,\Lambda$, thus an exact category.

By the universal property of $\mathcal{M}(\mathcal{E}_M)$, we have a monoid homomorphism $\mathcal{M}(\mathcal{E}_M) \to M$ induced by $\underline{\dim}$. Indeed $\mathcal{M}(\mathcal{E}_M)$ is a *thickening* of M by some non-cancellative part:

Proposition 3.7.4. *Let Λ be an artin algebra with n non-isomorphic simple modules and M a submonoid of $\mathbb{N}^n$. Then $\underline{\dim}\colon \mathcal{M}(\mathcal{E}_M) \to M$ induces an isomorphism $\mathcal{M}(\mathcal{E}_M)_{\mathsf{can}} \cong M$ of monoids.*

Proof. Let us introduce some notation. Write $\mathcal{E} := \mathcal{E}_M$ for simplicity. Let S_i denote the simple Λ-module with $\underline{\dim}\, S_i = e_i$ for $1 \le i \le n$, where e_i is the standard basis of $\mathbb{N}^n$. For an element $d \in \mathbb{N}^n$, we denote by S_d the unique semisimple Λ-module satisfying $\underline{\dim}\, S_d = d$, in other words, $S_d := \bigoplus_i S_i^{\oplus a_i}$ for $d = \sum_i a_i e_i$ with $a_i \in \mathbb{N}$.

For $d \in M$, clearly we have $S_d \in \mathcal{E}$. Since $\underline{\dim}\, S_d = d$ holds, the map $\underline{\dim}\colon \mathcal{M}(\mathcal{E}) \to M$ is a surjection.

Thus, to prove $\mathcal{M}(\mathcal{E})_{\mathsf{can}} \cong M$, it suffices to show that for two objects $X, Y \in \mathcal{E}$, we have $[X] \sim_{\mathsf{can}} [Y]$ in $\mathcal{M}(\mathcal{E})$ if and only if $\underline{\dim}\, X = \underline{\dim}\, Y$ holds (see Proposition 3.A.15). The "only if" part is clear since $\mathbb{N}^n$ (or M) is cancellative, so we will prove the "if" part. To do this, it suffices to show $[X] \sim_{\mathsf{can}} [S_d]$ for every $X \in \mathcal{E}$ with $\underline{\dim}\, X = d$.

Let $l_{\max}(X)$ be the maximum of lengths of indecomposable direct summands of X, and we show $[X] \sim_{\mathsf{can}} [S_d]$ by induction on $l := l_{\max}(X)$. If $l = 1$, then X is semisimple and $X \cong S_d$ holds. Suppose $l > 1$, and take a direct summand X_1 of X with $l(X_1) = l$. We have $X = X_1 \oplus X/X_1$, and take a simple submodule S of X_1. Note that S is a direct summand of S_d since S is a composition factor of X. Thus there are two exact sequences in $\mathsf{mod}\,\Lambda$:

$$0 \longrightarrow S \longrightarrow X_1 \longrightarrow X_1/S \longrightarrow 0, \quad \text{and}$$
$$0 \longrightarrow S_d/S \longrightarrow S_d/S \oplus X/X_1 \oplus S \longrightarrow X/X_1 \oplus S \longrightarrow 0,$$

where the second one is just a split exact sequence. By taking direct sum of these two sequences, we obtain the following exact sequence.

$$0 \longrightarrow S_d \longrightarrow X \oplus S_d \longrightarrow S \oplus X_1/S \oplus X/X_1 \longrightarrow 0$$

The dimension vectors of modules in this exact sequence is d, $2d$ and d respectively, so this is a conflation of $\mathcal{E}$. Therefore, we have $[X] + [S_d] = [X'] + [S_d]$ in $\mathcal{M}(\mathcal{E})$, hence $[X] \sim_{\mathsf{can}} [X']$ holds, where $X' = S \oplus X_1/S \oplus X/X_1$. Thus we can replace the direct summand X_1 of X with $S \oplus X_1/S$. By iterating this process to all direct summands of X with length l, we obtain an object Z in $\mathcal{E}$ such that $l_{\max}(Z) < l$ and $[X] \sim_{\mathsf{can}} [Z]$. Thus the claim follows from induction. $\square$

Thus our category $\mathcal{E}_M$ has a monoid M as a cancellative quotient of its Grothendieck monoid $\mathcal{M}(\mathcal{E}_M)$, but $\mathcal{M}(\mathcal{E}_M)$ is not cancellative in general. In the next subsection, we give an example of computation of $\mathcal{M}(\mathcal{E}_M)$.

3.7.2. The case of A_2 quiver and monoids with one generator. Throughout this subsection, we denote by k a field and by Q a quiver $1 \leftarrow 2$, and put $\Lambda := kQ$. Although the structure of the category $\mathsf{mod}\,\Lambda$ seems to be completely understood, this category has lots of extension-closed subcategories, and the computation of their Grothendieck monoids is rather hard as we shall see in this subsection. A particular case of this computation was considered in [BeGr, Section 3.4].

First we recall basic properties of the category $\mathsf{mod}\,\Lambda$. We have the complete list of indecomposable objects in $\mathsf{mod}\,\Lambda$: two simple modules S_1 and S_2, which are supported at vertices 1 and 2 respectively, and one non-simple projective injective module P. The Grothendieck group $\mathsf{K}_0(\mathsf{mod}\,\Lambda)$ is a free abelian group with basis $\{[S_1], [S_2]\}$, and we identify it with $\mathbb{Z}^2$.

Now we apply the construction $\mathcal{E}_M$ defined in the previous subsection to this case, where M is a submonoid of $\mathbb{N}^2$. We will compute the monoid $\mathcal{M}(\mathcal{E}_M)$ in the case that M is generated by one element. In such a case, by Proposition 3.7.4, $\mathcal{M}(\mathcal{E}_M)_{\mathsf{can}} \cong \mathbb{N}$ holds, thus $\mathcal{M}(\mathcal{E}_M)$ is a *thickening* of $\mathbb{N}$ in some sense.

To present the monoid structure, we will use the *Cayley quiver*, which is a monoid version of the Cayley graph of a group.

Definition 3.7.5. Let M be a monoid with a set of generators $A \subset M$. Then the *Cayley quiver* of M with respect to A is a quiver defined as follows:

- The vertex set is M.
- For each $a \in A$ and $m \in M$, we draw a (labelled) arrow $m \xrightarrow{a} m + a$.

For an atomic monoid M, the natural choice of A above is the set $\mathsf{Atom}\, M$ of all atoms of M. Now we can state our computation.

Proposition 3.7.6. *Let $M := \mathbb{N}(m, n)$ be a submonoid of $\mathbb{N}^2$ generated by $(m, n) \in \mathbb{N}^2$ with $(m, n) \neq (0, 0)$. Consider the exact category $\mathcal{E} := \mathcal{E}_M$ as in Definition 3.7.3. Then the following hold, where we set $l := \min\{m, n\}$.*

(1) *$\mathcal{E}$ has exactly $l + 1$ distinct simple objects $A_0, \cdots, A_l$, where*

$$A_i := P^i \oplus S_1^{m-i} \oplus S_2^{n-i}$$

for $0 \leq i \leq l$. We have $\dim A_i = (m, n)$ for every i.

(2) *Put $a_i := [A_i] \in \mathcal{M}(\mathcal{E})$ for $0 \leq i \leq l$, which are precisely the atoms of $\mathcal{M}(\mathcal{E})$. Then the Cayley quiver of $\mathcal{M}(\mathcal{E})$ with respect to $\mathsf{Atom}\,\mathcal{M}(\mathcal{E})$ is given as follows, where we draw arrows $\twoheadrightarrow$ to represent $l + 1$ arrows $a_0, \cdots, a_l$.*

- **(Case 1)** *The case $m \neq n$:*

$$\begin{tikzcd}
 & a_0 \arrow[dr, two heads] & & & \\
 & a_1 \arrow[r, two heads] & & & \\
0 \arrow[uur, "a_0"] \arrow[ur, "a_1"'] \arrow[dr, "a_l"'] & \vdots & 2a_0 \arrow[r, two heads] & 3a_0 \arrow[r, two heads] & \cdots \\
 & a_l \arrow[ur, two heads] & & &
\end{tikzcd}$$

In particular, if $m = 0$ or $n = 0$, then $\mathcal{M}(\mathcal{E}) \cong \mathbb{N}$ holds.

- **(Case 2)** *The case $m = n$:*

$$\begin{tikzcd}
 & a_0 \arrow[r, two heads] & 2a_0 \arrow[r, two heads] & 3a_0 \arrow[r, two heads] & \cdots \\
 & a_1 \arrow[ur, two heads] & & & \\
0 \arrow[uur, "a_0"] \arrow[ur, "a_1"'] \arrow[dr, "a_n"'] & \vdots & & & \\
 & a_n \arrow[r, "a_n"'] \arrow[uuur] & 2a_n \arrow[r, "a_n"'] \arrow[uuur] & 3a_n \arrow[r, "a_n"'] \arrow[uuur] & \cdots
\end{tikzcd}$$

Here non-labelled arrows $\to$ represent n arrows $a_0, \cdots, a_{n-1}$.

Proof. For simplicity, put $\mathcal{E} := \mathcal{E}_M$.

(1) It is clear that each A_i is a simple object in $\mathcal{E}$ since $\underline{\dim}\, A_i = (m, n)$, and that $A_i \cong A_j$ holds if and only if $i = j$. On the other hand, we claim that every object in $\mathcal{E}$ is a *direct sum* of A_i's, which clearly implies the assertion. In fact, take an object $X \in \mathcal{E}$ with $\underline{\dim}\, X = (Nm, Nn)$ for $N > 0$. Then X is isomorphic to $P^i \oplus S_1^{Nm-i} \oplus S_2^{Nn-i}$ for some $0 \leq i \leq Nl$. Take any integers $i_1, i_2, \cdots, i_N$ such that $0 \leq i_j \leq l$ for each j and $i = i_1 + \cdots + i_N$. Then it is straightforward to see that $X \cong A_{i_1} \oplus \cdots \oplus A_{i_N}$ holds.

(2) By Proposition 3.3.6, we have that $\mathsf{Atom}\,\mathcal{M}(\mathcal{E}) = \{a_0, a_1, \cdots, a_l\}$ holds for $a_i := [A_i]$, and all of them are distinct. On the other hand, we have a map $\varphi \colon \mathcal{M}(\mathcal{E}) \to \mathbb{N}$ which sends an object X with $\underline{\dim}\, X = (Nm, Nn)$ to N for $N \in \mathbb{N}$. This map is clearly a surjection (and actually this is the universal cancellative quotient of $\mathcal{M}(\mathcal{E})$ by Proposition 3.7.4, but this fact is not needed here). Denote by $\mathcal{M}(\mathcal{E})_N$ the inverse image $\varphi^{-1}(N)$, and our strategy to compute $\mathcal{M}(\mathcal{E})$ is to compute $\mathcal{M}(\mathcal{E})_N$. Clearly we have $\mathcal{M}(\mathcal{E})_0 = \{0 = [0]\}$ and $\mathcal{M}(\mathcal{E})_1 = \mathsf{Atom}\,\mathcal{M}(\mathcal{E}) = \{a_0, a_1, \cdots, a_l\}$.

Moreover, since every object of $\mathcal{E}$ is a direct sum of A_i's, we have $x \in \mathcal{M}(\mathcal{E})_N$ if and only if there exist integers $0 \le i_1, \cdots, i_N \le l$ satisfying $x = a_{i_1} + \cdots + a_{i_N}$. The key part of our computation is to show the following equality in $\mathcal{M}(\mathcal{E})$:

(Claim 1): *For $1 \le i \le l$ and $0 \le j \le l$, we have the equality $a_i + a_j = a_{i-1} + a_j$ in $\mathcal{M}(\mathcal{E})$, except the case $j = m = n$.*

Proof of Claim 1. First, note that we have the following exact sequence in $\mathsf{mod}\,\Lambda$.

$$0 \longrightarrow S_1 \longrightarrow P \longrightarrow S_2 \longrightarrow 0$$

Suppose that the equality $j = m = n$ does not hold, then one can easily check that either $j \le m-1$ or $j \le n-1$ (or both) holds. Suppose the former, and consider a split exact sequence below

$$0 \to P^j \oplus S_1^{m-1-j} \oplus S_2^{n-j} \to P^{i+j-1} \oplus S_1^{2m-i-j} \oplus S_2^{2n-i-j} \to P^{i-1} \oplus S_1^{m-i+1} \oplus S_2^{n-i} \to 0.$$

By taking direct sum of the above two exact sequences, one obtains a conflation in $\mathcal{E}$:

$$0 \longrightarrow A_j \longrightarrow A_i \oplus A_j \longrightarrow A_{i-1} \longrightarrow 0.$$

Thus $a_i + a_j = a_{i-1} + a_j$ holds in $\mathcal{M}(\mathcal{E})$.

Similarly, suppose that $j \le n-1$ holds. Then by considering the following split sequence

$$0 \to P^{i-1} \oplus S_1^{m-i} \oplus S_2^{n-i+1} \to P^{i+j-1} \oplus S_1^{2m-i-j} \oplus S_2^{2n-i-j} \to P^j \oplus S_1^{m-j} \oplus S_2^{n-1-j} \to 0,$$

as in the previous case, we obtain a conflation

$$0 \longrightarrow A_{i-1} \longrightarrow A_i \oplus A_j \longrightarrow A_j \longrightarrow 0,$$

which shows $a_i + a_j = a_{i-1} + a_j$ as well. This finishes the proof of Claim 1. $\square$

In what follows, we consider two cases.

(Case 1): *The case $m \neq n$.* For every i, j with $0 \le i \le l$ and $0 \le j \le l$, we have $a_i + a_j = a_{i-1} + a_j = \cdots = a_0 + a_j = a_0 + a_{j-1} = \cdots = a_0 + a_0 = 2a_0$ by Claim 1. Thus $\mathcal{M}(\mathcal{E})_2 = \{2a_0\}$ holds. We claim inductively that $\mathcal{M}(\mathcal{E})_N = \{Na_0\}$ for $N \ge 2$. This is because every element x in $\mathcal{M}(\mathcal{E})_{N+1}$ can be written as $x = a_i + y$ for some $0 \le i \le l$ and $y \in \mathcal{M}(\mathcal{E})_N$, thus $x = a_i + Na_0 = (a_i + a_0) + (N-1)a_0 = 2a_0 + (N-1)a_0 = (N+1)a_0$. Now that $\mathcal{M}(\mathcal{E})_N$ has been computed for all $N \ge 0$, the description of the Cayley quiver of $\mathcal{M}(\mathcal{E})$ easily follows.

(Case 2): *The case $m = n$.* Note that $l = m = n$ in this case, so we only use the letter n. For every i, j with $0 \le i \le n$ and $0 \le j \le n$, if $j \neq n$, then $a_i + a_j = a_{i-1} + a_j = \cdots = a_0 + a_j = a_0 + a_{j-1} = \cdots = 2a_0$ by Claim 1. Thus $\mathcal{M}(\mathcal{E})_2 = \{2a_0, 2a_n\}$ holds (and later we will show $2a_0 \neq 2a_n$). The same inductive argument as in (Case 1) shows that $\mathcal{M}(\mathcal{E})_N = \{Na_0, Na_n\}$ holds for $N \ge 2$, and it suffices to show that $Na_0 \neq Na_n$ in $\mathcal{M}(\mathcal{E})$ for $N \ge 2$.

Suppose that $Na_0 = Na_n$ holds in $\mathcal{M}(\mathcal{E})$, that is, $[A_0^N] = [A_n^N]$. Denote by $\mathcal{E}_P$ the full subcategory of $\mathcal{E}$ consisting of direct sums of $A_n = P^n$, or equivalently, the subcategory of $\mathcal{E}$ consisting of projective kQ-modules. Note that $A_n^N \in \mathcal{E}_P$ holds. Then the following claim holds:

(Claim 2): *In (Case 2), for every conflation*

$$0 \longrightarrow X \longrightarrow Y \xrightarrow{\pi} Z \longrightarrow 0$$

in $\mathcal{E}$, we have that $Y \in \mathcal{E}_P$ holds if and only if both $X \in \mathcal{E}_P$ and $Z \in \mathcal{E}_P$ hold (in other words, $\mathcal{E}_P$ is a Serre subcategory of $\mathcal{E}$).

Proof of Claim 2. If $X \in \mathcal{E}_P$ and $Z \in \mathcal{E}_P$ hold, then the above sequence splits since Z is projective, thus $Y \cong X \oplus Z \in \mathcal{E}_P$ holds. Conversely, suppose that $Y \in \mathcal{E}_P$ holds. Since π induces a surjection $\operatorname{top} Y \twoheadrightarrow \operatorname{top} Z$, and $\operatorname{top} Y$ is a direct sum of $\operatorname{top} P = S_2$, we have that so is $\operatorname{top} Z$. However, $\operatorname{top} A_i$ contains S_1 as a direct summand if $i < n$. Therefore, Z, which is a direct sum of appropriate A_i's, must be isomorphic to a direct sum of A_n. Thus $Z \in \mathcal{E}_P$ holds. Moreover, the above sequence splits since Z is projective, thus $Y \cong X \oplus Z$. Clearly $\mathcal{E}_P$ is closed under direct summands in $\mathcal{E}$, so X belongs to $\mathcal{E}_P$. $\square$

Let us return to our situation, that is, $[A_0^N] = [A_n^N]$ holds. Since $\mathcal{E}_P$ is a Serre subcategory of $\mathcal{E}$ by Claim 2 and A_n^N belongs to $\mathcal{E}_P$, Proposition 3.3.7 implies that A_0^N belongs to $\mathcal{E}_P$. This is a contradiction since A_0 is not projective. Therefore, we have $Na_0 \neq Na_n$, which completes the proof. $\square$

3.8. Counter-examples

In this section, we give examples of *bad behavior* of exact categories on several topics which we have studied in the previous sections.

3.8.1. On the poset of admissible subobjects. In this subsection, we collect some counter-examples on the poset $\mathsf{P}(X)$ (see Section 3.2.1 for the detail)

3.8.1.1. *Subobject posets which are not lattices.* The following example shows that $\mathsf{P}(X)$ *is not necessarily a lattice.*

EXAMPLE 3.8.1. Let k be a field and $\mathcal{E}$ a category of k-vector spaces whose dimensions are not equal to 1 and 3. Then $\mathcal{E}$ is an extension-closed subcategory of $\mathsf{mod}\, k$, thus an exact category. Denote by X the 6-dimensional vector space with basis $x_1, \cdots, x_6$, and put $A := \langle x_1, x_2, x_3, x_4\rangle$ and $B := \langle x_1, x_2, x_3, x_5\rangle$. Then since X/A and X/B has dimension 2, both A and B are admissible subobjects of X. Now the set-theoretic intersection of A and B is a 3-dimensional subspace $C := \langle x_1, x_2, x_3\rangle$, which does not belong to $\mathcal{E}$, thus lower bounds of A and B in $\mathsf{P}_{\mathcal{E}}(X)$ are precisely subspaces of C whose dimensions are exactly two or zero. Then A and B do not have the greatest lower bound, since there are many two-dimensional subspaces of C.

The following is also an example in which $\mathsf{P}(X)$ is not a lattice, although the ambient category is *pre-abelian.*

EXAMPLE 3.8.2. Let k be a field and Λ an algebra $\Lambda := k(1 \overset{\alpha}{\leftarrow} 2 \overset{\beta}{\leftarrow} 3)/(\beta\alpha)$. Put $\mathcal{E} := \mathsf{proj}\,\Lambda$, then $\mathcal{E}$ has the natural exact structure induced from $\mathsf{mod}\,\Lambda$, and every conflation splits in $\mathcal{E}$. It follows that a morphism $\iota\colon A \to B$ in $\mathcal{E}$ is an inflation if and only if ι is a section. Moreover, since $\operatorname{gl.dim}\Lambda = 2$ holds, kernels of morphisms between projective modules are projective, thus $\mathcal{E}$ is pre-abelian.

We claim that $\mathsf{P}_{\mathcal{E}}(\Lambda_\Lambda)$ is not a lattice. By the previous argument, we can identify $\mathsf{P}_{\mathcal{E}}(\Lambda)$ with the poset of submodules of Λ which are direct summands of Λ. We can write Λ as follows:

$$\Lambda_\Lambda = P_1 \oplus P_2 \oplus P_3 = 1 \oplus \substack{2\\1} \oplus \substack{3\\2} \tag{3.8.1}$$

Here P_i is an indecomposable projective modules corresponding to the vertex i, and the rightmost notation indicates composition series of modules.

Consider the following two maps:

$$\iota_1\colon P_1 \oplus P_2 \xrightarrow{\left[\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right]} P_1 \oplus P_2 \oplus P_3$$
$$\iota_2\colon P_1 \oplus P_2 \xrightarrow{\left[\begin{smallmatrix}1&0\\0&1\\0&\beta\end{smallmatrix}\right]} P_1 \oplus P_2 \oplus P_3$$

Here $\beta\colon P_2 \to P_3$ denotes the multiplication map $\beta\cdot(-)$. We can check by matrix elimination that these maps are sections, thus $N_i := \operatorname{Im}\iota_i$ belongs to $\mathsf{P}_{\mathcal{E}}(\Lambda)$ for each i. In the notation of (3.8.1), each N_i looks like the following:

$$N_1 = \{(a, \substack{b\\c}, \substack{0\\0}) \in P_1 \oplus P_2 \oplus P_3 \mid a, b, c \in k\}$$
$$N_2 = \{(a, \substack{b\\c}, \substack{0\\b}) \in P_1 \oplus P_2 \oplus P_3 \mid a, b, c \in k\}$$

We claim that N_1 and N_2 do not have the greatest lower bound in $\mathsf{P}_{\mathcal{E}}(\Lambda)$. Let N be a lower bound for N_1 and N_2 in $\mathsf{P}_{\mathcal{E}}(\Lambda)$. Then N must be contained in the *set-theoretic* intersection $N_1 \cap N_2$:

$$N_1 \cap N_2 = 1 \oplus {}_{1} \oplus 0 = \{(a, \substack{0\\c}, \substack{0\\0}) \in P_1 \oplus P_2 \oplus P_3 \mid a, c \in k\},$$

which is a two-dimensional vector space spanned by e_1 and α. Here $(a, \substack{0\\c}, \substack{0\\0})$ in the above notation corresponds to $ae_1 + c\alpha$. Consider the quotient module $\Lambda/N_1 \cap N_2$:

$$\Lambda/N_1 \cap N_2 = S_2 \oplus P_3 = 2 \oplus \substack{3\\2}$$

This is not projective, thus we have $N_1 \cap N_2 \notin \mathsf{P}_{\mathcal{E}}(\Lambda)$. Hence we have $N \subsetneq N_1 \cap N_2$, so $\dim N = 0$ or $\dim N = 1$ holds. Therefore, to show that the meet of N_1 and N_2 does not exist in $\mathsf{P}_{\mathcal{E}}(\Lambda)$, it suffices to show that there exist two distinct one-dimensional submodules of $N_1 \cap N_2$ which belong to $\mathsf{P}_{\mathcal{E}}(\Lambda)$. To show this, consider the following two maps:

$$i_1 \colon P_1 \xrightarrow{{}^t[1,0,0]} P_1 \oplus P_2 \oplus P_3$$
$$i_2 \colon P_1 \xrightarrow{{}^t[1,\alpha,0]} P_1 \oplus P_2 \oplus P_3$$

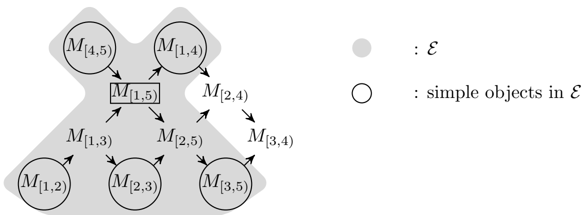
The images of these maps are one-dimensional submodules generated by e_1 and $e_1 + \alpha$ respectively. By matrix elimination, we can show that these are direct summands of Λ, thus both belong to $\mathsf{P}_{\mathcal{E}}(\Lambda)$. Therefore $\mathsf{P}_{\mathcal{E}}(\Lambda)$ is not a lattice.

3.8.1.2. *Subobject posets are lattices but not modular lattices.* Recall from Proposition 3.2.15 that if $\mathcal{E}$ is a quasi-abelian category with the maximal exact structure, subobject posets are in fact lattices. However, there exist examples which are quasi-abelian but these lattices are not necessarily modular. For such examples, we refer the reader to Section 3.8.2.1, since the modularity implies the unique length property (Corollary 3.2.19).

3.8.2. (JHP) and the unique length property. Next let us see some counter-examples on the unique factorization properties on exact categories.

3.8.2.1. *Lengths are not unique.* We collect some examples in which the unique length property fails (see Section 3.2.2). Example D in the introduction is one of such example, but is rather artificial and not idempotent complete. We give several idempotent complete examples (actually torsion-free classes, so quasi-abelian) which do not satisfy the unique length property. Actually we can obtain more examples in a quiver of type A by using the results in Section 3.6.

EXAMPLE 3.8.3. Let Q be a quiver $1 \leftarrow 2 \leftarrow 3 \rightarrow 4$. Then the Auslander-Reiten quiver of $\mathsf{mod}\, kQ$ is as follows. Here we use the notation introduced in Proposition 3.6.10.



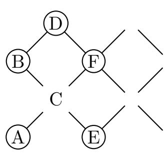
Define $\mathcal{E}$ as the additive subcategory of $\mathsf{mod}\, kQ$ corresponding to the gray region, then it is closed under extensions (and actually a torsion-free class) in $\mathsf{mod}\, kQ$. We can check that $\mathcal{E}$ has 5 simples indicated by circles: $\mathsf{sim}\, \mathcal{E} = \{M_{[1,2)}, M_{[2,3)}, M_{[3,5)}, M_{[4,5)}, M_{[1,4)}\}$. Now consider an object $X := M_{[1,5)}$. Roughly speaking, the gray region below X looks like a module category of an A_3 quiver, thus X seems to have length 3, and the gray region above X looks like a module category of an A_2 quiver, thus X seems to have length 2. In fact, we have the following two composition series of X in $\mathcal{E}$:

$$0 < M_{[4,5)} < X, \text{ and}$$
$$0 < M_{[1,2)} < M_{[1,3)} < X.$$

Thus lengths of X are not unique.

Here is another example, which we have already encountered.

EXAMPLE 3.8.4. Consider Example 3.6.17 and let w be the fourth one in Table 2, that is, $w = 42513$. Then $\mathcal{F}(w)$ is the additive subcategory corresponding to $\{A, B, C, D, E, F\}$ depicted as follows:



We have that $\mathsf{sim}\,\mathcal{F}(w) = \{A, B, D, E, F\}$. Consider $X := C \oplus D$. Then a conflation $A \rightarrowtail C \twoheadrightarrow E$ shows that C has length 2, hence $X = C \oplus D$ has length 3. On the other hand, a conflation $B \rightarrowtail X \twoheadrightarrow F$ implies that X has length 2. Thus lengths of X are not unique.

3.8.2.2. *Length are unique but (JHP) fails.* Here is an example of an exact category which satisfies the unique length property, but does not satisfy (JHP).

Example 3.8.5. Let $\mathcal{F}$ be a hereditary torsion-free class of length abelian categories. Then $\mathcal{F}$ satisfies the unique length property by Corollary 3.2.24. Typically, $\mathcal{F}$ arises in the following way: Take any artin algebra Λ and chose any set of simple modules $\mathcal{S} = \{S_1, \cdots, S_l\}$. Define $\mathcal{F}$ as follows:

$$\mathcal{F} := \{X \in \mathsf{mod}\,\Lambda \mid \mathrm{Hom}_\Lambda(X, S_i) = 0 \text{ for } 1 \leq i \leq l\}.$$

Then $\mathcal{F}$ is a torsion-free class, and the corresponding torsion class is the smallest Serre subcategory containing $\mathcal{S}$. Thus $\mathcal{F}$ is a hereditary torsion-free class.

We have already encountered such examples which do not satisfy (JHP): $\mathcal{E}_1$ in Example E, and $\mathcal{F}(c^2)$ and $\mathcal{F}(c^2 s_1)$ in Example 3.6.17.

3.8.2.3. *Examples which satisfy (JHP).* Finally, we collect examples which satisfy (JHP) for the convenience of the reader.

Example 3.8.6. The following examples satisfy (JHP).

(1) Krull-Schmidt categories *together with the split exact structure.* This follows by Proposition 3.7.1 and the fact that the uniqueness of direct sum decomposition holds.
(2) Length abelian categories. This is because the Jordan-Hölder theorem holds in any abelian categories, see e.g. [Ste, p.92].
(3) Torsion(-free) classes of $\mathsf{mod}\,\Lambda$ for a Nakayama algebra Λ (Corollary 3.5.19).
(4) Torsion-free classes of $\mathsf{mod}\,kQ$ for a quiver Q of type A which satisfies the condition given in Corollary 3.6.15. Explicit examples are given in Example 3.6.17 and $\mathcal{E}_1$ in Example E.
(5) The category $\mathcal{F}(\Delta)$ of modules with Δ-filtrations over a quasi-hereditary algebra, or more generally, over a standardly stratified algebra (see e.g. [PR, Proposition 1.2]). Simple objects in $\mathcal{F}(\Delta)$ are precisely standard modules.

3.8.3. Non-cancellative Grothendieck monoids. In this subsection, we will give some examples in which the Grothendieck monoids are *not finitely generated* or *not cancellative.*

3.8.3.1. *Functorially finite torsion-free class, but neither finitely generated nor cancellative.* In a length exact category, its Grothendieck monoid is atomic, thus it is not finitely generated as a monoid if and only if there exists infinitely many non-isomorphic simple objects (Proposition 3.4.8). We will give such an example.

Let k be an algebraically closed field and Q a Kronecker quiver, namely, $Q\colon 1 \leftleftarrows 2$. We refer the reader to [ARS, Section VIII.7] for the structure of $\mathsf{mod}\,kQ$. We denote by $\mathcal{E}$ the subcategory of $\mathsf{mod}\,kQ$ consisting of modules which does not contain S_2 (simple modules supported at 2) as a direct summand. Then $\mathcal{E}$ is closed under extensions, thus an exact category. Actually, $\mathcal{E}$ is a torsion-free class associated with an APR cotilting module $I_1 \oplus \tau S_2$, thus it is a functorially finite torsion-free class (with infinitely many indecomposables).

Recall that for each element $x = [x_1 : x_2] \in \mathbb{P}^1(k)$ in the projective line $\mathbb{P}^1(k)$, we have a regular module R_x with dimension vector $(1, 1)$, that is, R_x is the following representation of Q.

$$\begin{tikzcd} k & k \arrow[l, shift right, "x_1"'] \arrow[l, shift left, "x_2"] \end{tikzcd}$$

This assignment is injective, that is, we have $R_x \cong R_y$ if and only if $x = y$ in $\mathbb{P}^1(k)$. Now we claim the following:

$$\mathsf{sim}\,\mathcal{E} = \{S_1\} \cup \{R_x \mid x \in \mathbb{P}^1(k)\}.$$

Actually, for any indecomposable preprojective module X except S_1, it can be shown by explicit calculation that there exists an exact sequence

$$0 \to S_1 \to X \to R \to 0$$

such that R is a non-zero regular module. Thus such X is not simple in $\mathcal{E}$. Moreover, any indecomposable regular module is known to be written as a finite extension of R_x for some $x \in \mathbb{P}^1(k)$. Therefore, indecomposable modules except S_1 and R_x's are not simple. On the other hand, by considering dimension vectors, each R_x must be simple object, because $\underline{\dim}\, R_x = (1, 1)$ is an atom of $\underline{\dim}\, \mathcal{E}$. Thus R_x is simple in $\mathcal{E}$ for each $x \in \mathbb{P}^1(k)$.

Therefore, $\mathcal{M}(\mathcal{E})$ has infinitely many atoms (since k is an infinite field), so it is *not finitely generated.*

Furthermore, we claim that $\mathcal{M}(\mathcal{E})$ is not cancellative. In fact, for each $x = [x_1 : x_2] \in \mathbb{P}^1(k)$, we have an exact sequence

$$0 \longrightarrow S_1 \xrightarrow{\left[\begin{smallmatrix} -x_2 \\ x_1 \end{smallmatrix}\right]} P_2 \longrightarrow R_x \longrightarrow 0$$

where P_2 is an indecomposable projective module corresponding to $2 \in Q$, and the left map is an embedding of S_1 into the socle $S_1 \oplus S_1$ of P_2. This shows that $[S_1] + [R_x] = [P_2]$ holds for every $x \in \mathbb{P}^1(k)$. Since $[R_x] \neq [R_y]$ in $\mathcal{M}(\mathcal{E})$ for $x \neq y$, we must conclude that $\mathcal{M}(\mathcal{E})$ is *not cancellative.*

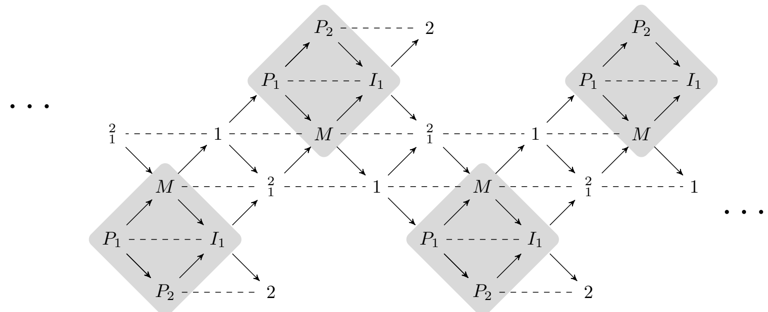
Moreover, this gives an example such that *non-isomorphic simples may represent the same element in the Grothendieck group.* By Proposition 3.5.9, we have an isomorphism $\underline{\dim}\colon \mathsf{K}_0(\mathcal{E}) \xrightarrow{\sim} \mathbb{Z}^2$. Thus for each $x, y \in \mathbb{P}^1(k)$, we have $[R_x] = [R_y]$ in $\mathsf{K}_0(\mathcal{E})$.

3.8.3.2. *Only finitely many indecomposables, but not cancellative.* In the previous example, the category we considered has infinitely many indecomposable objects. If an exact category $\mathcal{E}$ has finitely many indecomposables, then $\mathcal{M}(\mathcal{E})$ is finitely generated by a trivial reason, but it *may not be cancellative* as we shall see.

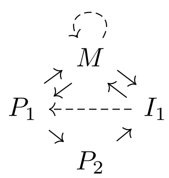
Let Λ be the following algebra, which is defined by the ideal quotient of the path algebra:

$$\Lambda := k \left(\begin{tikzcd} 1 \arrow[loop above, "\beta"] & 2 \arrow[l, "\alpha"'] \end{tikzcd} \right) / (\beta^2)$$

Then the Auslander-Reiten quiver of $\mathsf{mod}\,\Lambda$ is as follows:



where we write composition factors of modules, and $P_1 = \begin{smallmatrix}1\\1\end{smallmatrix}$, $P_2 = \begin{smallmatrix}2\\1\\1\end{smallmatrix}$, $I_1 = \begin{smallmatrix}2&&\\1&&2\\&1&\end{smallmatrix}$ and $M := \begin{smallmatrix}1&&2\\&1&\end{smallmatrix}$. Let $\mathcal{E}$ denote the additive subcategory of $\mathsf{mod}\,\Lambda$ corresponding to these gray regions: $\mathsf{ind}\,\mathcal{E} = \{P_1, P_2, I_1, M\}$. Then $\mathcal{E}$ is shown to be closed under extensions in $\mathsf{mod}\,\Lambda$ [2], and the Auslander-Reiten quiver of the exact category $\mathcal{E}$ is as follows:



[2]This does not seem to be so trivial. One categorical way to show this is to use results in [Eno1, Eno2]. By [Eno2, Corollary 3.10], one can endow $\mathcal{E}$ with the exact structure which corresponds to the Auslander-Reiten quiver of $\mathcal{E}$ drawn below. In this exact structure, $\mathcal{E}$ has a progenerator $P_1 \oplus P_2 = \Lambda$, and one can consider the Morita type embedding $\mathcal{E}(\Lambda, -)\colon \mathcal{E} \to \mathsf{mod}\,\Lambda$, which is nothing but the inclusion functor. Then by [Eno1, Proposition 2.8], its image, $\mathcal{E}$, is an extension-closed subcategory of $\mathsf{mod}\,\Lambda$.

By checking subobjects, it can be shown that all 4 indecomposable objects in $\mathcal{E}$ are simple objects in $\mathcal{E}$. However, by the Auslander-Reiten quiver of $\operatorname{mod}\Lambda$, it can be checked that we have the following conflations in $\mathcal{E}$:

$$\begin{aligned} 0 \longrightarrow P_1 \longrightarrow M \oplus P_2 \longrightarrow I_1 \longrightarrow 0, \\ 0 \longrightarrow M \longrightarrow P_1 \oplus I_1 \longrightarrow M \longrightarrow 0, \\ 0 \longrightarrow P_1 \longrightarrow P_2 \oplus P_2 \longrightarrow I_1 \longrightarrow 0. \end{aligned}$$

This implies the equality $[M] + [M] = [M] + [P_2] = [P_2] + [P_2]$ in $\mathcal{M}(\mathcal{E})$. Since $[M] \neq [P_2]$ by Proposition 3.3.6, we must have that $\mathcal{M}(\mathcal{E})$ is *not cancellative.*

3.9. Problems

In this section, we collect some open problems on several topics in this paper.

As we saw in Section 7, the computation of the Grothendieck monoid is rather difficult if it fails to be cancellative.

PROBLEM 3.9.1. *Let $\mathcal{E}$ be an exact category. Is there any criterion to check whether $\mathcal{M}(\mathcal{E})$ is cancellative?*

This leads to the following question.

PROBLEM 3.9.2. *Is there an example of an exact category $\mathcal{E}$ which satisfies the following conditions:*

(1) *$\mathcal{E}$ has finitely many indecomposable objects up to isomorphism.*
(2) *$\mathcal{E} = {}^{\perp}U$ for a cotilting module U over an artin algebra Λ, or more strongly, $\mathcal{E}$ is a torsion-free class of $\operatorname{mod}\Lambda$.*
(3) *$\mathcal{M}(\mathcal{E})$ is not cancellative.*

If we drop (1) or (2), then we have such an example (Section 3.8.3.1 and 3.8.3.2 respectively).

In what follows, let $\mathcal{F}$ be a functorially finite torsion-free class of $\operatorname{mod}\Lambda$ for an artin algebra Λ. The most general (and thus difficult) problem is the following:

PROBLEM 3.9.3. *Compute the Grothendieck monoid $\mathcal{M}(\mathcal{F})$, more precisely, draw the Cayley quiver as we did in Proposition 3.7.6.*

As a first approximation to $\mathcal{M}(\mathcal{F})$, the following problem naturally occurs, which is of interest in its own right.

PROBLEM 3.9.4. *For a given torsion-free class $\mathcal{F}$, classify (or count) simple objects in $\mathcal{F}$.*

Of course, we cannot expect the general classification of simples, but it may be done when Λ and $\mathcal{F}$ are given explicitly.

The cancellative quotient $\mathcal{M}(\mathcal{F})_{\mathsf{can}}$ is easier to handle with by Corollary 3.5.9: it is nothing but the monoid of dimension vectors $\underline{\dim}\,\mathcal{F}$. Moreover, it is a finitely generated submonoid of $\mathbb{N}^n$ by Proposition 3.5.16. Such a monoid is called a *positive affine monoid*, and this class is investigated in the theory of combinatorial commutative algebra and toric geometry via its monoid algebra. We refer the reader to [BrGu] for the details on affine monoids and affine monoid algebras.

PROBLEM 3.9.5. *For a given torsion-free class $\mathcal{F}$, investigate the combinatorial property of the affine monoid $\underline{\dim}\,\mathcal{F}$, the monoid of dimension vectors of modules in $\mathcal{F}$. In particular,*

- *Is this monoid normal?*
- *Describe the minimal generating set of it (this is related to simple objects in $\mathcal{F}$).*
- *When does this monoid homogeneous (this is related to the unique length property of $\mathcal{F}$)?*
- *Compute invariants of this monoid, such as extreme rays, the class group, support hyperplanes, and so on.*

Finally, we consider topics in Section 7. Let Q be a quiver of type A (or more generally, any Dynkin type, see Remark 3.6.19). Theorem 3.6.12 gives an explicit description of torsion-free classes of $\operatorname{mod} kQ$. Thus it is reasonable to expect that Problems 3.9.3 and 3.9.5 may be easier in this case. In addition, the author does not know when the uniqueness of lengths holds, except Example 3.8.5.

PROBLEM 3.9.6. *In the situation of Theorem 3.6.12, is there a combinatorial criterion for w to check whether $\mathcal{F}(w)$ has the unique length property?*

Also the following (purely combinatorial) problem is of interest.

PROBLEM 3.9.7. *For a Coxeter element c, is there any closed formula which gives the number of torsion-free classes satisfying (JHP), or equivalently, the number of c-sortable elements w such that $\# \mathsf{supp}(w) = \# \mathsf{Binv}(w)$ holds?*

3.A. Preliminaries on monoids

We collect some basic definitions and properties on monoids needed in this paper. Recall that *monoids are always assumed to be a commutative monoid with a unit*, and that we use an additive notation with unit 0.

3.A.1. Basic definitions. First, we collect some basic definitions on monoids.

DEFINITION 3.A.1. Let M be a monoid.

(1) M is *reduced* if $a + b = 0$ implies $a = b = 0$ for $a, b \in M$.
(2) M is *cancellative* if $a + x = a + y$ implies $x = y$ for $a, x, y \in M$.

There exists a natural pre-order $\leq$ on any monoid, defined as follows.

DEFINITION 3.A.2. Let M be a monoid. We define $x \leq y$ if there exists some $a \in M$ such that $y = x + a$.

It can be checked that $x \leq y$ if and only if $y \in x + M$ if and only if $y + M \subset x + M$. This pre-order is sometimes called *Green's pre-order* in semigroup theory, e.g. [Gri1, Gri2], or *divisibility pre-order* in the multiplicative theory of integral domains, e.g. [GH-K].

DEFINITION 3.A.3. A monoid M is called *naturally partially ordered* if the pre-order $\leq$ on M is a partial order, that is, $x \leq y$ and $y \leq x$ implies $x = y$.

The following properties can be easily checked.

PROPOSITION 3.A.4. *Let M be a monoid. Then M is reduced if and only if $0 \leq x \leq 0$ implies $x = 0$ for every $x \in M$. In particular, a naturally partially ordered monoid is reduced.*

Although we cannot naively take quotients of monoids as in abelian groups since there may not exist additive inverses, we can obtain some kind of *quotient monoids* by considering quotient sets with respect to *monoid congruences*, defined as follows.

DEFINITION 3.A.5. Let M be a monoid. The equivalence relation $\sim$ is called a *(monoid) congruence* if $x \sim y$ implies $a + x \sim a + y$ for every $a, x, y \in M$. In this case, the quotient set $M/\sim$ of equivalence classes naturally has the structure of a monoid.

We often use the smallest monoid congruence generated by some binary relation. See e.g. [Gri1, Propositions I.4.1, I.4.2] for the detail.

PROPOSITION 3.A.6. *Let M be a monoid and $\sim$ an arbitrary (not necessarily equivalence) binary relation on M. Then there exists the smallest monoid congruence $\approx$ which contains $\sim$, that is, $x \sim y$ implies $x \approx y$ for $x, y \in M$.*

3.A.2. Factorization properties on monoids. Let us define several notions on the *unique factorization property* on monoids. We refer the reader to [GH-K] for the details in this subsection (be aware that *monoids* in [GH-K] are assumed to be *cancellative*, but contents in this subsection hold in non-cancellative monoids). The most typical one is the freeness of monoids, which corresponds to (JHP) for exact categories by our main result (Theorem 3.4.9).

DEFINITION 3.A.7. Let M be a monoid.

(1) We say that a monoid is *finitely generated* if there exists a finite subset of M which generates M.
(2) For a subset A of M, we say that M is *free* on A if every element $x \in M$ can be written as a finite sum of elements in A in a unique way up to permutations. A monoid is called *free* if it is free on some subset of M.

For a set A, we denote by $\mathbb{N}^{(A)}$ the submonoid $\bigoplus_{a\in A} \mathbb{N}a$ of the free abelian group $\mathbb{Z}^{(A)} := \bigoplus_{a\in A} \mathbb{Z}a$ with basis A, which consists of finite sums of non-negative linear combinations of elements in A. It is easy to see that $\mathbb{N}^{(A)}$ is free on A and that a monoid is free if and only if it is isomorphic to $\mathbb{N}^{(A)}$ for some set A. If A is a finite set, then we often write $\mathbb{N}^A = \mathbb{N}^{(A)}$ and $\mathbb{Z}^A = \mathbb{Z}^{(A)}$. Moreover it is obvious that a free monoid is reduced.

Now let us consider elements of a monoid which cannot be decomposed into smaller ones. Here, for simplicity, we only consider *reduced* monoids (this is a reasonable assumption since Grothendieck monoids are reduced by Proposition 3.3.5).

Definition 3.A.8. Let M be a reduced monoid.

(1) A non-zero element x of M is called an *atom* if $x = y + z$ implies either $y = 0$ or $z = 0$ for $y, z \in M$. We denote by $\mathsf{Atom}\, M$ the set of all atoms in M.
(2) M is called *atomic* if $\mathsf{Atom}\, M$ generates M, that is, every element of M is a finite sum of atoms.
(3) M is called *factorial* if every element can be expressed as a finite sum of atoms, and this expression is unique up to permutations.
(4) M is called *half-factorial* if it is atomic, and for every element x and expressions

$$x = a_1 + \cdots + a_n$$

with $a_i \in \mathsf{Atom}\, M$, the number n depends only on x.

The following observations can be proved directly from the definitions.

Proposition 3.A.9. *Let M be a reduced monoid. Then the following hold.*

(1) *If M is generated by a subset A, then $\mathsf{Atom}\, M \subset A$ holds. In particular, $\mathsf{Atom}\, M$ is a finite set for a finitely generated monoid M.*
(2) *M is finitely generated and atomic if and only if M is atomic and $\mathsf{Atom}\, M$ is a finite set.*
(3) *If M is free on a subset A of M, then $A = \mathsf{Atom}\, M$ holds.*
(4) *If M is free, then it is atomic, cancellative and factorial.*
(5) *M is free if and only if M is factorial.*

3.A.3. Group completion. For a given monoid M, there is the universal construction which transforms M into a group. We call it a *group completion* in this paper.

Definition 3.A.10. Let M be a monoid. Then the *group completion* $\mathsf{gp}\, M$ is an abelian group $\mathsf{gp}\, M$ together with a map $\iota \colon M \to \mathsf{gp}\, M$ which satisfies the following universal property:

(1) ι is a monoid homomorphism.
(2) Every monoid homomorphism $f \colon M \to G$ into a group G factors uniquely ι, that is, there exists a unique group homomorphism $\overline{f} \colon \mathsf{gp}\, M \to G$ which satisfies $f = \iota\overline{f}$.

Example 3.A.11. Let M be a monoid which is free on A. Then $\mathsf{gp}\, M$ is a free abelian group with basis A, that is, $\mathsf{gp}(\mathbb{N}^{(A)}) = \mathbb{Z}^{(A)}$. The original basis A can be reconstructed from M since $A = \mathsf{Atom}\, M$, but this information is lost when taking the group completion, since bases of the free abelian group is far from unique. This is one reason why we consider monoids, not groups.

The explicit construction of the group completion is given as follows.

Proposition 3.A.12. *Let M be a monoid. Define an equivalence relation $\sim$ on the set $M \times M$ by*

$$(x_1, y_1) \sim (x_2, y_2) :\Leftrightarrow \text{ there exists an element } a \in M \text{ such that } x_1 + y_2 + a = x_2 + y_1 + a.$$

Then the quotient set $(M \times M)/\sim$, together with a map $M \to (M \times M)/\sim$ given by $x \mapsto (x, 0)$, is a group completion of M .

The cancellation property are related to the group completion as follows.

PROPOSITION 3.A.13. *Let M be a monoid. Then it is cancellative if and only if the natural map $\iota\colon M \to \mathsf{gp}\, M$ is an injection if and only if there is an injective monoid homomorphism into some group. In this case, every injective monoid homomorphism $\varphi\colon M \to G$ such that $\varphi(M)$ generates G is automatically a group completion of M.*

It follows that the image of the group completion is always cancellative. One of the difficulty of dealing with a monoid M is that M may not be cancellative. Thus this image is much easier to deal with than M, and it has more information on its group completion $\mathsf{gp}\, M$. This corresponds to the *positive part* of the Grothendieck group, see Section 3.3.2.

DEFINITION 3.A.14. Let M be a monoid. We denote by M_{can} the image of the group completion $\iota\colon M \to \mathsf{gp}\, M$, and call it the *cancellative quotient* of M.

We leave it the reader to check M_{can} is actually the largest cancellative quotient of M:

PROPOSITION 3.A.15. *Let M be a monoid and define an equivalence relation $\sim_{\mathsf{can}}$ on M by*

$$x \sim_{\mathsf{can}} y :\Leftrightarrow \text{ there exists an element } a \in M \text{ such that } x + a = y + a.$$

Then $\sim_{\mathsf{can}}$ is a monoid congruence on M, and we have an isomorphism of monoids $M_{\mathsf{can}} \cong M/\sim_{\mathsf{can}}$. Consequently, for every monoid homomorphism $\varphi\colon M \to N$ to a cancellative monoid N, there exists a unique monoid homomorphism $\overline{\varphi}\colon M_{\mathsf{can}} \to N$ such that $\varphi = \overline{\varphi}\iota$.

3.A.4. Length-like functions on monoids. We will introduce a kind of *length* on monoids, which corresponds to length-like functions on exact categories introduced in Definition 3.4.1.

DEFINITION 3.A.16. Let M be a monoid. A *length-like function* on M is a monoid homomorphism $\nu\colon M \to \mathbb{N}$ such that $\nu(x) = 0$ implies that $x = 0$.

The existence of a length-like function implies some nice properties.

PROPOSITION 3.A.17. *Let M be a monoid and suppose that there exists a length-like function $\nu\colon M \to \mathbb{N}$ on M. Then the following hold.*

(1) *M is naturally partially ordered, hence reduced.*
(2) *M is atomic.*

PROOF. (1) Suppose that $x \le y \le x$ holds for $x, y \in M$. Since ν is a monoid homomorphism, it follows immediately that $\nu(x) \le \nu(y) \le \nu(x)$ holds, thus $\nu(x) = \nu(y)$. On the other hand, we have $y = x + a$ for some $a \in M$ by $x \le y$. Thus $\nu(y) = \nu(x) + \nu(a)$, which implies that $\nu(a) = 0$. Since ν is a length-like function, $a = 0$ holds, hence $x = y$. Thus M is naturally partially ordered, so it is reduced by Proposition 3.A.4.

(2) Suppose that M is not atomic. Then take an element x such that:

(a) $x \neq 0$.
(b) x cannot be expressed as a finite sum of atoms.
(c) $\nu(x)$ is minimal among those x which satisfy (a) and (b).

Obviously x is not an atom by (b). Thus there is a decomposition $x = y + z$ with $y, z \neq 0$. Clearly either y or z satisfies (b), so let us assume that y satisfies (b). However, we have $\nu(x) = \nu(y) + \nu(z)$ with $\nu(y), \nu(z) \neq 0$ since ν is a length-like function. Therefore $\nu(y) < \nu(x)$ and y satisfies (a) and (b). This contradicts the minimality of x. □

For later use, we show the following characterization of half-factorial monoids.

LEMMA 3.A.18. *Let M be a monoid. Then the following are equivalent:*

(1) *M is a half-factorial monoid.*
(2) *M has a length-like function ν satisfying $\nu(a) = 1$ for every $a \in \mathsf{Atom}\, M$.*

PROOF. (1) ⇒ (2) Suppose that M is half-factorial. We define a monoid homomorphism $l\colon M \to \mathbb{N}$ as follows: For every x in $\mathcal{E}$, we can write $x = \sum_{i=1}^{l} a_i$ in $\mathcal{M}(\mathcal{E})$ with $a_i \in \mathsf{Atom}\, M$ for each i since M is atomic. We set $l(x) := l$. Since M is half-factorial, l does not depend on the choice of expressions, thus this map is well-defined. Furthermore, it is easy to see that l is a length-like function and that $l(a) = 1$ for every $a \in \mathsf{Atom}\, M$.

(2) ⇒ (1): First observe that M is reduced and atomic by Proposition 3.A.17. Let x be an element of M. Consider any expression

$$x = a_1 + \cdots + a_n$$

with $a_i \in \mathsf{Atom}\, M$. Then by (2), we have that $\nu(x) = \nu(a_1) + \cdots + \nu(a_n) = n$. Therefore, the number n depends only on x, so M is half-factorial. □

3.A.5. Characterizations of free monoids. In what follows, we give a criterion for a given monoid to be free. We use this results to check (JHP) in Section 4.

If a monoid M is free, then it has a length-like function (Lemma 3.A.18), its group completion is also free, and its rank coincides with the number of atoms. In general, we have the following inequality.

PROPOSITION 3.A.19. *Let M be a reduced atomic monoid and suppose that $\mathsf{gp}\, M$ is a free abelian group. Then the following inequality holds.*

$$\operatorname{rank}(\mathsf{gp}\, M) \leq \#\, \mathsf{Atom}\, M$$

PROOF. Let $\iota\colon M \to \mathsf{gp}\, M$ denote the group completion. As an abelian group, $\mathsf{gp}\, M$ is generated by ιM, so it is generated by $\iota(\mathsf{Atom}\, M)$. Thus $\operatorname{rank}(\mathsf{gp}\, M) \leq \#\, \mathsf{Atom}\, M$ holds. □

The following gives a kind of converse of this. This is an important characterization of free monoids, which is very useful to our setting.

THEOREM 3.A.20. *Let M be a monoid, and denote by $\iota\colon M \to \mathsf{gp}\, M$ the group completion of M. Then the following are equivalent:*

(1) *M is a free monoid.*
(2) *M has a length-like function, ι is injective on $\mathsf{Atom}\, M$, and $\mathsf{gp}\, M$ is a free abelian group with basis $\iota(\mathsf{Atom}\, M)$.*
(3) *M is reduced atomic, ι is injective on $\mathsf{Atom}\, M$, and $\mathsf{gp}\, M$ is a free abelian group with basis $\iota(\mathsf{Atom}\, M)$.*
(4) *M is reduced atomic, ι is injective on $\mathsf{Atom}\, M$, and all elements in $\iota(\mathsf{Atom}\, M)$ are linearly independent over $\mathbb{Z}$.*

PROOF. (1) ⇒ (2): M has a length-like function by Proposition 3.A.18. Since M is free, it is free on $\mathsf{Atom}\, M$ by Proposition 3.A.9 (3), and we have an isomorphism $M \cong \mathbb{N}^{(\mathsf{Atom}\, M)}$. Thus we have $\mathsf{gp}\, M \cong \mathbb{Z}^{(\mathsf{Atom}\, M)}$ by Example 3.A.11. Thus (2) follows.

(2) ⇒ (3): This follows from Proposition 3.A.17.

(3) ⇒ (4): This is trivial.

(4) ⇒ (1): Define a monoid homomorphism $\varphi\colon \mathbb{N}^{(\mathsf{Atom}\, M)} \to M$ by $\varphi(a) = a$ for each $a \in \mathsf{Atom}\, M$. We claim that this map is an isomorphism of monoids. It suffices to show that φ is a bijection.

Since M is reduced and atomic, every element of M is a finite sum of atoms. Thus φ is a surjection. On the other hand, consider the following commutative diagram of monoids

$$\begin{tikzcd} \mathbb{N}^{(\mathsf{Atom}\, M)} \arrow[r, two heads, "\varphi"] \arrow[d, hook, "i"'] & M \arrow[d, "\iota"] \\ \mathbb{Z}^{(\mathsf{Atom}\, M)} \arrow[r, "\overline{\varphi}"] & \mathsf{gp}\, M, \end{tikzcd}$$

where ι and i are group completions of M and $\mathbb{N}^{(\mathsf{Atom}\, M)}$ respectively, and $\overline{\varphi}$ is a group homomorphism induced by φ. Here i is obviously an injection. Moreover, since ι is injective on $\mathsf{Atom}\, M$ and all elements in $\iota(\mathsf{Atom}\, M)$ are linearly independent, $\overline{\varphi}$ is an injection. Then the above commutative diagram shows that so is φ. Therefore φ is a bijection. □

Under the assumption of finite generation, we have a more convenient characterization, in which we only have to count the number of atoms.

COROLLARY 3.A.21. *Let M be a monoid, and denote by $\iota\colon M \to \mathsf{gp}\, M$ the group completion of M. Then the following are equivalent:*

(1) *M is finitely generated and free as a monoid.*
(2) *M is a free monoid and $\#\,\mathsf{Atom}\, M$ is finite.*
(3) *M is reduced and atomic, ι is injective on $\mathsf{Atom}\, M$ and $\mathsf{gp}\, M$ is a free abelian group of finite rank with basis $\iota(\mathsf{Atom}\, M)$.*
(4) *The following hold:*
   (a) *M is reduced and atomic.*
   (b) *$\mathsf{gp}\, M$ is a free abelian group of finite rank.*
   (c) *$\#\,\mathsf{Atom}\, M = \mathrm{rank}(\mathsf{gp}\, M)$ holds, where* rank *denotes a rank as an abelian group.*

PROOF. (1) $\Rightarrow$ (2): This follows from Proposition 3.A.9 (1).

(2) $\Rightarrow$ (3): This follows from (1) $\Rightarrow$ (3) in Theorem 3.A.20.

(3) $\Rightarrow$ (4): Trivial.

(4) $\Rightarrow$ (1): We use the same notations and strategy as in the proof of (4) $\Rightarrow$ (1) in Theorem 3.A.20. It suffices to show that the natural map $\overline{\varphi}\colon \mathbb{Z}^{\mathsf{Atom}\, M} \to \mathsf{gp}\, M$ is an isomorphism. Since φ is a surjection, so is $\overline{\varphi}$. On the other hand, we have that $\mathsf{gp}\, M$ is free of rank $\#\,\mathsf{Atom}\, M$. By counting ranks, it follows that $\overline{\varphi}$ is an isomorphism. $\square$

CHAPTER 4

Bruhat inversions in Weyl groups and torsion-free classes over preprojective algebras

This chapter is based on [Eno5].

For an element w of the simply-laced Weyl group, Buan-Iyama-Reiten-Scott defined a subcategory $\mathcal{F}(w)$ of a module category over a preprojective algebra of Dynkin type. This paper aims at studying categorical properties of $\mathcal{F}(w)$ via its connection with the root system. We show that by taking dimension vectors, simple objects in $\mathcal{F}(w)$ bijectively correspond to Bruhat inversion roots of w. As an application, we obtain a combinatorial criterion for $\mathcal{F}(w)$ to satisfy the Jordan-Hölder property (JHP). To achieve this, we develop a method to find simple objects in a general torsion-free class by using a brick sequence associated to a maximal green sequence of it. For type A case, we give a diagrammatic construction of simple objects, and show that (JHP) can be characterized via a forest-like permutation, introduced by Bousquet-Mélou and Butler in the study of Schubert varieties.

4.1. Introduction

This paper focuses on the interplay between the preprojective algebras of Dynkin type and the root system. More precisely, we study a certain subcategory $\mathcal{F}(w)$ of the module category of the preprojective algebra via an inversion set in the root system.

4.1.1. Background. Let Φ be the simply-laced root system of type X and W the corresponding Weyl group. Let Q be a quiver of type X, that is, the underlying graph of Q is the Dynkin diagram X. Then the celebrated Gabriel's theorem gives a bijection between indecomposable representations of Q and positive roots in Φ, by taking dimension vectors.

A *preprojective algebra* Π of Φ is a finite-dimensional algebra which unifies the representation theory of all quivers of type X, and has a lot of symmetry compared to path algebras. This algebra has been one of the most important objects in the representation theory of algebras, for example, [AIRT, BIRS, GLS, IRRT, Miz], and also plays an important role in the theory of crystal bases of quantum groups, for example, [Lus, KaSa].

In this paper, we focus on a certain subcategory $\mathcal{F}(w)$ of $\operatorname{\mathsf{mod}}\Pi$ associated to an element w of W introduced by Buan-Iyama-Reiten-Scott [BIRS] (under the name $\mathcal{C}_w$). This category has a nice structure related to cluster algebras, that is, a stably 2-Calabi-Yau Frobenius category admitting a cluster-tilting object. Indeed, Geiss-Leclerc-Schröer later [GLS] proved that $\mathcal{F}(w)$ gives a categorification of the cluster algebra structure on the coordinate ring of the unipotent cell in the complex simple Lie group of Dynkin type X.

The category $\mathcal{F}(w)$ naturally arises also from the viewpoint of the representation theory of algebras, as well as the lattice theoretical study of the Weyl group. This category is a *torsion-free class* in $\operatorname{\mathsf{mod}}\Pi$, that is, closed under submodules and extensions. Mizuno [Miz] proved that the map $w \mapsto \mathcal{F}(w)$ is actually a bijection from W to the set $\operatorname{\mathsf{torf}}\Pi$ of all torsion-free classes in $\operatorname{\mathsf{mod}}\Pi$. He also proved that this bijection is an isomorphism of lattices, where we endow W with the right weak order and $\operatorname{\mathsf{torf}}\Pi$ the inclusion order. Via this isomorphism, lattice theoretical properties of W were investigated in [IRRT, DIRRT].

4.1.2. Main results. It is natural to expect that categorical properties of $\mathcal{F}(w)$ are related to combinatorial properties of w. In this direction, we prove the two main results: we classify *simple objects* in $\mathcal{F}(w)$ (Theorem A), and give a criterion for the validity of the *Jordan-Hölder type theorem* in $\mathcal{F}(w)$ (Theorem C).

A Π-module M in $\mathcal{F}(w)$ is called a *simple object* in $\mathcal{F}(w)$ if there is no non-trivial submodule L of M satisfying $L, M/L \in \mathcal{F}(w)$. This notion was introduced in the context of Quillen's exact categories, and has been investigated by several papers such as [Eno4, BHLR]. In [Eno4], the author classified simple objects in a torsion-free class in $\mathsf{mod}\, kQ$ for type A case, and the original motivation of this paper is to generalize this to other Dynkin types and to preprojective algebras.

Our strategy is to consider $\mathcal{F}(w)$ via the root system Φ. For $M \in \mathsf{mod}\,\Pi$, we can regard its dimension vector as a vector in the ambient space of Φ naturally. Let $\mathsf{inv}(w)$ be the set of *inversions* of w, positive roots which are sent to negative by w^{-1}. Then any $M \in \mathcal{F}(w)$, its dimension vector $\underline{\dim}\, M$ is a non-negative integer linear combination of inversions of w (Corollary 4.4.5). In parallel with simples in $\mathcal{F}(w)$, it is natural to consider an inversion of w which cannot be written as a sum of inversions of w non-trivially. We call such a root a *Bruhat inversion* (Definition 4.2.11, Theorem 4.2.17). Then the first main result of this paper is the following:

Theorem J (= Corollary 4.4.9). *Let Π be the preprojective algebra of Dynkin type and w an element of the corresponding Weyl group W. Then by taking dimension vectors, we have the bijection between the following two sets:*

(1) *The set of isomorphism classes of simple objects in $\mathcal{F}(w)$.*
(2) *The set of Bruhat inversions of w.*

This immediately deduces the similar result for the case of the path algebra kQ. A torsion-free class in $\mathsf{mod}\, kQ$ bijectively corresponds to a *c_Q-sortable element* of W by [IT]. For such an element w of W, we have a torsion-free class $\mathcal{F}_Q(w)$ in $\mathsf{mod}\, kQ$, which is actually equal to the restriction of $\mathcal{F}(w)$ to $\mathsf{mod}\, kQ$ (Proposition 4.5.4). Then we have the same result for $\mathcal{F}_Q(w)$:

Theorem K (= Theorem 4.5.5). *Let Q be the Dynkin quiver, w a c_Q-sortable element of the Weyl group W and $\mathcal{F}_Q(w)$ the corresponding torsion-free class in $\mathsf{mod}\, kQ$. Then by taking dimension vectors, we have the bijection between the following two sets:*

(1) *The set of isomorphism classes of simple objects in $\mathcal{F}_Q(w)$.*
(2) *The set of Bruhat inversions of w.*

As an application, we can characterize the validity of the *Jordan-Hölder property (JHP)* in $\mathcal{F}(w)$ or $\mathcal{F}_Q(w)$. We say that a torsion-free class $\mathcal{F}$ *satisfies (JHP)* if for any M in $\mathcal{F}$, any relative composition series of M in $\mathcal{F}$ are equivalent (see Definition 4.4.10 for the precise definition). By using the characterization of (JHP) obtained in [Eno5], we prove the following second main result.

Theorem L (= Theorem 4.4.15, Proposition 4.A.6). *Let Π be the preprojective algebra of Dynkin type and w an element of the corresponding Weyl group W. Then the following are equivalent:*

(1) *$\mathcal{F}(w)$ satisfies (JHP).*
(2) *Bruhat inversions of w are linearly independent.*
(3) *The number of Bruhat inversions of w is equal to that of supports of w.*

Moreover, for the type A case, the above statements are equivalent to the following:

(4) *w is forest-like in the sense of* [BMB]*, that is, its Bruhat inversion graph is acyclic.*

Here a *support* of w is a vertex i in the Dynkin diagram X such that the reduced expression of w contains the simple reflection s_i. See Appendix 4.A.2 for the details on forest-like permutations. The same result also holds for the case of path algebras (Corollary 4.5.6).

To show these results, we develop a method to find simple objects in a given torsion-free class $\mathcal{F}$ by using a *brick sequence* associated to a maximal green sequence of $\mathcal{F}$. A *maximal green sequence* of $\mathcal{F}$ is just a saturated chain $0 = \mathcal{F}_0 \lessdot \mathcal{F}_1 \lessdot \cdots \lessdot \mathcal{F}_l = \mathcal{F}$ of torsion-free classes. We can associate to it a sequence of bricks (modules B such that $\operatorname{End}_\Lambda(B)$ is a division ring) by using the brick labeling introduced in [DIRRT]. Once we can compute one brick sequence of $\mathcal{F}$, the following gives a way to determine all simple objects in $\mathcal{F}$:

PROPOSITION M (= Corollary 4.3.12). *Let Λ be a finite-dimensional algebra and $\mathcal{F}$ a torsion-free class in $\mathsf{mod}\,\Lambda$. Suppose that there is a maximal green sequence of $\mathcal{F}$, and let $B_1, \dots, B_l$ be the associated brick sequence. Then the following hold:*

(1) *Every simple object in $\mathcal{F}$ is isomorphic to B_i for some i.*
(2) *For $1 \leq i \leq l$, the following are equivalent:*
    (a) *B_i is a simple object in $\mathcal{F}$.*
    (b) *Every morphism $B_i \to B_j$ with $i < j$ is either zero or injective.*
    (c) *There is no surjection $B_i \twoheadrightarrow B_j$ with $i < j$.*

4.1.3. Organization. This paper is organized as follows. In Section 4.2, we give root-theoretical preliminaries and results. More precisely, we characterize Bruhat inversions (Theorem 4.2.17). Next we introduce a root sequence associated to a reduced expression of an element of W, which plays an important role later. In Section 4.3, we develop a general theory of simple objects in a torsion-free class, and prove Proposition M. In Section 4.4, we focus on torsion-free classes over preprojective algebras of Dynkin type. We show that a brick sequence of $\mathcal{F}(w)$ categorifies a root sequence of w, and prove Theorem J, K. In Section 4.5, we deduce several results on path algebras using preprojective algebras, and prove Theorem L. In the appendix, we give two combinatorial interpretation of the results for type A case. In Section 4.A.1, we give a diagrammatic construction of simple objects in $\mathcal{F}(w)$ using arc diagrams, and in Section 4.A.2, we show that $\mathcal{F}(w)$ satisfies (JHP) if and only if w is a forest-like permutation.

4.1.4. Conventions and notation. Throughout this paper, *we assume that all categories are skeletally small*, that is, the isomorphism classes of objects form a set. In addition, *all subcategories are assumed to be full and closed under isomorphisms.* For a Krull-Schmidt category $\mathcal{E}$, we denote by $\mathsf{ind}\,\mathcal{E}$ the set of isomorphism classes of indecomposable objects in $\mathcal{E}$. We denote by $|X|$ the number of non-isomorphic indecomposable direct summands of X.

For a poset P and two elements $a, b \in P$ with $a \leq b$, we denote by $[a, b]$ the *interval poset* $[a, b] := \{x \in P \mid a \leq x \leq b\}$ with the obvious partial order. For $x, y \in P$, we say that *x covers y* if $x > y$ holds and there exists no $z \in P$ with $x > z > y$. In this case, we write $x \gtrdot y$ or $y \lessdot x$.

For a set A, we denote by $\#A$ the cardinality of A.

Acknowledgement. The author would like to thank his supervisor Osamu Iyama for many helpful comments and discussions. He is also grateful to Yuya Mizuno for explaining to him arc diagrams and bricks for type A case. This work is supported by JSPS KAKENHI Grant Number JP18J21556.

4.2. Preliminaries on root system

In this section, we give some results on root systems which we need later. In particular, we give a definition and characterization of Bruhat inversions, and introduce the notion of a root sequence.

4.2.1. Basic definitions. First we recall some basic definitions and properties of root systems. We refer the reader to [Hum1, Hum2] for the details.

Let V be the Euclidean space, that is, a finite-dimensional $\mathbb{R}$-vector space with the positive definite symmetric bilinear form $(-,-)\colon V \times V \to \mathbb{R}$.

For two vectors $\alpha, \beta \in V$ with $\alpha \neq 0$, we put

$$\langle \beta, \alpha \rangle := 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}.$$

Note that $\langle -, \alpha \rangle$ is linear but $\langle \alpha, - \rangle$ is not. For $\alpha \in V$ with $\alpha \neq 0$, we denote by $t_\alpha \colon V \to V$ the reflection with respect to α, that is,

$$t_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha.$$

DEFINITION 4.2.1. A subset Φ of the Euclidean space V is called a *root system* if it satisfies the following axioms:

(R0) Φ is a finite subset of V which spans V as an $\mathbb{R}$-vector space, and does not contain 0.

(R1) $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for every $\alpha \in \Phi$.
(R2) $t_\alpha(\Phi) = \Phi$ for every $\alpha \in \Phi$.
(R3) $\langle \beta, \alpha \rangle \in \mathbb{Z}$ for every $\alpha, \beta \in \Phi$.

A root system is called *simply-laced* if it satisfies the following condition.

(R4) $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle$ for every $\alpha, \beta \in \Phi$.

Obviously (R4) is equivalent to that for every $\alpha, \beta \in \Phi$, if $(\alpha, \beta) \neq 0$, then α and β have the same length, that is, $(\alpha, \alpha) = (\beta, \beta)$.

Possible values of integers $\langle \alpha, \beta \rangle$ in (R3) are very limited as follows.

PROPOSITION 4.2.2 ([Hum1, 9.4]). *Let Φ be a root system, and let α and β be two roots in Φ with $(\alpha, \alpha) \leq (\beta, \beta)$. Then the following hold:*

(1) $\langle \alpha, \beta \rangle \cdot \langle \beta, \alpha \rangle \geq 0$ *and* $(|\langle \alpha, \beta \rangle|, |\langle \beta, \alpha \rangle|) \in \{(0,0), (1,1), (1,2), (1,3), (2,2)\}$.
(2) $\beta = \pm\alpha$ *if and only if* $(|\langle \alpha, \beta \rangle|, |\langle \beta, \alpha \rangle|) = (2,2)$.
(3) Φ *is simply-laced if and only if* $\langle \alpha, \beta \rangle \in \{-1, 0, 1\}$ *for every* $\alpha, \beta \in \Phi$ *with* $\alpha \neq \beta$.

Throughout this section, we will use the following notation:

- Φ is a root system in V.
- We fix a choice of simple roots Δ of Φ.
- Φ^+ (resp. Φ^-) is the set of positive roots (resp. negative roots) in Φ with respect to Δ.
- W is the Weyl group associated with Φ, that is, W is a subgroup of $\mathrm{GL}(V)$ generated by t_α with $\alpha \in \Phi$.
- $T \subset W$ is a set of reflections in W, that is, $T = \{t_\alpha \in W \mid \alpha \in \Phi\}$.
- We often write $s_\alpha = t_\alpha$ if α is a simple root.

It is well-known that W is generated by simple reflections. For $w \in W$, let

$$w = s_1 \cdots s_l$$

be such an expression. Then this expression is called a *reduced expression* if l is minimal among all such expressions. In this case, l is called the *length* of w and we write $\ell(w) := l$.

On the reduced expression, we will need the following *exchange property* later.

LEMMA 4.2.3 ([Hum2, 1.6, 1.7]). *Let $w = s_1 \cdots s_l$ be a reduced expression of $w \in W$, and let α be a simple roots. Then the following are equivalent:*

(1) $s_1 \cdots s_l s_\alpha$ *is not reduced, that is,* $\ell(ws_\alpha) < \ell(w)$.
(2) *There exists i such that* $s_1 \cdots s_l s_\alpha = s_1 \cdots \widehat{s_i} \cdots s_l$ *(s_i omitted).*
(3) $w(\alpha) \in \Phi^-$.

4.2.2. Inversion sets and root sequences. For $w \in W$, we define $\mathsf{inv}(w)$ by

$$\mathsf{inv}(w) := \Phi^+ \cap w(\Phi^-),$$

that is, $\mathsf{inv}(w)$ is the set of positive roots which are sent to negative roots by w^{-1}. We call an element of $\mathsf{inv}(w)$ an *inversion of w*. This set plays an important role in this paper, since it corresponds to the set of dimension vectors of bricks in a torsion-free class $\mathcal{F}(w)$ over the preprojective algebra (Corollary 4.4.5).

The following description of inversion sets is well-known.

PROPOSITION 4.2.4 ([Hum2, 1.7]). *Let w be an element of W. Take a reduced expression $w = s_{\alpha_1} \cdots s_{\alpha_l}$ of w with $\alpha_i \in \Delta$. Then we have*

$$\mathsf{inv}(w) = \{\alpha_1,\ s_{\alpha_1}(\alpha_2),\ s_{\alpha_1}s_{\alpha_2}(\alpha_3),\ \ldots,\ s_{\alpha_1} \cdots s_{\alpha_{l-1}}(\alpha_l)\},$$

and all the elements above are distinct. In particular, we have $\#\,\mathsf{inv}(w) = \ell(w)$.

By using this, we can easily show the following property.

LEMMA 4.2.5. *Let v and w elements of W satisfying $\ell(vw) = \ell(v) + \ell(w)$. Then we have* $\mathsf{inv}(vw) = \mathsf{inv}(v) \sqcup v(\mathsf{inv}(w))$.

If we choose a reduced expression of w, then Proposition 4.2.4 gives a sequence of positive roots. It turns out that the order of appearance are important for our purpose. This leads to the notion of *root sequences*.

DEFINITION 4.2.6. Let W be the Weyl group of a root system Φ.

(1) Let w be an element of W and $w = s_{\alpha_1} \cdots s_{\alpha_l}$ a reduced expression of w. Then a *root sequence of w associated to this expression* is a (ordered) sequence of positive roots

$$\alpha_1,\ s_{\alpha_1}(\alpha_2),\ s_{\alpha_1}s_{\alpha_2}(\alpha_3),\ \ldots,\ s_{\alpha_1}\cdots s_{\alpha_{l-1}}(\alpha_l).$$

(2) A *root sequence* is a sequence of roots which arises as a root sequence of some reduced expression of some element w in W. We call such a sequence a *root sequence of w*.

The notion of root sequences appeared in several papers: they are called *compatible (convex) orderings* in [Pap], *reflection orderings* in [Dye]. We borrowed the terminology *root sequences* from [GL, FS].

We will use the following characterization of inversion sets and root sequences due to Papi later.

THEOREM 4.2.7 ([Pap]). *A sequence $\mathcal{R}$ of positive roots is a root sequence if and only if the following two conditions are satisfied for any pair of positive roots α, β satisfying $\alpha + \beta \in \Phi^+$:*

(1) *If α and β appear in $\mathcal{R}$, then $\alpha + \beta$ appears between α and β in $\mathcal{R}$.*
(2) *If $\alpha + \beta$ appears in $\mathcal{R}$, then one of α and β appears and precedes $\alpha + \beta$ in $\mathcal{R}$.*

To give examples of root sequences (and its connection with *brick sequence* defined in the next section), it is convenient to introduce the *right weak order* $\leq_R$ on W and its *Hasse quiver*. Define a quiver $\mathsf{Hasse}(W, \leq_R)$ as follows:

- A vertex set of $\mathsf{Hasse}(W, \leq_R)$ is W.
- We draw an arrow $v \leftarrow vs$ if $\ell(vs) = \ell(v) + 1$ for $v \in W$ and a simple reflection $s \in W$.

Then define a partial order $\leq_R$ on W by $v \leq_R w$ if and only if there is a path from w to v in $\mathsf{Hasse}(W, \leq_R)$. It is known that $(W, \leq_R)$ is actually a lattice, see e.g. [BB, 3.2], and $\mathsf{Hasse}(W, \leq_R)$ is actually a Hasse quiver of $(W, \leq_R)$ defined later.

By construction, each reduced expression of $w = s_1 s_2 \cdots s_l$ gives a reverse path $e \leftarrow s_1 \leftarrow s_1 s_2 \leftarrow \cdots \leftarrow w$ from e to w in $\mathsf{Hasse}(W, \leq_R)$, and this correspondence is a bijection. We define the *root labeling* of arrows in $\mathsf{Hasse}(W, \leq_R)$ by attaching a positive root $v(\alpha)$ to an arrow $v \leftarrow v s_\alpha$. Then a root sequence associated to a reduced expression $w = s_1 \cdots s_l$ is obtained by reading labels of the corresponding reverse path $e \leftarrow s_1 \leftarrow \cdots \leftarrow w$.

EXAMPLE 4.2.8. Let Φ be the root system of type A_3 with its Dynkin graph 1—2—3. We denote by s_i (resp. α_i) the simple reflection (resp. simple root) associated to the vertex i for $1 \leq i \leq 3$. Consider $w = s_1 s_2 s_3 s_1 s_2 \in W$. Figure 1 shows all the reverse paths from e to w and its root labeling. Here we write $s_{1231} := s_1 s_2 s_3 s_1$ and $110 := \alpha_1 + \alpha_2$ for example. Then the left

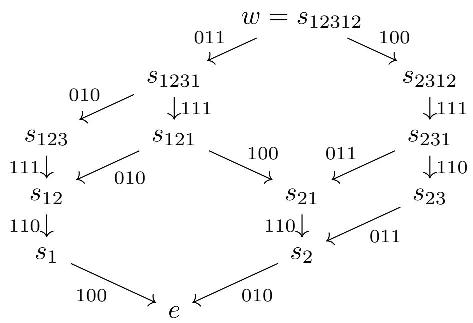


FIGURE 1. Root sequences of $w = s_{12312}$

most path corresponds to a reduced expression s_{12312}, and the right most corresponds to s_{23121}. Their associated root sequences are $100, 110, 111, 010, 011$ and $010, 011, 110, 111, 100$ respectively.

EXAMPLE 4.2.9. Let Φ be the root system of type D_4, whose Dynkin diagram is as follows:

$$\begin{array}{ccccc} & & 2 & & \\ & & \mid & & \\ 1 & - & 0 & - & 3 \end{array}$$

Consider $w = s_{012301230}$ (here we use the same notation as in Example 4.2.8). Figure 2 shows all the reduced expressions of w and its root sequence. For example, the right most path gives a reduced

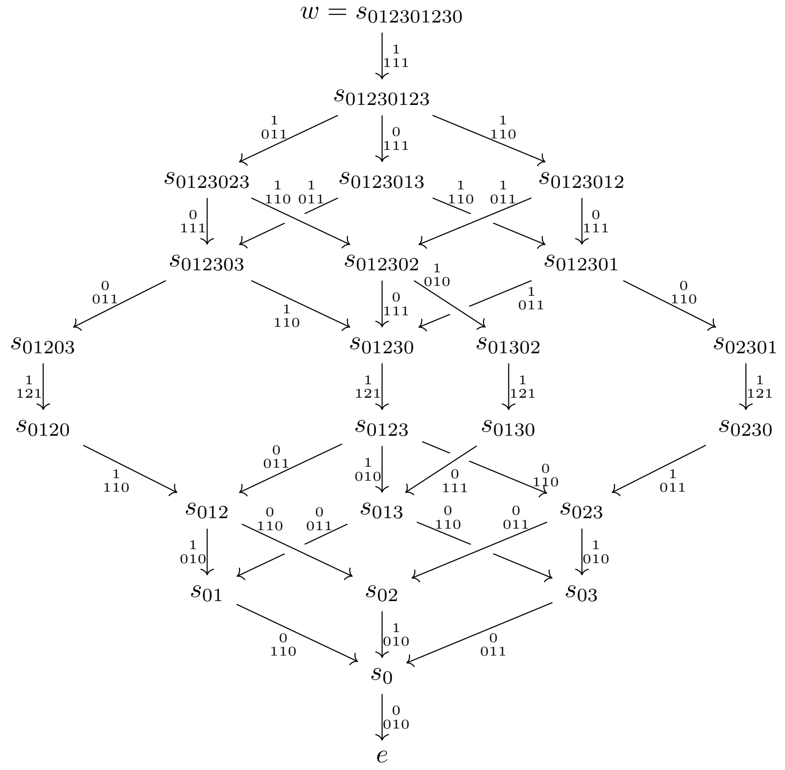


FIGURE 2. Root sequences for $w = s_{012301230}$

expression $w = s_{012030210}$ with its associated root sequence $\substack{0\\010}, \substack{0\\110}, \substack{1\\010}, \substack{1\\110}, \substack{1\\121}, \substack{0\\011}, \substack{0\\111}, \substack{1\\011}, \substack{1\\111}$ (here as before, $\substack{1\\011}$ denotes $\alpha_0 + \alpha_2 + \alpha_3$ for example).

4.2.3. Bruhat inversions. A simple root in a root system is *indecomposable in* Φ^+ in the sense that we cannot write it as a positive linear combination of positive roots in a non-trivial way.

We are interested, not in the whole set of positive roots, but in the inversion set $\mathsf{inv}(w)$ for a fixed $w \in W$. It is natural to ask which elements are *indecomposable* in the above sense. This leads to the notion of *Bruhat inversions.*

Let us begin with the property of reflections with respect to inversion roots of w.

PROPOSITION 4.2.10 ([Hum2, Theorem 5.8], [BB, Theorem 1.4.3]). *Let w be an element of W and β a positive root. Then the following hold:*

(1) *Then $\beta \in \mathsf{inv}(w)$ if and only if $\ell(t_\beta w) < \ell(w)$ holds.*
(2) *Let $s_1 s_2 \cdots s_l = w$ be a reduced expression with $s_i = s_{\alpha_i}$, and let $\beta_1, \beta_2, \ldots, \beta_l$ a root sequence associated to it. Then we have*

$$t_{\beta_i} w = s_1 s_2 \cdots \widehat{s_i} \cdots s_l.$$

Next we will define *Bruhat inversions* of elements in W. Recall that the *Bruhat order* on W is the transitive closure of the following relation: for every $t \in T$ and $w \in W$ satisfying $\ell(tw) < \ell(w)$, we have $tw \leq w$.

DEFINITION 4.2.11. Let w be an element of W and β a positive root. Then β is a *Bruhat inversion* of w if it satisfies $\ell(t_\beta w) = \ell(w) - 1$. We denote by $\mathsf{Binv}(w)$ the set of Bruhat inversions of w. We call an element of $\mathsf{inv}(w) \setminus \mathsf{Binv}(w)$ a *non-Bruhat inversion* of w.

By the description in Proposition 4.2.10, the number of Bruhat inversions can be computed as follows: fix a reduced expression $w = s_1 \cdots s_l$ of w, then $\# \mathsf{Binv}(w)$ is the number of i's such that deleting s_i from this expression still yields a reduced expression.

A Bruhat inversion is closely related to the covering relation in the Bruhat order as follows.

PROPOSITION 4.2.12. *For $w \in W$ and $\beta \in \Phi^+$, the following are equivalent:*

(1) *$\beta \in \mathsf{Binv}(w)$, that is, β is a Bruhat inversion of w.*
(2) *w covers $t_\beta w$ in the Bruhat order of w.*

PROOF. This easily follows from Proposition 4.2.10 and the chain property of the Bruhat order, see [BB, Theorem 2.2.6] for example. □

The following is an example of Bruhat inversions for type A case (we refer the reader to the appendix for the detail).

EXAMPLE 4.2.13. Let Φ be the standard root system of type A_n, and $\alpha_1, \dots, \alpha_n$ the simple roots. Then positive roots are of the form $\beta_{(i\ j)} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ for $1 \le i < j \le n+1$. We can identify W with the symmetric group S_{n+1}. For w in $W = S_{n+1}$, we have that $\beta_{(i,j)} \in \mathsf{inv}(w)$ if and only if (i,j) is the *classical inversion* of w, that is, $i < j$ and $w^{-1}(i) > w^{-1}(j)$ holds. Moreover, we can easily see that $\beta_{(i,j)} \in \mathsf{Binv}(w)$ if and only if $\beta_{(i,j)} \in \mathsf{inv}(w)$ and there is no $i < k < j$ with $\beta_{(i,k)}, \beta_{(k,j)} \in \mathsf{inv}(w)$. For example, if $w = 42153 \in S_5$, we have $\mathsf{inv}(w) = \{\beta_{(1,2}, \beta_{(1,4)}, \beta_{(2,4)}, \beta_{(3,4)}, \beta_{(3,5)}\}$ and $\mathsf{Binv}(w) = \mathsf{inv}(w) \setminus \{\beta_{(1,4)}\}$.

For a fixed element $w \in W$, we will give a characterization of Bruhat inversions of w among all inversions of w. To do this, we prepare some lemmas.

The following is known as the *lifting property* of the Bruhat order.

LEMMA 4.2.14 ([BB, Proposition 2.2.7]). *Let v and w be elements in W satisfying $v < w$, and let s be a simple reflection. If $\ell(sv) = \ell(v) + 1$ and $\ell(sw) = \ell(w) - 1$ holds, then we have $sv < w$ and $v < sw$.*

By using the lifting property, we can show the following technical lemma.

LEMMA 4.2.15. *Let w be an element of W and $\beta \in \mathsf{Binv}(w)$, and put $t = t_\beta \in W$. Then either one of the following holds:*

(1) *β is a simple root.*
(2) *There exists a simple reflection $s \in W$ which satisfies the following two conditions:*
    (a) *$\ell(sw) = \ell(w) + 1$.*
    (b) *$\ell(stw) = \ell(tw) + 1$.*

PROOF. Suppose that (2) does not hold. We will show that β must be simple.

Since (2) does not hold, for every simple reflection $s \in W$, we have that either (a) or (b) (or both) fails to hold. We will show the following claim.

(Claim): *There exists some simple reflection $s \in W$ such that* (a) *does not hold and* (b) *holds.*

Proof of (Claim). If this is not the case, then (b) does not hold for every simple reflection $s \in W$. This means that tw is the longest element in W, which contradicts to $\ell(w) = \ell(tw) + 1$. ■

Take such a simple reflection s. Then we have that $\ell(sw) = \ell(w) - 1$ and $\ell(stw) = \ell(tw) + 1$ hold, and that $tw < w$ by the assumption. Then by Lemma 4.2.14, we have that $tw \le sw < w$ holds. Since w covers tw in the Bruhat order, we must have $tw = sw$, hence $t = s$. Therefore β is a simple root. □

Now we can show that a Bruhat inversion can be transformed into a simple root:

PROPOSITION 4.2.16. *Let w be an element of W and β an inversion of w. Then the following are equivalent:*

(1) *β is a Bruhat inversion of w.*
(2) *There exists some element $v \in W$ which satisfies the following two conditions:*
  (a) *$\ell(vw) = \ell(v) + \ell(w)$ holds.*
  (b) *$v(\beta)$ is a simple root.*

PROOF. (1) $\Rightarrow$ (2): Suppose that $\beta \in \mathsf{Binv}(w)$ holds. If β is a simple root, then $v = e$ satisfies the conditions (2)(a) and (b).

From now on, we assume that β is not simple. Then by Lemma 4.2.15, there exists a simple reflection s such that $\ell(sw) = \ell(w)+1$ and $\ell(st_\beta w) = \ell(w)$ hold. Put $w' = sw$ and $\beta' := s(\beta)$, then $\beta' \in \mathsf{inv}(w')$ by Lemma 4.2.5. Moreover, $\ell(st_\beta w) = \ell(w)$ implies that $\ell(t_{\beta'} w') = \ell(st_\beta s \cdot sw) = \ell(w) = \ell(w') - 1$. Thus β' is a Bruhat inversion of w'. If β' is a simple root, then $v = s$ satisfies the conditions (2)(a) and (b).

If β' is not simple, then we can iterate this process by considering β' and w' instead of β and w. Moreover, this process must stop at some point since otherwise we would have $\ell(w) < \ell(w') < \cdots$, which contradicts to the existence of the longest element in W. Therefore, we have that the Bruhat inversion at this point, which can be written as $v(\beta)$ for some v, is a simple root.

(2) $\Rightarrow$ (1): Since $v(\beta)$ is a simple root and $v(\beta) \in \mathsf{inv}(vw)$ by Lemma 4.2.5, we have $\ell(t_{v(\beta)} vw) = \ell(vw) - 1 = \ell(v) + \ell(w) - 1$. On the other hand, since $t_{v(\beta)} vw = vt_\beta v^{-1} vw = vt_\beta w$ holds, we have $\ell(t_{v(\beta)} vw) \le \ell(v) + \ell(t_\beta w)$. Thus $\ell(w) - 1 \le \ell(t_\beta w)$. Since β is an inversion of w, we have $\ell(t_\beta w) \le \ell(w) - 1$ by Proposition 4.2.10. Therefore, we have $\ell(t_\beta w) = \ell(w) - 1$, that is, β is a Bruhat inversion of w. $\square$

The following is a main result in this section, which gives a characterization of Bruhat inversions.

THEOREM 4.2.17. *Let w be an element of the Weyl group W of Φ, and $\beta \in \mathsf{inv}(w)$. Then the following are equivalent:*

(1) *β is a* non-*Bruhat inversion of w.*
(2) *There exists $a_\gamma \in \mathbb{R}_{\ge 0}$ for each $\gamma \in \mathsf{inv}(w)$ which satisfies the following conditions:*
  - *$\beta = \sum_{\gamma \in \mathsf{inv}(w)} a_\gamma \gamma$ holds.*
  - *This expression is not of the form $\beta = 1 \cdot \beta$, that is, there exists some $\gamma \in \mathsf{inv}(w)$ such that $a_\gamma \ne \delta_{\beta,\gamma}$, where δ is the Kronecker delta.*
(3) *There exist $\gamma_1, \gamma_2 \in \mathsf{inv}(w)$ with $\gamma_1 \ne \gamma_2$ and $a_1, a_2 \in \mathbb{R}_{>0}$ such that $\beta = a_1\gamma_1 + a_2\gamma_2$ holds.*
(4) *There exist $\gamma_1, \gamma_2 \in \mathsf{inv}(w)$ with $\gamma_1 \ne \gamma_2$ and $n \in \{1, 2, 3\}$ such that $\beta = (\gamma_1 + \gamma_2)/n$ holds.*

Moreover, if Φ is simply-laced, then the above statements are equivalent to the following:

(5) *There exist $\gamma_1, \gamma_2 \in \mathsf{inv}(w)$ satisfying $\beta = \gamma_1 + \gamma_2$.*

PROOF. The implications (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) are clear.

(2) $\Rightarrow$ (1): Suppose that β is a Bruhat inversion. Then by Proposition 4.2.16, we have an element $v \in W$ such that $\ell(vw) = \ell(v) + \ell(w)$ and that $v(\beta)$ is a simple root. Note that Lemma 4.2.5 implies $v(\mathsf{inv}(w)) \subset \Phi^+$.

Now take $a_\gamma \in \mathbb{R}_{\ge 0}$ for $\gamma \in \mathsf{inv}(w)$ with $\beta = \sum_\gamma a_\gamma \gamma$ as claimed in (2). Then we have

$$v(\beta) = \sum_{\gamma \in \mathsf{inv}(w)} a_\gamma v(\gamma).$$

Since $v(\beta)$ is a simple root and all $v(\gamma)$'s are distinct positive roots, we must have that $a_\gamma = \delta_{\beta,\gamma}$, which contradicts to the condition in (2). Thus β is not a Bruhat inversion.

(1) $\Rightarrow$ (4),(5): Take any reduced expression $w = s_1 s_2 \cdots s_l$ of w, and let $\beta_1, \beta_2, \dots, \beta_l$ be the root sequence associated to it. Then $\beta = \beta_m$ for some m. Since we have $t_\beta w = s_1 \cdots \widehat{s_m} \cdots s_l$ by Proposition 4.2.10 and β is *not* a Bruhat inversion, the expression $s_1 \cdots \widehat{s_m} \cdots s_l$ is not reduced.

Take the minimal j such that $s_1 \cdots \widehat{s_m} \cdots s_j$ is *not* a reduced expression. Then by Lemma 4.2.3, there exists i such that $s_1 \cdots \widehat{s_m} \cdots s_{j-1} = s_1 \cdots \widehat{s_i} \cdots \widehat{s_m} \cdots s_j$. In this situation, we will prove the following claim.

(Claim): *We have $t_\beta(\beta_j) = -\beta_i$.*

Proof of (Claim). First we have the following equation:

$$\begin{aligned} t_\beta(\beta_j) &= t_{\beta_m}(\beta_j) \\ &= (s_1 \cdots s_{m-1} s_m s_{m-1} \cdots s_1)(\beta_j) \\ &= (s_1 \cdots s_{m-1} s_m s_{m-1} \cdots s_1)(s_1 \cdots s_{j-1})(\alpha_j) \\ &= (s_1 \cdots \widehat{s_m} \cdots s_{j-1})(\alpha_j), \end{aligned}$$

where α_j is a simple root with $s_{\alpha_j} = s_j$. On the other hand, since $s_1 \cdots \widehat{s_m} \cdots s_{j-1} s_j$ is not reduced, Lemma 4.2.3 implies that $(s_1 \cdots \widehat{s_m} \cdots s_{j-1})(\alpha_j)$ is a negative root. Thus $t_\beta(\beta_j)$ is a negative root.

Now consider the reflection with respect to $t_\beta(\beta_j)$:

$$\begin{aligned} t_{t_\beta(\beta_j)} &= t_\beta t_{\beta_j} t_\beta \\ &= (s_1 \cdots s_m \cdots s_1)(s_1 \cdots s_m \cdots s_j \cdots s_m \cdots s_1)(s_1 \cdots s_m \cdots s_1) \\ &= s_1 \cdots \widehat{s_m} \cdots s_{j-1} s_j s_{j-1} \cdots \widehat{s_m} \cdots s_1 \\ &= (s_1 \cdots \widehat{s_i} \cdots \widehat{s_m} \cdots s_j) s_j s_{j-1} \cdots \widehat{s_m} \cdots s_1 \\ &= s_1 \cdots s_i \cdots s_1 \\ &= t_{\beta_i}. \end{aligned}$$

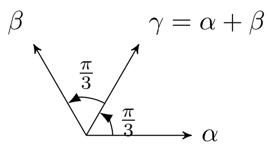
Thus the reflection with respect to $t_\beta(\beta_j)$ coincides with that along β_i. Since $t_\beta(\beta_j)$ is a negative root, we must have $t_\beta(\beta_j) = -\beta_i$. ■

Now by (Claim), we have $\beta_j - \langle \beta_j, \beta \rangle \beta = t_\beta(\beta_j) = -\beta_i$, thus $\langle \beta_j, \beta \rangle \beta = \beta_i + \beta_j$. Since β, β_i and β_j are positive roots, we must have $\langle \beta_j, \beta \rangle > 0$. We have $\langle \beta_j, \beta \rangle \in \{1, 2, 3\}$ by Proposition 4.2.2, so (4) holds.

Assume that Φ is simply-laced. Since $m < j$, we have that $\beta_m = \beta$ and β_j are distinct. Thus $\langle \beta_j, \beta \rangle = 1$ holds by Proposition 4.2.2, so (5) holds. □

By this theorem, a non-Bruhat inversion γ of w can be written as $\gamma = \alpha + \beta$ with $\alpha, \beta \in \mathsf{inv}(w)$ if Φ is simply-laced. This kind of equation gives a restriction of the relative position of α, β and γ as follows.

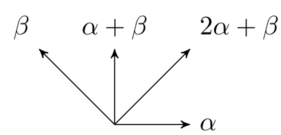
LEMMA 4.2.18. *Suppose that Φ is simply-laced, and that α and β in Φ satisfies $\gamma := \alpha + \beta \in \Phi$. Then $\langle \alpha, \beta \rangle = -1$, $\langle \alpha, \gamma \rangle = \langle \beta, \gamma \rangle = 1$ hold, thus they look as follows.*



PROOF. Clearly α, β, γ are distinct. Thus we have that $\langle \beta, \alpha \rangle, \langle \gamma, \alpha \rangle \in \{-1, 0, 1\}$ by Proposition 4.2.2. However, we have $\langle \gamma, \alpha \rangle = \langle \alpha + \beta, \alpha \rangle = \langle \alpha, \alpha \rangle + \langle \beta, \alpha \rangle = 2 + \langle \beta, \alpha \rangle$. Thus we must have $\langle \beta, \alpha \rangle = -1$ and $\langle \gamma, \alpha \rangle = 1$. The equation $\langle \gamma, \beta \rangle = 1$ can be shown similarly.

Since $(\alpha, \beta), (\alpha, \gamma) \neq 0$ and Φ is simply-laced, α, β, γ has the same length. Then it easily follows from $\langle \beta, \alpha \rangle = -1$ that the angle between α and β is $\frac{2}{3}\pi$. Therefore the situation looks like the figure. □

REMARK 4.2.19. If Φ is not simply-laced, then a non-Bruhat inversion γ of w may not be written as a sum of other inversions. For example, let Φ be the root system of type B_2 with α a short simple root and β a long simple root:



Then we have four positive roots. Consider $w = s_\beta s_\alpha s_\beta$. Easy computation shows that its root sequence is $\beta, \alpha+\beta, 2\alpha+\beta$, thus we have $\mathsf{inv}(w) = \{\beta, \alpha+\beta, 2\alpha+\beta\}$. By using Theorem 4.2.17, one can conclude that $\alpha+\beta$ is not a Bruhat inversion of w since $\alpha+\beta = \beta/2+(2\alpha+\beta)/2$ holds (of course this can be deduced by checking that $s_\beta \widehat{s_\alpha} s_\beta$ is not reduced, which is trivial). Nevertheless, we cannot write $\alpha+\beta$ as a sum of other inversions of w.

4.3. Brick sequence of a torsion-free class

In this section, we define a brick sequence of a torsion-free class $\mathcal{F}$, which is associated to a maximal green sequence of $\mathcal{F}$ (a saturated chain of torsion-free classes between 0 and $\mathcal{F}$). In particular, we focus on the relation between brick sequences and simple objects in $\mathcal{F}$.

Throughout this section, let Λ be a finite-dimensional k-algebra over a field k. We denote by $\mathsf{mod}\,\Lambda$ the category of finitely generated right Λ-modules. A module always means a finitely generated right modules.

4.3.1. Brick labeling. We briefly recall the lattice structure of torsion-free classes in $\mathsf{mod}\,\Lambda$, following [DIRRT]. Note that in [DIRRT], *torsion classes*, the dual notion of torsion-free classes, were mainly studied, but the same theory works also for torsion-free classes by the standard duality.

A subcategory $\mathcal{F}$ of $\mathsf{mod}\,\Lambda$ is a *torsion-free class* if it is closed under extensions and submodules in $\mathsf{mod}\,\Lambda$. We denote by $\mathsf{torf}\,\Lambda$ the set of all torsion-free classes in $\mathsf{mod}\,\Lambda$. Then $\mathsf{torf}\,\Lambda$ is a poset with respect to inclusion. Moreover, since intersection of any torsion-free classes is also a torsion-free class, $\mathsf{torf}\,\Lambda$ is a complete lattice.

For a poset P, its *Hasse quiver* $\mathsf{Hasse}\,P$ is a quiver defined as follows:

- A vertex set of $\mathsf{Hasse}\,P$ is P.
- We draw a unique arrow $x \to y$ in $\mathsf{Hasse}\,P$ if $x \gtrdot y$, that is, x covers y in a poset P.

Now $\mathsf{Hasse}(\mathsf{torf}\,\Lambda)$ has the additional structure called the *brick labeling*, established in [DIRRT]. Let us introduce some terminologies to state it. A Λ-module M in $\mathsf{mod}\,\Lambda$ is called a *brick* if $\mathrm{End}_\Lambda(M)$ is a division ring. For a collection $\mathcal{C}$ of Λ-modules, we will use the following notations:

- $\mathsf{add}\,\mathcal{C}$ denotes the category of direct summands of finite direct sums of modules in $\mathcal{C}$.
- $\mathsf{Sub}\,\mathcal{C}$ denotes the category of modules X such that there exists an injection from X to a module in $\mathsf{add}\,\mathcal{C}$.
- $\mathsf{Filt}\,\mathcal{C}$ denotes the category of modules X such that there exists a filtration $0 = X_0 \leq X_1 \leq \cdots \leq X_n = X$ of submodules of X satisfying $X_i/X_{i-1} \in \mathcal{C}$ for each i.
- $\mathsf{F}(\mathcal{C})$ denotes the smallest torsion-free class containing $\mathcal{C}$, or equivalently, $\mathsf{F}(\mathcal{C}) = \mathsf{Filt}(\mathsf{Sub}\,\mathcal{C})$.
- ${}^\perp\mathcal{C}$ denotes the category of modules X satisfying $\mathrm{Hom}_\Lambda(X, \mathcal{C}) = 0$.
- $\mathcal{C}^\perp$ denotes the category of modules X satisfying $\mathrm{Hom}_\Lambda(\mathcal{C}, X) = 0$.
- $\mathsf{ind}\,\mathcal{C}$ denotes the set of isomorphism classes of indecomposable modules in $\mathcal{C}$.
- $\mathsf{brick}\,\mathcal{C}$ denotes the set of isomorphism classes of bricks in $\mathcal{C}$.

For two collections $\mathcal{C}$ and $\mathcal{D}$ of Λ-modules, $\mathcal{C} * \mathcal{D}$ denotes the category of modules X such that there exist an exact sequence

$$0 \longrightarrow C \longrightarrow X \longrightarrow D \longrightarrow 0$$

with $C \in \mathcal{C}$ and $D \in \mathcal{D}$. From the classical torsion theory, we have $\mathsf{mod}\,\Lambda = {}^\perp\mathcal{F} * \mathcal{F}$ for a torsion-free class $\mathcal{F}$ in $\mathsf{mod}\,\Lambda$, see e.g. [ASS, VI.1]. It can be easily checked that the operation $*$ is associative, that is, we have $(\mathcal{C} * \mathcal{D}) * \mathcal{E} = \mathcal{C} * (\mathcal{D} * \mathcal{E})$ holds for collections $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ of Λ-modules. Thus we omit parentheses and just write as $\mathcal{C} * \mathcal{D} * \mathcal{E}$.

In [DIRRT], the following basic observation was established.

Proposition 4.3.1 ([DIRRT, Theorems 3.3, 3.4]). *Let $\mathcal{G} \subset \mathcal{F}$ be two torsion-free classes in $\mathsf{mod}\,\Lambda$. Then there exists an arrow $q\colon \mathcal{F} \to \mathcal{G}$ in $\mathsf{Hasse}(\mathsf{torf}\,\Lambda)$ if and only if $\mathsf{brick}({}^\perp\mathcal{G} \cap \mathcal{F})$ contains exactly one element B_q. In this case, we have ${}^\perp\mathcal{G} \cap \mathcal{F} = \mathsf{Filt}(B_q)$, $\mathcal{F} = \mathsf{Filt}(B_q) * \mathcal{G}$ and $\mathcal{G} = \mathcal{F} \cap B_q^\perp$.*

By this, to each arrow q in $\mathsf{Hasse}(\mathsf{torf}\,\Lambda)$ we can associate a brick B_q, which we call the *label of q*. This labeling is called the *brick labeling* of $\mathsf{torf}\,\Lambda$.

4.3.2. Brick sequence associated to a maximal green sequence. To study the structure of a fixed torsion-free class $\mathcal{F}$, a *brick sequence* associated to a *maximal green sequence* of $\mathcal{F}$ plays an important role later. Let us introduce these notions.

DEFINITION 4.3.2. Let $\mathcal{F}$ be a torsion-free class in $\mathsf{mod}\,\Lambda$. Then a *maximal green sequence of* $\mathcal{F}$ is a finite path in $\mathsf{Hasse}(\mathsf{torf}\,\Lambda)$ which starts at $\mathcal{F}$ and ends at 0. Or equivalently, a mximal green sequence of $\mathcal{F}$ is a saturated chain in $\mathsf{torf}\,\Lambda$ of the form $0 = \mathcal{F}_0 \lessdot \mathcal{F}_1 \lessdot \cdots \lessdot \mathcal{F}_l = \mathcal{F}$.

Maximal green sequences were introduced by Keller in the context of quiver mutations in cluster algebras, and have been investigated from various viewpoints. We refer the reader to the recent article [DK] and the reference therein for the details on this notion. The above definition is a straightforward generalization of maximal green sequences of abelian categories, which was introduced in [BST].

To a maximal green sequence of $\mathcal{F}$, we can associate a sequence of bricks in $\mathcal{F}$ as follows. This is analogous to the root sequence associated to a reduced expression in the Weyl group (and actually gives a categorification as we shall see in Proposition 4.4.3).

DEFINITION 4.3.3. Let $0 = \mathcal{F}_0 \leftarrow \mathcal{F}_1 \leftarrow \ldots \leftarrow \mathcal{F}_l = \mathcal{F}$ be a maximal green sequence of a torsion-free class $\mathcal{F}$ in $\mathsf{mod}\,\Lambda$. Then a *brick sequence associated to it* is a sequence $B_1, B_2, \ldots, B_l$ of bricks where each B_i is the label of the arrow $\mathcal{F}_i \leftarrow \mathcal{F}_{i-1}$ in $\mathsf{Hasse}(\mathsf{torf}\,\Lambda)$. We simply call a brick sequence associated to some maximal green sequence of $\mathcal{F}$ a *brick sequence of* $\mathcal{F}$.

As in the case of root sequences, we take the appearance order of bricks into account. For a fixed $\mathcal{F}$, brick sequences of $\mathcal{F}$ heavily depend on the choice of maximal green sequences. In general, the lengths of brick sequences may differ, as the following example shows.

EXAMPLE 4.3.4. Let Q be a quiver $1 \leftarrow 2$. Then $\mathsf{Hasse}(\mathsf{torf}\,kQ)$ and its brick labeling are as follows:

$$\begin{tikzcd}
& \mathsf{mod}\,kQ = \mathsf{add}\{1, \begin{smallmatrix}2\\1\end{smallmatrix}, 2\} \arrow[dl, "2"'] \arrow[dr, "1"] & \\
\mathsf{add}\{1, \begin{smallmatrix}2\\1\end{smallmatrix}\} \arrow[d, "\begin{smallmatrix}2\\1\end{smallmatrix}"'] & & \mathsf{add}\{2\} \arrow[ddl, "2"] \\
\mathsf{add}\{1\} \arrow[dr, "1"'] & & \\
& 0 &
\end{tikzcd}$$

Here we write composition series to indicate kQ-modules. Thus there are exactly two brick sequences of $\mathsf{mod}\,kQ$, namely, $1, \begin{smallmatrix}2\\1\end{smallmatrix}, 2$ and $2, 1$.

The fundamental property of a brick sequence is the following.

THEOREM 4.3.5. *Let $B_1, \ldots, B_l$ be a brick sequence of a torsion-free class $\mathcal{F}$ in $\mathsf{mod}\,\Lambda$. Then the following hold.*

(1) $\mathrm{Hom}_\Lambda(B_j, B_i) = 0$ *for* $j > i$.
(2) $B_1, \ldots, B_l$ *are pairwise non-isomorphic.*
(3) $\mathcal{F} = \mathsf{Filt}(B_l) * \mathsf{Filt}(B_{l-1}) * \cdots * \mathsf{Filt}(B_2) * \mathsf{Filt}(B_1)$ *holds. In particular, we have* $\mathcal{F} = \mathsf{Filt}(B_1, \ldots, B_l)$.

PROOF. Let $0 = \mathcal{F}_0 \leftarrow \mathcal{F}_1 \leftarrow \ldots \leftarrow \mathcal{F}_l = \mathcal{F}$ be a maximal green sequence of $\mathcal{F}$ which gives a brick sequence $B_1, \ldots, B_l$.

(1) By Proposition 4.3.1 and the definition of the brick labeling, each B_i is contained in ${}^\perp\mathcal{F}_{i-1} \cap \mathcal{F}_i$. Thus $B_j \in {}^\perp\mathcal{F}_{j-1} \subset {}^\perp\mathcal{F}_i$ holds since $\mathcal{F}_{j-1} \supset \mathcal{F}_i$ for $j > i$. Therefore $\mathrm{Hom}_\Lambda(B_j, B_i) = 0$ by $B_i \in \mathcal{F}_i$.

(2) This is clear from (1).

(3) We will prove by backward induction on i that ${}^\perp\mathcal{F}_{i-1} \cap \mathcal{F} = \mathsf{Filt}(B_l) * \cdots * \mathsf{Filt}(B_{i+1}) * \mathsf{Filt}(B_i)$ holds for $1 \leq i \leq l$. For $i = l$, this holds by Proposition 4.3.1.

Suppose this holds for $i = j+1$ with $1 \leq j < l$, and take $M \in {}^{\perp}\mathcal{F}_{j-1} \cap \mathcal{F}$. Since ${}^{\perp}\mathcal{F}_j * \mathcal{F}_j = \mathsf{mod}\,\Lambda$ holds, we have an exact sequence in $\mathsf{mod}\,\Lambda$

$$0 \longrightarrow T \longrightarrow M \longrightarrow F \longrightarrow 0$$

with $T \in {}^{\perp}\mathcal{F}_j$ and $F \in \mathcal{F}_j$. Since $\mathcal{F}$ is closed under submodules and $M \in \mathcal{F}$, we have $T \in {}^{\perp}\mathcal{F}_j \cap \mathcal{F} = \mathsf{Filt}(B_l) * \cdots * \mathsf{Filt}(B_{j+1})$ by the induction hypothesis. On the other hand, since ${}^{\perp}\mathcal{F}_{j-1}$ is closed under quotients, we have $F \in {}^{\perp}\mathcal{F}_{j-1} \cap \mathcal{F}_j = \mathsf{Filt}(B_j)$. Thus we have $M \in \mathsf{Filt}(B_l) * \cdots * \mathsf{Filt}(B_{j+1}) * \mathsf{Filt}(B_j)$. □

We remark that this statement appeared in [Tre, Corollary 6.5] for the case $\mathcal{F} = \mathsf{mod}\,\Lambda$, and the filtration given in (3) above is called the *Harder-Narasimhan filtration* associated to a maximal green sequence. Also this filtration was considered in [Tat, Theorem 6.8] in the setting of quasi-abelian categories. Since torsion-free classes are quasi-abelian, we can apply his result to our setting to obtain this theorem.

4.3.3. Simple objects in a torsion-free class. We introduce the notion of *simple objects* in a torsion-free class.

Definition 4.3.6. Let $\mathcal{F}$ be a torsion-free class in $\mathsf{mod}\,\Lambda$.

(1) $M \in \mathcal{F}$ is a *simple object in* $\mathcal{F}$ if $M \neq 0$ and for any short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

of Λ-modules with $L, N \in \mathcal{F}$, we have $L = 0$ or $M = 0$.

(2) We denote by $\mathsf{sim}\,\mathcal{F}$ the set of isomorphism classes of simple objects in $\mathcal{F}$.

Note that whether a given module M is simple object or not depends on the torsion-free class which contains M. Originally, the notion of simple objects was introduced and studied in the context of exact categories by several papers such as [Eno4, BHLR]. Since we are only interested in torsion-free classes, we will not work in full generality.

For a torsion-free class, a simple object can be described by the following property.

Lemma 4.3.7. *Let $\mathcal{F}$ be a torsion-free class in $\mathsf{mod}\,\Lambda$ and M a non-zero object in $\mathcal{F}$. Then the following are equivalent:*

(1) *M is simple in $\mathcal{F}$.*

(2) *Every morphism $M \to F$ with $F \in \mathcal{F}$ is either zero or injective.*

Proof. (1) $\Rightarrow$ (2): Take a non-zero morphism $\varphi \colon M \to F$ with $F \in \mathcal{F}$. Then we have an exact sequence

$$0 \longrightarrow \operatorname{Ker}\varphi \longrightarrow M \longrightarrow \operatorname{Im}\varphi \longrightarrow 0$$

in $\mathsf{mod}\,\Lambda$. Since $\operatorname{Im}\varphi$ is a submodule of $F \in \mathcal{F}$, we have $\operatorname{Im}\varphi \in \mathcal{F}$. Similarly $\operatorname{Ker}\varphi \in \mathcal{F}$ holds since so is M. On the other hand, $\operatorname{Im}\varphi \neq 0$ by assumption. Therefore, $\operatorname{Ker}\varphi = 0$ holds by the simplicity of M.

(2) $\Rightarrow$ (1): Take a short exact sequence

$$0 \longrightarrow L \longrightarrow M \xrightarrow{\pi} N \longrightarrow 0$$

with $L, N \in \mathcal{F}$. Then by assumption, π should be zero or injection. We have $N = 0$ in the former case, and $L = 0$ in the latter. Thus M is a simple object in $\mathcal{F}$. □

We will investigate the relation between simple objects in $\mathcal{F}$ and brick sequences of $\mathcal{F}$. One of the remarkable property is that simples always appear in every brick sequence of $\mathcal{F}$:

Proposition 4.3.8. *Let $\mathcal{F}$ be a torsion-free class in $\mathsf{mod}\,\Lambda$ and S a simple object in $\mathcal{F}$. Then S appears exactly once in every brick sequence of it (if exists).*

Proof. Let $B_1, \dots, B_l$ be a brick sequence of $\mathcal{F}$. Then we have $\mathcal{F} = \mathsf{Filt}(B_1, \dots, B_l)$ holds by Theorem 4.3.5. This means that S has a filtration consisting of $B_1, \dots, B_l$, but since S is a simple object in $\mathcal{F}$, clearly $S \cong B_i$ holds for some i. By Theorem 4.3.5, all the bricks in this brick sequence are pairwise non-isomorphic, thus S must appear exactly once. □

By this, to find a simple object in $\mathcal{F}$, we only have to work in a fixed brick sequence of $\mathcal{F}$. In order to give a criterion for a brick in a brick sequence to be simple, we will introduce some technical condition and lemmas.

Definition 4.3.9. Let M be a Λ-module and $\mathcal{C}$ a collection of Λ-modules. Then we say that $\mathcal{C}$ has the *zero-mono property for M* if every morphism $M \to C$ with $C \in \mathcal{C}$ is either zero or injective in $\mathsf{mod}\,\Lambda$.

Then Lemma 4.3.7 amounts to that M is simple in a torsion-free class $\mathcal{F}$ if and only if $\mathcal{F}$ has the zero-mono property for M.

The important advantage of the zero-mono property is that this property is closed under extensions in the following sense:

Lemma 4.3.10. *Let M be a Λ-module and $\mathcal{C}, \mathcal{D}$ collections of Λ-modules. If $\mathcal{C}$ and $\mathcal{D}$ have the zero-mono property for M, then so does $\mathcal{C} * \mathcal{D}$.*

Proof. Consider any short exact sequence

$$0 \longrightarrow C \xrightarrow{\iota} X \xrightarrow{\pi} D \longrightarrow 0$$

with $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Take any morphism $\varphi \colon M \to X$, and it suffices to show that φ is either zero or injective.

Consider the following diagram.

$$\begin{tikzcd} & & M \arrow[dl, dashed, "\overline{\varphi}"'] \arrow[d, "\varphi"] & & \\ 0 \arrow[r] & C \arrow[r, "\iota"] & X \arrow[r, "\pi"] & D \arrow[r] & 0 \end{tikzcd}$$

Since $\mathcal{D}$ has the zero-mono property for M, either $\pi\varphi$ is an injection or $\pi\varphi = 0$. In the former case, φ is an injection, so suppose the latter. Then there exists a morphism $\overline{\varphi} \colon M \to C$ with $\varphi = \overline{\varphi}\iota$. Since $\mathcal{C}$ has the zero-mono property for M, we have $\overline{\varphi}$ is either zero or injective. Thus φ is either zero or injective respectively. $\square$

As a first application of this corollary, we can show the following.

Corollary 4.3.11. *Let $\mathcal{F}$ be a torsion-free class in $\mathsf{mod}\,\Lambda$. Then taking labels gives an injection*

$$\{\textit{Arrows in } \mathsf{Hasse}(\mathsf{torf}\,\Lambda) \textit{ starting at } \mathcal{F}\} \hookrightarrow \mathsf{sim}\,\mathcal{F}.$$

Proof. Let $\mathcal{F} \to \mathcal{F}'$ be an arrow in $\mathsf{Hasse}(\mathsf{torf}\,\Lambda)$ with B its label. We will show that B is simple in $\mathcal{F}$, or equivalently, $\mathcal{F}$ has the zero-mono property for B by Lemma 4.3.7.

By Proposition 4.3.1, we have $\mathcal{F} = \mathsf{Filt}(B) * \mathcal{F}'$ and $\mathcal{F}' = \mathcal{F} \cap B^{\perp}$. Therefore, according to Lemma 4.3.10, it suffices to show that B and $\mathcal{F}'$ have the zero-mono property for B. Clearly B has the zero-mono property for B since B is a brick. On the other hand, since $\mathcal{F}' \subset B^{\perp}$, every morphism from B to $F \in \mathcal{F}'$ should be zero. Thus $\mathcal{F}'$ has the zero-mono property. $\square$

Another application of Lemma 4.3.10 is the following complete description of a brick in a given brick sequence to be simple.

Corollary 4.3.12. *Let $B_1, \dots, B_l$ be a brick sequence of a torsion-free class $\mathcal{F}$ in $\mathsf{mod}\,\Lambda$. Then any simple objects in $\mathcal{F}$ are contained in $\{B_1, \dots, B_l\}$, and the following are equivalent for $1 \le i \le l$.*

(1) *B_i is simple in $\mathcal{F}$.*
(2) *Every morphism $B_i \to B_j$ is either zero or injective for each $j \neq i$, or equivalently, $j > i$.*
(3) *$\{B_1, \dots, B_l\}$ has the zero-mono property for B_i.*
(4) *There is no surjection $B_i \twoheadrightarrow B_j$ with $i < j$.*

Proof. Since we have $\mathrm{Hom}_{\Lambda}(B_i, B_j) = 0$ for $j < i$ by Theorem 4.3.5, the two conditions in (2) are equivalent.

(1) $\Rightarrow$ (2): This is clear by Lemma 4.3.7 since $B_j \in \mathcal{F}$ for every j.

(2) $\Rightarrow$ (3): It suffices to recall that B_i has the zero-mono property for B_i since B_i is a brick.

(3) $\Rightarrow$ (1): Lemma 4.3.10 shows that $\mathsf{Filt}(B_1, \dots, B_l)$ has the zero-mono property for B_i. Since $\mathcal{F} = \mathsf{Filt}(B_1, \dots, B_l)$ holds by Theorem 4.3.5, we have that B_i is simple in $\mathcal{F}$ by Lemma 4.3.7.

(2) $\Rightarrow$ (4): This is clear since B_i and B_j with $j > i$ are non-isomorphic by Theorem 4.3.5.

(4) $\Rightarrow$ (2): Suppose that there are some $j > i$ and a map $\varphi \colon B_i \to B_j$ which is neither zero nor injective. Take minimal j with this property. We claim that φ is actually a surjection.

Let $0 = \mathcal{F}_0 \leftarrow \mathcal{F}_1 \leftarrow \cdots \leftarrow \mathcal{F}_l = \mathcal{F}$ be a maximal green sequence of $\mathcal{F}$ corresponding to the brick sequence $B_1, \dots, B_l$. Then we have $\mathcal{F}_j = \mathsf{Filt}(B_j) * \mathcal{F}_{j-1}$ by Proposition 4.3.1. Moreover, since (1)-(3) are equivalent and $B_1, \dots, B_{j-1}$ is a brick sequence of $\mathcal{F}_{j-1}$, the minimality of j implies that B_i is a simple object in $\mathcal{F}_{j-1}$, that is, $\mathcal{F}_{j-1}$ has the zero-mono property for B_i.

We have $\operatorname{Im}\varphi \in \mathcal{F}_j$ since it is a submodule of $B_j \in \mathcal{F}_j$. Thus by $\mathcal{F}_j = \mathsf{Filt}(B_j) * \mathcal{F}_{j-1}$, we have the following exact sequence

$$\begin{tikzcd}
 & & B_i \arrow[r, "\varphi"] \arrow[d, two heads, "\pi"] & B_j & \\
0 \arrow[r] & X \arrow[r] & \operatorname{Im}\varphi \arrow[r] \arrow[ur, hook] & F \arrow[r] & 0
\end{tikzcd}$$

with $X \in \mathsf{Filt}(B_j)$ and $F \in \mathcal{F}_{j-1}$. Suppose that $X = 0$, then $\operatorname{Im}\varphi \cong F \in \mathcal{F}_{j-1}$ holds. Thus π is either zero or injective. Since $\varphi \neq 0$, we have that π is injective, hence so is φ, which is a contradiction. Thus $X \neq 0$. It follows from $X \in \mathsf{Filt}(B_i)$ and $X \neq 0$ that $\dim B_i \leq \dim X$ holds. On the other hand, since we have injections $X \hookrightarrow \operatorname{Im}\varphi \hookrightarrow B_i$, we have $\dim X \leq \dim(\operatorname{Im}\varphi) \leq \dim B_i$. It follows that $\dim(\operatorname{Im}\varphi) = \dim B_i$, hence the inclusion map $\operatorname{Im}\varphi \hookrightarrow B_i$ is an isomorphism. Therefore, φ is a surjection. $\square$

Thus, the problem to determine simple objects in a torsion-free class $\mathcal{F}$ can be solved by the following process in principle.

(1) Find and fix a maximal green sequence of $\mathcal{F}$ (if exists).
(2) Compute a brick sequence $B_1, \dots, B_l$ associated to it.
(3) For each i, check whether every morphism $B_i \to B_j$ with $i < j$ is either zero or injective (or equivalently, check whether there is no surjection $B_i \twoheadrightarrow B_j$ with $i < j$).
(4) $\mathsf{sim}\,\mathcal{F}$ consists of B_i with such a property.

Probably the most non-trivial part of the above computation is (1) and (2). If $\mathcal{F}$ is functorially finite, then (1) can be computed by using mutations of support τ-tilting modules in [AIR], and (2) can be computed by using the description of brick labels associated to mutations due to [Asa1].

Example 4.3.13. We will give two examples of computation of simple objects by considering algebras borrowed from [DIRRT, Example 1.14]. Let Λ_1 and Λ_2 be k-algebras defined by

$$\Lambda_1 = k\Big(\begin{tikzcd} 1 \arrow[loop left, "u"] \arrow[r] & 2 \end{tikzcd} \Big)/(u^2) \quad \text{and} \quad \Lambda_2 = k\Big(\begin{tikzcd} 1 \arrow[r] & 2 \arrow[loop right, "v"] \end{tikzcd} \Big)/(v^2).$$

Then $\mathsf{torf}\,\Lambda_i$ and their brick labeling for $i = 1, 2$ is as follows:

$$\begin{tikzcd}[column sep=small]
 & \mathcal{F}_1 \arrow[dl, "2"'] & \mathcal{F}_2 \arrow[l, "\substack{1\\2}"'] & \mathcal{F}_3 \arrow[l, "\substack{1\\1\\2}"'] & \\
0 & & & & \mathsf{mod}\,\Lambda_1 \arrow[ul, "1"'] \arrow[dll, "2"] \\
 & & \mathcal{F}_4 \arrow[ull, "1"] & &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}[column sep=small]
 & \mathcal{G}_1 \arrow[dl, "2"'] & \mathcal{G}_2 \arrow[l, "\substack{1\\2\\2}"'] & \mathcal{G}_3 \arrow[l, "\substack{1\\2}"'] & \\
0 & & & & \mathsf{mod}\,\Lambda_2 \arrow[ul, "1"'] \arrow[dll, "2"] \\
 & & \mathcal{G}_4 \arrow[ull, "1"] & &
\end{tikzcd}$$

We omit the description of each torsion-free class. By using Corollary 4.3.12, one can obtain the following table of simple objects. Note that $\mathsf{torf}\,\Lambda_1$ and $\mathsf{torf}\,\Lambda_2$ are isomorphic as posets, and $\mathcal{F}_2$ corresponds to $\mathcal{G}_2$. However, the number of simple objects in $\mathcal{F}_2$ differs from that in $\mathcal{G}_2$. This shows that to determine (the number of) simple objects, the poset structure of $\mathsf{torf}\,\Lambda$ is not enough, and we indeed have to compute brick labels and homomorphisms between labels in general.

Fortunately, in the case of preprojective algebras (or path algebras) of Dynkin type, we can make use of root-theoretical properties developed in the previous section to determine simples, as we shall see in the next section.

torsion-free class in $\mathsf{mod}\,\Lambda_1$	simple objects	torsion-free class in $\mathsf{mod}\,\Lambda_1$	simple objects
0	$\varnothing$	0	$\varnothing$
$\mathcal{F}_1$	2	$\mathcal{G}_1$	2
$\mathcal{F}_2$	$2, \substack{1\\2}$	$\mathcal{G}_2$	$2, \substack{1\\2\\2}$
$\mathcal{F}_3$	$2, \substack{1\\2}, \substack{1\\1\\2}$	$\mathcal{G}_3$	$2, \substack{1\\2}$
$\mathcal{F}_4$	1	$\mathcal{G}_4$	1
$\mathsf{mod}\,\Lambda_1$	$1, 2$	$\mathsf{mod}\,\Lambda_1$	$1, 2$

TABLE 1. Simple objects in torsion-free classes in $\mathsf{mod}\,\Lambda_1$ and $\mathsf{mod}\,\Lambda_2$

We end this section by the following small lemma, which we need later.

LEMMA 4.3.14. *Let $\mathcal{F}$ be a torsion-free class in* $\mathsf{mod}\,\Lambda$ *such that the interval* $[0, \mathcal{F}]$ *in* $\mathsf{torf}\,\Lambda$ *is finite. Then every brick B in $\mathcal{F}$ appears at least one brick sequence of $\mathcal{F}$.*

PROOF. Consider $\mathsf{F}(B)$, the smallest torsion-free class containing $\mathcal{F}$. Then $\mathsf{F}(B) \subset \mathcal{F}$ holds, that is, $\mathsf{F}(B) \in [0, \mathcal{F}]$.

Since $[0, \mathcal{F}]$ is a finite lattice, clearly there exists a maximal green sequence of $\mathcal{F}$ which is of the form $0 \leftarrow \dots \leftarrow \mathsf{F}(B) \leftarrow \dots \leftarrow \mathcal{F}$. In [DIRRT, Theorem 3.4], it was shown that there is only one arrow which starts at $\mathsf{F}(B)$ in $\mathsf{Hasse}(\mathsf{torf}\,\Lambda)$, and that its label is B. Therefore, this maximal sequence gives the desired brick sequence. $\square$

4.4. Torsion-free classes over preprojective algebras of Dynkin type

In this section, we will classify simples in torsion-free classes in preprojective algebras of Dynkin type, by using root systems and brick sequences.

4.4.1. Notation and preliminary results. Let us briefly recall the definition of preprojective algebras of Dynkin type, with emphasis on their relation to root systems.

Let Φ be the simply-laced root system of the Dynkin type X, namely, $X \in \{A_n, D_n, E_6, E_7, E_8\}$. We denote by W its Weyl group, and we fix a choice of simple roots of Φ.

Let $Q = (Q_0, Q_1)$ be a Dynkin quiver of the same type X, that is, Q is a quiver whose underlying graph is the Dynkin diagram of type X. Then we may identify Q_0 with the index set of simple roots of Φ. For $i \in Q_0$, we denote by α_i the simple root of Φ corresponding to the vertex i in the Dynkin diagram, and by $s_i = s_{\alpha_i} \in W$ the simple reflection with respect to α_i.

Let $\overline{Q}$ be a *double quiver* of Q, which is obtained from Q by adding an arrow $a^* \colon j \to i$ for each arrow $a \colon i \to j$ in Q. The *preprojective algebra* of Q is defined by

$$\Pi = \Pi_\Phi := k\overline{Q} \Bigg/ \left(\sum_{a \in Q_1} aa^* - a^*a \right).$$

It is known that Π is a finite-dimensional k-algebra, and it only depends on Φ and does not depend on the choice of Q (the choice of orientations of the Dynkin diagram) up to isomorphism. Thus we call it a *preprojective algebra of* Φ.

It is convenient to consider the dimension vector of Π-modules inside the ambient space of Φ. Let V be the ambient space of Φ, then V has a basis $\{\alpha_i \mid i \in Q_0\}$ as a $\mathbb{R}$-vector space. Define a group homomorphism $\underline{\dim} \colon \mathsf{K}_0(\mathsf{mod}\,\Pi) \to V$ by $[S_i] \mapsto \alpha_i$, where $\mathsf{K}_0(\mathsf{mod}\,\Pi)$ denotes the Grothendieck group of $\mathsf{mod}\,\Pi$ and S_i denotes the simple Π-module corresponding to the vertex $i \in Q_0$. We simply write $\underline{\dim}\, M := \underline{\dim}[M]$ for $M \in \mathsf{mod}\,\Pi$.

To sum up, *throughout this section, we keep the following notation:*

- Φ is a simply-laced root system of Dynkin type X in the ambient space V.
- Q is a Dynkin quiver of the same type X.
- W is the Weyl group of Φ.
- Π is the preprojective algebra of Φ (or Q).
- S_i is the simple Π-module corresponding to the vertex $i \in Q_0$.

- $\underline{\dim}\colon \mathsf{K}_0(\mathsf{mod}\,\Pi) \to V$ is a map defined by $[S_i] \mapsto \alpha_i$ for $i \in Q_0$.

Torsion-free classes in $\mathsf{mod}\,\Pi$ was completely classified by Mizuno in [Miz], and they are in bijection with elements in W. Let us briefly explain his result in our context.

For a vertex $i \in Q_0$, we denote by e_i the corresponding idempotent of Π. We denote by I_i the two-sided ideal of Π generated by $1 - e_i$. For an element $w \in W$, we can define a two-sided ideal I_w of Π as follows: Take any reduced expression $w = s_{u_1} s_{u_2} \cdots s_{u_l}$ of w in W. Then $I(w)$ is defined by

$$I(w) = I_{u_l \dots u_2 u_1} := I_{u_l} I_{u_{l-1}} \cdots I_{u_2} I_{u_1}.$$

This construction does not depend on the choice of reduced expressions of w by [BIRS, Theorem III.1.9].

The following result of Mizuno gives the key connection between the representation theory of Π and its root system Φ.

THEOREM 4.4.1 ([Miz, Theorem 2.30]). *Let Φ be a simply-laced root system of Dynkin type, W its Weyl group and Π its preprojective algebra.*

(1) $\mathcal{F}(w) := \mathsf{Sub}(\Pi/I(w))$ *is a torsion-free class in* $\mathsf{mod}\,\Pi$.
(2) *The map $w \mapsto \mathcal{F}(w)$ gives a bijection*

$$W \xrightarrow{\sim} \mathsf{torf}\,\Pi.$$

Moreover, this bijection is actually an isomorphism of finite lattices, where we endow W with the right weak order.

4.4.2. Brick sequences in preprojective algebras. We begin with studying the relation between brick sequences of $\mathcal{F}(w)$ and root sequences of w.

By Theorem 4.4.1, we can identify a maximal green sequence of $\mathcal{F}(w)$ with a saturated chain in the interval $[e, w]$ in $(W, \leq_R)$, which in turn is identified with a particular choice of reduced expression of w (see e.g. [BB, Proposition 3.1.2]). In this way, we can talk about a *brick sequence of $\mathcal{F}(w)$ associated to a reduced expression of w.* More precisely, let $w = s_{u_1} \cdots s_{u_l}$ be a reduced expression of $w \in W$. Then we have a maximal green sequence $0 = \mathcal{F}(e) \leftarrow \mathcal{F}(s_{u_1}) \leftarrow \mathcal{F}(s_{u_1} s_{u_2}) \leftarrow \cdots \leftarrow \mathcal{F}(s_{u_1} \cdots s_{u_l}) = \mathcal{F}(w)$ of $\mathcal{F}(w)$, thus we obtain the corresponding brick sequence of $\mathcal{F}(w)$. We remark that this sequence coincides with a sequence of *layer modules* considered in [AIRT].

By a lattice isomorphism in Theorem 4.4.1, an arrow in $\mathsf{Hasse}(\mathsf{torf}\,\Pi)$ is of the form $\mathcal{F}(w) \leftarrow \mathcal{F}(ws_i)$ for some $w \in W$ and $i \in Q_0$ satisfying $\ell(w) < \ell(ws_i)$. In this case, we have $I(ws_i) \subset I(w)$, and the following holds.

PROPOSITION 4.4.2 ([IRRT, Theorem 4.1]). *The brick label of $\mathcal{F}(w) \leftarrow \mathcal{F}(ws_i)$ is given by $I(w)/I(ws_i)$.*

Next we will compute dimension vectors of brick labels following [AIRT].

PROPOSITION 4.4.3. *Let $w = s_{u_1} \dots s_{u_l}$ be a reduced expression of $w \in W$ and $B_1, B_2, \dots, B_l$ its associated brick sequence of $\mathcal{F}(w)$. Then the following equality holds in V for $1 \leq m \leq l$:*

$$\underline{\dim}\, B_m = s_{u_1} \cdots s_{u_{m-1}}(\alpha_{u_m}).$$

In particular, $\{\underline{\dim}\, B_1, \dots, \underline{\dim}\, B_l\} = \mathsf{inv}(w) \subset \Phi^+$ holds, and $\underline{\dim}\, B_1, \underline{\dim}\, B_2, \dots, \underline{\dim}\, B_l$ is a root sequence associated to the above reduced expression of w, defined in Definition 4.2.6.

PROOF. In [AIRT, Theorem 2.7], it was shown that

$$[B_m] = [I(s_{u_{m-1}} \cdots s_{u_1})/I(s_{u_m} \cdots s_{u_1})] = R_{u_1} \cdots R_{u_{m-1}}[S_{u_j}]$$

holds in $\mathsf{K}_0(\mathsf{mod}\,\Pi)$, where $R_i\colon \mathsf{K}_0(\mathsf{mod}\,\Pi) \to \mathsf{K}_0(\mathsf{mod}\,\Pi)$ for $i \in Q_0$ is a group homomorphism defined by

$$R_i([S_j]) := [S_j] - (2\delta_{ij} - m_{ij})[S_i]$$

on the free basis $\{[S_j] \,|\, j \in Q_0\}$ of $\mathsf{K}_0(\mathsf{mod}\,\Pi)$. Here δ is the Kronecker delta, and m_{ij} is the number of edges in the Dynkin diagram of Φ which connect i and j.

Therefore, it suffices to check that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{K}_0(\mathsf{mod}\,\Pi) \arrow[r, "R_i"] \arrow[d, "\underline{\dim}"] & \mathsf{K}_0(\mathsf{mod}\,\Pi) \arrow[d, "\underline{\dim}"] \\
V \arrow[r, "s_i"] & V
\end{tikzcd}$$

We only have to check on the basis $[S_j]$ for $j \in Q_0$. This follows from the following equality:

$$2\delta_{ij} - m_{ij} = \langle \alpha_j, \alpha_i \rangle = \begin{cases} 2 & \text{if } i = j, \\ 0 & \text{if } i \text{ and } j \text{ are not connected by an arrow in } Q, \\ -1 & \text{if } i \text{ and } j \text{ are connected by an arrow in } Q, \end{cases} \tag{4.4.1}$$

which can be checked directly. □

Therefore, a brick sequence of $\mathcal{F}(w)$ serves as a categorification of a root sequence of w. Note that different bricks may have the same dimension vector.

EXAMPLE 4.4.4. Consider an element w in Example 4.2.8. Then Figure 3 is the Hasse quiver of the interval $[0, \mathcal{F}(w)]$ with its brick labels. For example, two bricks $\begin{smallmatrix}1\\2\end{smallmatrix}$ and $\begin{smallmatrix}2\\1\end{smallmatrix}$ are non-isomorphic

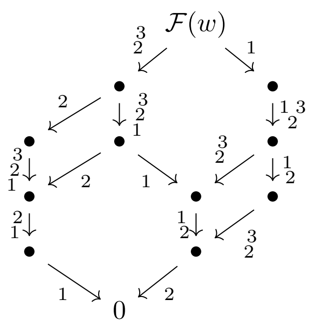


FIGURE 3. Brick sequences of $w = s_{12312}$

but have the same dimension vector $\alpha_1 + \alpha_2$.

The same computation can be done for Example 4.2.9, and actually Figure 3 can be seen as a Hasse quiver of $[0, \mathcal{F}(w)]$ with dimension vectors of its brick labels.

COROLLARY 4.4.5. *Let w be an element of W, then the following hold.*

(1) *For any $M \in \mathcal{F}(w)$, we have*

$$\underline{\dim}\, M = \sum_{\beta \in \mathsf{inv}(w)} n_\beta \beta,$$

where n_β is some non-negative integer for each $\beta \in \mathsf{inv}(w)$.

(2) *For any $B \in \mathsf{brick}\,\mathcal{F}(w)$, we have $\underline{\dim}\, B \in \mathsf{inv}(w)$. Thus $\underline{\dim}$ induces a surjection $\underline{\dim}\colon \mathsf{brick}\,\mathcal{F}(w) \to \mathsf{inv}(w)$.*

PROOF. (1) Let $B_1, \dots, B_l$ be a brick sequence of $\mathcal{F}(w)$. Then by Proposition 4.4.3, we have $\mathsf{inv}(w) = \{\underline{\dim}\, B_1, \dots, \underline{\dim}\, B_l\}$.

Let $M \in \mathcal{F}(w)$. Then Theorem 4.3.5 implies that $M \in \mathsf{Filt}(B_1, \dots, B_l)$. Thus by taking the dimension vector, the assertion immediately follows.

(2) Let B be a brick in $\mathcal{F}(w)$. Then Lemma 4.3.14 implies that there is a brick sequence of $\mathcal{F}(w)$ which contains B, since $[0, \mathcal{F}(w)] \subset \mathsf{torf}\,\Pi \cong W$ is a finite lattice. Now Proposition 4.4.3 implies that $\underline{\dim}\, B \in \mathsf{inv}(w)$. Thus the map $\underline{\dim}\colon \mathsf{brick}\,\mathcal{F}(w) \to \mathsf{inv}(w)$ is well-defined. Moreover, it is clearly surjective by Proposition 4.4.3. □

4.4.3. Simple objects versus Bruhat inversions. By Proposition 4.3.8, to find simple objects in $\mathcal{F}(w)$, it suffices to check whether B_i is simple or not for a fixed brick sequence $B_1, \dots, B_l$ of $\mathcal{F}(w)$. For this, the *homological lemma* due to Crawley-Boevey is useful. To state this, let us introduce the symmetric bilinear form $\langle -, - \rangle_\Pi$ on $\mathsf{K}_0(\mathsf{mod}\,\Pi)$ defined by

$$\langle \sum_i a_i[S_i], \sum_j b_j[S_j] \rangle_\Pi = 2 \sum_{i \in Q_0} a_i b_i - \sum_{i \to j \in Q_1} (a_i b_j + a_j b_i).$$

This coincides with the standard homological symmetric bilinear form associated to the quiver Q (or twice of it, see e.g. [ASS, VII.4]). It can also be interpreted as the restriction of the Euler form of the preprojective algebra $\widehat{\Pi}$ of the extended Dynkin type corresponding to Π, see e.g. [IRRT, Section 3]. Then we have the following formula due to Crawley-Boevey.

LEMMA 4.4.6 ([C-B, Lemma 1]). *Let $M, N \in \mathsf{mod}\,\Pi$. Then we have the following equation:*

$$\langle [M], [N] \rangle_\Pi = \dim \operatorname{Hom}_\Pi(M, N) + \dim \operatorname{Hom}_\Pi(N, M) - \dim \operatorname{Ext}^1_\Pi(M, N).$$

We remark that this formula can be shown easily by using the 2-Calabi-Yau property of the preprojective algebra $\widehat{\Pi}$ of the extended Dynkin type.

We can check the compatibility of the bilinear form on $\mathsf{K}_0(\mathsf{mod}\,\Pi)$ and the value $\langle \alpha, \beta \rangle$ in $\alpha, \beta \in V$ defined in Section 2 as follows.

LEMMA 4.4.7. *Suppose that $M, N \in \mathsf{mod}\,\Pi$ satisfy $\underline{\dim}\, M, \underline{\dim}\, N \in \Phi^+$. Then we have*

$$\langle [M], [N] \rangle_\Pi = \langle \underline{\dim}\, M, \underline{\dim}\, N \rangle.$$

PROOF. Since $\underline{\dim}\, M$ and $\underline{\dim}\, N$ are positive roots, we can write as $\underline{\dim}\, M = \sum_i m_i \alpha_i$ and $\underline{\dim}\, N = \sum_j n_j \alpha_j$ for non-negative integers m_i, n_j. Then we have

$$\begin{aligned}
\langle \underline{\dim}\, M, \underline{\dim}\, N \rangle &= \langle \sum_i m_i \alpha_i, \sum_j n_j \alpha_j \rangle \\
&= \sum_i \left(m_i \langle \alpha_i, \sum_j n_j \alpha_j \rangle \right) \\
&= \sum_i \left(m_i \langle \sum_j n_j \alpha_j, \alpha_i \rangle \right) \\
&= \sum_i \sum_j m_i n_j \langle \alpha_j, \alpha_i \rangle.
\end{aligned}$$

The first equality follows from definition, the second and the last follow since $\langle -, - \rangle$ is linear with respect to the first variable, and the third follows since Φ is simply-laced and both vectors inside $\langle -, - \rangle$ are roots. By using the equation (4.4.1), we can compute this as follows:

$$\sum_i \sum_j m_i n_j \langle \alpha_j, \alpha_i \rangle = 2 \sum_{i \in Q_0} m_i m_i - \sum_{i \to j \in Q} (m_i n_j + m_i n_j)$$

Thus we have the assertion. □

By using this, we can show the following main result of this paper.

THEOREM 4.4.8. *Let w be an element of W and $B_1, \dots, B_l$ a brick sequence of $\mathcal{F}(w)$. Then the following are equivalent for $1 \le m \le l$*

(1) *$\underline{\dim}\, B_m \in \mathsf{Binv}(w)$ holds.*
(2) *B_m is a simple object in $\mathcal{F}(w)$.*

PROOF. (1) ⇒ (2) Suppose that B_m is not a simple object in $\mathcal{F}(w)$. Then there exists an exact sequence

$$0 \longrightarrow L \longrightarrow B_m \longrightarrow N \longrightarrow 0$$

with $L, N \neq 0$. By applying $\underline{\dim}$, we obtain

$$\underline{\dim}\, B_m = \underline{\dim}\, L + \underline{\dim}\, N$$

with $\underline{\dim}\, L, \underline{\dim}\, N \neq 0$. By Corollary 4.4.5, both $\underline{\dim}\, L$ and $\underline{\dim}\, N$ are non-negative integer linear combinations of inversions of w such that at least one of the coefficients should be strictly positive. Thus Theorem 4.2.17 implies that $\underline{\dim}\, B_m$ is a non-Bruhat inversion.

(2) $\Rightarrow$ (1): Suppose that $\underline{\dim}\, B_m$ is a non-Bruhat inversion of w. We will use Lemma 4.3.7 to show that B_m is not a simple object in $\mathcal{F}(w)$.

By Theorem 4.2.17, there exists some $\alpha, \beta \in \mathsf{inv}(w)$ such that $\alpha + \beta = \underline{\dim}\, B_m$ since Φ is simply-laced. Moreover, these satisfy $\langle \underline{\dim}\, B_m, \alpha \rangle = \langle \underline{\dim}\, B_m, \beta \rangle = 1$ by Lemma 4.2.18.

On the other hand, since $\{\underline{\dim}\, B_1, \dots, \underline{\dim}\, B_l\} = \mathsf{inv}(w)$ by Proposition 4.4.3, there are i and j such that $\underline{\dim}\, B_i = \alpha$ and $\underline{\dim}\, B_j = \beta$. By exchanging α and β if necessary, we may assume that $i < j$. Moreover, since $\underline{\dim}\, B_1, \dots, \underline{\dim}\, B_l$ is a root sequence and $\underline{\dim}\, B_m = \underline{\dim}\, B_i + \underline{\dim}\, B_j$ holds, Theorem 4.2.7 implies $i < m < j$.

To summarize, we have found i and j with $i < m < j$ such that $\underline{\dim}\, B_m = \underline{\dim}\, B_i + \underline{\dim}\, B_j$ and $\langle \underline{\dim}\, B_m, \underline{\dim}\, B_j \rangle = 1$ hold. Then Lemma 4.4.7 implies

$$\langle [B_m], [B_j] \rangle_\Pi = \langle \underline{\dim}\, B_m, \underline{\dim}\, B_j \rangle = 1.$$

By combining this with Lemma 4.4.6, we have

$$\dim \mathrm{Hom}_\Pi(B_m, B_j) + \dim \mathrm{Hom}_\Pi(B_j, B_m) = 1 + \dim \mathrm{Ext}^1_\Pi(B_j, B_m) \geq 1.$$

On the other hand, since $m < j$, we must have $\mathrm{Hom}_\Pi(B_j, B_m) = 0$ by Theorem 4.3.5. Therefore, we have $\dim \mathrm{Hom}_\Pi(B_m, B_j) \geq 1$, that is, $\mathrm{Hom}_\Pi(B_m, B_j) \neq 0$.

Take any non-zero morphism $\varphi \colon B_m \to B_j$. Since $\underline{\dim}\, B_m = \underline{\dim}\, B_k + \underline{\dim}\, B_m$, we have $\dim B_m > \dim B_k$, hence φ cannot be an injection. Therefore Lemma 4.3.7 implies that B_m is not a simple object in $\mathcal{F}(w)$. $\square$

In conclusion, we have the following classification of simple objects.

COROLLARY 4.4.9. *Let w be an element of W. Then the following hold.*

(1) *A brick B in $\mathcal{F}(w)$ is a simple object in $\mathcal{F}(w)$ if and only if $\underline{\dim}\, B \in \mathsf{Binv}(w)$ holds.*
(2) *The map $\underline{\dim} \colon \mathsf{brick}\, \mathcal{F}(w) \to \mathsf{inv}(w)$ in Corollary 4.4.5 restricts to a bijection*

$$\underline{\dim} \colon \mathsf{sim}\, \mathcal{F}(w) \xrightarrow{\sim} \mathsf{Binv}(w).$$

In other words, simple objects in $\mathcal{F}(w)$ bijectively correspond to Bruhat inversions of w by taking dimension vectors.

PROOF. (1) Lemma 4.3.14 implies that B appears in some brick sequence of $\mathcal{F}(w)$. Then (1) is obvious by Theorem 4.4.8.

(2) Fix a brick sequence $B_1, \dots, B_l$ of $\mathcal{F}(w)$. If S is a simple object in $\mathcal{F}(w)$, then $S \cong B_j$ for some j by Proposition 4.3.8. Thus $\underline{\dim}\, S \in \mathsf{Binv}(w)$ holds by Theorem 4.4.8. Thus we obtain a map $\underline{\dim} \colon \mathsf{sim}\, \mathcal{F} \to \mathsf{Binv}(w)$.

We claim that this map is a bijection. Let $\gamma \in \mathsf{Binv}(w)$. Then since $\{\underline{\dim}\, B_1, \dots, \underline{\dim}\, B_l\} = \mathsf{inv}(w) \supset \mathsf{Binv}(w)$ by Proposition 4.4.3, there exists j with $\underline{\dim}\, B_j = \gamma$. This B_j is simple in $\mathcal{F}(w)$ by Theorem 4.4.8. Thus $\underline{\dim} \colon \mathsf{sim}\, \mathcal{F} \to \mathsf{Binv}(w)$ is surjective.

On the other hand, let S and S' be simple objects in $\mathcal{F}(w)$ satisfying $\underline{\dim}\, S = \underline{\dim}\, S'$. Then by the above argument, we have $S \cong B_i$ and $S' \cong B_j$ for some i and j, hence $\underline{\dim}\, B_i = \underline{\dim}\, B_j$ holds. Since elements in $\{\underline{\dim}\, B_1, \dots, \underline{\dim}\, B_l\} = \mathsf{inv}(w)$ are pairwise distinct by Proposition 4.2.4, we have $i = j$, which shows $S \cong S'$. Thus $\underline{\dim} \colon \mathsf{sim}\, \mathcal{F} \to \mathsf{Binv}(w)$ is injective. $\square$

4.4.4. Characterization of the Jordan-Hölder property. Next we will characterize when $\mathcal{F}(w)$ satisfies the Jordan-Hölder property in the sense of [Eno4] in terms of the combinatorics of w. Let us recall some related definitions and results from [Eno4].

DEFINITION 4.4.10. Let $\mathcal{F}$ be a torsion-free class in $\mathsf{mod}\, \Lambda$ for a finite-dimensional k-algebra Λ.

(1) For M in $\mathcal{F}$, a *composition series of M in $\mathcal{F}$* is a series of submodules of M
$$0 = M_0 \subset M_1 \subset \cdots \subset M_m$$
such that M_i/M_{i-1} is a simple object in $\mathcal{F}$ for each i.

(2) For M in $\mathcal{F}$, let $0 = M_0 \subset \cdots \subset M_m = M$ and $0 = M'_1 \subset \cdots \subset M'_n = M$ be two composition series of M in $\mathcal{F}$. We say that these are *equivalent* if $m = n$ holds and there exists a permutation σ of the set $\{1, 2, \dots, n\}$ such that $M_i/M_{i-1} \cong M'_{\sigma(i)}/M'_{\sigma(i)-1}$ holds for each i.

(3) We say that $\mathcal{F}$ satisfies the *Jordan-Hölder property*, abbreviated by *(JHP)*, if any composition series of M are equivalent for every object M in $\mathcal{F}$.

In [Eno4, Theorem 5.10], the author gives a numerical criterion for (JHP). To rephrase his result in our context, we introduce the *support* of modules or torsion-free classes.

Definition 4.4.11. Let Λ be a finite-dimensional k-algebra and $\mathsf{sim}(\mathsf{mod}\,\Lambda)$ the set of isomorphism classes of simple Λ-modules.

(1) For a module M, the *support of M* is a set of simple Λ-modules defined by
$$\mathsf{supp}\,M := \{S \in \mathsf{sim}(\mathsf{mod}\,\Lambda) \mid S \text{ is a composition factor of } M\}.$$

(2) For a collection $\mathcal{C}$ of modules, the *support of $\mathcal{C}$* is a set of simple Λ-modules defined by
$$\mathsf{supp}\,\mathcal{C} := \bigcup_{M \in \mathcal{C}} \mathsf{supp}\,M.$$

Then the following gives a numerical criterion for (JHP). Here for a set A, we denote by $\mathbb{Z}^{(A)}$ the free abelian group with basis A. We simply write $\mathbb{Z}^A := \mathbb{Z}^{(A)}$ if A is a finite set.

Theorem 4.4.12. *Let $\mathcal{F}$ be a torsion-free class in $\mathsf{mod}\,\Lambda$ for a finite-dimensional k-algebra Λ. Suppose that $\mathcal{F} = \mathsf{Sub}\,M$ holds for some $M \in \mathcal{F}$. Then the following are equivalent:*

(1) *$\mathcal{F}$ satisfies (JHP).*

(2) *The natural map $\mathbb{Z}^{(\mathsf{sim}\,\mathcal{F})} \to \mathsf{K}_0(\mathsf{mod}\,\Lambda)$ which sends $M \in \mathsf{sim}\,\mathcal{F}$ to $[M]$ is an injection.*

(2′) *The natural map $\mathbb{Z}^{(\mathsf{sim}\,\mathcal{F})} \to \mathbb{Z}^{(\mathsf{supp}\,\mathcal{F})}$ is an isomorphism, where we identify $\mathbb{Z}^{(\mathsf{supp}\,\mathcal{F})}$ with a subgroup of $\mathsf{K}_0(\mathsf{mod}\,\Lambda)$ generated by $[S]$ with $S \in \mathsf{supp}\,\mathcal{F}$.*

(3) *$\#\,\mathsf{sim}\,\mathcal{F} = \#\,\mathsf{supp}\,\mathcal{F}$ holds.*

Moreover, the map in (2′) is always surjective.

Proof. We give a proof using τ-tilting theory and the Grothendieck group $\mathsf{K}_0(\mathcal{F})$ of the exact category $\mathcal{F}$, for which we refer to [AIR] and [Eno4] respectively. It is shown in [Eno4, Theorem 4.12, Corollary 5.14] that the following are equivalent:

(i) $\mathcal{F}$ satisfies (JHP).

(ii) The natural map $\mathbb{Z}^{(\mathsf{sim}\,\mathcal{F})} \to \mathsf{K}_0(\mathcal{F})$, which is always surjective, is an isomorphism.

(iii) $\#\,\mathsf{sim}\,\mathcal{F} = |U|$ holds, where U is a support τ^--tilting module with $\mathcal{F} = \mathsf{Sub}\,U$ and $|U|$ is a number of non-isomorphic indecomposable direct summands of U.

On the other hand, [AIR, Proposition 2.2] implies $|U| = \#\,\mathsf{supp}\,U$. Since $\mathcal{F} = \mathsf{Sub}\,U$, clearly $\mathsf{supp}\,\mathcal{F} = \mathsf{supp}\,U$ holds, hence we have $|U| = \#\,\mathsf{supp}\,\mathcal{F}$. Therefore, (1) and (3) are equivalent.

To see that (2) and (2′) are also equivalent, let us consider $\mathsf{K}_0(\mathcal{F})$. By using [Eno4, Lemma 5.7], one can show that the natural map $\mathsf{K}_0(\mathcal{F}) \to \mathsf{K}_0(\mathsf{mod}\,\Lambda)$ is an injection, and that its image is precisely $\mathbb{Z}^{(\mathsf{supp}\,\mathcal{F})}$. Thus all the conditions are equivalent. □

To describe a characterization of (JHP) for $\mathcal{F}(w)$, we introduce the *support of $w \in W$*. Recall that simple roots and simple reflections are parametrized by Q_0 in our setting.

Definition 4.4.13. Let w be an element of W. Then *its support* is a subset $\mathsf{supp}(w)$ of Q_0 defined as follows:
$$\mathsf{supp}(w) = \{i \in Q_0 \mid \text{there is a reduced expression of } w \text{ which contains } s_i\}$$

Then the support of w coincides with the support of $\mathcal{F}(w)$ in the following sense:

PROPOSITION 4.4.14. *Let w be an element in W. Then a natural bijection $Q_0 \xrightarrow{\sim} \mathsf{sim}(\mathsf{mod}\,\Pi)$, which sends i to the simple Π-module corresponding to i, restricts to a bijection*

$$\mathsf{supp}(w) \xrightarrow{\sim} \mathsf{supp}\,\mathcal{F}(w).$$

PROOF. For a positive root $\beta \in \Phi^+$, we can write $\beta = \sum_{i \in Q_0} n_i \alpha_i$ with $n_i \geq 0$ in a unique way. Denote by $\mathsf{supp}(\beta)$ the set of $i \in Q_0$ with $n_i > 0$. Then Corollary 4.4.5 clearly implies that the bijection $Q_0 \to \mathsf{sim}(\mathsf{mod}\,\Pi)$ restricts to a bijection

$$\bigcup_{\beta \in \mathsf{inv}(w)} \mathsf{supp}(\beta) \xrightarrow{\sim} \mathsf{supp}\,\mathcal{F}(w).$$

Thus it suffices to show $\bigcup_{\beta \in \mathsf{inv}(w)} \mathsf{supp}(\beta) = \mathsf{supp}(w)$.

Let $w = s_{u_1} \cdots s_{u_l}$ be a reduced expression of w, and $\beta_1, \dots, \beta_l$ its associated root sequence. Then we have $\mathsf{inv}(w) = \{\beta_1, \dots, \beta_l\}$. First suppose that i belongs to $\bigcup_{\beta \in \mathsf{inv}(w)} \mathsf{supp}(\beta)$, then $i \in \mathsf{supp}(\beta_m)$ for some $1 \leq m \leq l$. Recall that $\beta_m = s_{u_1} \cdots s_{u_{m-1}}(\alpha_{u_m})$, and that $\{\alpha_u \,|u \in Q_0\}$ is a basis of V. Since each $s_u \colon V \to V$ changes only the α_u-component of roots with respect to this basis, i should appear in $\{u_1, u_2, \dots, u_m\}$. Thus $i \in \mathsf{supp}(w)$ holds.

Conversely, suppose $i \in \mathsf{supp}(w)$. Take the minimal m such that $u_m = i$ holds. We claim $i \in \mathsf{supp}(\beta_m)$. Indeed, we have $\beta_m = s_{u_1} s_{u_2} \cdots s_{u_{m-1}}(\alpha_i)$, and i does not appear in $u_1, \dots, u_{m-1}$ by the minimality of m. Since s_u changes only the α_u-component, the α_i-component of β_m is 1, hence $i \in \mathsf{supp}(\beta_m)$ holds. $\square$

Now the following immediately follows from these observations.

THEOREM 4.4.15. *Let w be an element of W. Then the following are equivalent:*

(1) *$\mathcal{F}(w)$ satisfies the Jordan-Hölder property.*
(2) *A map $\varphi_w \colon \mathbb{Z}^{\mathsf{Binv}(w)} \to \mathbb{Z}^{\mathsf{supp}(w)}$ defined by $\varphi_w(\sum_i n_i \alpha_i) = \sum_i n_i e_i$ for $\sum_i n_i \alpha_i \in \mathsf{Binv}(w)$ is a bijection, where e_i denotes the basis of $\mathbb{Z}^{\mathsf{supp}(w)}$ corresponding to $i \in \mathsf{supp}(w)$.*
(3) *Elements in $\mathsf{Binv}(w)$ are linearly independent in V.*
(4) *$\#\,\mathsf{Binv}(w) = \#\,\mathsf{supp}(w)$ holds.*

Moreover, the map in (2) *is always surjective.*

PROOF. Since $\mathcal{F}(w) = \mathsf{Sub}(\Pi/I_w)$ by definition, we can apply Theorem 4.4.12 to $\mathcal{F}(w)$. Hence the following are equivalent:

(i) $\mathcal{F}(w)$ satisfies (JHP).
(ii) The map $\mathbb{Z}^{(\mathsf{sim}\,\mathcal{F}(w))} \to \mathbb{Z}^{\mathsf{supp}\,\mathcal{F}(w)}$, which is always a surjection, is an isomorphism.
(iii) $\#\,\mathsf{sim}\,\mathcal{F}(w) = \#\,\mathsf{supp}\,\mathcal{F}(w)$ holds.

By identifying Q_0 with simple Π-modules, $\mathsf{supp}\,\mathcal{F}(w)$ bijectively corresponds to $\mathsf{supp}(w)$ by Proposition 4.4.14. Moreover, $\mathsf{sim}\,\mathcal{F}(w)$ bijectively corresponds to $\mathsf{Binv}(w)$ by taking dimension vectors by Corollary 4.4.9. Thus the map in (ii) are exactly same as φ_w in the assertion under the identification $\mathbb{Z}^{(\mathsf{sim}\,\mathcal{F}(w))} \cong \mathbb{Z}^{\mathsf{Binv}(w)}$ and $\mathbb{Z}^{\mathsf{supp}\,\mathcal{F}(w)} \cong \sum_{i \in \mathsf{supp}(w)} \mathbb{Z}\alpha_i$.

The left hand side in (iii) is equal to $\#\,\mathsf{Binv}(w)$ by Corollary 4.4.9, and the right hand side is equal to $\#\,\mathsf{supp}(w)$ by Proposition 4.4.14, thus (1), (2) and (4) are equivalent. Moreover, it is clear that (2) is equivalent to (3) since φ_w is always surjective. $\square$

We will use the map φ above later in the appendix to relate our characterization of (JHP) to *forest-like permutations* defined in [BMB] and the Schubert variety X_w for type A case.

REMARK 4.4.16. The equality $\#\,\mathsf{Binv}(w) = \#\,\mathsf{supp}(w)$ naturally arises when one consider the Bruhat interval in W and its Poincaré polynomial. Let w be an element of W and $[e, w]$ the interval with respect to the Bruhat order. A *Pincaré polynomial $P_w(q)$ of w* is defined by

$$P_w(q) := \sum_{v \in [e,w]} q^{\ell(v)}.$$

Let us write $P_w(q) = \sum_{i=0}^{\ell(w)} a_i q^i$. Then we have $\#\,\mathsf{supp}(w) = a_1$ and $\#\,\mathsf{Binv}(w) = a_{l-1}$, since supports of w are precisely simple reflections which are below w in the Bruhat order, and Bruhat inversions of w are in bijection with elements which are covered by w in the Bruhat order by Proposition 4.2.12. Thus our criterion is equivalent to $a_1 = a_{l-1}$.

EXAMPLE 4.4.17. Consider an element $w = s_{12312}$ in Example 4.2.8. Then there are three Bruhat inversions of w, namely, 100, 010 and 011. This number is equal to the number of $\mathsf{supp}(w) = \{1, 2, 3\}$, thus $\mathcal{F}(w)$ satisfies (JHP).

On the other hand, consider an element $w = s_{012301230}$ in Example 4.2.9. Then we have $\mathsf{supp}(w) = \{0, 1, 2, 3\}$, but a computation shows $\mathsf{Binv}(w) = \mathsf{inv}(w) \setminus \{\substack{1\\121}\}$ (for example, this follows from the fact that deleting any letter from $s_{012301230}$ yields a reduced expression except for the middle 0). so there are eight Bruhat inversions of w. Thus $\mathcal{F}(w)$ does not satisfy (JHP).

4.4.5. Conjectures. In this subsection, we give some natural conjectures on the existence of the particular kind of short exact sequences related to Theorem 4.4.8.

The most non-trivial part of the proof of Theorem 4.4.8 is to show that B_m is non-simple if $\underline{\dim}\, B_m$ is non-Bruhat. If $\underline{\dim}\, B_m$ is non-Bruhat, then as in the proof, there is a brick sequence $B_1, \dots, B_i, \dots, B, \dots, B_j, \dots, B_l$ of $\mathcal{F}(w)$ such that $\underline{\dim}\, B_i + \underline{\dim}\, B_j = \underline{\dim}\, B$. Then the following conjecture naturally occurs.

CONJECTURE 4.4.18. *Let w be an element of W. Take a brick sequence $B_1, \dots, B_l$ of $\mathcal{F}(w)$, and suppose $\underline{\dim}\, B_m = \underline{\dim}\, B_i + \underline{\dim}\, B_j$ for $1 \le i < m < j \le l$. Then there is an exact sequence*

$$0 \longrightarrow B_i \longrightarrow B_m \longrightarrow B_j \longrightarrow 0.$$

In fact, in the proof of Theorem 4.4.8, we only construct a non-zero non-injection $\varphi \colon B_m \to B_j$, which is enough for our purpose. This conjecture can be seen as a natural generalization of the result of Proposition 4.5.7, where the path algebra case was shown over an algebraically closed field.

We have another conjecture on non-simple objects. A *semibrick* $\mathcal{S}$ in $\mathsf{mod}\,\Pi$ is a set of bricks in $\mathsf{mod}\,\Pi$ such that $\mathrm{Hom}_\Pi(S, T) = 0$ holds for every two distinct elements $S, T \in \mathcal{S}$.

CONJECTURE 4.4.19. *Let w be an element of W and B a non-simple object in $\mathcal{F}(w)$. Then there is a semibrick $\{S, T\}$ in $\mathcal{F}(w)$ and an exact sequence*

$$0 \longrightarrow S \longrightarrow B \longrightarrow T \longrightarrow 0.$$

This conjecture is closely related to the lattice property (forcing order) of the interval $[e, w]$ or $[0, \mathcal{F}(w)]$, and the root-theoretical combinatorial property of the inversion set (contractibility of inversion triples defined in [GL]). The author has obtained the proof of Conjecture 4.4.19 for type A_n, D_n using combinatorics of (signed) permutations and E_6 using computer.

Conjecture 4.4.19 can be shown to be equivalent to the following conjecture. Recall that a simple object in a torsion-free class $\mathcal{F}$ appears in every brick sequence of $\mathcal{F}$ by Proposition 4.3.8. Then it is natural to ask whether the converse holds:

CONJECTURE 4.4.20. *Let w be an element of W. If a brick B appears in every brick sequence of $\mathcal{F}(w)$, then B is a simple object in $\mathcal{F}(w)$.*

This conjecture makes sense for any torsion-free classes over any finite-dimensional algebras, but this fails in general. For example, consider $\mathcal{G}_3$ in Example 4.3.13. Then there is only one brick sequence of $\mathcal{G}_3$, namely, $2, \substack{1\\2\\2}, \substack{1\\2}$. However, $\substack{1\\2\\2}$ is non-simple in $\mathcal{G}_3$.

4.5. Torsion-free classes over path algebras of Dynkin type

In this section, we use the results in the previous section to study torsion-free classes over path algebras of Dynkin type. *Throughout this section, let Q be a Dynkin quiver, and we use the same notation as in Section 4.4.1.* We have a natural surjection of algebras $\Pi \twoheadrightarrow kQ$, defined by annihilating all arrows in $\overline{Q}$ which do not appear in Q. Thereby we have an embedding $\mathsf{mod}\, kQ \hookrightarrow \mathsf{mod}\,\Pi$, and we often identify $\mathsf{mod}\, kQ$ with a subcategory of $\mathsf{mod}\,\Pi$.

Let us recall the celebrated theorem of Gabriel:

THEOREM 4.5.1. *The assignment $M \mapsto \underline{\dim} M$ for $M \in \mathsf{mod}\, kQ$ induces a bijection*

$$\underline{\dim}\colon \mathsf{ind}(\mathsf{mod}\, kQ) \xrightarrow{\sim} \Phi^+.$$

In other words, indecomposable kQ-modules bijectively correspond to positive roots by taking dimension vectors.

4.5.1. Coxeter-sortable elements and torsion-free classes. We begin with introducing some terminology which we need to give a description of $\mathsf{torf}\, kQ$. Put $n := \# Q_0$. Then a *Coxeter element c_Q of Q* is an element $c_Q = s_{u_1} \cdots s_{u_n}$ of W with $Q_0 = \{u_1, \dots, u_n\}$ which satisfies the following condition: if there is an arrow $i \leftarrow j$ in Q, then s_i appears before s_j in this expression of c_Q.

Let $c = c_Q$ be a Coxeter element of Q, and w an element of W. We say that w is *c-sortable* if there exists a reduced expression of the form $w = c^{(0)} c^{(1)} \cdots c^{(m)}$ such that each $c^{(i)}$ is a subword of c satisfying $\mathsf{supp}(c^{(0)}) \supset \mathsf{supp}(c^{(1)}) \supset \cdots \supset \mathsf{supp}(c^{(m)})$. We call such an expression a *c-sorting word of w.*

Now we can state the classification of torsion-free classes in $\mathsf{mod}\, kQ$, which was first established by [IT], and then generalized to any acyclic quiver by [AIRT] and [Tho].

THEOREM 4.5.2 ([IT, Theorem 4.3]). *Let Q be a Dynkin quiver and W its Weyl group. For $w \in W$, define a subcategory $\mathcal{F}_Q(w)$ of $\mathsf{mod}\, kQ$ by*

$$\mathcal{F}_Q(w) := \mathsf{add}\{M \in \mathsf{ind}(\mathsf{mod}\, kQ) \mid \underline{\dim} M \in \mathsf{inv}(w)\}.$$

Then the assignment $w \mapsto \mathcal{F}_Q(w)$ gives a bijection

$$\{w \mid w \text{ is } c_Q\text{-sortable}\} \xrightarrow{\sim} \mathsf{torf}\, kQ.$$

4.5.2. Simple objects versus Bruhat inversions. Let w be a c_Q-sortable element, then we have a torsion-free class $\mathcal{F}_Q(w)$ in $\mathsf{mod}\, kQ$ and a torsion-free class $\mathcal{F}(w)$ in $\mathsf{mod}\, \Pi$. The relation between these two was stated implicitly in [AIRT] and the proof was involved, thus we present a brief explanation of it.

We begin with the following observation.

PROPOSITION 4.5.3 ([AIRT, Theorem 3.3]). *Let w be a c_Q-sortable element in W and $B_1, \dots, B_l$ a brick sequence of $\mathcal{F}(w)$ associated with a c_Q-sorting word of w. Then we have $B_i \in \mathsf{mod}\, kQ$ for each i.*

Using this, we obtain the following description of $\mathcal{F}_Q(w)$ via a brick sequence (c.f. [AIRT, Theorem 3.11]).

PROPOSITION 4.5.4. *Let w be a c_Q-sortable element in W, and let $B_1, \dots, B_l$ be a brick sequence of $\mathcal{F}(w)$ associated with a c_Q-sorting word of w. Then we have $\mathcal{F}_Q(w) = \mathsf{add}\{B_1, \dots, B_l\}$. Moreover, $\mathcal{F}_Q(w) = \mathcal{F}(w) \cap \mathsf{mod}\, kQ$ holds.*

PROOF. By Proposition 4.5.3, we have $B_1, \dots, B_l \in \mathsf{mod}\, kQ$, and $\{\underline{\dim} B_1, \dots, \underline{\dim} B_l\} = \mathsf{inv}(w)$ holds by Proposition 4.4.3. Since Theorem 4.5.1 implies that there exists exactly one indecomposable kQ-module which has a fixed dimension vector, every indecomposable kQ-module M with $\underline{\dim} M \in \mathsf{inv}(w)$ should appear in $\{B_1, \dots, B_l\}$. Therefore, by the definition of $\mathcal{F}_Q(w)$, we have $\mathcal{F}_Q(w) = \mathsf{add}\{B_1, \dots, B_l\}$.

We will prove $\mathcal{F}_Q(w) = \mathcal{F}(w) \cap \mathsf{mod}\, kQ$. Since each B_i belongs to $\mathcal{F}(w) \cap \mathsf{mod}\, kQ$, we have $\mathcal{F}_Q(w) \subset \mathcal{F}(w) \cap \mathsf{mod}\, kQ$. Conversely, let $M \in \mathcal{F}(w) \cap \mathsf{mod}\, kQ$. Then we have $M \in \mathsf{Filt}(B_1, \dots, B_l)$ by Theorem 4.3.5, where Filt is considered inside $\mathsf{mod}\, \Pi$. On the other hand, since $\mathsf{mod}\, kQ \subset \mathsf{mod}\, \Pi$ is closed under subquotients, clearly we have $M \in \mathsf{Filt}_{kQ}(B_1, \dots, B_l)$, where Filt_{kQ} means we consider it inside $\mathsf{mod}\, kQ$. However, $\mathsf{add}\{B_1, \dots, B_l\} = \mathcal{F}_Q(w)$ is known to be closed under extensions in $\mathsf{mod}\, kQ$ since it is a torsion-free class by Theorem 4.5.2. Thus $M \in \mathcal{F}_Q(w)$ holds. $\square$

Now we can state our classification of simple objects in $\mathcal{F}_Q(w)$.

THEOREM 4.5.5. *Let w be a c_Q-sortable element of W. Then a bijection $\underline{\dim}\colon \mathsf{ind}\, \mathcal{F}_Q(w) \to \mathsf{inv}(w)$ restricts to a bijection*

$$\underline{\dim}\colon \mathsf{sim}\, \mathcal{F}_Q(w) \xrightarrow{\sim} \mathsf{Binv}(w).$$

In other words, simple objects in $\mathcal{F}_Q(w)$ bijectively correspond to Bruhat inversions of w by taking dimension vectors.

PROOF. Let $B_1, \dots, B_l$ be a brick sequence of $\mathcal{F}(w)$ (not $\mathcal{F}_Q(w)$!) associated to a c_Q-sorting word of w. Then Proposition 4.5.4 says that $\mathcal{F}_Q(w) = \{B_1, \dots, B_l\}$. Thus it suffices to show the following:

(Claim): *The following are equivalent for $1 \le m \le l$:*

(1) $\underline{\dim}\, B_m \in \mathsf{Binv}(w)$ *holds.*
(2) B_m *is a simple object in* $\mathcal{F}(w)$
(3) B_m *is a simple object in* $\mathcal{F}_Q(w)$.

The equivalence of (1) and (2) is nothing but Theorem 4.4.8, thus it suffices to show that (2) and (3) are equivalent. This immediately follows from the fact that $\mathcal{F}_Q(w) = \mathcal{F}(w) \cap \mathsf{mod}\, kQ$ holds by Proposition 4.5.4 and that $\mathsf{mod}\, kQ$ is closed under subquotients in $\mathsf{mod}\, \Pi$. $\square$

As in the case of $\mathcal{F}(w)$, we can characterize the validity of the Jordan-Hölder property as follows.

COROLLARY 4.5.6. *Let w be an element of W. Then $\mathcal{F}_Q(w)$ satisfies (JHP) if and only if $\#\,\mathsf{Binv}(w) = \#\,\mathsf{supp}(w)$ holds.*

PROOF. Immediate from Theorems 4.4.12 and 4.5.5, once we observe that $\mathsf{supp}\,\mathcal{F}(w)$ are in bijection with $\mathsf{supp}(w)$, which can be proved similarly to Proposition 4.4.14. $\square$

These results generalize the results in [Eno4], in which the type A case was proved by direct computation of representations of Q.

If we assume that the base field k is algebraically closed, then we can use the following result of [DR] to give a more quick proof of Theorem 4.5.5 without using preprojective algebras.

PROPOSITION 4.5.7 ([DR]). *Let k be an algebraically closed field and Q a Dynkin quiver. Take indecomposable kQ-modules L, M, N such that $\underline{\dim}\, L + \underline{\dim}\, N = \underline{\dim}\, M$ holds in Φ^+. Then by interchanging L and N if necessary, there is an exact sequence in $\mathsf{mod}\, kQ$ of the following form:*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

Note that the proof of this given in [DR] is algebro-geometric. The author do not know whether the same method can be used to study preprojective algebras, and whether this kind of exact sequence always exists in the preprojective algebra case (see Conjecture 4.4.18).

Now we can give another proof of Theorem 4.5.5 provided that k is algebraically closed.

PROOF OF THEOREM 4.5.5. Recall that we have a bijection $\underline{\dim}\colon \mathsf{ind}\,\mathcal{F}_Q(w) \to \mathsf{inv}(w)$. We will show that $M \in \mathsf{ind}\,\mathcal{F}_Q(w)$ is simple in $\mathcal{F}_Q(w)$ if and only if $\underline{\dim}\, M \in \mathsf{Binv}(w)$. If M is not simple, then Theorem 4.2.17 clearly implies that $\underline{\dim}\, M$ is a non-Bruhat inversion of w by taking dimension vectors. Thus it suffices to show that if $\underline{\dim}\, M$ is a non-Bruhat inversion of w, then M is not a simple object in $\mathcal{F}_Q(w)$.

By Theorem 4.2.17, there are $\alpha, \beta \in \mathsf{inv}(w)$ such that $\underline{\dim}\, M = \alpha + \beta$ holds since Φ is simply-laced. Take indecomposable kQ-modules M_α and M_β with $\underline{\dim}\, M_\alpha = \alpha$ and $\underline{\dim}\, M_\beta = \beta$, which exist by Theorem 4.5.1. By definition we have $M_\alpha, M_\beta \in \mathcal{F}_Q(w)$ holds. Then Proposition 4.5.7 implies that there exists a short exact sequence

$$0 \longrightarrow M_\alpha \longrightarrow M \longrightarrow M_\beta \longrightarrow 0$$

by interchanging α and β if necessary. Clearly this implies that M is not a simple object in $\mathcal{F}_Q(w)$. $\square$

4.A. Description and enumeration for type A

In this appendix, we focus on type A case and give more explicit and combinatorial description of results in this paper. First, we give an explicit diagrammatic construction of simple objects in $\mathcal{F}(w)$ by using a *Bruhat inversion graph* G_w. Next, we characterize elements w such that $\mathcal{F}(w)$ satisfies (JHP) in terms of G_w, and deduce some numerical consequences.

Throughout this appendix, we will use the following notation:

- Q is a quiver whose underlying graph is the following Dynkin diagram of type A_n:
$$1 \text{ ——— } 2 \text{ ——— } \cdots \text{ ——— } n.$$
- Φ is a standard root system of type A_n in $V := \mathbb{R}^{n+1}$, that is, $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n+1\}$, where ε_i denotes the standard basis of V.
- We fix simple roots by $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq n$.
- W is the Weyl group of Φ, and we often identify $W = S_{n+1}$ with the symmetric group S_{n+1} acting on the set $[n+1] := \{1, 2, \dots, n, n+1\}$ so that $w(\varepsilon_i) = \varepsilon_{w(i)}$ holds.
- For $i, j \in [n+1]$, we denote by $(i\ j) \in S_{n+1}$ the transposition of the letter i and j, and put $\beta_{(i,j)} := \varepsilon_i - \varepsilon_j$. Then $(i\ j)$ is identified with the reflection with respect to $\beta_{(i,j)}$.
- For $w \in S_{n+1}$, we often use the *one-line notation for* w, that is, we write as $w = w(1)w(2)\cdots w(n+1)$.
- Π is the preprojective algebra of Φ.
- $\mathcal{F}(w) \in \mathsf{torf}\,\Pi$ is the torsion-free class in $\mathsf{mod}\,\Pi$ defined in Theorem 4.4.1.

First, let us introduce the combinatorial variants of (Bruhat) inversion sets.

DEFINITION 4.A.1. Let w be an element of S_{n+1}.

(1) $\mathsf{Inv}(w)$ consists of a pair (i, j) with $1 \leq i < j \leq n+1$ such that the letter j appears left to i in the one-line notation for w.
(2) $\mathsf{BInv}(w)$ consists of a pair (i, j) with $1 \leq i < j \leq n+1$ such that the letter j appears left to i and there is no k with $i < k < j$ such that the letter k appears between j and i.

This notation is justified by the following, which can be proved by direct calculation.

PROPOSITION 4.A.2. *Let w be an element of $W = S_{n+1}$ and $i, j \in [n+1]$. Then the following hold.*

(1) $\beta_{(i,j)} \in \mathsf{inv}(w)$ *if and only if* $(i, j) \in \mathsf{Inv}(w)$.
(2) $\beta_{(i,j)} \in \mathsf{Binv}(w)$ *if and only if* $(i, j) \in \mathsf{BInv}(w)$.

Let us introduce a way to visualize Bruhat inversions, a *Bruhat inversion graph*. Let w be an element of $W = S_{n+1}$. Consider a square array of boxes with $(n+1)$ rows and $(n+1)$ columns. We name (i, j) to the box in the i-th row and j-th column, and put a dot in $(i, w(i))$ for each $1 \leq i \leq n+1$. We call it a *diagram of* w. A *Bruhat inversion graph* G_w is obtained by connecting every two dots in the diagram of w which correspond to the Bruhat inversion of w, that is, we connect $(w^{-1}(i), i)$ and $(w^{-1}(j), j)$ if $(i, j) \in \mathsf{BInv}(w)$ holds. Figure 4 is examples for G_w for

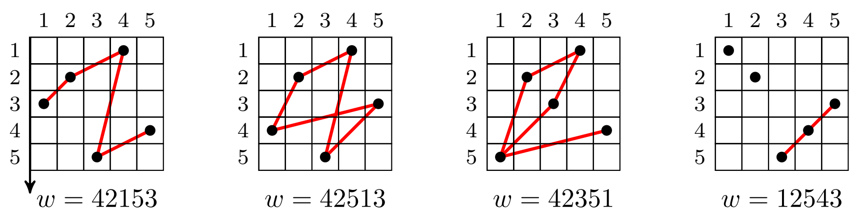
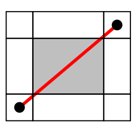


FIGURE 4. Bruhat inversion graphs G_w

elements w in S_5. It is clear from definition that G_w is obtained by connecting every two dots in the diagram of w which look as follows,



such that there are no dots in the gray region. It is also clear from definition that an edge in G_w bijectively corresponds to a Bruhat inversion of w.

4.A.1. Diagrammatic description of simples in $\mathcal{F}(w)$. In this subsection, we will give a combinatorial description of simple objects in $\mathcal{F}(w)$ using *arc diagrams* introduced in [Rea2] and the description of bricks given in [Asa2]. The author would like to thank Y. Mizuno for explaining to him the interpretation of the description in [Asa2] in terms of arc diagrams.

Let w be an element of $S_{n+1} = W$. We will construct a Π-module B_e for each edge e in G_π in the following way:

(1) Remove all the edges in G_w except e.
(2) Move down all the dots into a single horizontal line, allowing e to curve, but not to pass through any dots. We call this diagram an *arc diagram* of e.
(3) Draw n vertical dashed lines between adjacent dots in the arc diagram, and name these lines as $1, 2, \dots, n$ from left to right.
(4) Define a (not necessarily full) subquiver $Q(e)$ of $\overline{Q}$ by the following rule:
   - The vertex set of $Q(e)$ consists of $i \in \overline{Q}_0$ such that e and the line i intersect in the arc diagram of e.
   - Suppose that we have $i, i+1 \in Q(e)_0$. If the segment of e cut by the lines i and $i+1$ is above the unique dot between these lines, then we put an arrow $i \to i+1$, and put $i \leftarrow i+1$ if the segment is below the dot.

   We call $Q(e)$ *the defining quiver of B_e*.
(5) Define a Π-module B_e as follows, where we construct B_e as a representation of $\overline{Q}$.
   - To each $i \in \overline{Q}_0$, we assign k if $i \in Q(e)_0$, and 0 otherwise.
   - To each arrow $i \to j \in \overline{Q}$, we assign the identity map if $i \to j \in Q(e)_1$, and 0 otherwise.

   Since two cycles in $k\overline{Q}$ annihilates B_e by construction, we can regard B_e as a Π-module.

EXAMPLE 4.A.3. Figure 5 is an example of this construction for $w = 42513$ and all the edges in G_w. The middle part is an arc diagram of three edges, and the right part shows defining quivers of B_e corresponding to magenta, green, red, gray and blue edges from top to bottom.

FIGURE 5. Example of B_e for $w = 42513$

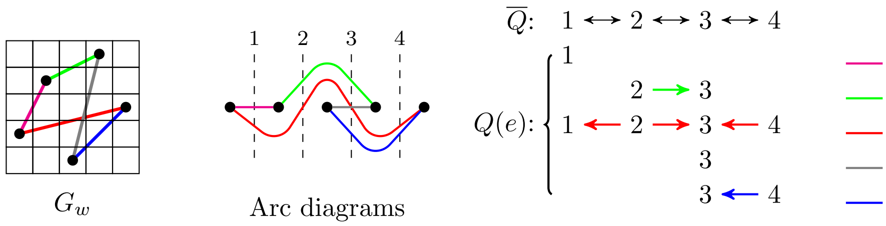
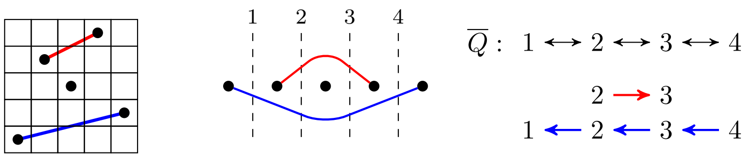


Figure 6 is an example for $w = 42351$ and two particular edges in G_w. Note that the orientations of edges between 2 and 3 in $Q(e)$ may differ as in this example.

FIGURE 6. Example of B_e for $w = 42351$

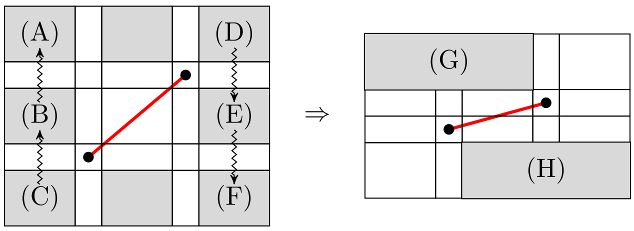


The above construction of arc diagrams is due to [Rea2]. More precisely, in [Rea2], arcs were assigned only to *descents* of w, which are inversions $(i, j) \in \mathsf{Inv}(w)$ such that w is of the form $\cdots ji \cdots$. Similarly, our construction of Π-modules B_e is a generalization of the one given in [Asa2, Theorem 4.6], where B_e was given (without using arc diagrams) for elements w with unique descent.

We will confirm that B_e is the simple object in $\mathcal{F}(w)$ associated with the Bruhat inversion corresponding to e.

PROPOSITION 4.A.4. *Let w be an element of $W = S_{n+1}$. Take $(i,j) \in \mathsf{BInv}(w)$, and let e denotes the edge in G_w corresponding to it. Then B_e is the unique simple object in $\mathcal{F}(w)$ with* $\underline{\dim}\, B_e = \beta_{(i,j)}$.

PROOF. First, we will construct another element w_e of W with a unique descent. The following picture illustrates the construction, where all the dots lie in the gray regions.



⇒ Sort (G) and (H) ⇒ w_e

The leftmost diagram is the diagram of w, and the red edge indicates e. Then perform the following procedure, requiring that all the diagrams in each step are diagrams of some elements in W:

(1) Move all the dots in (B) and (C) to (A), and those in (D) and (E) to (F).
(2) Sort all the dots in (G) and (H) so that the column number increases from top to bottom.

Denote by w_e the resulting element. Then it is clear from the construction that w_e has the unique descent. See Figure 7 for the example of this process, where $w = 56723814$.

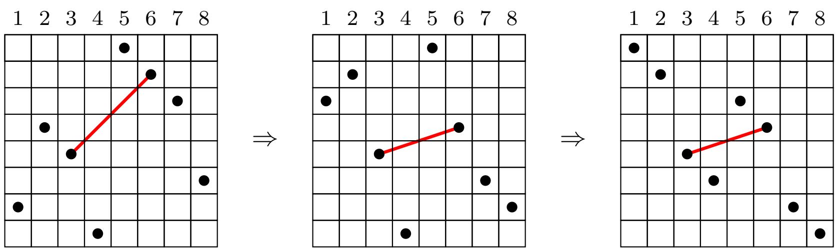
Alternatively, in terms of the one-line notation, we can describe w_e as follow. Let $w = \cdots (a) \cdots j \cdots (b) \cdots i \cdots (c) \cdots$ be the one-line notation for w. First, move all the numbers in (b) and (c) which are smaller than i to (a), and all the numbers in (a) and (b) which are larger than j to (c). Then we obtain an element of the form $\cdots (a) \cdots ji \cdots (c) \cdots$, since (i,j) is a Bruhat inversion of w. Next, sort the part (a) and (c) in ascending order, and denote by w_e the resulting element. Then $w_e = \cdots \underline{ji} \cdots$ has the unique descent at the underlined part.

It is straightforward to see that the module B_e is the same as the module $B_{w_e} := B_{e'}$, where e' is the (unique) edge in the Bruhat inversion graph of w_e. Moreover, it is easily checked that B_{w_e} is nothing but the brick constructed in [Asa2, Theorem 4.6] associated to w_e, which has the unique descent.

It is shown in [Asa2, Theorem 3.1] that B_{w_e} is the label of the unique arrow starting at $\mathcal{F}(w_e)$. In particular, we have $B_{w_e} \in \mathsf{brick}\,\mathcal{F}(w_e)$. On the other hand, by construction, it is straightforward to check that $\mathsf{Inv}(w_e) \subset \mathsf{Inv}(w)$ holds, which implies $w_e \leq w$ in the right weak order in W, see e.g. [BB, Proposition 3.1.3]. Thus we have $\mathcal{F}(w_e) \subset \mathcal{F}(w)$ in $\mathsf{torf}\,\Pi$ by Theorem 4.4.1. Hence we have $B_{w_e} \in \mathsf{brick}\,\mathcal{F}(w)$. Therefore, Corollary 4.4.9 implies that B_{w_e} is the unique simple object in $\mathcal{F}(w)$ with its dimension vector $\beta_{(i,j)}$, since $\beta_{(i,j)} \in \mathsf{Binv}(w)$. □

By this, we can obtain all the simple objects in $\mathcal{F}(w)$ by computing B_e for each edge e in G_w, as we have done in Figure 5.

FIGURE 7. Example for w_e in Proposition 4.A.4



$w = 56723814$ 52163784 $w_e = 12563478$

4.A.2. Forest-like permutations and the Jordan-Hölder property. Next, we will investigate elements $w \in W = S_{n+1}$ such that $\mathcal{F}(w)$ satisfy the Jordan-Hölder property. By Theorem 4.4.15, this is equivalent to that $\#\,\mathsf{supp}(w) = \#\,\mathsf{Binv}(w)$. By using [Sage], we calculated the number of such elements in S_{n+1} and obtained a sequence $2, 6, 22, 89, 379, 1661, \dots$. This coincides with [OEIS, A111053], a sequence of the number of *forest-like permutations* defined in [BMB]. These conditions are equivalent (up to a multiplication by the longest element), as we shall see later.

The following is the result of [BMB] in our context.

Theorem 4.A.5 ([BMB, Theorem 1.1]). *Let w be an element of $W = S_{n+1}$. Then the following are equivalent:*

(1) *The Bruhat inversion graph G_w is a forest, that is, it does not contain any cycles as an undirected graph.*
(2) *Define a map $L_w \colon \mathbb{Z}^n \to \mathbb{Z}^{\mathsf{Binv}(w)}$ by $L_w(e_i) = \sum\{\beta \mid \beta \in \mathsf{Binv}(w), i \in \mathsf{supp}(\beta)\}$, where e_i denotes the i-th standard basis of $\mathbb{Z}^n$. Then this map is surjective.*
(3) *w avoids the patterns 4231 and $3\overline{41}2$ with Bruhat restriction $4 \leftrightarrow 1$, that is, there exist no $1 \leq a < b < c < d \leq n+1$ such that the one-line notation for w is of the form $w = \cdots d \cdots b \cdots c \cdots a \cdots$ or $w = \cdots c \cdots da \cdots b \cdots$.*

We call w *forest-like* if G_w is a forest. We remark that w is forest-like in our sense if and only if $w_0 w$ is forest-like in the sense of [BMB], where $w_0 = (n+1)n \cdots 21$ is the longest element.

This turns out to be equivalent to our characterization of w such that $\mathcal{F}(w)$ satisfies (JHP):

Proposition 4.A.6. *Let w be an element of $W = S_{n+1}$. Then the following are equivalent:*

(1) *$\mathcal{F}(w)$ satisfies (JHP), that is, $\#\,\mathsf{supp}(w) = \#\,\mathsf{BInv}(w)$ holds.*
(2) *w is forest-like.*

Proof. We will show that the surjectivity of $L_w \colon \mathbb{Z}^n \to \mathbb{Z}^{\mathsf{Binv}(w)}$ is equivalent to the bijectivity of the map $\varphi_w \colon \mathbb{Z}^{\mathsf{Binv}(w)} \to \mathbb{Z}^{\mathsf{supp}(w)}$ in Theorem 4.4.15.

Since $\mathsf{supp}(w)$ is a subset of $Q_0 = \{1, 2, \dots, n\}$, we have the natural inclusion $\iota \colon \mathbb{Z}^{\mathsf{supp}(w)} \hookrightarrow \mathbb{Z}^n$. Then it is clear from the definition of L_w that L_w is surjective if and only if so is $L'_w := L_w \circ \iota \colon \mathbb{Z}^{\mathsf{supp}(w)} \to \mathbb{Z}^{\mathsf{Binv}(w)}$.

Now it is straightforward to check that L'_w is nothing but the $\mathbb{Z}$-dual of φ_w by calculating matrix representations of L'_w and φ_w: the transpose of the matrix representing L'_w coincides with the matrix representing φ_w. In particular, since φ_w is always surjective, L'_w should be injective. Therefore, the surjectivity of L_w is equivalent to the bijectivity of L'_w, which is in turn equivalent to the bijectivity of φ_w, since $\mathbb{Z}$-dual $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ is a duality of the category of finitely generated free $\mathbb{Z}$-modules. $\square$

The motivation of forest-like permutations in [BMB] comes from the study of Schubert varieties in the flag variety. Consider the flag variety $\mathrm{SL}_{n+1}(\mathbb{C})/B$, where B is the subgroup of upper triangular matrices. For $w \in S_{n+1}$, let e_w denote the permutation matrix for w. Then the *Schubert variety* X_w is the Zariski closure of the B-orbit of e_w in the flag variety.

It is known that X_w is a projective variety, which is not necessarily smooth. There are various studies on the relation between algebro-geometric properties of X_w and combinatorial properties of w. For example, X_w is smooth if and only if w avoids the patterns 4231 and 3412. Since there are many excellent papers and books on Schubert varieties, we only refer the reader to the recent survey article [AbBi] for the details.

We say that a variety is *factorial* if the local ring at every point is a unique factorization domain. In [WY, Proposition 2], it was proved that X_w is factorial if and only if the map L_w in Theorem 4.A.5 is surjective. Thus Theorem 4.A.5 characterizes the factoriality of X_w in a combinatorial way.

By combining this result to Theorems 4.4.15 and 4.A.5, Proposition 4.A.6 and [BMB, Theorem 3.1], we immediately obtain the following summary:

Corollary 4.A.7. *Let w be an element of $W = S_{n+1}$. Then the following are equivalent:*

(1) *G_w is forest.*
(2) *$\# \mathsf{supp}(w) = \# \mathsf{BInv}(w)$ holds.*
(3) *Elements in $\mathsf{Binv}(w)$ are linearly independent.*
(4) *$\mathcal{F}(w)$ satisfies (JHP).*
(5) *X_w is factorial.*
(6) *w avoids the pattern 4231 and $3\overline{41}2$ with Bruhat restriction $4 \leftrightarrow 1$.*

We remark that there is an explicit formula for the generation function of the number of forest-like permutations in [BMB, (2)], and the number grows exponentially, which says that although there are lots of forest-like permutations, the number is relatively small compared to $\# S_{n+1} = (n+1)!$.

Theorem 4.4.15 shows that (2)-(4) above are equivalent for all Dynkin types. It seems that they are also equivalent to (5) for other Dynkin types, by the same argument as in Proposition 4.A.6 using the maps φ_w in Theorem 4.4.15 and L_w in Theorem 4.A.5 and the Monk-Chevalley formula. Since the author does not have appropriate knowledge on algebraic geometries and Schubert varieties, we state it in the following conjecture:

CONJECTURE 4.A.8. *Let G be a simple algebraic group over $\mathbb{C}$, B a Borel subgroup, Φ the associated root system and W the Weyl group. Fix $w \in W$, and denote by X_w the Schubert variety in G/B. Then the following are equivalent:*

(1) *X_w is factorial.*
(2) *$\# \mathsf{supp}(w) = \# \mathsf{Binv}(w)$ holds, or equivalently, Bruhat inversions of w are linearly independent, or equivalently, the torsion-free class $\mathcal{F}(w)$ in $\mathsf{mod}\, \Pi_\Phi$ satisfies (JHP).*

For type D case, we can use the realization of the Weyl group as a group of signed permutations with even number of negatives. In particular, it seems to be an interesting problem to find an analogue of Corollary 4.A.7 for type D case.

CHAPTER 5

Monobrick, a uniform approach to torsion-free classes and wide subcategories

This chapter is based on [Eno6]

For a length abelian category, we show that all torsion-free classes can be classified by using only the information on bricks, including non functorially-finite ones. The idea is to consider the set of simple objects in a torsion-free class, which has the following property: it is a set of bricks where every non-zero map between them is an injection. We call such a set a monobrick. In this paper, we provide a uniform method to study torsion-free classes and wide subcategories via monobricks. We show that monobricks are in bijection with left Schur subcategories, which contains all subcategories closed under extensions, kernels and images, thus unifies torsion-free classes and wide subcategories. Then we show that torsion-free classes bijectively correspond to cofinally closed monobricks. Using monobricks, we deduce several known results on torsion(-free) classes and wide subcategories (e.g. finiteness result and bijections) in length abelian categories, without using τ-tilting theory. For Nakayama algebras, left Schur subcategories are the same as subcategories closed under extensions, kernels and images, and we show that its number is related to the large Schröder number.

5.1. Introduction

For a finite-dimensional algebra Λ, several classes of subcategories of $\mathsf{mod}\,\Lambda$ have been investigated in the representation theory of algebras. Among them, *torsion classes* and *torsion-free classes* have been central, together with their connection to the tilting theory and various triangulated categories.

Recently, Adachi-Iyama-Reiten's paper [AIR] made a major breakthrough in a classification of these subcategories. They show that functorially finite torsion-free classes can be classified using *support τ^--tilting modules*. Their method is to consider Ext-injective objects.

In this paper, we take a different approach, which enables us to classify *all* torsion-free classes in *any length abelian categories*. Our method is to consider *simple objects* (Definition 5.2.2). We observe that every simple object in a torsion-free class is a brick (a module with a division endomorphism ring), and we classify torsion-free classes using only the information on bricks:

Theorem N (= Theorem 5.3.15). *Let $\mathcal{A}$ be a length abelian category. Then we have a bijection between the following sets.*

(1) *The set of all torsion-free classes $\mathcal{F}$ in $\mathcal{A}$.*
(2) *$\{\mathcal{M} \mid \mathcal{M}$ is a set of bricks in $\mathcal{A}$ satisfying the following two conditions: $\}$*
 (MB) *Every non-zero map between objects in $\mathcal{M}$ is injective.*
 (CC) *If there is an injection $N \hookrightarrow M$ for a brick $N \notin \mathcal{M}$ and $M \in \mathcal{M}$, then there is a non-zero non-injection $N \to M'$ for some $M' \in \mathcal{M}$.*

The map from (1) *to* (2) *is given by the set $\mathsf{sim}\,\mathcal{F}$ of simple objects in $\mathcal{F}$, and from* (2) *to* (1) *is given by taking the extension closure $\mathsf{Filt}\,\mathcal{M}$ of $\mathcal{M}$.*

Example 5.1.1 (= Example 5.7.4). Let Λ be *any* Nakayama algebra whose quiver is given by $1 \rightleftarrows 2$. Then there are 4 bricks in $\mathsf{mod}\,\Lambda$, namely, $1, 2, {}^{1}_{2}, {}^{2}_{1}$. Only from the information on (non-)injections between them, we can combinatorially list up a set $\mathcal{M}$ satisfying (MB) and (CC) above, namely, $\varnothing, \{1\}, \{2\}, \{1, {}^{2}_{1}\}, \{2, {}^{1}_{2}\}, \{1, 2\}$. Hence there are 6 torsion-free classes in $\mathsf{mod}\,\Lambda$, namely, $0 = \mathsf{Filt}\,\varnothing, \mathsf{Filt}\{1\}, \mathsf{Filt}\{2\}, \mathsf{Filt}\{1, {}^{2}_{1}\}, \mathsf{Filt}\{2, {}^{1}_{2}\}, \mathsf{Filt}\{1, 2\} = \mathsf{mod}\,\Lambda$.

In this paper, we call a set $\mathcal{M}$ of bricks satisfying (MB) a *monobrick*. A well-known example is a *semibrick*, a pairwise Hom-orthogonal set of bricks. It is classical that semibricks in $\mathcal{A}$ are in bijection with wide subcategories in $\mathcal{A}$ by the same maps as in Theorem N (c.f. [Rin1, 1.2]). The aim of this paper is to provide a uniform theory to study monobricks and several kinds of subcategories including torsion-free classes and wide subcategories, thereby giving a systematic framework for studying these subcategories.

Our starting point is the bijection between $\mathsf{mbrick}\,\mathcal{A}$, the set of monobricks in $\mathcal{A}$, and $\mathsf{Schur_L}\,\mathcal{A}$, the set of *left Schur subcategories*. A left Schur subcategory is a category whose simple objects satisfy the one-sided Schur's lemma (see Definition 5.2.5). The class of left Schur subcategories contains any subcategories of $\mathcal{A}$ which are closed under extensions, kernels and images, thus unifies torsion-free classes and wide subcategories.

Theorem O (= Theorems 5.2.11, 5.3.15, 5.4.5). *Let $\mathcal{A}$ be a length abelian category. Then we have bijections between the set of left Schur subcategories and monobricks in $\mathcal{A}$:*

$$\mathsf{Schur_L}\,\mathcal{A} \underset{\mathsf{Filt}}{\overset{\mathsf{sim}}{\rightleftarrows}} \mathsf{mbrick}\,\mathcal{A}$$

Moreover, this bijection restricts to the following bijections:

- $\mathsf{wide}\,\mathcal{A} \rightleftarrows \mathsf{sbrick}\,\mathcal{A}$ *between the set of wide subcategories of $\mathcal{A}$ and semibricks in $\mathcal{A}$, and*
- $\mathsf{torf}\,\mathcal{A} \rightleftarrows \mathsf{mbrick_{c.c.}}\,\mathcal{A}$ *between the set of torsion-free classes in $\mathcal{A}$ and cofinally closed monobricks in $\mathcal{A}$ (monobricks satisfying* (CC) *in Theorem N).*

In the case of Nakayama algebras, we show that left Schur subcategories are precisely subcategories closed under extensions, kernels and images (Theorem 5.6.1), and the number of left Schur subcategories is related to the large Schröder number (Theorem 5.6.12).

We establish Theorem O by using two natural maps $\mathsf{W}\colon \mathsf{Schur_L}\,\mathcal{A} \twoheadrightarrow \mathsf{wide}\,\mathcal{A}$ and $\mathsf{F}\colon \mathsf{Schur_L}\,\mathcal{A} \twoheadrightarrow \mathsf{torf}\,\mathcal{A}$, where $\mathsf{W}(\mathcal{E})$ is the same as in [IT, MS] (Definition 5.4.3) and $\mathsf{F}(\mathcal{E})$ is the smallest torsion-free class containing $\mathcal{E}$. We describe these maps in terms of a natural *poset structure* of monobricks $\mathcal{M}$, namely, $L \le M$ in $\mathcal{M}$ if there is an injection $L \hookrightarrow M$. Then semibricks and cofinally closed monobricks can be characterized by this poset structure (see Proposition 5.4.2 and Definition 5.3.2). Now the maps W and F are easily described in terms of the poset structure as follows.

Theorem P (= Theorems 5.3.15, 5.4.5). *Let $\mathcal{A}$ be a length abelian category.*

(1) *The following diagram commutes, and the horizontal maps are bijections.*

$$\begin{tikzcd} \mathsf{Schur_L}\,\mathcal{A} \arrow[r, shift left, "\mathsf{sim}"] \arrow[d, two heads, "\mathsf{W}"] & \mathsf{mbrick}\,\mathcal{A} \arrow[l, shift left, "\mathsf{Filt}"] \arrow[d, two heads, "\mathsf{max}"] \\ \mathsf{wide}\,\mathcal{A} \arrow[r, shift left, "\mathsf{sim}"] & \mathsf{sbrick}\,\mathcal{A} \arrow[l, shift left, "\mathsf{Filt}"] \end{tikzcd}$$

Here $\mathsf{max}\,\mathcal{M}$ for a monobrick $\mathcal{M}$ denotes the set of maximal elements of $\mathcal{M}$.

(2) *The following diagram commutes, and the horizontal maps are bijections.*

$$\begin{tikzcd} \mathsf{Schur_L}\,\mathcal{A} \arrow[r, shift left, "\mathsf{sim}"] \arrow[d, two heads, "\mathsf{F}"] & \mathsf{mbrick}\,\mathcal{A} \arrow[l, shift left, "\mathsf{Filt}"] \arrow[d, two heads, "\overline{(-)}"] \\ \mathsf{torf}\,\mathcal{A} \arrow[r, shift left, "\mathsf{sim}"] & \mathsf{mbrick_{c.c.}}\,\mathcal{A} \arrow[l, shift left, "\mathsf{Filt}"] \end{tikzcd}$$

Here $\overline{\mathcal{M}}$ for a monobrick $\mathcal{M}$ denotes the cofinal closure of $\mathcal{M}$ (Definition 5.3.5).

As an application, we can quickly prove the finiteness result in [DIJ]: $\mathsf{torf}\,\mathcal{A}$ is a finite set if and only if there are only finitely many bricks (Theorem 5.5.5). In addition, we can easily deduce the following bijections between $\mathsf{torf}\,\mathcal{A}$ and $\mathsf{wide}\,\mathcal{A}$ using only some combinatorial observation on posets. This was proved in [MS] in the case of finite-dimensional algebras by using τ-tilting theory.

Corollary Q (= Corollary 5.5.4). *Let $\mathcal{A}$ be a length abelian category. Suppose that $\mathsf{torf}\,\mathcal{A}$ is a finite set. Then the maps $\mathsf{W}\colon \mathsf{torf}\,\mathcal{A} \rightleftarrows \mathsf{wide}\,\mathcal{A}\colon \mathsf{F}$ are mutually inverse bijections.*

Comparison to τ-tilting theory. For the convenience of the reader, we summarize advantages and disadvantages of monobricks compared to τ-tilting theory.

(Advantages)

- τ-tilting theory uses Ext-projectives, while we use simple objects. This enables us to work with any length abelian categories, where there may not be any projective objects.
- τ-tilting theory cannot classify non functorially finite cases, while monobricks can. This is because non-functorially finite subcategories may not have Ext-projectives.
- Using monobricks, we can study both wide subcategories and torsion-free classes in the same framework, and the relation between them become more transparent.
- Left Schur subcategories, or subcategories closed under extensions, kernels and images, seem to be new objects to study. Our enumerative result on Nakayama algebras via the Schröder number suggests that there are more hidden combinatorics in other algebras.

(Disadvantages)

- In general, left Schur subcategories are complicated to deal with. Actually, there are left Schur subcategories which are not even closed under direct summands (Example 5.7.2).
- One of the benefits of τ-tilting theory is a *mutation*, which provides a way to create various torsion-free classes starting from $\mathsf{mod}\,\Lambda$. So far, we have no such theory for monobricks.
- We cannot investigate functorial finiteness by monobricks. More precisely, two isomorphic monobricks (as posets) can correspond to functorially finite and non-functorially finite torsion-free classes (Example 5.7.5).

Finally, we should mention the relation of this paper to [Asa1], where a bijection between functorially finite torsion-free classes and semibricks satisfying some conditions was established. Although we cannot reprove his result (due to the last diasadvantage), his map can be easily described via monobricks: $\mathcal{F} \mapsto \mathsf{max}(\mathsf{sim}\,\mathcal{F})$. See Remark 5.4.6 for the more detail.

ORGANIZATION. This paper is organized as follows. In Section 5.2, we study basic properties of left Schur subcategories and monobricks, and establish a bijection between them. In Section 5.3, we study the cofinal closure $\overline{\mathcal{M}}$ and show its relation to torsion-free classes. In Section 5.4, we study the map W and its relation to $\mathsf{max}\,\mathcal{M}$. In Section 5.5, we apply previous results to show results on torsion-free classes and wide subcategories. In Section 5.6, we give a combinatorial classification of monobricks over Nakayama algebras, and enumerate its number. In Section 5.7, we show some examples of the classification of monobricks and the computation of $\overline{\mathcal{M}}$ and $\mathsf{max}\,\mathcal{M}$.

CONVENTIONS AND NOTATION. Throughout this paper, *we assume that all categories are skeletally small*, that is, the isomorphism classes of objects form a set. In addition, *all subcategories are assumed to be full and closed under isomorphisms.* We often identify an isomorphism class in a category with its representative. *We always denote by $\mathcal{A}$ a skeletally small length abelian category*, that is, an abelian category in which every object has finite length. For a collection $\mathcal{C}$ of objects in $\mathcal{A}$, we denote by $\mathsf{add}\,\mathcal{C}$ the subcategory of $\mathcal{A}$ consisting of direct summands of finite direct sums of objects in $\mathcal{C}$. For a finite-dimensional algebra Λ, we denote by $\mathsf{mod}\,\Lambda$ the category of finitely generated right Λ-modules. For a set A, we denote by $\#A$ its cardinality.

5.2. Bijection between monobricks and left Schur subcategories

First we introduce a *monobrick* in a length abelian category $\mathcal{A}$. Recall that a *brick* in $\mathcal{A}$ is an object M such that $\operatorname{End}_{\mathcal{A}}(M)$ is a division ring.

DEFINITION 5.2.1. Let $\mathcal{M}$ be a set of isomorphism classes of bricks in $\mathcal{A}$.

1. $\mathcal{M}$ is called a *monobrick* if every morphism between elements of $\mathcal{M}$ is either zero or an injection in $\mathcal{A}$. We denote by $\mathsf{mbrick}\,\mathcal{A}$ the set of monobricks in $\mathcal{A}$.
2. $\mathcal{M}$ is called a *semibrick* if every morphism between elements of $\mathcal{M}$ is either zero or an isomorphism. We denote by $\mathsf{sbrick}\,\mathcal{A}$ the set of semibricks in $\mathcal{A}$.

Note the assumption that a monobrick $\mathcal{M}$ consists of bricks is automatically satisfied if we require the above property, since every non-zero endomorphism of M in $\mathcal{M}$ should be an injection, thus an isomorphism since M has finite length.

Clearly every semibrick is a monobrick, thus $\mathsf{sbrick}\,\mathcal{A} \subset \mathsf{mbrick}\,\mathcal{A}$ holds. Next we introduce *left Schur subcategories* of $\mathcal{A}$. Roughly speaking, it is an extension-closed subcategory of $\mathcal{A}$ such that the "one-sided Schur's lemma" holds. Let us define some notations.

DEFINITION 5.2.2. Let $\mathcal{E}$ be a subcategory of $\mathcal{A}$.

(1) $\mathcal{E}$ is *closed under extensions* or *extension-closed* in $\mathcal{A}$ if it satisfies the following condition: for every short exact sequence
$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$
in $\mathcal{A}$, if X and Z belong to $\mathcal{E}$, then so does Y.

(2) Suppose that $\mathcal{E}$ is extension-closed in $\mathcal{A}$. Then a non-zero object M in $\mathcal{E}$ is a *simple object in* $\mathcal{E}$ if there is *no* exact sequence of the form
$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$
in $\mathcal{A}$ satisfying $L, M, N \in \mathcal{E}$ and $L, N \neq 0$. We denote by $\mathsf{sim}\,\mathcal{E}$ the set of isomorphism classes of simple objects in $\mathcal{E}$.

Clearly $\mathsf{sim}\,\mathcal{A}$ is nothing but the set of the usual simple objects in an abelian category $\mathcal{A}$. Thus $\mathsf{sim}\,\mathcal{E}$ is an analogue of simple objects *inside* $\mathcal{E}$. Simple objects can be considered as one of the invariants of extension-closed subcategories, and has an application such as a characterization of the Jordan-Hölder type property in $\mathcal{E}$ [Eno4].

To define and study left Schur subcategories, the following terminology is useful.

DEFINITION 5.2.3. Let $\mathcal{C}$ be a collection of objects in $\mathcal{A}$. Then a non-zero object $M \in \mathcal{A}$ is *left Schurian for* $\mathcal{C}$ if every morphism $M \to C$ with $C \in \mathcal{C}$ is either zero or an injection in $\mathcal{A}$.

Note that we do not require that M belongs to $\mathcal{C}$. It is clear that a collection $\mathcal{M}$ of non-zero objects in $\mathcal{A}$ is a monobrick if and only if every object in $\mathcal{M}$ is left Schurian for $\mathcal{M}$. Simple objects in $\mathcal{A}$ (in the usual sense) are left Schurian for any collection $\mathcal{C}$.

The fundamental relation between left Schurian objects and simple objects is as follows.

PROPOSITION 5.2.4. *Let $\mathcal{E}$ be an extension-closed subcategory of $\mathcal{A}$. Then the following hold.*

(1) $\{M \in \mathcal{E} \mid M$ *is left Schurian for* $\mathcal{E}\}$ *is a monobrick.*

(2) *If M in $\mathcal{E}$ is left Schurian for $\mathcal{E}$, then M is simple in $\mathcal{E}$.*

PROOF. (1) Suppose that M and N in $\mathcal{E}$ are left Schurian for $\mathcal{E}$. Then since M is left Schurian for $\mathcal{E}$, every morphism $M \to N$ is either zero or an injection. Thus the assertion holds.

(2) Suppose that M in $\mathcal{E}$ is left Schurian for $\mathcal{E}$, and take an exact sequence
$$0 \longrightarrow L \longrightarrow M \xrightarrow{\pi} N \longrightarrow 0$$
in $\mathcal{A}$ with $L, N \in \mathcal{E}$. Then π should be either zero or an injection. In the former case, we have $N = 0$, and in the latter, we have $L = 0$. Thus M is simple in $\mathcal{E}$. $\square$

Then we can define a left Schur subcategory as follows.

DEFINITION 5.2.5. A subcategory $\mathcal{E}$ of $\mathcal{A}$ is *left Schur* if it satisfies the following conditions:

(1) $\mathcal{E}$ is closed under extensions in $\mathcal{A}$.

(2) Every simple object in $\mathcal{E}$ is left Schurian for $\mathcal{E}$, that is, for a simple object M in $\mathcal{E}$, every morphism $M \to X$ with $X \in \mathcal{E}$ is either zero or an injection in $\mathcal{A}$.

We denote by $\mathsf{Schur}_{\mathsf{L}}\,\mathcal{A}$ the set of left Schur subcategories of $\mathcal{A}$.

The following immediately follows from Proposition 5.2.4:

COROLLARY 5.2.6. *Let $\mathcal{E}$ be a left Schur subcategory of $\mathcal{A}$ and M an object in $\mathcal{E}$. Then M is simple in $\mathcal{E}$ if and only if M is left Schurian for $\mathcal{E}$. Moreover,* $\mathsf{sim}\,\mathcal{E}$ *is a monobrick.*

We are going to show that all wide subcategories and torsion-free classes are left Schur. Let us recall the definitions of these subcategories.

Definition 5.2.7. Let $\mathcal{E}$ be a subcategory of $\mathcal{A}$.

(1) $\mathcal{E}$ is *closed under kernels (resp. cokernels, images)* if for every morphism $X \to Y$ in $\mathcal{E}$, we have $\operatorname{Ker} f$ (resp. $\operatorname{Coker} f$, $\operatorname{Im} f$) belongs to $\mathcal{E}$.
(2) $\mathcal{E}$ is *closed under subobjects (resp. quotients)* if every subobject (resp. quotient object) of X belongs to $\mathcal{E}$ for every X in $\mathcal{E}$.
(3) $\mathcal{E}$ is a *torsion-free class (resp. torsion class) in* $\mathcal{A}$ if it is closed under extensions and subobjects (resp. extensions and quotients) in $\mathcal{A}$. We denote by $\mathsf{torf}\,\mathcal{A}$ the set of torsion-free classes in $\mathcal{A}$.
(4) $\mathcal{E}$ is a *wide subcategory of* $\mathcal{A}$ if it is closed under extensions, kernels and cokernels. We denote by $\mathsf{wide}\,\mathcal{A}$ the set of wide subcategories of $\mathcal{A}$.

It can be easily shown that every wide subcategory or every torsion-free class in $\mathcal{A}$ is closed under extensions, kernels and images. We prove that this condition implies left Schur, thus $\mathsf{wide}\,\mathcal{A} \subset \mathsf{Schur}_{\mathsf{L}}\,\mathcal{A}$ and $\mathsf{torf}\,\mathcal{A} \subset \mathsf{Schur}_{\mathsf{L}}\,\mathcal{A}$ hold.

Proposition 5.2.8. *Let $\mathcal{E}$ be a subcategory of $\mathcal{A}$ which is closed under extensions, kernels and images in $\mathcal{A}$. Then $\mathcal{E}$ is a left Schur subcategory of $\mathcal{A}$.*

Proof. Let M be a simple object in $\mathcal{E}$ and $f\colon M \to X$ be a morphism with $X \in \mathcal{E}$. Then we have the following exact sequence in $\mathcal{A}$:

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow M \longrightarrow \operatorname{Im} f \longrightarrow 0$$

Since $\mathcal{E}$ is closed under kernels and images, we have $\operatorname{Ker} f, \operatorname{Im} f \in \mathcal{E}$. Thus either $\operatorname{Ker} f = 0$ or $\operatorname{Im} f = 0$ since M is simple in $\mathcal{E}$. In the former case, we have that f is an injection in $\mathcal{A}$, and in the latter, we have $f = 0$. Thus M is left Schurian for $\mathcal{E}$, hence $\mathcal{E}$ is left Schur. $\square$

Example 5.2.9. Let k be a field and Q be a quiver $1 \leftarrow 2 \leftarrow 3$. Then the Auslander-Reiten quiver of $\mathsf{mod}\,kQ$ is as follows:

$$\begin{tikzcd}
 & & \begin{smallmatrix}3\\2\\1\end{smallmatrix} \arrow[dr] & & \\
 & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[dr] & \\
1 \arrow[ur] \arrow[rr, dashed, no head] & & 2 \arrow[ur] \arrow[rr, dashed, no head] & & 3
\end{tikzcd}$$

Now $\mathcal{E}_1 = \mathsf{add}\{1, \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\\1\end{smallmatrix}, 2\}$ is a torsion-free class, and $\mathsf{sim}\,\mathcal{E}_1 = \{1, 2, \begin{smallmatrix}3\\2\\1\end{smallmatrix}\}$. It can be checked that every simple object in $\mathcal{E}_1$ is left Schurian for $\mathcal{E}_1$, thus is a left Schur subcategory (this follows also from Proposition 5.2.8). On the other hand, consider $\mathcal{E}_2 = \mathsf{add}\{\begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\\1\end{smallmatrix}, 2\}$. This subcategory is closed under extensions, and all the three indecomposables are simple objects in $\mathcal{E}_2$. However, we have a non-zero non-injection $\begin{smallmatrix}2\\1\end{smallmatrix} \twoheadrightarrow 2$. Thus $\mathcal{E}_2$ is not a left Schur subcategory.

For a left Schur subcategory $\mathcal{E}$ of $\mathcal{A}$, we have a monobrick $\mathsf{sim}\,\mathcal{E}$ by Corollary 5.2.6. Conversely, for a given monobrick $\mathcal{M}$, we will construct a left Schur subcategory whose simples are $\mathcal{M}$. To do this, we will use the following operation.

Definition 5.2.10. Let $\mathcal{C}$ be a collection of objects in $\mathcal{A}$. Then $\mathsf{Filt}\,\mathcal{C}$ denotes the subcategory of $\mathcal{A}$ consisting of objects X such that there is a chain

$$0 = X_0 < X_1 < \cdots < X_n = X$$

of subobjects of X such that X_i/X_{i-1} is in $\mathcal{C}$ for each i. We call such a chain a *$\mathcal{C}$-filtration* of X, and n a *length* of this $\mathcal{C}$-filtration.

It follows from the Noether isomorphism theorem that $\mathsf{Filt}\,\mathcal{C}$ is extension-closed, and it is obvious from the construction that it is the smallest extension-closed subcategory of $\mathcal{A}$ containing $\mathcal{C}$.

Now we can state our first main result of this paper.

Theorem 5.2.11. *Let $\mathcal{A}$ be a length abelian category. Then* sim *and* Filt *give mutually inverse bijections between left Schur subcategories of $\mathcal{A}$ and monobricks in $\mathcal{A}$, which extends the bijection between wide subcategories and semibricks:*

$$\begin{tikzcd} \mathsf{sim}\colon \ \mathsf{Schur}_{\mathsf{L}}\,\mathcal{A} \arrow[r, leftrightarrow, "\sim"] & \mathsf{mbrick}\,\mathcal{A} \ : \ \mathsf{Filt} \\ \mathsf{wide}\,\mathcal{A} \arrow[u, hook] \arrow[r, leftrightarrow, "\sim"] & \mathsf{sbrick}\,\mathcal{A} \arrow[u, hook] \end{tikzcd}$$

We need some preparation to prove it. For two collections $\mathcal{C}$ and $\mathcal{D}$ of objects in $\mathcal{A}$, we denote by $\mathcal{C} * \mathcal{D}$ the subcategory of $\mathcal{A}$ consisting of objects X such that there is an exact sequence

$$0 \longrightarrow C \longrightarrow X \longrightarrow D \longrightarrow 0$$

in $\mathcal{A}$ with $C \in \mathcal{C}$ and $D \in \mathcal{D}$. As for this, the following lemma in [Eno5, Lemma 3.10] is useful. We give a proof for the convenience of the reader.

Lemma 5.2.12. *Let M be an object in $\mathcal{A}$. If M is left Schurian for two collections $\mathcal{C}$ and $\mathcal{D}$ of objects in $\mathcal{A}$, then M is left Schurian also for $\mathcal{C} * \mathcal{D}$.*

Proof. Take a short exact sequence in $\mathcal{A}$

$$0 \longrightarrow C \xrightarrow{\iota} X \xrightarrow{\pi} D \longrightarrow 0$$

with $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Let $\varphi\colon M \to X$ be any morphism, and we will prove that φ is either zero or an injection in $\mathcal{A}$. Consider the following commutative diagram.

$$\begin{tikzcd} & & M \arrow[dl, dashed, "\overline{\varphi}"'] \arrow[d, "\varphi"] & & \\ 0 \arrow[r] & C \arrow[r, "\iota"] & X \arrow[r, "\pi"] & D \arrow[r] & 0 \end{tikzcd}$$

Since M is left Schurian for $\mathcal{D}$, either $\pi\varphi$ is an injection or $\pi\varphi = 0$. In the former case, φ is an injection, so suppose the latter. Then there exists a morphism $\overline{\varphi}\colon M \to C$ with $\varphi = \overline{\varphi}\iota$. Since M is left Schurian for $\mathcal{C}$, we have that $\overline{\varphi}$ is either zero or injective. Thus φ is either zero or injective respectively. □

Now we are ready to prove Theorem 5.2.11.

Proof of Theorem 5.2.11. For a left Schur subcategory $\mathcal{E}$ of $\mathcal{A}$, we have $\mathsf{sim}\,\mathcal{E} \in \mathsf{mbrick}\,\mathcal{A}$ by Corollary 5.2.6, thus we have a map $\mathsf{sim}\colon \mathsf{Schur}_{\mathsf{L}}\,\mathcal{A} \to \mathsf{mbrick}\,\mathcal{A}$.

For the converse direction, let $\mathcal{M}$ be a monobrick in $\mathcal{A}$. Then $\mathsf{Filt}\,\mathcal{M}$ is closed under extensions in $\mathcal{A}$. We will prove that $\mathsf{Filt}\,\mathcal{M}$ is a left Schur subcategory of $\mathcal{A}$. We show the following claim:

(Claim): Let $\mathcal{M}$ be a monobrick. Then the following are equivalent for $M \in \mathsf{Filt}\,\mathcal{M}$.

(1) M is simple in $\mathsf{Filt}\,\mathcal{M}$.
(2) M is in $\mathcal{M}$.
(3) M is left Schurian for $\mathsf{Filt}\,\mathcal{M}$.

Proof of (Claim).

(1) ⇒ (2): By the construction of $\mathsf{Filt}\,\mathcal{M}$ and the definition of a simple object, every simple object in $\mathsf{Filt}\,\mathcal{M}$ should belong to $\mathcal{M}$.

(2) ⇒ (3): Let $M \in \mathcal{M}$. Since $\mathcal{M}$ is a monobrick, M is left Schurian for $\mathcal{M}$. Then by using Lemma 5.2.12 repeatedly, M is left Schurian for $\mathsf{Filt}\,\mathcal{M}$.

(3) ⇒ (1): This follows from Proposition 5.2.4. ■

In particular, the implication (1) ⇒ (3) implies that $\mathsf{Filt}\,\mathcal{M}$ is left Schur. Therefore we obtain a map $\mathsf{Filt}\colon \mathsf{mbrick}\,\mathcal{A} \to \mathsf{Schur}_{\mathsf{L}}\,\mathcal{A}$. We will prove that these maps are mutually inverse to each other. Since $\mathcal{A}$ is length, it can be easily shown by induction on lengths that $\mathcal{E} = \mathsf{Filt}(\mathsf{sim}\,\mathcal{E})$ holds for any extension-closed subcategory $\mathcal{E}$ of $\mathcal{A}$. Thus $\mathsf{Filt} \circ \mathsf{sim}\colon \mathsf{Schur}_{\mathsf{L}}\,\mathcal{A} \to \mathsf{Schur}_{\mathsf{L}}\,\mathcal{A}$ is the identity. Conversely, (Claim) implies $\mathsf{sim}(\mathsf{Filt}\,\mathcal{M}) = \mathcal{M}$ for a monobrick $\mathcal{M}$. Therefore, we have mutually inverse bijections $\mathsf{sim}\colon \mathsf{Schur}_{\mathsf{L}}\,\mathcal{A} \rightleftarrows \mathsf{mbrick}\,\mathcal{A}\colon \mathsf{Filt}$.

Finally, we claim that this bijections $\mathsf{sim}\colon \mathsf{Schur}_{\mathsf{L}}\,\mathcal{A} \xrightarrow{\sim} \mathsf{mbrick}\,\mathcal{A}\colon \mathsf{Filt}$ restricts to bijections $\mathsf{wide}\,\mathcal{A} \leftrightarrow \mathsf{sbrick}\,\mathcal{A}$. Ringel's result [Rin1, 1.2] implies that $\mathsf{Filt}\,\mathcal{M}$ is a wide subcategory of $\mathcal{A}$ if $\mathcal{M}$

is a semibrick. Conversely, let $\mathcal{E}$ be a wide subcategory of $\mathcal{A}$. Then $\mathcal{E}$ is an abelian category, and it is easy to see that simple objects in $\mathcal{E}$ coincides with usual simple objects in an abelian category $\mathcal{E}$. Thus the Schur's lemma in $\mathcal{E}$ implies that $\mathsf{sim}\,\mathcal{E}$ is a semibrick. □

5.3. Maps to torsion-free classes and cofinally closed monobricks

In this section, we will show that a left Schur subcategory $\mathcal{E}$ of $\mathcal{A}$ is a torsion-free class if and only if $\mathsf{sim}\,\mathcal{E}$ is a *cofinally closed monobrick*. Then we construct a map from $\mathsf{mbrick}\,\mathcal{A}$ to the set of cofinally closed monobricks, taking the *cofinal closure*, which corresponds to the map $\mathsf{F}(\mathcal{E})$ which sends $\mathcal{E}$ to the smallest torsion-free class containing $\mathcal{E}$.

5.3.1. Cofinal extension and cofinal closure of monobricks. First we observe that each monobrick has a natural poset structure, which will play a central role in this paper.

Definition 5.3.1. Let $\mathcal{M}$ be a monobrick in $\mathcal{A}$. For M, N in $\mathcal{M}$, we define $M \leq N$ if there is an injection $M \to N$ in $\mathcal{A}$. Since $\mathcal{A}$ is length, it is easily checked that $\leq$ is actually a partial order on $\mathcal{M}$. We call this order the *submodule order on* $\mathcal{M}$.

We introduce a notion of *cofinal extension* between monobricks, and *cofinally closed* monobricks.

Definition 5.3.2. Let $\mathcal{M}$ and $\mathcal{N}$ be two monobricks in $\mathcal{A}$. Then we say that $\mathcal{N}$ is a *cofinal extension of* $\mathcal{M}$, or $\mathcal{M}$ is *cofinal in* $\mathcal{N}$, if it satisfies the following conditions:

(1) $\mathcal{M} \subset \mathcal{N}$ holds.
(2) For every $N \in \mathcal{N}$, there exists $M \in \mathcal{M}$ satisfying $N \leq M$ in $\mathcal{N}$.

We say that a monobrick $\mathcal{M}$ is *cofinally closed* if there is no proper cofinal extension of $\mathcal{M}$. We denote by $\mathsf{mbrick}_{\mathsf{c.c.}}\,\mathcal{A}$ the set of cofinally closed monobricks in $\mathcal{A}$.

Note that this is a purely poset theoretical notion, and has nothing to do with the actual module structure of each brick.

Cofinal extensions of $\mathcal{M}$ is closed under unions in the following sense:

Proposition 5.3.3. *Let $\mathcal{M}$ be a monobrick in $\mathcal{A}$. Suppose that $\{\mathcal{N}_i \mid i \in I\}$ is a family of cofinal extensions of $\mathcal{M}$. Then $\bigcup_{i\in I} \mathcal{N}_i$ is a cofinal extension of $\mathcal{M}$ (in particular, it is a monobrick).*

Proof. Clearly, we only have to see that $\mathcal{N} := \bigcup_{i\in I} \mathcal{N}_i$ is actually a monobrick. Take N_1 and N_2 in $\mathcal{N}$ with $N_1 \in \mathcal{N}_{i_1}$ and $N_2 \in \mathcal{N}_{i_2}$, and let $f\colon N_1 \to N_2$ be any map. Since $\mathcal{N}_{i_2}$ is a cofinal extension of $\mathcal{N}$, there is an injection $\iota\colon N_2 \hookrightarrow M$ with $M \in \mathcal{M}$. Then the composition $\iota f\colon N_1 \to M$ is a map between elements in $\mathcal{N}_{i_1}$, thus it should be either zero or an injection since $\mathcal{N}_{i_1}$ is a monobrick. Then the injectivity of ι implies that f is either zero or an injection. □

This immediately implies the existence of the *largest cofinal extension* of a given monobrick, which is cofinally closed.

Corollary 5.3.4. *Let $\mathcal{M}$ be a monobrick in $\mathcal{A}$. Then the union $\overline{\mathcal{M}}$ of all cofinal extensions of $\mathcal{M}$ satisfies the following properties:*

(1) *$\overline{\mathcal{M}}$ is a cofinal extension of $\mathcal{M}$.*
(2) *For every cofinal extension $\mathcal{N}$ of $\mathcal{M}$, we have $\mathcal{M} \subset \mathcal{N} \subset \overline{\mathcal{M}}$.*
(3) *$\overline{\mathcal{M}}$ is cofinally closed. Moreover, if $\mathcal{N}$ is a cofinal extension of $\mathcal{M}$ which is cofinally closed, then $\mathcal{N} = \overline{\mathcal{M}}$ holds.*

Proof. (1), (2) Clear from Proposition 5.3.6 and the definition of $\overline{\mathcal{M}}$.

(3) Let $\mathcal{M}'$ be a cofinal extension of $\overline{\mathcal{M}}$. Then it is easy to see that $\mathcal{M}'$ is also a cofinal extension of $\mathcal{M}$. Thus (2) implies $\mathcal{M}' \subset \overline{\mathcal{M}}$, thus $\mathcal{M}' = \overline{\mathcal{M}}$. Therefore, $\overline{\mathcal{M}}$ is cofinally closed. On the other hand, let $\mathcal{N}$ be a cofinal extension of $\mathcal{M}$ which is cofinally closed. Then we have $\mathcal{M} \subset \mathcal{N} \subset \overline{\mathcal{M}}$ holds by (2). It is obvous that $\overline{\mathcal{M}}$ is a cofinal extension of $\mathcal{N}$, thus we have $\mathcal{N} = \overline{\mathcal{M}}$ by the definition of the cofinal closedness. □

Definition 5.3.5. Let $\mathcal{M}$ be a monobrick. We denote by $\overline{\mathcal{M}}$ the union of all cofinal extensions of $\mathcal{M}$, and call it the *cofinal closure of* $\mathcal{M}$. Then $\overline{\mathcal{M}}$ is the unique cofinal extension of $\mathcal{M}$ which is cofinally closed by Corollary 5.3.4.

Taking the cofinal closure defines a map $\overline{(-)}\colon \mathsf{mbrick}\,\mathcal{A} \twoheadrightarrow \mathsf{mbrick}_{\mathsf{c.c.}}\,\mathcal{A}$, which is the identity on $\mathsf{mbrick}_{\mathsf{c.c.}}\,\mathcal{A}$ by the definition of cofinal closedness. Similarly, we can check that a monobrick $\mathcal{M}$ is cofinally closed if and only if $\overline{\mathcal{M}} = \mathcal{M}$ holds.

Next, we will characterize the cofinal closure as in the theory of integral extensions of commutative rings: an integral closure of a ring is given by the set of all elements which is integral over the base ring. To this purpose, we will introduce the following notion.

Definition 5.3.6. Let $\mathcal{M}$ be a monobrick in $\mathcal{A}$. We say that a brick N in $\mathcal{A}$ is *cofinal over* $\mathcal{M}$ if it satisfies the following conditions:

(1) There exist $M \in \mathcal{M}$ and an injection $N \hookrightarrow M$ in $\mathcal{A}$.
(2) Every map $N \to M'$ with $M' \in \mathcal{M}$ is either zero or an injection.

Proposition 5.3.7. *Let $\mathcal{M}$ be a monobrick in $\mathcal{A}$ and N a brick in $\mathcal{A}$. Then N is cofinal over $\mathcal{M}$ if and only if $\mathcal{M} \cup \{N\}$ is a cofinal extension of $\mathcal{M}$.*

Proof. The "if" part is clear. Conversely, suppose that N is cofinal over $\mathcal{M}$, and we claim that $\mathcal{M} \cup \{N\}$ is a cofinal extension of $\mathcal{M}$. Obviously it suffices to show that $\mathcal{M} \cup \{N\}$ is a monobrick.

Clearly we only have to show that every map $f\colon M \to N$ with $M \in \mathcal{M}$ is either zero or an injection. By the assumption, there is an injection $\iota\colon N \hookrightarrow M'$ with $M' \in \mathcal{M}$. Then the composition $\iota f\colon M \to M'$ is a map between elements in $\mathcal{M}$, thus is either zero or an injection. Since ι is injective, this implies that f is either zero or an injection. ☐

Now we can describe a cofinal extension of a monobrick in terms of elements:

Corollary 5.3.8. *Let $\mathcal{M}$ be a monobrick in $\mathcal{A}$ and $\mathcal{N}$ a set of bricks in $\mathcal{A}$ satisfying $\mathcal{M} \subset \mathcal{N}$ (we do not require that $\mathcal{N}$ is a monobrick). Then the following are equivalent:*

(1) *$\mathcal{N}$ is a cofinal extension of $\mathcal{M}$ (in particular, $\mathcal{N}$ is a monobrick).*
(2) *Every element in $\mathcal{N}$ is cofinal over $\mathcal{M}$.*

Proof. (1) $\Rightarrow$ (2): It follows immediately from definition.

(2) $\Rightarrow$ (1): By Proposition 5.3.7, we have that $\mathcal{M} \cup \{N\}$ is a cofinal extension of $\mathcal{M}$ for each $N \in \mathcal{N}$. Since we have $\mathcal{N} = \bigcup_{N \in \mathcal{N}} (\mathcal{M} \cup \{N\})$, Proposition 5.3.6 implies that $\mathcal{N}$ is a cofinal extension of $\mathcal{M}$. ☐

Similarly, we have the following description of the cofinal closure.

Corollary 5.3.9. *Let $\mathcal{M}$ be a monobrick in $\mathcal{A}$. Then we have*

$$\overline{\mathcal{M}} = \{N \in \mathsf{brick}\,\mathcal{A} \mid N \text{ is cofinal over } \mathcal{M}\}.$$

In particular, $\mathcal{M}$ is cofinally closed if and only if the following condition is satisfied:

(CC) *If a brick N has an injection $N \hookrightarrow M$ of some $M \in \mathcal{M}$ and $N \notin \mathcal{M}$, then there is some non-zero non-injection $N \to M'$ to $M' \in \mathcal{M}$.*

Proof. Since $\overline{\mathcal{M}}$ is a cofinal extension of $\mathcal{M}$, every element in $\overline{\mathcal{M}}$ is cofinal over $\mathcal{M}$. Conversely, suppose that a brick N is cofinal over $\mathcal{M}$. Then $\mathcal{M} \cup \{N\}$ is a cofinal extension of $\mathcal{M}$ by Proposition 5.3.7. Thus we have $\mathcal{M} \cup \{N\} \subset \overline{\mathcal{M}}$ by Corollary 5.3.4, hence $N \in \overline{\mathcal{M}}$. ☐

5.3.2. Torsion-free classes and cofinally closed monobricks. In this subsection, we will show that the map $\mathsf{F}\colon \mathsf{Schur}_{\mathsf{L}}\,\mathcal{A} \twoheadrightarrow \mathsf{torf}\,\mathcal{A}$ corresponds to the map $\overline{(-)}\colon \mathsf{mbrick}\,\mathcal{A} \twoheadrightarrow \mathsf{mbrick}_{\mathsf{c.c.}}\,\mathcal{A}$ defined in the previous subsection.

First of all, we can construct a torsion-free class from any collection of objects in $\mathcal{A}$ as follows.

Definition 5.3.10. Let $\mathcal{C}$ be a collection of objects in $\mathcal{A}$.

- We denote by $\mathsf{sub}\,\mathcal{C}$ the collection of all subobjects of objects in $\mathcal{C}$ (where subobjects mean the usual subobjects in an abelian category $\mathcal{A}$).

- We denote by $\mathsf{F}(\mathcal{C}) := \mathsf{Filt}(\mathsf{sub}\,\mathcal{C})$.

LEMMA 5.3.11. *Let $\mathcal{C}$ be a collection of objects in $\mathcal{A}$. Then $\mathsf{F}(\mathcal{C})$ is the smallest torsion-free class containing $\mathcal{C}$.*

PROOF. Although this is well-known (e.g. [MS, Lemma 3.1]), we give a proof here for the convenience. Clearly it suffices to show that $\mathsf{F}(\mathcal{C})$ is a torsion-free class in $\mathcal{A}$. Since $\mathsf{F}(\mathcal{C}) = \mathsf{Filt}(\mathsf{sub}\,\mathcal{C})$ is extension-closed in $\mathcal{A}$, it is enough to show that $\mathsf{F}(\mathcal{C})$ is closed under subobjects.

Take any $M \in \mathsf{F}(\mathcal{C})$ and its subobject $X \hookrightarrow M$. We will show $X \in \mathsf{F}(\mathcal{C})$ by the induction of a $(\mathsf{sub}\,\mathcal{C})$-filtration length n of M. If $n = 1$, then $M \in \mathsf{sub}\,\mathcal{C}$ holds, thus M is a subobject of some $C \in \mathcal{C}$. Then it follows that X is also a subobject of C, which proves $X \in \mathsf{sub}\,\mathcal{C} \subset \mathsf{F}(\mathcal{C})$.

Now suppose $n > 1$. Then there is a short exact sequence $0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$ with $L, N \in \mathsf{F}(\mathcal{C})$ such that L and N has $(\mathsf{sub}\,\mathcal{C})$-filtrations of length smaller than n. We can obtain the following exact commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & L \cap X \arrow[r] \arrow[d, hook] & X \arrow[r] \arrow[d, hook] & \pi(X) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & L \arrow[r, "\iota"] & M \arrow[r, "\pi"] & N \arrow[r] & 0
\end{tikzcd}$$

where all the vertical maps are injections. By the induction hypothesis, we have $L \cap X, \pi(X) \in \mathsf{F}(\mathcal{C})$. Thus $X \in \mathsf{F}(\mathcal{C})$ holds since $\mathsf{F}(\mathcal{C})$ is extension-closed. $\square$

The following basic observation is used later.

LEMMA 5.3.12. *Let $\mathcal{C}$ be a collection of objects in $\mathcal{A}$. Then $\mathsf{F}(\mathcal{C}) = \mathsf{F}(\mathsf{Filt}\,\mathcal{C})$ holds.*

PROOF. Since $\mathcal{C} \subset \mathsf{Filt}\,\mathcal{C} \subset \mathsf{F}(\mathsf{Filt}\,\mathcal{C})$, we have $\mathsf{F}(\mathcal{C}) \subset \mathsf{F}(\mathsf{Filt}\,\mathcal{C})$ by the minimality of $\mathsf{F}(\mathcal{C})$. On the other hand, $\mathsf{F}(\mathcal{C}) = \mathsf{Filt}(\mathsf{sub}\,\mathcal{C}) \supset \mathsf{Filt}\,\mathcal{C}$ holds, thus $\mathsf{F}(\mathcal{C}) \supset \mathsf{F}(\mathsf{Filt}\,\mathcal{C})$. $\square$

Thus we have the following commutative diagram.

$$\begin{tikzcd}
\mathsf{mbrick}\,\mathcal{A} \arrow[r, "\sim", "\mathsf{Filt}"'] \arrow[dr, "\mathsf{F}"'] \arrow[dd, dashed] & \mathsf{Schur_L}\,\mathcal{A} \arrow[d, two heads, "\mathsf{F}"] \\
 & \mathsf{torf}\,\mathcal{A} \arrow[d, hook] \\
\mathsf{mbrick}\,\mathcal{A} & \mathsf{Schur_L}\,\mathcal{A} \arrow[l, "\sim"', "\mathsf{sim}"]
\end{tikzcd}$$

The following claims that the dotted map is nothing but taking the cofinal closure.

PROPOSITION 5.3.13. *Let $\mathcal{M}$ be a monobrick in $\mathcal{A}$. Then we have $\overline{\mathcal{M}} = \mathsf{sim}\,\mathsf{F}(\mathcal{M})$.*

PROOF. First, we will prove $\mathcal{M} \subset \mathsf{sim}\,\mathsf{F}(\mathcal{M})$, which is equivalent to that every object in $\mathcal{M}$ is left Schurian for $\mathsf{F}(\mathcal{M})$ by Corollary 5.2.6. Take any $M \in \mathcal{M}$. Then M is left Schurian for $\mathcal{M}$ since $\mathcal{M}$ is a monobrick. Since every object in $\mathsf{sub}\,\mathcal{M}$ admits an injection into some object in $\mathcal{M}$, it is easily checked that M is left Schurian also for $\mathsf{sub}\,\mathcal{M}$. Then Lemma 5.2.12 implies that M is left Schurian for $\mathsf{Filt}(\mathsf{sub}\,\mathcal{M}) = \mathsf{F}(\mathcal{M})$.

Next, we will prove that $\mathcal{M} \subset \mathsf{sim}\,\mathsf{F}(\mathcal{M})$ is a cofinal extension. Let X be a simple object in $\mathsf{F}(\mathcal{M}) = \mathsf{Filt}(\mathsf{sub}\,\mathcal{M})$. Then clearly we must have $X \in \mathsf{sub}\,\mathcal{M}$. It follows that there is an injection $X \hookrightarrow M$ with $M \in \mathcal{M}$. This shows that $\mathsf{sim}\,\mathsf{F}(\mathcal{M})$ is a cofinal extension of $\mathcal{M}$.

Therefore, we have $\mathsf{sim}\,\mathsf{F}(\mathcal{M}) \subset \overline{\mathcal{M}}$ by Corollary 5.3.4. On the other hand, since $\overline{\mathcal{M}}$ is a monobrick, every object in $\overline{\mathcal{M}}$ is left Schurian for $\overline{\mathcal{M}}$, thus so is for $\mathcal{M}$. Then the same argument as the first part implies $\overline{\mathcal{M}} \subset \mathsf{sim}\,\mathsf{F}(\mathcal{M})$. Hence $\mathsf{sim}\,\mathsf{F}(\mathcal{M}) = \overline{\mathcal{M}}$ holds. $\square$

As a corollary, we have the following description of simple objects in $\mathsf{F}(\mathcal{E})$ for a left Schur category $\mathcal{E}$.

COROLLARY 5.3.14. *Let $\mathcal{E}$ be a left Schur subcategory of $\mathcal{A}$. Then $\mathsf{sim}\,\mathsf{F}(\mathcal{E})$ consists of bricks N in $\mathcal{A}$ which satisfy the following conditions:*

(1) *There is an injection* $N \hookrightarrow M$ *with* $M \in \mathsf{sim}\,\mathcal{E}$.
(2) *Every map* $N \to M'$ *with* $M' \in \mathsf{sim}\,\mathcal{E}$ *is either zero or an injection.*

Proof. This immediately follows from Corollary 5.3.9, since we have $\mathsf{sim}\,\mathsf{F}(\mathcal{E}) = \overline{\mathsf{sim}\,\mathcal{E}}$ by Proposition 5.3.13 and Theorem 5.2.11. $\square$

Now we can state our characterization of torsion-free classes via monobricks.

Theorem 5.3.15. *Let* $\mathcal{A}$ *be a length abelian category. Then we have the following commutative diagram, and all the horizontal maps are bijective.*

$$\begin{tikzcd}
\mathsf{torf}\,\mathcal{A} \arrow[r, shift left, "\mathsf{sim}"] \arrow[d, hook] \arrow[dd, bend right=60, "1"'] & \mathsf{mbrick}_{\mathsf{c.c.}}\,\mathcal{A} \arrow[l, shift left, "\mathsf{Filt}"] \arrow[d, hook] \arrow[dd, bend left=60, "1"] \\
\mathsf{Schur}_{\mathsf{L}}\,\mathcal{A} \arrow[r, shift left, "\mathsf{sim}"] \arrow[d, two heads, "\mathsf{F}"] & \mathsf{mbrick}\,\mathcal{A} \arrow[l, shift left, "\mathsf{Filt}"] \arrow[d, two heads, "\overline{(-)}"] \\
\mathsf{torf}\,\mathcal{A} \arrow[r, shift left, "\mathsf{sim}"] & \mathsf{mbrick}_{\mathsf{c.c.}}\,\mathcal{A} \arrow[l, shift left, "\mathsf{Filt}"]
\end{tikzcd}$$

Proof. Clearly it suffices to show that the following are equivalent for a monobrick $\mathcal{M}$ in $\mathcal{A}$:
(1) $\mathcal{M}$ is cofinally closed.
(2) $\mathsf{Filt}\,\mathcal{M}$ is a torsion-free class.

(1) $\Rightarrow$ (2): If $\mathcal{M}$ is cofinally closed, then $\overline{\mathcal{M}} = \mathcal{M}$ holds. Therefore, we have $\mathsf{Filt}\,\mathcal{M} = \mathsf{Filt}\,\overline{\mathcal{M}} = \mathsf{Filt}(\mathsf{sim}\,\mathsf{F}(\mathcal{M})) = \mathsf{F}(\mathcal{M})$ by Proposition 5.3.13, thus $\mathsf{Filt}\,\mathcal{M}$ is a torsion-free class in $\mathcal{A}$.

(2) $\Rightarrow$ (1): Since $\mathsf{Filt}\,\mathcal{M}$ is a torsion-free class, we have $\mathsf{F}(\mathcal{M}) = \mathsf{F}(\mathsf{Filt}\,\mathcal{M}) = \mathsf{Filt}\,\mathcal{M}$ holds by Lemma 5.3.12. Thus we have $\overline{\mathcal{M}} = \mathsf{sim}\,\mathsf{F}(\mathcal{M}) = \mathsf{sim}(\mathsf{Filt}\,\mathcal{M}) = \mathcal{M}$ by Proposition 5.3.13. Therefore $\mathcal{M}$ is cofinally closed. $\square$

We can obtain the following characterization of left Schur subcategories:

Corollary 5.3.16. *Let* $\mathcal{E}$ *be a subcategory of* $\mathcal{A}$. *Then* $\mathcal{E}$ *is left Schur if and only if there exist a torsion-free class* $\mathcal{F}$ *of* $\mathcal{A}$ *and a subset* $\mathcal{M}$ *of* $\mathsf{sim}\,\mathcal{F}$ *such that* $\mathcal{E} = \mathsf{Filt}\,\mathcal{M}$ *holds.*

Proof. Since $\mathsf{sim}\,\mathcal{F}$ (and its subset) is a monobrick for a torsion-free class $\mathcal{F}$, the "if" part is clear. Conversely, let $\mathcal{E}$ be a left Schur subcategory of $\mathcal{A}$. Then we have $\mathsf{sim}\,\mathcal{E} \subset \overline{\mathsf{sim}\,\mathcal{E}} = \mathsf{sim}\,\mathsf{F}(\mathcal{E})$ by Proposition 5.3.13. Thus $\mathcal{F} := \mathsf{F}(\mathcal{E})$ satisfies the desired condition. $\square$

As a similar result, we can prove the following.

Corollary 5.3.17. *Let* $\mathcal{M}$ *be a set of isomorphism classes of bricks in* $\mathcal{A}$. *Then* $\mathcal{M}$ *is a monobrick if and only if there exist a cofinally closed monobrick* $\mathcal{N}$ *of* $\mathcal{A}$ *such that* $\mathcal{M}$ *is a subset of* $\mathcal{N}$.

Proof. The "if" part is clear, and take $\mathcal{N} := \overline{\mathcal{M}}$ for the "only if" part. $\square$

5.4. Maps to semibricks and wide subcategories

We construct maps $\mathsf{max}\colon \mathsf{mbrick}\,\mathcal{A} \to \mathsf{sbrick}\,\mathcal{A}$ and $\mathsf{W}\colon \mathsf{Schur}_{\mathsf{L}}\,\mathcal{A} \to \mathsf{wide}\,\mathcal{A}$, which are the identities if restricted to $\mathsf{sbrick}\,\mathcal{A}$ and $\mathsf{wide}\,\mathcal{A}$ respectively. These maps correspond to each other under Theorem 5.2.11.

First we consider the following operation on monobricks.

Proposition 5.4.1. *Let* $\mathcal{M}$ *be a monobrick in* $\mathcal{A}$. *Then the set* $\mathsf{max}\,\mathcal{M}$ *of maximal elements for the submodule order on* $\mathcal{M}$ *is a semibrick.*

Proof. It suffices to show that every map $f\colon M \to N$ with $M, N \in \mathsf{max}\,\mathcal{M}$ is zero if $M \neq N$. Suppose that f is non-zero. Then f should be an injection in $\mathcal{A}$ since $\mathcal{M}$ is a monobrick. It follows that $M \leq N$, which contradicts to the maximality of M. $\square$

Thus we obtain the map $\mathsf{max}\colon \mathsf{mbrick}\,\mathcal{A} \to \mathsf{sbrick}\,\mathcal{A}$. Actually we have the following characterization of a semibrick in terms of the poset structure.

PROPOSITION 5.4.2. *Let $\mathcal{S}$ be a monobrick in $\mathcal{A}$. Then the following are equivalent:*

(1) *$\mathcal{S}$ is a semibrick.*
(2) *$\mathcal{S}$ is a discrete poset, that is, $M \leq N$ in $\mathcal{S}$ implies $M = N$.*
(3) *$\mathsf{max}\,\mathcal{S} = \mathcal{S}$ holds.*

PROOF. This is immediate from definitions and Proposition 5.4.1. □

Next we introduce a map $\mathsf{W}\colon \mathsf{Schur}_{\mathsf{L}}\,\mathcal{A} \to \mathsf{wide}\,\mathcal{A}$. This extends the map $\mathsf{W}\colon \mathsf{torf}\,\mathcal{A} \to \mathsf{wide}\,\mathcal{A}$ defined by Marks-Šťovíček [MS].

DEFINITION 5.4.3. Let $\mathcal{E}$ be a left Schur subcategory of $\mathcal{A}$. Then $\mathsf{W}(\mathcal{E})$ is a subcategory of $\mathcal{E}$ consisting of objects $W \in \mathcal{E}$ satisfying the following condition: For every map $f\colon W \to X$ with $X \in \mathcal{E}$, we have $\operatorname{Coker} f \in \mathcal{E}$, where $\operatorname{Coker} f$ denotes the cokernel of f in $\mathcal{A}$.

The following is a key lemma to show that $\mathsf{W}(\mathcal{E})$ is actually a wide subcategory.

LEMMA 5.4.4. *Let $\mathcal{E}$ be a left Schur subcategory of $\mathcal{A}$. Then the following holds.*

(1) *For $M \in \mathsf{sim}\,\mathcal{E}$, the following are equivalent:*
   (a) *M belongs to $\mathsf{W}(\mathcal{E})$.*
   (b) *M is maximal in the submodule order on $\mathsf{sim}\,\mathcal{E}$.*
   (c) *Every non-zero morphism $f\colon M \to X$ with $X \in \mathcal{E}$ is an injection in $\mathcal{A}$ and satisfies $\operatorname{Coker} f \in \mathcal{E}$.*
(2) *If we have a short exact sequence*

$$0 \longrightarrow L \xrightarrow{\ i\ } M \xrightarrow{\ p\ } N \longrightarrow 0$$

in $\mathcal{A}$ with $L, M, N \in \mathcal{E}$, then M is in $\mathsf{W}(\mathcal{E})$ if and only if both L and N are in $\mathsf{W}(\mathcal{E})$.

PROOF. (1) Let M be a simple object in $\mathcal{E}$.

(a) $\Rightarrow$ (b): Suppose that M is not maximal in $\mathsf{sim}\,\mathcal{E}$. Then we have a proper injection $\iota\colon M \hookrightarrow M'$ with $M' \in \mathsf{sim}\,\mathcal{E}$. Consider the following exact sequence in $\mathcal{A}$:

$$0 \longrightarrow M \xrightarrow{\ \iota\ } M' \longrightarrow \operatorname{Coker}\iota \longrightarrow 0$$

Since ι is not an isomorphism, $\operatorname{Coker}\iota$ is non-zero. Then $\operatorname{Coker}\iota$ does not belong to $\mathcal{E}$, since otherwise M' would not be simple in $\mathcal{E}$. This implies that M does not belong to $\mathsf{W}(\mathcal{E})$.

(b) $\Rightarrow$ (c) Let $f\colon M \to X$ be a non-zero map with $X \in \mathcal{E}$. Then f is an injection in $\mathcal{A}$ since $\mathcal{E}$ is left Schur. We will prove $\operatorname{Coker} f \in \mathcal{E}$ by induction on a $(\mathsf{sim}\,\mathcal{E})$-filtration length of X.

If X belongs to $\mathsf{sim}\,\mathcal{E}$, then the maximality of M clearly implies that f should be an isomorphism. Suppose that X has $(\mathsf{sim}\,\mathcal{E})$-filtration of length $n > 1$. Then we have a short exact sequence $0 \to L \xrightarrow{\iota} X \xrightarrow{\pi} N \to 0$ with $L \in \mathsf{sim}\,\mathcal{E}$ and N has $(\mathsf{sim}\,\mathcal{E})$-filtration of length $n-1$ (in particular, $L, N \in \mathcal{E}$). Consider the following diagram:

$$\begin{tikzcd}
 & & M \arrow[d, "f"] & & \\
0 \arrow[r] & L \arrow[r, "\iota"] & X \arrow[r, "\pi"] & N \arrow[r] & 0
\end{tikzcd}$$

We consider two cases.

(Case 1): $\pi f = 0$. In this case, there is a map $\overline{f}\colon M \to L$ satisfying $f = \iota\overline{f}$. By the induction hypothesis, $\overline{f}$ is either zero or an injection with $\operatorname{Coker}\overline{f} \in \mathcal{E}$. In the former case, we have $f = 0$, which is a contradiction. Thus the latter holds. Then we obtain the following exact commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & M \arrow[r, equal] \arrow[d, "\overline{f}"] & M \arrow[d, "f"] & & \\
0 \arrow[r] & L \arrow[r, "\iota"] \arrow[d] & X \arrow[r, "\pi"] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{Coker} \overline{f} \arrow[r] \arrow[d] & \operatorname{Coker} f \arrow[r] \arrow[d] & N \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $\operatorname{Coker} \overline{f}$ and N belong to $\mathcal{E}$, so does $\operatorname{Coker} f$ since $\mathcal{E}$ is extension-closed.

(Case 2): $\pi f \neq 0$. In this case, πf is an injection with $\operatorname{Coker}(\pi f) \in \mathcal{E}$ by the induction hypothesis. Then we obtain the following exact commutative diagram.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & M \arrow[r, equal] \arrow[d, "f"] & M \arrow[d, "\pi f"] & \\
0 \arrow[r] & L \arrow[r, "\iota"] \arrow[d, equal] & X \arrow[r, "\pi"] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] & \operatorname{Coker} f \arrow[r] \arrow[d] & \operatorname{Coker}(\pi f) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since L and $\operatorname{Coker}(\pi f)$ belong to $\mathcal{E}$, so does $\operatorname{Coker} f$ since $\mathcal{E}$ is extension-closed.

(c) $\Rightarrow$ (a): Clear from the definition of $\mathsf{W}(\mathcal{E})$.

(2) Suppose that L and N belong to $\mathsf{W}(\mathcal{E})$, and we will prove $M \in \mathsf{W}(\mathcal{E})$. Take any map $f\colon M \to X$ with $X \in \mathcal{E}$. Then we obtain the following exact commutative diagram, where $\overline{f}$ is a map induced from the universality of the cokernel N of i.

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "i"] \arrow[d, equal] & M \arrow[r, "p"] \arrow[d, "f"] & N \arrow[r] \arrow[d, "\overline{f}"] & 0 \\
 & L \arrow[r, "fi"] & X \arrow[r] & \operatorname{Coker}(fi) \arrow[r] & 0
\end{tikzcd}$$

Since L is in $\mathsf{W}(\mathcal{E})$ and X is in $\mathcal{E}$, we have $\operatorname{Coker}(fi) \in \mathcal{E}$. Therefore, we have $\operatorname{Coker} \overline{f} \in \mathcal{E}$ since N is in $\mathsf{W}(\mathcal{E})$. On the other hand, it can be shown that the right square is a pushout diagram. Thus $\operatorname{Coker} f \cong \operatorname{Coker} \overline{f}$ holds, which proves $\operatorname{Coker} f \in \mathcal{E}$. Therefore $M \in \mathsf{W}(\mathcal{E})$ holds.

Conversely, suppose that M belongs to $\mathsf{W}(\mathcal{E})$, and we will show that L and N belong to $\mathsf{W}(\mathcal{E})$. First we will prove $L \in \mathsf{W}(\mathcal{E})$. Take any map $f\colon L \to X$ with $X \in \mathcal{E}$. Then by taking the pushout, we obtain the following exact commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "i"] \arrow[d, "f"] & M \arrow[r, "p"] \arrow[d, "\overline{f}"] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r] & E \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Since the left square is pushout, we have $\operatorname{Coker} f \cong \operatorname{Coker} \overline{f}$. On the other hand, we have $E \in \mathcal{E}$ since $\mathcal{E}$ is extension-closed and $X, N \in \mathcal{E}$. Thus $\operatorname{Coker} \overline{f} \in \mathcal{E}$ holds by $M \in \mathsf{W}(\mathcal{E})$. Therefore $\operatorname{Coker} f \in \mathcal{E}$, which proves $L \in \mathsf{W}(\mathcal{E})$.

Next we will prove $N \in \mathsf{W}(\mathcal{E})$. Take any map $f\colon N \to X$ with $X \in \mathcal{E}$. Then since p is a surjection, $\operatorname{Coker} f \cong \operatorname{Coker}(fp)$ holds. On the other hand, $\operatorname{Coker}(fp) \in \mathcal{E}$ holds by $M \in \mathsf{W}(\mathcal{E})$ and $X \in \mathcal{E}$. Therefore $\operatorname{Coker} f \in \mathcal{E}$, which proves $N \in \mathsf{W}(\mathcal{E})$. □

Now we are ready to prove the main result in this section.

Theorem 5.4.5. *Let $\mathcal{A}$ be a length abelian category. Then the following hold.*

(1) $\mathsf{W}(\mathcal{W}) = \mathcal{W}$ *holds for a wide subcategory* $\mathcal{W}$ *of* $\mathcal{A}$.
(2) $\mathsf{W}(\mathcal{E})$ *is a wide subcategory of* $\mathcal{A}$ *for a left Schur subcategory* $\mathcal{E}$ *of* $\mathcal{A}$.
(3) *The following diagram commutes, and all the horizontal maps are bijective.*

$$\begin{tikzcd}
\mathsf{wide}\,\mathcal{A} \arrow[r, shift left, "\mathsf{sim}"] \arrow[d, hook] \arrow[dd, bend right=60, "1"'] & \mathsf{sbrick}\,\mathcal{A} \arrow[l, shift left, "\mathsf{Filt}"] \arrow[d, hook] \arrow[dd, bend left=60, "1"] \\
\mathsf{Schur_L}\,\mathcal{A} \arrow[r, shift left, "\mathsf{sim}"] \arrow[d, two heads, "\mathsf{W}"] & \mathsf{mbrick}\,\mathcal{A} \arrow[l, shift left, "\mathsf{Filt}"] \arrow[d, two heads, "\mathsf{max}"] \\
\mathsf{wide}\,\mathcal{A} \arrow[r, shift left, "\mathsf{sim}"] & \mathsf{sbrick}\,\mathcal{A} \arrow[l, shift left, "\mathsf{Filt}"]
\end{tikzcd}$$

Proof. (1) Clear from the definition of $\mathsf{W}(\mathcal{W})$ since $\mathcal{W}$ is closed under cokernels.

(2), (3) Let $\mathcal{E}$ be a left Schur subcategory of $\mathcal{A}$ and put $\mathcal{M} := \mathsf{sim}\,\mathcal{E}$. By Theorem 5.2.11, it clearly suffices to show that $\mathsf{W}(\mathcal{E}) = \mathsf{Filt}(\mathsf{max}\,\mathcal{M})$, since $\mathsf{max}\,\mathcal{M}$ is a semibrick by Proposition 5.4.1. By Lemma 5.4.4 (1), we have $\mathsf{max}\,\mathcal{M} \subset \mathsf{W}(\mathcal{E})$. Since $\mathcal{E}$ is extension-closed, it can be easily checked that Lemma 5.4.4 (2) implies $\mathsf{Filt}(\mathsf{max}\,\mathcal{M}) \subset \mathsf{W}(\mathcal{E})$.

Conversely, take any M in $\mathsf{W}(\mathcal{E})$. We will prove $M \in \mathsf{Filt}(\mathsf{max}\,\mathcal{M})$ by the induction on a $\mathcal{M}$-filtration length n of M.

If $n = 1$, we have $M \in \mathcal{M} = \mathsf{sim}\,\mathcal{E}$. Thus Lemma 5.4.4 (1) implies $M \in \mathsf{max}\,\mathcal{M}$, hence in particular $M \in \mathsf{Filt}(\mathsf{max}\,\mathcal{M})$. Suppose $n > 1$, then there is a short exact sequence in $\mathcal{A}$

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0.$$

such that L and N has $\mathcal{M}$-filtrations of length smaller than n. In particular, we have $L, N \in \mathcal{E}$. Then Lemma 5.4.4 (2) implies that $L, N \in \mathsf{W}(\mathcal{E})$ by $M \in \mathsf{W}(\mathcal{E})$. By the induction hypothesis, we have $L, N \in \mathsf{Filt}(\mathsf{max}\,\mathcal{M})$, which shows $M \in \mathsf{Filt}(\mathsf{max}\,\mathcal{M})$. □

Remark 5.4.6. Let $\mathcal{F}$ be a torsion-free class in $\mathcal{A}$. Then $\mathsf{max}(\mathsf{sim}\,\mathcal{F}) = \mathsf{sim}\,\mathsf{W}(\mathcal{F})$ holds by Theorem 5.4.5. In [AP, Proposition 6.5], it is shown that this coincides with the set of *brick labels of arrows* starting at $\mathcal{F}$, which is introduced in [DIRRT]. Therefore, in our context, considering the brick labels of $\mathcal{F}$ is nothing but taking the maximal element of the simple objects in $\mathcal{F}$. In [Asa1], the bijection between functorially finite torsion-free classes and semibricks satisfying some condition was established, and its bijection is given by taking brick labels of arrows starting at $\mathcal{F}$, thus coincides with $\mathsf{max}(\mathsf{sim}\,\mathcal{F})$.

5.5. Applications

In this section, we give an application of the theory of monobricks to torsion-free classes and wide subcategories. We give new proofs of several results on these subcategories, such as Demonet-Iyama-Jasso's finiteness results [DIJ] and Marks-Šťovíček's bijection [MS], and make the relation between torsion-free classes and wide subcategories more transparent, which can be applied to any length abelian categories, without using any τ-tilting theory.

5.5.1. Maps between torsion-free classes and wide subcategories via monobricks. In this section, we consider the restrictions of our maps W and F to $\mathsf{W}\colon \mathsf{torf}\,\mathcal{A} \to \mathsf{wide}\,\mathcal{A}$ and $\mathsf{F}\colon \mathsf{wide}\,\mathcal{A} \to \mathsf{torf}\,\mathcal{A}$. By using monobricks, we can reprove and generalize a Marks-Šťovíček's result by using only the easy poset theoretical argument.

Proposition 5.5.1. *Let $\mathcal{A}$ be a length abelian category. Then the following hold.*

(1) *Let $\mathcal{M}$ be a monobrick in $\mathcal{A}$ and $\mathcal{N}$ a cofinal extension of $\mathcal{M}$. Then* $\mathsf{max}\,\mathcal{M} = \mathsf{max}\,\mathcal{N}$ *holds. In particular, we have* $\mathsf{max}\,\overline{\mathcal{M}} = \mathsf{max}\,\mathcal{M}$ *holds.*

(2) *Let* $\mathcal{S}$ *be a semibrick. Then we have* $\max \overline{\mathcal{S}} = \mathcal{S}$. *Thus the composition* $\max \circ \overline{(-)} \colon \mathsf{sbrick}\, \mathcal{A} \to \mathsf{sbrick}\, \mathcal{A}$ *is the identity.*
(3) [MS, Proposition 3.3] *The composition* $\mathsf{W} \circ \mathsf{F} \colon \mathsf{wide}\, \mathcal{A} \to \mathsf{wide}\, \mathcal{A}$ *is the identity.*

Proof. (1) Let M be a maximal element of $\mathcal{M}$. If M is not maximal in $\mathcal{N}$, then there is some $N \in \mathcal{N}$ with $M < N$. However, since $\mathcal{N}$ is cofinal in $\mathcal{M}$, there is some $M' \in \mathcal{M}$ with $N \leq M'$, which implies $M < M'$. This is a contradiction, thus we have $\max \mathcal{M} \subset \max \mathcal{N}$. Conversely, let N be a maximal element of $\mathcal{N}$. Then since $\mathcal{N}$ is cofinal in $\mathcal{M}$, there is some $M \in \mathcal{M}$ with $N \leq M$. Then the maximality implies $N = M \in \mathcal{M}$, thus $N \in \max \mathcal{M}$ holds.

(2) Obvious from (1).

(3) This follows from (2) and Theorems 5.3.15 and 5.4.5. $\square$

In general, the map $\mathsf{F} \colon \mathsf{wide}\, \mathcal{A} \to \mathsf{torf}\, \mathcal{A}$ is not a surjection, and its image is studied in [AP]. We give a description of its image in terms of monobricks.

Proposition 5.5.2. *Let* $\mathcal{F}$ *be a torsion-free class in* $\mathcal{A}$. *Then the following are equivalent.*
(1) $\mathcal{F} = \mathsf{F}(\mathsf{W}(\mathcal{F}))$ *holds.*
(2) *There is a wide subcategory* $\mathcal{W}$ *satisfying* $\mathcal{F} = \mathsf{F}(\mathcal{W})$.
(3) *There is a semibrick* $\mathcal{S}$ *satisfying* $\mathsf{sim}\, \mathcal{F} = \overline{\mathcal{S}}$.
(4) $\mathsf{sim}\, \mathcal{F}$ *is a cofinal extension of some semibrick.*
(5) $\max(\mathsf{sim}\, \mathcal{F})$ *is cofinal in* $\mathsf{sim}\, \mathcal{F}$, *that is, for every element* M *in* $\mathsf{sim}\, \mathcal{F}$, *there is an element* $S \in \mathsf{sim}\, \mathcal{F}$ *such that* $M \leq S$ *holds and* S *is maximal in* $\mathsf{sim}\, \mathcal{F}$.

Proof. (1) $\Rightarrow$ (2): Obvious.

(2) $\Rightarrow$ (3): Clear from Theorem 5.4.5 and Theorem 5.3.15.

(3) $\Rightarrow$ (4): This is clear since $\mathsf{sim}\, \mathcal{F} = \overline{\mathcal{S}}$ is a cofinal extension of $\mathcal{S}$.

(4) $\Rightarrow$ (5): Let $\mathcal{S}$ be a semibrick such that $\mathsf{sim}\, \mathcal{F}$ is a cofinal extension of $\mathcal{S}$. Then we have $\mathcal{S} = \max \mathcal{S} = \max(\mathsf{sim}\, \mathcal{F})$ holds by Propositions 5.4.2 and 5.5.1. Thus $\max(\mathsf{sim}\, \mathcal{F})$ is cofinal in $\mathsf{sim}\, \mathcal{F}$.

(5) $\Rightarrow$ (1): (5) implies that $\mathsf{sim}\, \mathcal{F}$ is a cofinal extension of $\max(\mathsf{sim}\, \mathcal{F})$, and $\mathsf{sim}\, \mathcal{F}$ is cofinally closed by Theorem 5.3.15. Thus we have $\overline{\max(\mathsf{sim}\, \mathcal{F})} = \mathsf{sim}\, \mathcal{F}$ holds by Corollary 5.3.4. This is nothing but (1) under the bijections in Theorems 5.3.15 and Theorem 5.4.5. $\square$

Example 5.5.3. Let us consider the 2-Kronecker case, see Example 5.7.5 for the detail and notations. By checking which cofinally closed monobrick $\mathcal{M}$ satisfies that $\max \mathcal{M}$ is cofinal in $\mathcal{M}$, any monobricks except case (M2) satisfy this. Thus any torsion-free classes except (M2) (torsion-free classes consisting of all preprojective modules) belong to the image of $\mathsf{F} \colon \mathsf{wide}\, \mathcal{A} \to \mathsf{torf}\, \mathcal{A}$.

As a corollary, we can quickly prove a Marks-Šťovíček's bijection (c.f. [MS, Corollary 3.11]).

Corollary 5.5.4. *Let* $\mathcal{A}$ *be a length abelian category. Then the maps* $\mathsf{F} \colon \mathsf{wide}\, \mathcal{A} \rightleftarrows \mathsf{torf}\, \mathcal{A} \colon \mathsf{W}$ *induces a bijection between* $\mathsf{wide}\, \mathcal{A}$ *and* $\mathsf{F}(\mathsf{wide}\, \mathcal{A})$. *If* $\mathcal{A}$ *has only finitely many torsion-free classes, then* $\mathsf{F}(\mathsf{wide}\, \mathcal{A}) = \mathsf{torf}\, \mathcal{A}$ *holds, thus* F *and* W *are mutually inverse bijections between* $\mathsf{wide}\, \mathcal{A}$ *and* $\mathsf{torf}\, \mathcal{A}$.

Proof. Since the composition $\mathsf{wide}\, \mathcal{A} \xrightarrow{\mathsf{F}} \mathsf{torf}\, \mathcal{A} \xrightarrow{\mathsf{W}} \mathsf{wide}\, \mathcal{A}$ is the identity by Proposition 5.5.1, it suffices to show the last assertion. Suppose that $\mathcal{A}$ has finitely many torsion-free classes, and it suffices to prove that $\max \mathcal{M}$ is cofinal in $\mathcal{M}$ for every (cofinally closed) monobricks by Proposition 5.5.2. We will see in Theorem 5.5.5 that there are only finitely many bricks in $\mathcal{A}$ up to isomorphism. Therefore, every monobrick $\mathcal{M}$ is a finite poset, thus clearly $\max \mathcal{M}$ is cofinal in $\mathcal{M}$. $\square$

5.5.2. Finiteness conditions. In this subsection, we study several finiteness conditions on monobricks. First we consider when $\mathsf{torf}\, \mathcal{A}$ or $\mathsf{wide}\, \mathcal{A}$ or $\mathsf{mbrick}\, \mathcal{A}$ is finite. We denote by $\mathsf{brick}\, \mathcal{A}$ the set of isomorphism classes of bricks in $\mathcal{A}$.

Theorem 5.5.5. *Let* $\mathcal{A}$ *be a length abelian category. Then the following are equivalent:*
(1) $\mathsf{brick}\, \mathcal{A}$ *is finite, that is, there are only finitely many bricks in* $\mathcal{A}$ *up to isomorphism.*

(2) $\mathsf{mbrick}\,\mathcal{A}$ *is finite.*
(2)′ $\mathsf{Schur}_{\mathsf{L}}\,\mathcal{A}$ *is finite.*
(3) $\mathsf{mbrick}_{\mathsf{c.c.}}\,\mathcal{A}$ *is finite.*
(3)′ $\mathsf{torf}\,\mathcal{A}$ *is finite.*
(4) $\mathsf{sbrick}\,\mathcal{A}$ *is finite.*
(4)′ $\mathsf{wide}\,\mathcal{A}$ *is finite.*
(5) *There are only finitely many subcategories of $\mathcal{A}$ which are closed under extensions, kernels and images.*

PROOF. First, note that (i) and (i)′ are equivalent for $i = 2, 3, 4$ by Theorems 5.2.11 and 5.3.15.

(1) ⇒ (2): This is clear since $\mathsf{mbrick}\,\mathcal{A}$ is a subset of $2^{\mathsf{brick}\,\mathcal{A}}$, the power set of $\mathsf{brick}\,\mathcal{A}$.

(2) ⇒ (3): This is clear by $\mathsf{mbrick}_{\mathsf{c.c.}}\,\mathcal{A} \subset \mathsf{mbrick}\,\mathcal{A}$.

(3) ⇒ (4): This is clear by an injection $\overline{(-)}\colon \mathsf{sbrick}\,\mathcal{A} \hookrightarrow \mathsf{mbrick}_{\mathsf{c.c.}}\,\mathcal{A}$ shown in Proposition 5.5.1.

(4) ⇒ (1): A map $\mathsf{brick}\,\mathcal{A} \to \mathsf{sbrick}\,\mathcal{A}$ defined by $S \mapsto \{S\}$ is clearly injective.

(2)′ ⇒ (5): This is clear since every subcategory of $\mathcal{A}$ closed under extensions, kernels and images is left Schur by Proposition 5.2.8.

(5) ⇒ (3)′: This is clear since every torsion-free class in $\mathcal{A}$ is closed under extensions, kernels and images. □

DEFINITION 5.5.6. We call a length abelian category $\mathcal{A}$ *brick-finite* if it satisfies the equivalent conditions of Theorem 5.5.5.

REMARK 5.5.7. In the case $\mathcal{A} = \mathsf{mod}\,\Lambda$ for a finite-dimensional algebra Λ, the equivalence of (1) and (3)′ is a particular case of [DIJ, Theorems 3.8, 4.2], and such an algebra is called *τ-tilting finite*. Actually it was shown in [DIJ] that $\mathsf{mod}\,\Lambda$ is brick-finite if and only if there are only finitely many *functorially finite* torsion-free classes, a little stronger result than ours.

Next we consider when each monobrick consists of finitely many bricks. We begin with the following general observation on posets. A subset X of a poset P is called a *chain* if X is totally ordered, and an *antichain* if every two distinct elements in X are incomparable. For an element m of a poset P, we put $\downarrow m := \{x \in P \mid x \leq m\}$.

LEMMA 5.5.8. *Let P be a poset such that every chain in $\downarrow m$ is finite for every m in P. Then P is finite if and only if it satisfies the following two conditions.*

(1) *$\mathsf{max}\,P$ is cofinal in P, that is, every element is below some maximal element.*
(2) *Every antichain of P is a finite set.*

PROOF. If P is finite, then it clearly satisfies (1) and (2).

Conversely, suppose that P is an infinite set. Since $\mathsf{max}\,P$ is an antichain of P, it is a finite set by (2). By (1), we have $P = \bigcup_{m \in \mathsf{max}\,P} (\downarrow m)$. Since P is infinite and $\mathsf{max}\,P$ is finite, we may assume that $\downarrow m_1$ is an infinite set for $m_1 \in \mathsf{max}\,P$.

Put $P_1 := (\downarrow m_1) \setminus \{m_1\} = \{x \in P \mid x < m_1\}$. Clearly P_1 also satisfies (2), since every antichain of P_1 is also an antichain of P. Suppose that there is an element $x \in P_1$ which is not below any maximal elements in P_1. Since x is not maximal in P_1, there is some $x < x_1$ with $x_1 \in P_1$, and x_1 is not below any maximal elements in P_1. By iterating this, we obtain an infinite chain inside $P_1 \subset (\downarrow m_1)$, which is a contradiction. Thus P_1 satisfies (1).

Now we can repeat the same process as P to find an element $m_2 \in P_1$ such that $\downarrow m_2$ is infinite. Then apply the same process to $P_2 := \{x \in P \mid x < m_2\}$. We can iterate this procedure, and obtain an infinite chain $m_1 > m_2 > m_3 > \cdots$ in $\downarrow m_1$. This is a contradiction. □

By using this, we can prove the following criterion of the finiteness of a monobrick.

PROPOSITION 5.5.9. *Let $\mathcal{M}$ be a monobrick in $\mathcal{A}$. Then $\mathcal{M}$ is a finite set if and only if it satisfies the following two conditions:*

(1) *Every element in $\mathcal{M}$ is below some maximal element in $\mathcal{M}$.*
(2) *Every semibrick $\mathcal{S}$ with $\mathcal{S} \subset \mathcal{M}$ is a finite set.*

Proof. First we will check that a poset $\mathcal{M}$ satisfies the assumption and the conditions in Lemma 5.5.8. Take any $M \in \mathcal{M}$ and consider $\downarrow M$. Clearly $0 \leq l(X) < l(M)$ holds for every element X in $\downarrow M$, where $l(-)$ denotes the lengths of objects in $\mathcal{A}$. If $X < X'$ in $\mathcal{M}$, then $l(X) < l(X')$ holds. Thus clearly $\downarrow M$ cannot contain any infinite chains.

The condition (1) and (2) in Lemma 5.5.8 is nothing but (1) and (2) in this proposition respectively. Moreover, we can check that a subset $\mathcal{S}$ of $\mathcal{M}$ is an antichain if and only if $\mathcal{S}$ is a semibrick. Thus the assertion holds. □

As an application, we have the following criterion of the finiteness of the number of simple objects in a given torsion-free class.

Corollary 5.5.10. *Let $\mathcal{F}$ be a torsion-free class in $\mathcal{A}$. Then $\mathsf{sim}\,\mathcal{F}$ is a finite set if and only if $\mathcal{F}$ satisfies the following conditions.*

(1) *$\mathcal{F} = \mathsf{F}(\mathcal{W})$ holds for some wide subcategory $\mathcal{W}$ of $\mathcal{A}$.*
(2) *Every semibrick $\mathcal{S}$ satisfying $\mathcal{S} \subset \mathsf{sim}\,\mathcal{F}$ is finite.*

The following is the fundamental relation between brick-finiteness and the finiteness of each monobrick.

Theorem 5.5.11. *Let $\mathcal{A}$ be a length abelian category. Then the following are equivalent.*

(1) *Every monobrick in $\mathcal{A}$ is a finite set.*
(2) *Every cofinally closed monobrick in $\mathcal{A}$ is a finite set.*
(2)′ *$\#\,\mathsf{sim}\,\mathcal{F}$ is finite for every torsion-free class $\mathcal{F}$ in $\mathcal{A}$.*
(3) *Every semibrick in $\mathcal{A}$ is a finite set, and the map $\mathsf{F}\colon \mathsf{wide}\,\mathcal{A} \hookrightarrow \mathsf{torf}\,\mathcal{A}$ is surjective.*

Moreover, if $\mathcal{A} = \mathsf{mod}\,\Lambda$ for a finite-dimensional algebra Λ, then the following are also equivalent.

(4) *$\mathsf{mod}\,\Lambda$ is brick-finite, that is, there are only finitely many bricks in $\mathsf{mod}\,\Lambda$.*

Proof. (1) ⇒ (2): Obvious.

(2) ⇒ (1): Let $\mathcal{M}$ be a monobrick in $\mathcal{A}$. Then $\mathcal{M} \subset \overline{\mathcal{M}}$ holds for the cofinal closure of $\mathcal{M}$. Since $\overline{\mathcal{M}}$ is cofinally closed, it is finite by (2), thus so is $\mathcal{M}$.

(2) ⇔ (2)′: Clear from Theorem 5.3.15.

(1) + (2)′ ⇒ (3): The surjectivity of the map $\mathsf{F}\colon \mathsf{wide}\,\mathcal{A} \hookrightarrow \mathsf{torf}\,\mathcal{A}$ follows from Corollary 5.5.10. Since every semibrick is a monobrick, it is finite by (1).

(3) ⇒ (2)′: Clear from Corollary 5.5.10.

Now we have shown the equivalence of (1), (2), (2)′ and (3). From now on, suppose that Λ is a finite-dimensional algebra and $\mathcal{A} = \mathsf{mod}\,\Lambda$.

(2)′ ⇒ (4): Suppose that $\mathsf{mod}\,\Lambda$ is not brick-finite. Then by [DIJ, Theorem 3.8], there is a torsion-free class $\mathcal{F}$ in $\mathsf{mod}\,\Lambda$ which is not functorially finite. Put $\mathcal{F}_0 := 0 \in \mathsf{torf}(\mathsf{mod}\,\Lambda)$. Then [DIJ, Theorem 3.1] implies that there is a functorially finite torsion-free class $\mathcal{F}_1$ satisfying $\mathcal{F}_0 \subsetneq \mathcal{F}_1 \subset \mathcal{F}$. Since $\mathcal{F}$ is not functorially finite, we have $\mathcal{F}_1 \subsetneq \mathcal{F}$. By repeating this process, we obtain a strictly ascending chain $0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots$ of torsion-free classes. Put $\mathcal{G} := \bigcup_{i \geq 0} \mathcal{F}_i$. Then it is clearly a torsion-free class, and $\mathsf{sim}\,\mathcal{G}$ is finite by (2)′. Therefore, there is some i such that $\mathsf{sim}\,\mathcal{G} \subset \mathcal{F}_i$ holds. Since $\mathcal{F}_i$ is extension-closed, this would imply $\mathcal{G} = \mathsf{Filt}(\mathsf{sim}\,\mathcal{G}) \subset \mathcal{F}_i \subset \mathcal{G}$, thus $\mathcal{F}_i = \mathcal{F}_{i+1} = \cdots = \mathcal{G}$, which is a contradiction.

(4) ⇒ (1): Clear. □

We propose the following conjecture related to this, which is of interest in its own.

Conjecture 5.5.12. *Let Λ be a finite-dimensional algebra. If every semibrick in $\mathsf{mod}\,\Lambda$ is a finite set, then $\mathsf{mod}\,\Lambda$ is brick-finite, that is, Λ is τ-tilting finite.*

Roughly speaking, Proposition 5.5.9 and Theorem 5.5.11 says that in order to show brick-finiteness, we have to show the finiteness of antichains (semibricks) and chains of bricks. Thus this conjecture is roughly equivalent to the following question: if every monobrick has *finite width* (finite antichains), then does every monobrick have a *finite height*?

Regarding this, it was recently shown in [ST, Theorem 1.1] that the finiteness of *height* implies brick-finiteness. More precisely, it was shown that if there is an upper bound on the lengths of bricks, then $\mathsf{mod}\,\Lambda$ is brick-finite.

5.6. Monobricks over Nakayama algebras

For a finite-dimensional algebra Λ, we put $\mathsf{mbrick}\,\Lambda := \mathsf{mbrick}(\mathsf{mod}\,\Lambda)$ and so on. In this section, we investigate monobricks and left Schur subcategories of $\mathsf{mod}\,\Lambda$ for a Nakayama algebra Λ. For the details on Nakayama algebras, we refer the reader to standard texts such as [ASS, V.3].

First of all, we show that left Schur subcategories are precisely categories closed under extensions, kernels and images.

THEOREM 5.6.1. *Let Λ be a Nakayama algebra and $\mathcal{E}$ a subcategory of $\mathsf{mod}\,\Lambda$. Then $\mathcal{E}$ is left Schur if and only if $\mathcal{E}$ is closed under extensions, kernels and images. In particular, we have the bijection between the following two sets:*

(1) $\mathsf{mbrick}\,\Lambda$, *the set of monobricks in* $\mathsf{mod}\,\Lambda$.
(2) *The set of subcategories of* $\mathsf{mod}\,\Lambda$ *closed under extensions, kernels and images.*

The maps are given by Filt *and* sim.

PROOF. We use the result in [Eno4, Corollary 5.19]: every torsion-free class in $\mathsf{mod}\,\Lambda$ satisfies the Jordan-Hölder type property. We refer the reader to [Eno4] for the details on this property.

By Proposition 5.2.8, we only have to show that every left Schur subcategory $\mathcal{E}$ of $\mathsf{mod}\,\Lambda$ is closed under kernels and images. By Theorem 5.2.11, there is a monobrick $\mathcal{M}$ satisfying $\mathcal{E} = \mathsf{Filt}\,\mathcal{M}$. Consider the cofinal closure $\overline{\mathcal{M}}$ of $\mathcal{M}$ and put $\mathcal{F} := \mathsf{F}(\mathcal{E})$. Then we have $\overline{\mathcal{M}} = \mathsf{sim}\,\mathcal{F} \supset \mathcal{M}$ by Proposition 5.3.13. Let $f\colon X \to Y$ be a map in $\mathcal{E}$. Since $\mathcal{F}$ is closed under kernels and images in $\mathsf{mod}\,\Lambda$, we obtain the following short exact sequence in $\mathcal{F}$:

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow X \longrightarrow \operatorname{Im} f \longrightarrow 0.$$

Since $\mathcal{F}$ satisfies the Jordan-Hölder property, we can speak of *composition factors inside* $\mathcal{F}$. Since X is in $\mathcal{E} = \mathsf{Filt}\,\mathcal{M}$, all the composition factors of X inside $\mathcal{F}$ belongs to $\mathcal{M}$ by $\mathcal{M} \subset \mathsf{sim}\,\mathcal{F}$. Therefore, all the composition factors of $\operatorname{Im} f$ and $\operatorname{Ker} f$ must be in $\mathcal{M}$, since the above short exact sequence is a conflation in $\mathcal{F}$. This implies $\operatorname{Im} f$ and $\operatorname{Ker} f$ belong to $\mathsf{Filt}\,\mathcal{M} = \mathcal{E}$. □

Our next aim is to give a combinatorial classification of monobricks for Nakayama algebras. The following basic observation on quotient algebras and monobricks is useful. Recall that for a two-sided ideal I of a finite-dimensional algebra Λ, we have the natural fully faithful functor $\mathsf{mod}(\Lambda/I) \hookrightarrow \mathsf{mod}\,\Lambda$, and its essential image consists of Λ-modules M satisfying $MI = 0$. By this, we may identify $\mathsf{mod}(\Lambda/I)$ with the subcategory of $\mathsf{mod}\,\Lambda$ consisting of such modules.

PROPOSITION 5.6.2. *Let Λ be a finite-dimensional algebra and I its two-sided ideal. Then by identifying* $\mathsf{mod}(\Lambda/I)$ *with the subcategory of* $\mathsf{mod}\,\Lambda$, *we have*

$$\mathsf{mbrick}(\Lambda/I) = \mathsf{mod}(\Lambda/I) \cap \mathsf{mbrick}\,\Lambda$$

PROOF. This follows from the fact that the natural functor $\mathsf{mod}(\Lambda/I) \hookrightarrow \mathsf{mod}\,\Lambda$ is fully faithful and that a morphism in $\mathsf{mod}(\Lambda/I)$ is an injection in $\mathsf{mod}(\Lambda/I)$ if and only if so is in $\mathsf{mod}\,\Lambda$. □

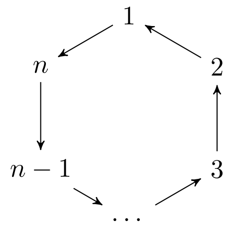
By this, the classification of monobricks over Λ/I is reduced to that of Λ in principle. Keeping this in mind, it suffices to consider the following two Nakayama algebras.

DEFINITION 5.6.3. Let n be a positive integer. Then we define two algebras A_n and B_n as follows:

(1) A_n is the path algebra of the following quiver.

$$1 \longleftarrow 2 \longleftarrow \cdots \longleftarrow n$$

(2) B_n is the quotient of the path algebra of the following by the ideal generated by all the paths of lengths n.



Note that we have the natural identification $B_n/\langle e_n\rangle \cong A_{n-1}$, where e_n is the primitive idempotent of B_n corresponding to n.

The following shows that to classify monobricks over Nakayama algebras, it suffices to consider A_n and B_n.

Proposition 5.6.4. *Let Λ be a basic connected Nakayama algebra with $\#\,\mathsf{sim}(\mathsf{mod}\,\Lambda) = n$.*

(1) *If the quiver of Λ is acyclic, then $\Lambda \cong A_n/I$ for some I, thus $\mathsf{mbrick}\,\Lambda \subset \mathsf{mbrick}\,A_n$ holds.*
(2) *If the quiver of Λ is cyclic, then there exist a Nakayama algebra B' and two algebra surjections $B' \twoheadrightarrow \Lambda$ and $B' \twoheadrightarrow B_n$ such that $\mathsf{mbrick}\,\Lambda \subset \mathsf{mbrick}\,B_n$ holds inside $\mathsf{mbrick}\,B'$.*

Proof. (1) This is well-known, e.g. [ASS, Theorem V.3.2]

(2) The existence of a Nakayama algebra B' such that Λ and B_n are quotients of B' is obvious (consider the path algebra of the cyclic quiver and annihilate sufficiently large paths), thus it suffices to see that every brick M in $\mathsf{mod}\,B'$ is contained in $\mathsf{mod}\,B_n$. This is clear since if indecomposable module M does not belong to B_n, then it is easily checked that M has non-zero endomorphism which is not an isomorphism. $\square$

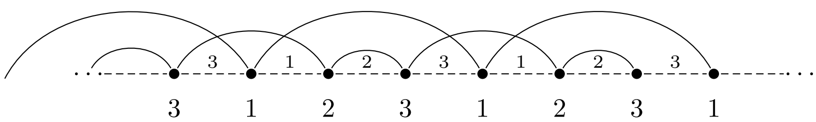
To deal with modules over A_n and B_n, we will use the following combinatorial description.

Definition 5.6.5. Let n be a positive integer.

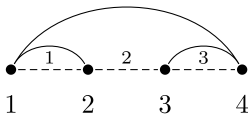
- We put $[n] := \{1, 2, \dots, n\}$.
- For two elements i, j in $[n]$, we calculate $i+j \in [n]$ and $i-j \in [n]$ modulo n, for example, $n+1 = 1$ and $1-1 = n$.
- An *arc on* $[n]$ is an element of $[n] \times [n]$.
- An *admissible arc on* $[n]$ is an arc (i, j) satisfying $i < j$.
- For an arc $\alpha = (i, j)$ on $[n]$, we call i its *starting point* and j its *ending point*.
- The *socle series* of an arc $\alpha = (i, j)$ on $[n]$ is a sequence of elements in $[n]$ defined by $(i, i+1, \dots, j-1)$.
- An *arc diagram* $\mathcal{D}$ *on* $[n]$ is a set of arcs, that is, a subset of $[n] \times [n]$.
- An arc diagram $\mathcal{D}$ is *admissible* if every arc in $\mathcal{D}$ is admissible.

We represent arcs and arc diagrams on $[n]$ as follows: Consider the Euclidean plane $\mathbb{R}^2$ and put i on $(i, 0) + \mathbb{Z}(n, 0)$ for each $i \in [n]$. Then for an arc $\alpha = (i, j)$, we draw "arcs" in the upper half-plane which connect each i and the first j which appear right to this i.

For example, the following is an arc diagram $\mathcal{D} = \{(1, 1), (2, 3), (3, 2)\}$ on [3]. These three arcs have socle series $(1, 2, 3), (2), (3, 1)$ respectively.



As this figure, it is convenient to draw a dashed line on the x-axis and label each line segment as above, so that the socle series of α is the sequence of labels surrounded by α. Also, we often draw an admissible arc diagram by omitting the repeated part, for example, the following is a picture of an admissible arc diagram $\{(1,2),(1,4),(3,4)\}$ on $[4]$.



We say that a sequence $(n_1,\dots,n_a)$ is a *partial sequence* of a sequence $(m_1,\dots,m_b)$ if there is some integer i with $1 \le i \le b-a+1$ satisfying $n_1 = m_i, n_2 = m_{i+1},\dots,n_a = m_{i+a-1}$ holds. For example, $(3,1)$, (2) and $(2,3,1)$ are subsequences of $(2,3,1)$, but $(1,2)$, $(2,1)$ and $(3,1,2)$ are not.

Definition 5.6.6. We say that a pair $\{\alpha,\beta\}$ of two different arcs α and β on $[n]$ is a *weakly non-crossing pair* if *either* of the following conditions is satisfied:

- The socle series of α is a partial sequence of that of β.
- The socle series of β is a partial sequence of that of α.
- The socle series of α and β are disjoint, that is, there exists no element in $[n]$ which appears in the both socle series.

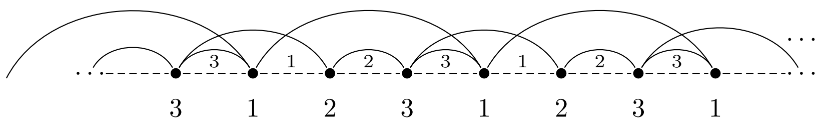
Moreover, for a weakly non-crossing pair $\{\alpha,\beta\}$, we define the following.

(1) It is a *mono-crossing pair* if α and β have the same starting point.
(2) It is a *epi-crossing pair* if α and β have the same ending point.
(3) It is a *non-crossing pair* if it is neither mono-crossing nor epi-crossing.

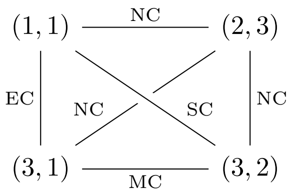
We say that $\{\alpha,\beta\}$ is a *strictly crossing pair* if they are not weakly non-crossing.

Intuitively, a pair of two arcs is weakly non-crossing if they do not *cross* in the half-plane model except at their starting points or ending points, and is non-crossing if in addition they do not share neither the stating points nor the ending points.

Example 5.6.7. Consider an arc diagram $\{(1,1),(2,3),(3,1),(3,2)\}$ on $[3]$:



Then the crossing relations between four arcs are as follows:



Here NC, EC, MC and SC means non-crossing, epi-crossing, mono-crossing and strictly crossing respectively.

Remark 5.6.8. Suppose that $\alpha = (a,b)$ and $\beta = (c,d)$ are distinct *admissible* arcs. Then it is straightforward to see that $\{\alpha,\beta\}$ is strictly crossing if and only if $a < c < b < d$ or $c < a < d < b$.

Definition 5.6.9. Let n be a positive integer and $\mathcal{D}$ an arc diagram on $[n]$.

(1) $\mathcal{D}$ is *non-crossing* if every distinct pair of arcs in $\mathcal{D}$ is a non-crossing pair.
(2) $\mathcal{D}$ is *mono-crossing* if every distinct pair of arcs in $\mathcal{D}$ is either a mono-crossing or a non-crossing pair.

Now let us return to the algebraic side.

Definition 5.6.10. Let n be a positive integer and $\alpha = (i,j)$ be an arc on $[n]$. Then we denote by M_α the unique indecomposable B_n-module satisfying $\operatorname{soc} M_\alpha = i$ and $\operatorname{top} M_\alpha = j-1$. If α is admissible, that is, $i < j$, then we may regard M_α as an A_{n-1}-module by the surjection $B_n \twoheadrightarrow B_n/\langle e_n\rangle \cong A_{n-1}$.

Now by the standard description of indecomposable modules and morphisms between them over Nakayama algebras (e.g. [ASS, Theorem V.3.5]), it is easy to show the following.

PROPOSITION 5.6.11. *Let n be a positive integer. Then the assignment $\alpha \mapsto M_\alpha$ induces a bijection between the set of arcs on $[n]$ and $\mathsf{brick}\, B_n$, and a bijection between the set of admissible arcs on $[n]$ and $\mathsf{brick}\, A_{n-1}$. Moreover, the following hold for two arcs α and β on $[n]$.*

(1) *$\{\alpha, \beta\}$ is a non-crossing pair if and only if $\{M_\alpha, M_\beta\}$ is a semibrick.*
(2) *$\{\alpha, \beta\}$ is a mono-crossing pair if and only if $\{M_\alpha, M_\beta\}$ is a monobrick and not a semibrick.*

Therefore, we have the following bijections, where $\mathcal{M}_\mathcal{D} := \{M_\alpha \mid \alpha \in \mathcal{D}\}$ for an arc diagram $\mathcal{D}$:

$$\begin{tikzcd}
\{\textit{mono-crossing arc diagrams on } [n]\} \arrow[r, "\mathcal{M}_{(-)}", "\sim"'] & \mathsf{mbrick}\, B_n \\
\{\textit{non-crossing arc diagrams on } [n]\} \arrow[r, "\sim"] \arrow[u, hook] & \mathsf{sbrick}\, B_n \arrow[u, hook]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\{\textit{mono-crossing admissible arc diagrams on } [n]\} \arrow[r, "\mathcal{M}_{(-)}", "\sim"'] & \mathsf{mbrick}\, A_{n-1} \\
\{\textit{non-crossing admissible arc diagrams on } [n]\} \arrow[r, "\sim"] \arrow[u, hook] & \mathsf{sbrick}\, A_{n-1}. \arrow[u, hook]
\end{tikzcd}$$

By combining this with Theorem 5.6.1, the problem of classifying all the subcategories closed under extensions, kernels and images is reduced to purely combinatorial problem, namely, classifying all the mono-crossing (admissible) arc diagrams on $[n]$.

In the rest of this section, we give an explicit formula on $\#\,\mathsf{mbrick}\, A_n$ and $\#\,\mathsf{mbrick}\, B_n$. Note that the formula on $\#\,\mathsf{sbrick}\, A_n$ and $\#\,\mathsf{sbrick}\, B_n$ is given by Asai [Asa1, Lemmas 3.4, 3.7]:

$$\#\,\mathsf{sbrick}\, A_{n-1} = \frac{1}{n+1}\binom{2n}{n} \qquad \text{(the } n\text{-th Catalan number, [OEIS, A000108])}$$

$$\#\,\mathsf{sbrick}\, B_n = \binom{2n}{n} \qquad \text{([OEIS, A000984])}$$

Here $\binom{n}{i}$ denotes the binomial coefficient. We can also compute $\#\,\mathsf{sbrick}\, A_{n-1}$ using Proposition 5.6.11. Non-crossing admissible arc diagrams on $[n]$ clearly correspond to the classical *non-crossing partition on* $[n]$ (see [Rin2, N.4.1] for the detail). Therefore, its number is equal to the number of non-crossing partitions, which is well-known to be equal to the Catalan number.

Our enumeration of monobricks is based on the same idea: find a bijection between the set of mono-crossing arc diagrams and some combinatorial sets, whose number has already been computed by combinatorialists.

The following is our enumerative result.

THEOREM 5.6.12. *Let n be a positive integer. Then the following equalities hold.*

$$\#\,\mathsf{mbrick}\, A_n = \text{[OEIS, A006318]}(n) = \sum_{i=0}^{n} \frac{1}{i+1}\binom{n}{i}\binom{n+i}{i}, \ \textit{the } n\textit{-the large Schröder number.} \tag{5.6.1}$$

$$\#\,\mathsf{mbrick}\, B_n = \text{[OEIS, A002003]}(n) = 2\sum_{i=0}^{n} \binom{n-1}{i}\binom{n+i}{i}. \tag{5.6.2}$$

PROOF. By Proposition 5.6.11, it suffices to count the numbers of

(1) mono-crossing admissible arc diagrams on $[n+1]$, and
(2) mono-crossing arc diagrams on $[n]$.

(1) We will show that the number of mono-crossing admissible arc diagrams on $[n]$ is equal to the $(n-1)$-the large Schröder number. The outline of the enumeration is as follows: we will show

that mono-crossing admissible arc diagrams are in bijection with *non-crossing linked partitions* introduced in [Dyk], whose number was known to be the large Schröder number.

A *non-crossing linked partition of* $[n]$ is a set π of non-empty subsets of $[n]$ satisfying the following conditions.

(NCL1) $[n] = \bigcup_{E\in\pi} E$ holds.
(NCL2) For every $E, F \in \pi$ with $E \neq F$, there exists no $a < b < c < d$ satisfying $a, c \in E$ and $b, d \in F$.
(NCL3) We have $\#(E \cap F) \leq 1$ for every distinct $E, F \in \pi$, and if $j \in E \cap F$, then either $j = \min E$, $\#E > 1$ and $j \neq \min F$ hold, or the converse $j = \min F$, $\#F > 1$ and $j \neq \min E$ hold. In particular, if $E \cap F \neq \varnothing$, then $\#E, \#F > 1$ holds.

Then the number of non-crossing linked partitions of $[n]$ is equal to the $(n-1)$-the large Schröder number by [Dyk]. We will prove the equality (5.6.1) by constructing a bijection from the set of non-crossing linked partitions of $[n]$ to the set of mono-crossing admissible arc diagrams. Our construction is essentially the same as the graphical presentation given in [CWY].

Let π be a non-crossing linked partition of $[n]$. For each $E \in \pi$ and $j \in E$ with $j \neq \min E$, we draw an arc $(\min E, j)$. By this, we obtain an admissible arc diagram $\mathcal{D}_\pi$.

We claim that $\mathcal{D}_\pi$ is actually a mono-crossing arc diagram. Let $\{\alpha, \beta\}$ be a pair of arcs in $\mathcal{D}_\pi$ with $\alpha \neq \beta$. By (NCL2) and Remark 5.6.8, this pair is weakly non-crossing. Thus it suffices to show $\{\alpha, \beta\}$ is not epi-crossing. Assume that $\{\alpha, \beta\}$ is epi-crossing, then α and β have the same ending point, that is, we can write as $\alpha = (i_\alpha, j)$ and $\beta = (i_\beta, j)$ with $i_\alpha \neq i_\beta$. By the construction of $\mathcal{D}_\pi$, there is E and F in π satisfying $\{i_\alpha, j\} \subset E$, $\{i_\beta, j\} \subset F$, $i_\alpha = \min E$ and $i_\beta = \min F$. Then $E \neq F$ holds by $i_\alpha \neq i_\beta$. Now we have $j \in E \cap F$ but $j \neq \min E, \min F$, which contradicts to (NCL3). Therefore, $\{\alpha, \beta\}$ is not epi-crossing, thus $\mathcal{D}_\pi$ is a mono-crossing arc diagram.

Conversely, let $\mathcal{D}$ be a mono-crossing admissible arc diagram on $[n]$. For each i in $[n]$, define $E_i \subset [n]$ as follows:

$$E_i := \begin{cases} \{i\} \cup \{j \mid (i,j) \in \mathcal{D}\} & \text{if there is some arc starting at } i, \\ \{i\} & \text{if there is no arc either starting or ending at } i, \\ \varnothing & \text{otherwise} \end{cases}$$

Note that $i = \min E_i$ holds if $E_i \neq \varnothing$, thus E_i's are pairwise distinct.

Put $\pi_\mathcal{D} := \{E_i \mid 1 \leq i \leq n, E_i \neq \varnothing\}$. We claim that $\pi_\mathcal{D}$ is a non-crossing linked partition of $[n]$. Clearly $\pi_\mathcal{D}$ satisfies (NCL1). Assume that $\pi_\mathcal{D}$ does not satisfy (NCL2), that is, there is some $E_{i_1}, E_{i_2} \in \pi_\mathcal{D}$ and $a, c \in E_{i_1}$, $b, d \in E_{i_2}$ satisfying $a < b < c < d$. Then we have $i_1 < b < c < d$ by $i_1 = \min E_{i_1}$. We consider two cases $i_1 < i_2$ and $i_2 < i_1$.

If $i_1 < i_2$, then $i_1 < i_2 \leq b < c < d$ holds by $i_2 = \min E_{i_2}$. Now $c \in E_{i_1}$ and $d \in E_{i_2}$ imply $(i_1, c) \in \mathcal{D}$ and $(i_2, d) \in \mathcal{D}$. From this, (i_1, c) and (i_2, d) is strictly crossing by $i_1 < i_2 < c < d$, which is a contradiction. If $i_2 < i_1$, then $i_2 < i_1 < b < c$ holds. Now $c \in E_{i_1}$ and $b \in E_{i_2}$ imply $(i_1, c) \in \mathcal{D}$ and $(i_2, b) \in \mathcal{D}$. Since these two arcs are strictly crossing, this is a contradiction. Thus (NCL2) holds.

Next we will show that $\pi_\mathcal{D}$ satisfies (NCL3). Suppose that $E_{i_1} \cap E_{i_2} \neq \varnothing$ for $i_1 \neq i_2$ and take $j \in E_{i_1} \cap E_{i_2}$. If $\#E_{i_1} = 1$, then $j = i_1$ and there is no arc either starting or ending at i_1. However, $i_1 \in E_{i_2}$ and $i_1 \neq i_2$ implies that $(i_2, i_1) \in \mathcal{D}$, which is a contradiction. Thus $\#E_{i_1} > 1$ and $\#E_{i_2} > 1$ hold. Now if $j \neq i_1$ and $j \neq i_2$, then $(i_1, j), (i_2, j) \in \mathcal{D}$ holds. This is a contradiction since these two arcs are epi-crossing. Thus either $j = i_1$ or $j = i_2$ holds. In the former case, we have $j = i_1 = \min E_{i_1}$ and $j = i_1 \neq i_2 = \min E_{i_2}$, and in the latter we have $j = \min E_{i_2}$ and $j \neq \min E_{i_1}$. Therefore (NCL3) is satisfied.

Now we have shown that $\pi_\mathcal{D}$ is a non-crossing linked partition of $[n]$ if $\mathcal{D}$ is a mono-crossing admissible arc diagram. It is quite straightforward to see that $\mathcal{D} = \mathcal{D}_{\pi_\mathcal{D}}$ holds for a mono-crossing admissible arc diagram, so we omit the proof.

Finally we show that $\pi = \pi_{\mathcal{D}_\pi}$ holds for a non-crossing linked partition π of $[n]$. Let $E \in \pi$, and we will show $E = E_i$ for $i := \min E$. We consider two cases:

(Case 1): $E = \{i\}$. Suppose that there is some arc (i, j) in $\mathcal{D}_\pi$. By the construction of $\mathcal{D}_\pi$, there is some $F \in \pi$ with $i = \min F$ and $j \in F$. This contradicts to (NCL3) since $i \in E \cap F$

and $\#E = 1$. It follows that there is no arc starting at i. Similarly, suppose that there is some arc (j, i) in $\mathcal{D}_\pi$. Then there is some $F \in \pi$ with $j = \min F$ and $i \in F$. This contradicts to (NCL3) by $i \in E \cap F$ and $\#E = 1$. Therefore, there is no arc either starting or ending at i, hence $E_i = \{i\} = E$ holds.

(Case 2): $\#E > 1$. In this case, there is some arc starting at i in $\mathcal{D}_\pi$. By construction, $E \subset E_i$ holds. Conversely, take $j \in E_i$. Then $(i, j) \in \mathcal{D}_\pi$ holds, thus there is some $F \in \pi$ with $i = \min F$ and $j \in F$. Since $i \in E \cap F$ and $i = \min E = \min F$, we must have $E = F$ by (NCL3). Thus $j \in F = E$ holds, hence $E = E_i$.

We have shown $\pi \subset \pi_{\mathcal{D}_\pi}$. Conversely, take $E_i \in \pi_{\mathcal{D}_\pi}$. We consider two cases.

(Case 1): $E_i = \{i\}$. In this case, by construction, there is no arc either starting or ending at i in $\mathcal{D}_\pi$. This means that there is no $F \in \pi$ with $\#F > 1$ which contains i. Thus $\{i\} \in \pi$ should hold by (NCL1), that is, $E_i \in \pi$.

(Case 2): $\#E_i > 1$. By construction, there is some arc (i, j) in $\mathcal{D}_\pi$, thus there is some $E \in \pi$ satisfying $i = \min E$ and $j \in E$. It suffices to show $E = E_i$. If $j' \in E$ with $j' \neq i$, then $(i, j') \in \mathcal{D}_\pi$ holds by construction. Thus $j' \in E_i$ holds, and we obtain $E \subset E_i$. Conversely, suppose $j' \in E_i$ with $j' \neq i$. Then $(i, j') \in \mathcal{D}_\pi$, so there is some $E' \in \pi$ with $i = \min E'$ and $j' \in E'$. Then $i \in E \cap E'$ satisfies $i = \min E = \min E'$, which implies $E = E'$ by (NCL3). Thus $j' \in E' = E$ holds. Therefore, we have $E_i = E \in \pi$.

Hence we obtain $\pi = \pi_{\mathcal{D}_\pi}$, which completes the proof.

(2) We will show the equality (5.6.2) by calculating the generating function using (5.6.1). Put $a_n := \# \operatorname{\mathsf{mbrick}} A_n$, $b_n := \# \operatorname{\mathsf{mbrick}} B_n$ and consider the following generating functions.

$$f(t) := \sum_{n=1}^{\infty} a_n t^m,$$
$$g(t) := \sum_{n=1}^{\infty} b_n t^n.$$

By using (5.6.1), it is known that the following hold (see e.g. [Bru, Theorem 8.5.7]):

$$f(t) = \frac{1 - t - \sqrt{1 - 6t + t^2}}{2}$$

To compute $g(t)$, we claim the following relation between a_n and b_n.

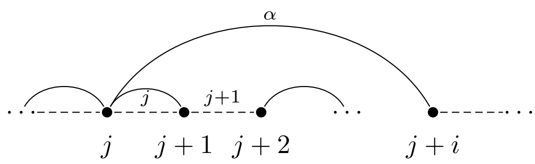
(Claim): *The equality* $b_n = a_n + \sum_{i=1}^{n} i\, a_i\, a_{n+1-i}$ *holds.*

Proof of (Claim): Let MD_n be the set of mono-crossing arc diagram on $[n]$, so $\#\mathsf{MD}_n = b_n$. For $0 \leq i \leq n$, we define $\mathsf{MD}_n(i)$ as follows: $\mathsf{MD}_n(0)$ consists of $\mathcal{D} \in \mathsf{MD}_n$ such that there is no arc in $\mathcal{D}$ whose socle series contains n, and for $1 \leq i \leq n$, $\mathsf{MD}_n(i)$ consists of $\mathcal{D} \in \mathsf{MD}_n$ there is an arc in $\mathcal{D}$ whose socle series contains n, and the minimum length of such arcs is i. Here the length of the arc (i, j) is defined to be $j - i \in [n]$. Then clearly we have the following decomposition, and we will count the number of $\mathsf{MD}_n(i)$.

$$\mathsf{MD}_n = \bigsqcup_{i=0}^{n} \mathsf{MD}_n(i),$$

For $i = 0$, clearly elements in $\mathsf{MD}_n(0)$ are precisely mono-crossing *admissible* arc diagrams on $[n]$. Thus $\#\mathsf{MD}_n(0) = a_n$ holds.

Let $1 \leq i \leq n$. There are i arcs whose socle series contain n and whose lengths are i, that is, $(n - i + 1, 1), (n - i + 2, 2), \ldots, (n, n + i)$. Elements in $\mathsf{MD}_n(i)$ contains precisely one such arc, since any two such arcs are strictly crossing. Fix one such arc $\alpha = (j, j + i)$, and we will count the number of elements in $\mathsf{MD}_n(i)$ which contains α. Let $\mathcal{D}$ be such an element. Then consider the restriction of $\mathcal{D}$ to the part $\{j, j + 1, \ldots, j + i\}$, more precisely, consider the set of arcs whose socle series are partial sequences of that of α.



By shifting $-i$, these arcs except α gives mono-crossing admissible arc diagram on $[i]$ (note that it is not $[i-1]$, since arcs cannot share the endpoint with α). Conversely, any mono-crossing admissible arc diagram on $[i]$ can be occur in this way by shifting $+i$.

In a similar way, consider the set of arcs whose socle series are disjoint from that of α. By shifting $-(j+i)$, these arcs gives mono-crossing admissible arc diagram on $[n-i+1]$, and vice versa. Therefore, there are $i \cdot a_i \cdot a_{n-i+1}$ possible arc diagrams in $\mathsf{MD}_n(i)$. ■

Now, by using (Claim), we obtain the following equality.

$$
\begin{aligned}
g(t) &= \sum_{n=1}^{\infty} b_n t^n \\
&= (a_1 + 1a_1a_1)t + (a_2 + 1a_1a_2 + 2a_2a_1)t^2 + (a_3 + 1a_1a_3 + 2a_2a_2 + 3a_3a_1)t^3 + \cdots \\
&= (a_1 t + a_2 t^2 + a_3 t^3 + \cdots) \cdot (1 + a_1 + 2a_2 t + 3a_3 t^2 + \cdots) \\
&= f(t) \cdot (1 + \frac{df(t)}{dt}) \\
&= \frac{1 - t - \sqrt{1-6t+t^2}}{2} \cdot \frac{1}{2}\left(\frac{3-t}{\sqrt{1-6t+t^2}} + 1 \right) \\
&= \frac{1}{2}\left(\frac{1+t}{\sqrt{1-6t+t^2}} - 1 \right)
\end{aligned}
$$

Since this coincides with the generating function of [OEIS, A002003], we have done. □

Remark 5.6.13. In the paper [Eno7], we will compute the number of monobricks in $\mathsf{mod}\, A_n$ by using the completely different method. In fact, the number of subcategories in $\mathsf{mod}\, kQ$ closed under extensions, kernels and images (thus left Schur) is determined in [Eno7] for a Dynkin quiver Q. If we choose Q to be an A_n quiver with the linear orientation, A_n is nothing but kQ which is Nakayama, thus the number of monobricks in $\mathsf{mod}\, A_n$ is equal to the number of such subcategories by Theorem 5.6.1. Moreover, in [Eno7], it is shown that the number appeared in the right hand side of (5.6.1) for a fixed i is equal to the number of monobricks with i elements.

5.7. Examples of computations

In what follows, we fix an algebraically closed field k. For several finite-dimensional algebras Λ, we show the lists of monobricks and left Schur subcategories, and the behavior of the maps $\mathsf{W}\colon \mathsf{Schur_L}\, \Lambda \twoheadrightarrow \mathsf{wide}\, \Lambda$ and $\mathsf{F}\colon \mathsf{Schur_L}\, \Lambda \twoheadrightarrow \mathsf{torf}\, \Lambda$ in terms of their counterparts $\mathsf{max}\colon \mathsf{mbrick}\, \Lambda \twoheadrightarrow \mathsf{sbrick}\, \Lambda$ and $(\overline{-})\colon \mathsf{mbrick}\, \Lambda \twoheadrightarrow \mathsf{mbrick_{c.c.}}\, \Lambda$.

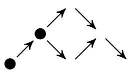
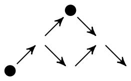
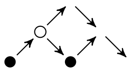
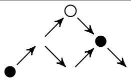
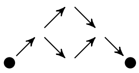
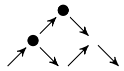
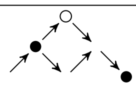
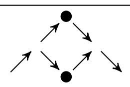
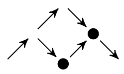
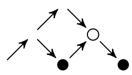
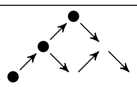
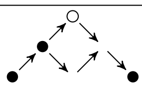
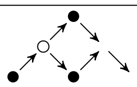
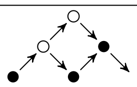
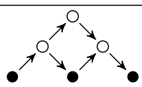
Example 5.7.1. Let Q be a quiver $1 \leftarrow 2 \leftarrow 3$, then the AR quiver of $\mathsf{mod}\, kQ$ is given in Table 1. By Theorem 5.6.12, we have $\#\, \mathsf{mbrick}\, kQ = 22$, the third Schröder number. There are $1 + 6$ monobricks $\mathcal{M}$ with $\#\mathcal{M} \leq 1$, namely, an empty set, and a singleton consisting of each indecomposable kQ-modules.

In Table 2, we list the remaining 15 monobricks, together with their poset structure. For example, the notation $1 < {}^2_1, 3$ means that this poset consists of the disjoint union of two chains $1 < {}^2_1$ and 3. For each monobrick $\mathcal{M}$, we write the corresponding left Schur subcategory $\mathsf{Filt}\, \mathcal{M}$ in the AR quiver, where the black vertices are $\mathcal{M}$, and the white are the rest. If $\mathsf{Filt}\, \mathcal{M}$ is not a wide subcategory, then we write the monobrick corresponding to $\mathsf{W}(\mathsf{Filt}\, \mathcal{M})$, which is equal to $\mathsf{max}\, \mathcal{M}$ by Theorem 5.4.5. Similarly, if $\mathsf{Filt}\, \mathcal{M}$ is not a torsion-free class, then we write the monobrick corresponding to $\mathsf{F}(\mathsf{Filt}\, \mathcal{M})$, which is equal to the cofinal closure $\overline{\mathcal{M}}$ by Theorem 5.3.15.

TABLE 1. The Auslander-Reiten quiver of $\mathsf{mod}\, k[1 \leftarrow 2 \leftarrow 3]$

$$\begin{tikzcd}
 & & \begin{smallmatrix}3\\2\\1\end{smallmatrix} \arrow[dr] & & \\
 & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[dr] & \\
1 \arrow[ur] & & 2 \arrow[ur] & & 3
\end{tikzcd}$$

TABLE 2. Monobricks $\mathcal{M}$ over $k[1 \leftarrow 2 \leftarrow 3]$ with $\#\mathcal{M} \geq 2$

$\mathcal{M}$ (as a poset)	left Schur subcats	wide?	$\max \mathcal{M}$	torsion-free?	$\overline{\mathcal{M}}$
$1 < \begin{smallmatrix}2\\1\end{smallmatrix}$		No	$\begin{smallmatrix}2\\1\end{smallmatrix}$	Yes	itself
$1 < \begin{smallmatrix}3\\2\\1\end{smallmatrix}$		No	$\begin{smallmatrix}3\\2\\1\end{smallmatrix}$	No	$1 < \begin{smallmatrix}2\\1\end{smallmatrix} < \begin{smallmatrix}3\\2\\1\end{smallmatrix}$
$1, 2$		Yes	itself	Yes	itself
$1, \begin{smallmatrix}3\\2\end{smallmatrix}$		Yes	itself	No	$1, 2 < \begin{smallmatrix}3\\2\end{smallmatrix}$
$1, 3$		Yes	itself	Yes	itself
$\begin{smallmatrix}2\\1\end{smallmatrix} < \begin{smallmatrix}3\\2\\1\end{smallmatrix}$		No	$\begin{smallmatrix}3\\2\\1\end{smallmatrix}$	No	$1 < \begin{smallmatrix}2\\1\end{smallmatrix} < \begin{smallmatrix}3\\2\\1\end{smallmatrix}$
$\begin{smallmatrix}2\\1\end{smallmatrix}, 3$		Yes	itself	No	$1 < \begin{smallmatrix}2\\1\end{smallmatrix}, 3$
$2, \begin{smallmatrix}3\\2\\1\end{smallmatrix}$		Yes	itself	No	$1 < \begin{smallmatrix}3\\2\\1\end{smallmatrix}, 2$
$2 < \begin{smallmatrix}3\\2\end{smallmatrix}$		No	$\begin{smallmatrix}3\\2\end{smallmatrix}$	Yes	itself
$2, 3$		Yes	itself	Yes	itself
$1 < \begin{smallmatrix}2\\1\end{smallmatrix} < \begin{smallmatrix}3\\2\\1\end{smallmatrix}$		No	$\begin{smallmatrix}3\\2\\1\end{smallmatrix}$	Yes	itself
$1 < \begin{smallmatrix}2\\1\end{smallmatrix}, 3$		No	$\begin{smallmatrix}2\\1\end{smallmatrix}, 3$	Yes	itself
$1 < \begin{smallmatrix}3\\2\\1\end{smallmatrix}, 2$		No	$2, \begin{smallmatrix}3\\2\\1\end{smallmatrix}$	Yes	itself
$1, 2 < \begin{smallmatrix}3\\2\end{smallmatrix}$		No	$1, \begin{smallmatrix}3\\2\end{smallmatrix}$	Yes	itself
$1, 2, 3$		Yes	itself	Yes	itself

Now let us see some specific examples of computation of $\mathsf{max}\,\mathcal{M}$ and $\overline{\mathcal{M}}$. For a given monobrick $\mathcal{M}$, it is easy to describe its poset structure (we have $L \le M$ in $\mathcal{M}$ whenever there is a non-zero map $L \to M$). Thus its maximal element $\mathsf{max}\,\mathcal{M}$ can be easily computed. For example, consider $\mathcal{M} = \{1, \begin{smallmatrix}2\\1\end{smallmatrix}, 3\}$. Then since we have an injection $1 \hookrightarrow \begin{smallmatrix}2\\1\end{smallmatrix}$ and there are no other non-zero homomorphism, its poset structure is $1 < \begin{smallmatrix}2\\1\end{smallmatrix}, 3$, hence $\mathsf{max}\,\mathcal{M} = \{\begin{smallmatrix}2\\1\end{smallmatrix}, 3\}$.

The computation of $\overline{\mathcal{M}}$ is a little harder than $\mathsf{max}\,\mathcal{M}$, but not so difficult. Recall from Corollary 5.3.9 that $\overline{\mathcal{M}}$ consists of all bricks N which satisfies the following two conditions:

(1) N is a submodule of some $M \in \mathcal{M}$.
(2) Every map $N \to M'$ with $M' \in \mathcal{M}$ is either zero or an injection.

Thus, to compute $\overline{\mathcal{M}}$, first list up all submodules of elements in $\mathcal{M}$ which are bricks and not in $\mathcal{M}$, then check whether the condition (2) above holds. For example, let $\mathcal{M} = \{2, \begin{smallmatrix}3\\2\\1\end{smallmatrix}\}$. Then proper submodules which are bricks are exactly $\begin{smallmatrix}2\\1\end{smallmatrix}$ and 1. However, there is a non-zero non-injection $\begin{smallmatrix}2\\1\end{smallmatrix} \twoheadrightarrow 2$, thus we exclude $\begin{smallmatrix}2\\1\end{smallmatrix}$. In this way we obtain $\overline{\mathcal{M}} = \mathcal{M} \cup \{1\}$.

Next consider the path algebra of an A_3 quiver with another orientation.

EXAMPLE 5.7.2. Let Q be a quiver $1 \to 2 \leftarrow 3$. There are $1+6$ monobricks $\mathcal{M}$ with $\#\mathcal{M} \le 1$, namely, an empty set, and a singleton consisting of each indecomposable kQ-modules. It turns out that $\#\,\mathsf{mbrick}\,\Lambda = 26$, which is different from $1 \leftarrow 2 \leftarrow 3$. This means that *the number of left Schur subcategories (or monobricks) depends on the orientation of the quiver.*

In Table 4, we list the remaining 19 monobricks and their maximal elements and cofinal closures. Wide subcategories are categories in which $\mathsf{max}\,\mathcal{M}$ is "itself," and torsion-free classes are categories in which $\overline{\mathcal{M}}$ is "itself." In this case, there are several examples which are not closed under direct summands, kernels or images. Subcategories $\mathcal{E}$ with (*) are not closed under direct summands (hence is closed under neither images nor kernels), and the white vertices in $\mathcal{E}$ indicate indecomposables of $\mathsf{add}\,\mathcal{E}$ which do not belong to $\mathcal{E}$. The only one category with (**) is closed under images, thus closed under direct summands, but is not closed under kernels. The remaining categories are all closed under kernels and images, and there are 22 such subcategories. This number coincides with the previous example, and see the next remark for the explanation.

REMARK 5.7.3. In [Eno7], it is shown that the number of subcategories of $\mathsf{mod}\,kQ$ which are closed under extensions, kernels and images does not depend on the orientation of the underlying graph for a Dynkin quiver Q, although the number of monobricks does depend as we have seen. In particular, if Q is of type A_n, then the number of such subcategories is equal to the n-th large Schröder number by Theorem 5.6.12.

The next example is non-hereditary case, which already appeared in the introduction.

EXAMPLE 5.7.4. Let Λ be *any* Nakayama algebra whose quiver is $1 \rightleftarrows 2$. Then there are four bricks in $\mathsf{mod}\,\Lambda$, namely, $\mathsf{brick}\,\Lambda = \{1, 2, \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}2\\1\end{smallmatrix}\}$. By using this (and without any consideration of other modules), we obtain the list of monobricks Table 5.

Finally, we consider representation-infinite case.

EXAMPLE 5.7.5. Let Q be a 2-Kronecker quiver, that is, $Q = [1 \leftleftarrows 2]$. Then a complete classification of indecomposable kQ-module is known, see e.g. [ARS, Section VIII.7]. By using this, we obtain the following three classes of bricks.

(1) Indecomposable preprojective modules $\{P_1, P_2, P_3, \dots\}$.
(2) Regular simple modules $\{R_\lambda\}_{\lambda \in \mathbb{P}^1(k)}$.
(3) Indecomposable preinjective modules $\{I_1, I_2, I_3, \dots\}$.

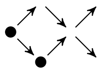
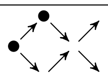
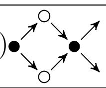
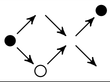
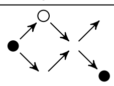
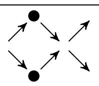
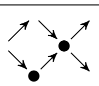
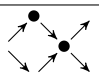
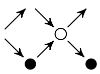
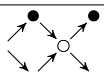
Here $P_1 = P(1), P_2 = P(2), P_3 = \tau^- P_1, P_4 = \tau^- P_2, P_5 = \tau^- P_3, \dots$ and $I_1 = I(2), I_2 = I(1), I_3 = \tau I_1, I_4 = \tau I_2, \dots$, where $P(i)$ (resp. $I(i)$) is the indecomposable projective (resp. injective) module corresponding to the vertex i, and τ is the Auslander-Reiten translation.

To classify monobricks over kQ, we need to know the lists of pairs (B_1, B_2) of bricks such that there is a non-zero non-injection from B_1 to B_2, and pairs such that there is an injection but no non-zero non-injection from B_1 to B_2. This is summarized in Figure 1, where $B_1 \rightsquigarrow B_2$ (resp. $B_1 \hookrightarrow B_2$) indicates that there is a non-zero non-injection $B_1 \to B_2$ (resp. an injection).

TABLE 3. The Auslander-Reiten quiver of $\mathsf{mod}\, k[1 \to 2 \leftarrow 3]$

$$\begin{tikzcd} & {\substack{3\\2}} \arrow[dr] & & 1 \\ 2 \arrow[ur] \arrow[dr] & & {\substack{1\,3\\2}} \arrow[ur] \arrow[dr] & \\ & {\substack{1\\2}} \arrow[ur] & & 3 \end{tikzcd}$$

TABLE 4. Monobricks $\mathcal{M}$ over $k[1 \to 2 \leftarrow 3]$ with $\#\mathcal{M} \geq 2$

$\mathcal{M}$	$\mathcal{E}$	$\max \mathcal{M}$	$\overline{\mathcal{M}}$
$2 < \substack{1\\2}$		$\substack{1\\2}$	itself
$2 < \substack{3\\2}$		$\substack{3\\2}$	itself
$2 < \substack{1\,3\\2}$	$(*)$ 	$\substack{1\,3\\2}$	$\mathcal{N}$
$2, 1$		itself	itself
$2, 3$		itself	itself
$\substack{1\\2}, \substack{3\\2}$		itself	$\substack{3\\2} > 2 < \substack{1\\2}$
$\substack{1\\2} < \substack{1\,3\\2}$		$\substack{1\,3\\2}$	itself
$\substack{3\\2} < \substack{1\,3\\2}$		$\substack{1\,3\\2}$	itself
$\substack{1\\2}, 3$		itself	$2 < \substack{1\\2}, 3$
$\substack{3\\2}, 1$		itself	$2 < \substack{3\\2}, 1$

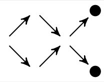
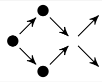
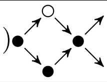
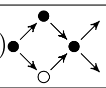
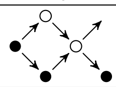
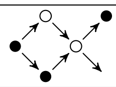
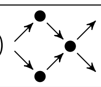
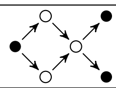
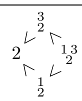
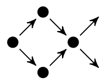
$\mathcal{M}$	$\mathcal{E}$	$\max \mathcal{M}$	$\overline{\mathcal{M}}$
$1, 3$		itself	itself
$\substack{3\\2} > 2 < \substack{1\\2}$		$\substack{1\\2}, \substack{3\\2}$	itself
$2 < \substack{1\\2} < \substack{1\,3\\2}$	$(*)$ 	$\substack{1\,3\\2}$	$\mathcal{N}$
$2 < \substack{3\\2} < \substack{1\,3\\2}$	$(*)$ 	$\substack{1\,3\\2}$	$\mathcal{N}$
$2 < \substack{1\\2}, 3$		$\substack{1\\2}, 3$	itself
$2 < \substack{3\\2}, 1$		$\substack{3\\2}, 1$	itself
$\substack{3\\2} < \substack{1\,3\\2} > \substack{1\\2}$	$(**)$ 	$\substack{1\,3\\2}$	$\mathcal{N}$
$1, 2, 3$		itself	itself
$\mathcal{N} =$ 		$\substack{1\,3\\2}$	itself

TABLE 5. Monobricks over cyclic Nakayama algebras with 2 simples

$\mathcal{M}$	$\max \mathcal{M}$	$\overline{\mathcal{M}}$
$\varnothing$	itself	itself
1	itself	itself
2	itself	itself
$\substack{1\\2}$	itself	$2 < \substack{1\\2}$

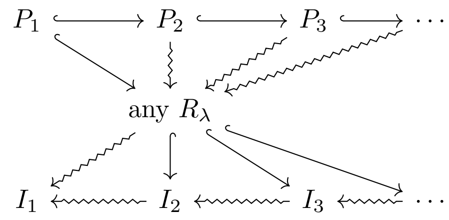
$\mathcal{M}$	$\max \mathcal{M}$	$\overline{\mathcal{M}}$
$\substack{2\\1}$	itself	$1 < \substack{2\\1}$
$1 < \substack{2\\1}$	$\substack{2\\1}$	itself
$2 < \substack{1\\2}$	$\substack{1\\2}$	itself
$1, 2$	itself	itself

Any other pairs can be deduced from the composition of arrows in Figure 1. Since there are lots of monobricks, we only consider cofinally closed monobricks. This is enough for classifying monobricks since a set of bricks is a monobrick if and only if it is a subset of some cofinally closed monobricks by Corollary 5.3.17.

The following are the list of all cofinally closed monobricks, or the list of simple objects in all torsion-free classes.

(M1) $\{P_1, P_2, \ldots, P_i\}$ for some i.

FIGURE 1. Structures of bricks in $\mathsf{mod}\, k[1 \leftleftarrows 2]$



(M2) $\{P_1, P_2, P_3, \dots\}$.
(M3) $\{P_1\} \cup \{R_\lambda\}_{\lambda \in X}$ for any non-empty subset $X \subset \mathbb{P}^1(k)$.
(M4) $\{P_1\} \cup \{R_\lambda\}_{\lambda \in \mathbb{P}^1(k)} \cup \{I_i\}$ for $i \geq 2$.
(M5) $\{I_1\}$.

In this list, finite monobricks are (M1), (M3) for a finite set X and (M5). The poset structure is as follows.

$(M1)$	$(M2)$	$(M3)$	$(M4)$	$(M5)$
$P_1 < P_2 < \cdots < P_i$	$P_1 < P_2 < \cdots$	$P_1 \hookrightarrow \boxed{R_\lambda \ \vdots \ R_{\lambda'}}_X$	$P_1 \hookrightarrow \boxed{R_\lambda \ \vdots \ R_{\lambda'}}_{\mathbb{P}^1(k)} \hookrightarrow I_i$	I_1

TABLE 6. The poset structure of each monobrick

Using this, we can easily compute $\mathsf{W}(\mathcal{F})$ for each torsion-free class, since $\mathsf{W}(\mathcal{F})$ is equal to $\mathsf{Filt}(\mathsf{max}(\mathsf{sim}\,\mathcal{F}))$ by Theorem 5.4.5. Moreover, $\mathsf{max}(\mathsf{sim}\,\mathcal{F})$ is nothing but the brick labels starting at $\mathcal{F}$ (Remark 5.4.6), we can compute the brick labels (c.f. [DIRRT, Example 3.6]). This can be summarized as follows.

$(M1)$	$(M2)$	$(M3)$	$(M4), (M5)$
$\{P_i\}$	$\varnothing$	$\{R_\lambda\}_{\lambda \in X}$	$\{I_i\}$

TABLE 7. The maximal element of each monobrick, or all the semibricks

Since $\mathsf{max}\colon \mathsf{mbrick}_{\mathsf{c.c.}}\,\Lambda \to \mathsf{sbrick}\,\Lambda$ is surjective by Proposition 5.5.1, this table can also be seen as a table of all semibricks.

We remark that if X consists of one element in (M3), then the monobrick is isomorphic to $P_1 < P_2$ as posets, although the former corresponds to non-functorially finite torsion-free class but the latter to functorially finite.

CHAPTER 6

Rigid modules and ICE-closed subcategories in quiver representations

This chapter is based on [Eno7].

We introduce image-cokernel-extension-closed (ICE-closed) subcategories of module categories. This class unifies both torsion classes and wide subcategories. We show that ICE-closed subcategories over the path algebra of Dynkin type are in bijection with basic rigid modules, and that the number does not depend on the orientation of the quiver. We give an explicit formula of this number for each Dynkin type, and in particular, it is equal to the large Schröder number for type A case.

6.1. Introduction

Let Λ be an artin algebra and $\mathsf{mod}\,\Lambda$ the category of finitely generated right Λ-modules. There are several kinds of subcategories of $\mathsf{mod}\,\Lambda$ which have been investigated in the representation theory of algebras, e.g. *wide* subcategories, *torsion classes*, *torsion-free classes*, and so on. These subcategories are defined by the property that they are closed under certain operations: e.g. taking kernels, cokernels, images, extensions, submodules, or quotients.

In this paper, we propose a new class of subcategories of $\mathsf{mod}\,\Lambda$, *ICE-closed* subcategories, which is a subcategory closed under Images, Cokernels and Extensions. Typical examples of ICE-closed subcategories are torsion classes and wide subcategories, but there are more than them.

Recently, there are lots of studies on the classification of *nice* subcategories in terms of *nice* modules. One of the most prominent results is the τ*-tilting theory* established in [AIR], which gives a bijection between functorially finite torsion classes and certain class of modules called *support* τ*-tilting modules*. Actually, it is a generalization of the *Ingalls-Thomas bijection* [IT], which classifies functorially finite torsion classes over hereditary algebras by *support tilting* modules.

The aim of this paper is to provide a similar classification of ICE-closed subcategories over hereditary algebras. More precisely, we show that such subcategories are in bijection with *rigid modules*, modules without self-extensions. The main result is summarized as follows:

THEOREM R (= Theorem 6.2.3). *Let Q be a Dynkin quiver. Then there is a bijection between the following two sets:*

(1) *The set of ICE-closed subcategories of* $\mathsf{mod}\,kQ$.
(2) *The set of isomorphism classes of basic rigid kQ-modules.*

For an ICE-closed subcategory $\mathcal{C}$, the corresponding rigid kQ-module is the basic Ext*-projective generator $P(\mathcal{C})$ of $\mathcal{C}$, and for a rigid kQ-module U, the corresponding ICE-closed subcategory is given by the category* $\mathsf{cok}\,U$ *consisting of cokernels of maps in* $\mathsf{add}\,U$.

In addition, we show that a subcategory closed under cokernels and extensions is automatically closed under images, thus is ICE-closed. Actually, this theorem holds for any representation-finite hereditary artin algebras, and can be generalized to representation-infinite case by restricting the class of ICE-closed subcategories, see Theorem 6.2.3 for the precise statement.

In the appendix, we will count the number of ICE-closed subcategories by using several results in other papers. The result is summarized as follows.

THEOREM S. *Let Q be a Dynkin quiver. Then the number of ICE-closed subcategories only depends on the underlying Dynkin graph, not on the choice of an orientation (Theorem 6.A.3).*

Moreover, we have an explicit formula of this number for each Dynkin type, and if Q is of type A_n, then it is equal to the n-th large Schröder number (Corollay 6.A.15).

We expect that there is a hidden theory which generalizes this paper to non-hereditary case, as [AIR] generalizes [IT].

Organization. This paper is organized as follows. In Section 6.2, we give basic definitions and state a main result Theorem 6.2.3. In Section 6.3, we give a proof of Theorem 6.2.3. In Section 6.4, we consider the relation between ICE-closed subcategories, torsion classes and wide subcategories in detail via rigid modules. In the appendix, we count the number of ICE-closed subcategories for each Dynkin type.

Conventions and notation. Throughout this paper, *all subcategories are assumed to be full and closed under isomorphisms, direct sums and direct summands.* An *artin R-algebra* is an R-algebra over a commutative artinian ring R which is finitely generated as an R-module. We often omit the base ring R, and simply call it an *artin algebra*.

For an artin algebra Λ, we denote by $\mathsf{mod}\,\Lambda$ the category of finitely generated right Λ-modules. All modules are finitely generated right modules unless otherwise stated. For a collection $\mathcal{C}$ of Λ-modules, we denote by $\mathsf{add}\,\mathcal{C}$ the subcategory of $\mathsf{mod}\,\Lambda$ consisting of direct summands of finite direct sums of objects in $\mathcal{C}$. A module M is called *basic* if there is a decomposition $M = \bigoplus_{i=1}^{n} M_i$ such that each M_i is indecomposable and pair-wise non-isomorphic. For a module M, we denote by $|M|$ the number of non-isomorphic indecomposable direct summands of M.

6.2. Basic definitions and the main result

In this section, we give basic definitions and introduce some notation, and state our main result. First of all, recall that a module $M \in \mathsf{mod}\,\Lambda$ over an artin algebra Λ is *rigid* if $\operatorname{Ext}^1_\Lambda(M, M) = 0$ holds. Then we define several conditions on the subcategory of $\mathsf{mod}\,\Lambda$.

Definition 6.2.1. Let Λ be an artin algebra and $\mathcal{C}$ a subcategory of $\mathsf{mod}\,\Lambda$.

(1) $\mathcal{C}$ is *closed under extensions* if for every short exact sequence in $\mathsf{mod}\,\Lambda$

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0,$$

we have that $L, N \in \mathcal{C}$ implies $M \in \mathcal{C}$

(2) $\mathcal{C}$ is *closed under quotients (resp. submodules)* if for every short exact sequence in $\mathsf{mod}\,\Lambda$

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0,$$

we have that $M \in \mathcal{C}$ implies $N \in \mathcal{C}$ (resp. $L \in \mathcal{C}$).

(3) $\mathcal{C}$ is *closed under cokernels (resp. images)* if for every map $f \colon M \to N$ with $M, N \in \mathcal{C}$, we have $\operatorname{Coker} f \in \mathcal{C}$ (resp. $\operatorname{Im} f \in \mathcal{C}$).

(4) $\mathcal{C}$ is a *torsion class* if $\mathcal{C}$ is closed under quotients and extensions.

(5) $\mathcal{C}$ is a *wide subcategory* if $\mathcal{C}$ is closed under kernels, cokernels and extensions.

(6) $\mathcal{C}$ is *image-cokernel-extension-closed*, abbreviated by *ICE-closed*, if $\mathcal{C}$ is closed under images, cokernels and extensions.

(7) $\mathcal{C}$ is *cokernel-extension-closed*, abbreviated by *CE-closed*, if $\mathcal{C}$ is closed under cokernels and extensions.

Then clearly all torsion classes and wide subcategories are (I)CE-closed, thus ICE-closed subcategories can be seen as a generalization of these two classes.

To an extension-closed subcategory of $\mathsf{mod}\,\Lambda$, we can associate a rigid module by taking the Ext-*progenerator.*

Definition 6.2.2. Let Λ be an artin algebra and $\mathcal{C}$ an extension-closed subcategory of $\mathsf{mod}\,\Lambda$.

(1) An object $X \in \mathcal{C}$ is Ext-*projective in* $\mathcal{C}$ if $\operatorname{Ext}^1_\Lambda(X, \mathcal{C}) = 0$ holds. We denote by $\mathcal{P}(\mathcal{C})$ the subcategory of $\mathcal{C}$ consisting of all the Ext-projective objects in $\mathcal{C}$.

(2) $\mathcal{C}$ has *enough* Ext*-projectives* if for every object $X \in \mathcal{C}$, there is a short exact sequence

$$0 \longrightarrow Y \longrightarrow P \longrightarrow X \longrightarrow 0$$

with $P \in \mathcal{P}(\mathcal{C})$ and $Y \in \mathcal{C}$.

(3) An object $P \in \mathcal{C}$ is an Ext*-progenerator* if $\mathsf{add}\, P = \mathcal{P}(\mathcal{C})$ and $\mathcal{C}$ has enough Ext-projectives.

Now we are ready to state our main result. Throughout this paper, we will use the following notations for an artin algebra Λ.

- $\mathsf{rigid}\, \Lambda$ denotes the set of isomorphism classes of basic rigid Λ-modules.
- $\mathsf{ice}\, \Lambda$ denotes the set of ICE-closed subcategories of $\mathsf{mod}\, \Lambda$.
- $\mathsf{ice_p}\, \Lambda$ denotes the set of ICE-closed subcategories of $\mathsf{mod}\, \Lambda$ with enough Ext-projectives.
- For an extension-closed subcategory $\mathcal{C}$ of $\mathsf{mod}\, \Lambda$ with an Ext-progenerator, we denote by $P(\mathcal{C})$ the unique basic Ext-progenerator of $\mathcal{C}$.
- For a Λ-module U, we denote by $\mathsf{cok}\, U$ the subcategory of $\mathsf{mod}\, \Lambda$ consisting of cokernels of maps in $\mathsf{add}\, U$.
- For a collection $\mathcal{U}$ of Λ-modules, we denote by $\mathsf{Fac}\mathcal{U}$ (resp. $\mathsf{Sub}\mathcal{U}$) the subcategory of $\mathsf{mod}\, \Lambda$ consisting of quotients (resp. submodules) of objects in $\mathsf{add}\mathcal{U}$.

THEOREM 6.2.3. *Let Λ be a hereditary artin algebra. Then we have the following bijections*

$$\mathsf{rigid}\, \Lambda \underset{P}{\overset{\mathsf{cok}}{\rightleftarrows}} \mathsf{ice_p}\, \Lambda.$$

Moreover, if Λ is representation-finite, then every CE-closed subcategory is automatically ICE-closed, and $\mathsf{ice_p}\, \Lambda = \mathsf{ice}\, \Lambda$ holds.

This bijection extends a bijection between support (τ-)tilting modules and functorially finite torsion classes given in [IT] or [AIR] in the following sense. Let Λ be a hereditary artin algebra. It is known that a torsion class is functorially finite if and only if it has enough Ext-projectives. Then the following diagram commutes, and the horizontal maps are bijective.

$$\begin{tikzcd}
\mathsf{rigid}\, \Lambda \arrow[r, shift left, "\mathsf{cok}"] & \mathsf{ice_p}\, \Lambda \arrow[l, shift left, "P"] \\
\mathsf{stilt}\, \Lambda \arrow[u, hook] \arrow[r, shift left, "\mathsf{Fac}"] & \mathsf{f\text{-}tors}\, \Lambda \arrow[u, hook] \arrow[l, shift left, "P"]
\end{tikzcd}$$

Here the bottom bijections were those given in [IT] or [AIR]. See Proposition 6.4.4 for the detail.

REMARK 6.2.4. ICE-closed subcategories generalize both torsion classes and wide subcategories. Another generalization of these two classes was introduced in [Eno6], *right Schur subcategories* (actually the dual was studied in the paper). Every ICE-closed subcategory is a right Schur, but the converse does nod hold in general. If Λ is Nakayama, then these coincides by [Eno6, Theorem 6.1]. In [Eno6], we classify right Schur subcategories in any length abelian category by using *simple objects* in them. This is in contrast with our use of Ext-projectives.

6.3. Proof of the main theorem

In this section, we give a proof of Theorem 6.2.3. *Throughout this section, we denote by Λ a hereditary artin algebra.*

First we give a map $P(-)\colon \mathsf{ice_p}\, \Lambda \to \mathsf{rigid}\, \Lambda$.

PROPOSITION 6.3.1. *Let $\mathcal{C}$ be an CE-closed subcategory of $\mathsf{mod}\, \Lambda$. Then the following hold.*

(1) *Every* Ext*-projective object in $\mathcal{C}$ is rigid.*
(2) *There are only finitely many indecomposable* Ext*-projective objects in $\mathcal{C}$ up to isomorphism.*
(3) *If $\mathcal{C}$ has enough* Ext*-projectives, then it has an* Ext*-progenerator $P(\mathcal{C})$, and $\mathcal{C} = \mathsf{cok}\, P(\mathcal{C})$ holds.*

Proof. (1) Clear from definition.

(2) We claim that there are at most $|\Lambda|$ indecomposable Ext-projectives in $\mathcal{C}$. Let $M_1, \dots, M_m$ be pairwise non-isomorphic Ext-projectives in $\mathcal{C}$. Then clearly $M := M_1 \oplus \cdots \oplus M_m$ is basic rigid, or partial tilting since Λ is hereditary. Then by taking the Bongartz completion, there is a Λ-module N such that $M \oplus N$ is a basic tilting Λ-module (see [ASS, Lemma VI.2.4] for the detail). It follows that $m = |M| \le |M \oplus N| = |\Lambda|$.

(3) Since $\mathcal{C}$ has enough projectives, (2) implies that $\mathcal{C}$ has an Ext-progenerator $P(\mathcal{C})$. Since $\mathcal{C}$ is closed under cokernels and $P(\mathcal{C}) \in \mathcal{C}$, clearly $\mathcal{C} \supset \mathsf{cok}\, P(\mathcal{C})$ holds. Conversely, we have $\mathcal{C} \subset \mathsf{cok}\, P(\mathcal{C})$ since $\mathcal{C}$ has enough Ext-projectives. $\square$

The following lemma is essential in our proof. This says that $\mathsf{add}\, U$ *is closed under images if* Λ *is herediatry and* U *is rigid.*

Lemma 6.3.2. *Let Λ be a hereditary artin algebra and U a rigid Λ-module. Then $\mathsf{Fac}\, U \cap \mathsf{Sub}\, U = \mathsf{add}\, U$ holds.*

Proof. Clearly we have $\mathsf{add}\, U \subset \mathsf{Fac}\, U \cap \mathsf{Sub}\, U$. Conversely, let $X \in \mathsf{Fac}\, U \cap \mathsf{Sub}\, U$. Take a left $(\mathsf{add}\, U)$-approximation $\varphi \colon X \to U^X$ with $U^X \in \mathsf{add}\, U$, which is an injection by $X \in \mathsf{Sub}\, U$. Then we have the following commutative exact diagram in $\mathsf{mod}\, \Lambda$:

$$\begin{tikzcd} & U_0 \arrow[d, two heads] & & & \\ 0 \arrow[r] & X \arrow[r, "\varphi"] & U^X \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

By applying $\mathrm{Hom}(-, U)$, we obtain an exact sequence

$$\mathrm{Hom}_\Lambda(U^X, U) \xrightarrow{(-)\circ\varphi} \mathrm{Hom}_\Lambda(X, U) \longrightarrow \mathrm{Ext}^1_\Lambda(C, U) \longrightarrow \mathrm{Ext}^1_\Lambda(U^X, U).$$

Since φ is a left $(\mathsf{add}\, U)$-approximation, $(-)\circ\varphi$ is a surjection. In addition, $\mathrm{Ext}^1_\Lambda(U^X, U)$ vanishes since U is rigid, hence $\mathrm{Ext}^1_\Lambda(C, U) = 0$. On the other hand, since Λ is hereditary, we have an exact sequence

$$\mathrm{Ext}^1_\Lambda(C, U_0) \longrightarrow \mathrm{Ext}^1_\Lambda(C, X) \longrightarrow 0.$$

Since we have $\mathrm{Ext}^1_\Lambda(C, U_0) = 0$, we obtain $\mathrm{Ext}^1_\Lambda(C, X) = 0$. It follows that the short exact sequence $0 \to X \to U^X \to C \to 0$ splits, which implies $X \in \mathsf{add}\, U$. $\square$

Our next aim is to show that $\mathsf{cok}\, U$ is ICE-closed if U is rigid. We will make use of the subcategory $\mathcal{X}_U$ associated to U, which was introduced by Auslander-Reiten [AR2].

Definition 6.3.3. Let Λ be an artin algebra and U a Λ-module with $\mathrm{Ext}^{>0}_\Lambda(U, U) = 0$. Then we denote by $\mathcal{X}_U$ a subcategory of $\mathsf{mod}\, \Lambda$ consisting of modules X such that there is an exact sequence

$$\cdots \xrightarrow{f_2} U_1 \xrightarrow{f_1} U_0 \xrightarrow{f_0} X \to 0$$

with $\mathrm{Ext}^{>0}_\Lambda(U, \mathrm{Im}\, f_i) = 0$ for all $i \ge 0$.

We borrow the following lemma from [AR2].

Lemma 6.3.4 ([AR2, Proposition 5.1]). *Let Λ be an artin algebra and U a Λ-module with $\mathrm{Ext}^{>0}_\Lambda(U, U) = 0$. Then $\mathcal{X}_U$ is closed under extensions, has an* Ext*-progenerator U, and is closed under mono-cokernels, that is, for every short exact sequence*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

in $\mathsf{mod}\, \Lambda$, if L and M belong to $\mathcal{X}_U$, then so does N.

The following is basic properties of $\mathcal{X}_U$ in our hereditary setting. In particular, we have $\mathcal{X}_U = \mathsf{cok}\, U$ for a rigid module U over a hereditary algebra Λ.

PROPOSITION 6.3.5. *Let Λ be a hereditary artin algebra and U a rigid Λ-module.*

(1) *The following are equivalent for $X \in \mathsf{mod}\,\Lambda$.*
  (a) *X belongs to $\mathcal{X}_U$.*
  (b) *X belongs to $\mathsf{cok}\,U$.*
  (c) *There is an short exact sequence of the following form with U_1 and U_0 in $\mathsf{add}\,U$.*

$$0 \longrightarrow U_1 \longrightarrow U_0 \longrightarrow X \longrightarrow 0$$

(2) *$\mathsf{cok}\,U$ is an ICE-closed subcategory of $\mathsf{mod}\,\Lambda$.*
(3) *$\mathsf{cok}\,U$ has an Ext-progenerator U.*

PROOF. (1) Clearly (a) implies (b) by $\mathcal{X}_U \subset \mathsf{cok}\,U$. Also the implication (b) $\Rightarrow$ (c) follows immediately from Lemma 6.3.2 since Λ is hereditary and U is rigid, and Lemma 6.3.4 shows (c) $\Rightarrow$ (a).

(2) By Lemma 6.3.4, we only have to show that $\mathcal{X}_U = \mathsf{cok}\,U$ is closed under images, because this will immediately imply that $\mathcal{X}_U$ is closed under cokernels since $\mathcal{X}_U$ is closed under mono-cokernels.

Take any $\varphi\colon X \to Y$ with $X, Y \in \mathcal{X}_U$. Since $X \in \mathcal{X}_U \subset \mathsf{Fac}\,U$, we may assume $X \in \mathsf{add}\,U$ to show $\operatorname{Im}\varphi \in \mathcal{X}_U$. Since Y belongs to $\mathcal{X}_U$, there is a short exact sequence $0 \to U_1 \to U_0 \to Y \to 0$ with $U_1, U_0 \in \mathsf{add}\,U$ by (1). By taking pullback, we obtain the following exact commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & U_1 \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\text{p.b.}"] & X \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & U_1 \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\text{p.b.}"] & \operatorname{Im}\varphi \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & U_1 \arrow[r] & U_0 \arrow[r] & Y \arrow[r] & 0
\end{tikzcd}$$

Now we have $\mathrm{Ext}^1_\Lambda(X, U_1) = 0$ by $X \in \mathsf{add}\,U$, which implies that the top short exact sequence splits. Thus $E \cong U_1 \oplus X \in \mathsf{add}\,U$ and $Z \in \mathsf{Fac}\,U \cap \mathsf{Sub}\,U$ hold. Therefore we get $Z \in \mathsf{add}\,U$ by Lemma 6.3.2. Then the middle horizontal short exact sequence implies $\operatorname{Im}\varphi \in \mathcal{X}_U$.

(3) Obvious from the short exact sequence in (1)(a). $\square$

Now we are ready to prove Theorem 6.2.3.

PROOF OF THEOREM 6.2.3. Proposition 6.3.1 gives a map $P(-)\colon \mathsf{ice_p}\,\Lambda \to \mathsf{rigid}\,\Lambda$, and Proposition 6.3.5 gives a map $\mathsf{cok}\colon \mathsf{rigid}\,\Lambda \to \mathsf{ice_p}\,\Lambda$. These propositions also show that these maps are mutually inverse to each other.

Finally we prove statements for the representation-finite case. Let Λ be a representation-finite hereditary artin algebra, that is, $\mathsf{mod}\,\Lambda$ has finitely many indecomposables up to isomorphism. Then [Eno2, Corollary 3.15] implies that every subcategory of $\mathsf{mod}\,\Lambda$ closed under extensions has enough Ext-projectives. In particular, we have $\mathsf{ice}\,\Lambda = \mathsf{ice_p}\,\Lambda$. Moreover, if $\mathcal{C}$ is a CE-closed subcategory of $\mathsf{mod}\,\Lambda$, then Proposition 6.3.1 implies that $\mathcal{C} = \mathsf{cok}\,P(\mathcal{C})$ holds. Since $P(\mathcal{C})$ is rigid, $\mathcal{C}$ is automatically closed under images by Proposition 6.3.5. $\square$

6.4. Maps to torsion classes and wide subcategories

The class of ICE-closed subcategories contain both the classes of torsion classes and wide subcategories, so it is natural to ask the relation between these three classes. The aim of this section is to introduce two natural maps from the set of ICE-closed subcategories to the set of torsion classes and wide subcategories, and to investigate these maps via rigid modules.

Let us introduce some notation. For an artin algebra Λ, we denote by $\mathsf{tors}\,\Lambda$ (resp. $\mathsf{f\text{-}tors}\,\Lambda$) the set of torsion classes (resp. functorially finite torsion classes) in $\mathsf{mod}\,\Lambda$. Similarly, we denote by $\mathsf{wide}\,\Lambda$ (resp. $\mathsf{f\text{-}wide}\,\Lambda$) the set of wide subcategories (resp. functorially finite wide subcategories) in $\mathsf{mod}\,\Lambda$.

First, we construct two maps $\mathsf{T}\colon\ \mathsf{ice}\,\Lambda \twoheadrightarrow \mathsf{tors}\,\Lambda$ and $\mathsf{W}\colon\ \mathsf{ice}\,\Lambda \twoheadrightarrow \mathsf{wide}\,\Lambda$ which are the identities on $\mathsf{tors}\,\Lambda$ and $\mathsf{wide}\,\Lambda$ respectively.

Definition 6.4.1. Let Λ be an artin algebra and $\mathcal{C}$ an ICE-subcategory of $\mathsf{mod}\,\Lambda$.

(1) $\mathsf{T}(\mathcal{C})$ denotes the smallest torsion class containing $\mathcal{C}$.
(2) $\mathsf{W}(\mathcal{C})$ is a subcategory of $\mathcal{C}$ defined as follows:

$$\mathsf{W}(\mathcal{C}) = \{W \in \mathcal{C} \mid \operatorname{Ker}\varphi \in \mathcal{C} \text{ for any map } \varphi\colon C \to W \text{ with } C \in \mathcal{C}\}$$

Clearly $\mathsf{T}(\mathcal{T}) = \mathcal{T}$ for $\mathcal{T} \in \mathsf{tors}\,\Lambda$ and $\mathsf{W}(\mathcal{W}) = \mathcal{W}$ for $\mathcal{W} \in \mathsf{wide}\,\Lambda$. It is non-trivial that W actually defines a map $\mathsf{W}\colon \mathsf{ice}\,\Lambda \to \mathsf{wide}\,\Lambda$, as we shall see below.

Proposition 6.4.2. *Let Λ be an artin algebra and $\mathcal{C}$ an ICE-closed subcategory of $\mathsf{mod}\,\Lambda$. Then $\mathsf{W}(\mathcal{C})$ is a wide subcategory of $\mathsf{mod}\,\Lambda$.*

Proof. This a special case of [Eno6, Theorem 4.5], but here we will give a proof using the general result in [KaSc, Exercise 8.23]. According to it, we say that an object $X \in \mathsf{mod}\,\Lambda$ is *$\mathcal{C}$-coherent* if $X \in \mathsf{Fac}\,\mathcal{C}$ and $\operatorname{Ker}\varphi \in \mathcal{C}$ for every map $C \to X$ with $C \in \mathcal{C}$. Since $\mathcal{C}$ is closed under cokernels, it is easy to check that every $\mathcal{C}$-coherent object belongs to $\mathcal{C}$, namely, $\mathsf{W}(\mathcal{C})$ coincides with the category of $\mathcal{C}$-coherent objects. Then since $\mathcal{C}$ is extension-closed, [KaSc, Exercise 8.23] implies that $\mathsf{W}(\mathcal{C})$ is a wide subcategory of $\mathsf{mod}\,\Lambda$. $\square$

If Λ is hereditary, then T is equal to Fac, as the following general proposition shows.

Proposition 6.4.3 (c.f. [IT, Proposition 2.13]). *Let Λ be a hereditary artin algebra and $\mathcal{C}$ an extension-closed subcategory of $\mathsf{mod}\,\Lambda$. Then $\mathsf{Fac}\,\mathcal{C} = \mathsf{T}(\mathcal{C})$ holds, namely, $\mathsf{Fac}\,\mathcal{C}$ is a torsion class.*

Proof. We only have to show that $\mathsf{Fac}\,\mathcal{C}$ is closed under extensions. Take a short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

with $L, N \in \mathsf{Fac}\,\mathcal{C}$, and take surjections $\pi_L\colon C_L \twoheadrightarrow L$ and $\pi_N\colon C_N \twoheadrightarrow N$. Since Λ is hereditary, the induced map $\operatorname{Ext}^1_\Lambda(N, C_L) \to \operatorname{Ext}^1_\Lambda(N, L)$ is a surjection. Thus we have the following exact commutative diagram, where we in addition take pullback along π_N.

$$\begin{tikzcd}
0 \arrow[r] & C_L \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\text{p.b.}"] & C_N \arrow[r] \arrow[d, two heads, "\pi_N"] & 0 \\
0 \arrow[r] & C_L \arrow[r] \arrow[d, two heads, "\pi_L"'] \arrow[dr, phantom, "\text{p.o.}"] & E \arrow[r] \arrow[d, two heads] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Since $\mathcal{C}$ is closed under extensions, we have $F \in \mathcal{C}$. Thus we obtain $M \in \mathsf{Fac}\,\mathcal{C}$. $\square$

Next we will consider counterparts of the maps Fac and W in terms of rigid modules. Let Λ be a hereditary artin algebra. We denote by $\mathsf{stilt}\,\Lambda$ the set of isomorphism classes of basic support tilting Λ-modules. If U a rigid Λ-module, then $\mathsf{Fac}\,U$ is closed under extension by [AS, Proposition 5.5, Corollary 5.9], thus it is a torsion class. By the result of [IT] or [AIR], there is a unique basic support tilting module $\overline{U}$ satisfying $\mathsf{Fac}\,\overline{U} = \mathsf{Fac}\,U$. We call $\overline{U}$ the *co-Bongartz completion* of U.

Now the following proposition can immediately follows from definition, so we omit the proof.

Proposition 6.4.4. *Let Λ be a hereditary artin algebra. Then the following diagram commutes, and the dashed maps are given by taking the co-Bongartz completion.*

$$\begin{tikzcd}
\mathsf{stilt}\,\Lambda \arrow[r, shift left, "\mathsf{Fac}"] \arrow[d, hook] \arrow[dd, bend right=60, "1"'] & \mathsf{f\text{-}tors}\,\Lambda \arrow[l, shift left, "P"] \arrow[d, hook] \arrow[dd, bend left=60, "1"] \\
\mathsf{rigid}\,\Lambda \arrow[r, shift left, "\mathsf{cok}"] \arrow[d, dashed, "\overline{(-)}"'] \arrow[dr, "\mathsf{Fac}"] & \mathsf{ice_p}\,\Lambda \arrow[l, shift left, "P"] \arrow[d, two heads, "\mathsf{Fac}"] \\
\mathsf{stilt}\,\Lambda \arrow[r, shift left, "\mathsf{Fac}"] & \mathsf{f\text{-}tors}\,\Lambda \arrow[l, shift left, "P"]
\end{tikzcd}$$

Next we will investigate the map $\mathsf{W}\colon \mathsf{ice_p}\,\Lambda \to \mathsf{wide}\,\Lambda$, which is more non-trivial than T. To do this, we introduce the Fac-*minimality*, *covers* and *split projectives*, following [AS].

DEFINITION 6.4.5. Let $\mathcal{P}$ be a subcategory of $\mathsf{mod}\,\Lambda$. We say that $\mathcal{P}$ is Fac-*minimal* if there is no proper subcategory $\mathcal{P}'$ of $\mathcal{P}$ satisfying $\mathcal{P} \subset \mathsf{Fac}\,\mathcal{P}'$. We say that $U \in \mathsf{mod}\,\Lambda$ is Fac-*minimal* if $\mathsf{add}\,U$ is Fac-minimal.

Note that *subcategories* are required to be closed under direct sums and direct summands. If $U \in \mathsf{mod}\,\Lambda$ is basic and $U = \bigoplus_{i \in I} U_i$ with each U_i indecomposable, then U is Fac-minimal if and only if there is no proper subset J of I satisfying $U \in \mathsf{Fac}(\bigoplus_{j \in J} U_j)$. Fac-minimal basic modules are called *covering-indecomposable modules* in [AS].

DEFINITION 6.4.6. Let $\mathcal{C}$ be a subcategory of $\mathsf{mod}\,\Lambda$. Then an object P in $\mathcal{C}$ is *split projective* if every surjection $C \twoheadrightarrow P$ in $\mathsf{mod}\,\Lambda$ with $C \in \mathcal{C}$ splits. We denote by $\mathcal{P}_0(\mathcal{C})$ the subcategory of $\mathcal{C}$ consisting of all split projective objects in $\mathcal{C}$.

It can be shown $\mathcal{P}_0(\mathcal{C})$ is closed under direct sums and direct summands. Clearly we have the inclusion $\mathcal{P}_0(\mathcal{C}) \subset \mathcal{P}(\mathcal{C})$ for an extension-closed subcategory $\mathcal{C}$ of $\mathsf{mod}\,\Lambda$.

Next we recall the notion of *covers* of a category, introduced in [AS].

DEFINITION 6.4.7. Let $\mathcal{C}$ be a subcategory of $\mathsf{mod}\,\Lambda$ and $\mathcal{P}$ a subcategory of $\mathcal{C}$.

(1) $\mathcal{P}$ is a *cover of* $\mathcal{C}$ if $\mathcal{C} \subset \mathsf{Fac}\,\mathcal{P}$ holds.
(2) $\mathcal{P}$ is a *minimal cover of* $\mathcal{C}$ if $\mathcal{P}$ is a cover of $\mathcal{C}$ and there is no proper subcategory $\mathcal{P}'$ of $\mathcal{P}$ which is a cover of $\mathcal{C}$.
(3) An object P in $\mathcal{C}$ is a (minimal) cover of $\mathcal{C}$ if so is $\mathsf{add}\,P$.
(4) $\mathcal{C}$ has a *finite (minimal) cover* if there is an object P in $\mathcal{C}$ which is a (minimal) cover of $\mathcal{C}$.

We will use some results in [AS] summarized as follows.

PROPOSITION 6.4.8 ([AS, Theorem 2.3, Corollary 2.4]). *Let Λ be an artin algebra and $\mathcal{C}$ a subcategory of $\mathsf{mod}\,\Lambda$. Then the following hold.*

(1) *Let $\mathcal{P}$ be a cover of $\mathcal{C}$. Then $\mathcal{P}$ is a minimal cover of $\mathcal{C}$ if and only if $\mathcal{P}$ is Fac-minimal if and only if $\mathcal{P} = \mathcal{P}_0(\mathcal{C})$. In particular, the minimal cover of $\mathcal{C}$ is unique if it exists.*
(2) *If $\mathcal{C}$ has a finite cover, then $\mathcal{C}$ has a finite minimal cover. Thus there is a unique basic Fac-minimal cover P of $\mathcal{C}$ up to isomorphism, which satisfies $\mathcal{P}_0(\mathcal{C}) = \mathsf{add}\,P$.*

By using this, we can define the following operation which yields a Fac-minimal basic module.

DEFINITION 6.4.9. Let Λ be an artin algebra and $M \in \mathsf{mod}\,\Lambda$. Then the Fac-*minimal version of* M is a basic Fac-minimal cover M_0 of $\mathsf{add}\,M$, which is unique up to isomorphism by Proposition 6.4.8.

If M is a cover of a subcategory $\mathcal{C}$ of $\mathsf{mod}\,\Lambda$, then Proposition 6.4.8 implies that its Fac-minimal version M_0 satisfies $\mathsf{add}\,M_0 = \mathcal{P}_0(\mathcal{C})$.

Now we will use the following general observation when Ext-projectives coincides with split projectives, which is of interest in its own.

PROPOSITION 6.4.10. *Let Λ be an artin algebra and $\mathcal{C}$ an extension-closed subcategory of $\mathsf{mod}\,\Lambda$ with enough* Ext*-projectives. Then the following are equivalent:*

(1) *$\mathcal{P}_0(\mathcal{C}) = \mathcal{P}(\mathcal{C})$ holds, that is, every* Ext*-projective object in $\mathcal{C}$ is split projective in $\mathcal{C}$.*
(2) *$\mathcal{P}(\mathcal{C})$ is Fac-minimal.*
(3) *$\mathcal{C}$ is closed under epi-kernels, that is, for every short exact sequence*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0,$$

if M and N belong to $\mathcal{C}$, then so does L.

PROOF. (1) $\Leftrightarrow$ (2): Immediate from Proposition 6.4.8, since $\mathcal{P}(\mathcal{C})$ is a cover of $\mathcal{C}$ by the enough Ext-projectivity of $\mathcal{C}$.

(1) $\Rightarrow$ (3): Suppose that we have a short exact sequence $0 \to L \to M \to N \to 0$ with $M, N \in \mathcal{C}$. Since $\mathcal{C}$ has enough Ext-projectives, there exists a short exact sequence $0 \to N' \to$

$P \to N \to 0$ with $P \in \mathcal{P}(\mathcal{C})$ and $N' \in \mathcal{C}$. Then by taking the pullback, we obtain the following exact commutative diagram.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & N' \arrow[r, equal] \arrow[d] & N' \arrow[d] & \\
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & E \arrow[r, "\varphi"] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $\mathcal{C}$ is extension-closed, the middle vertical exact sequence implies $E \in \mathcal{C}$. Since $P \in \mathcal{P}(\mathcal{C})$, it is split projective by (1). Thus φ splits, hence L is a direct summand of E. This implies $L \in \mathcal{C}$ since a subcategory $\mathcal{C}$ is assumed to be closed under direct summands.

(3) $\Rightarrow$ (1): Let P be an Ext-projective object in $\mathcal{C}$, and take any surjection $\pi\colon C \twoheadrightarrow P$ with $C \in \mathcal{C}$. Then we have a short exact sequence $0 \to \operatorname{Ker}\pi \to C \to P \to 0$, thus $\operatorname{Ker}\pi \in \mathcal{C}$ holds by (3). Therefore, since P is Ext-projective, this short exact sequence must split. This shows that P is split projective. $\square$

Now next we consider functorial finiteness of wide subcategories, which is of interest in its own. In particular, we will show that the functorial finiteness is equivalent to the existence of finite (co)cover, and to the contravariantly (covariantly) finiteness.

The following describes the relation between finite covers and covariantly finiteness.

Lemma 6.4.11 ([AS, Theorem 4.5, Proposition 3.7]). *Let Λ be an artin algebra and $\mathcal{C}$ a subcategory of $\mathsf{mod}\,\Lambda$ closed under images. Then the following are equivalent.*

(1) *$\mathcal{C}$ is covariantly finite.*
(2) *$\mathcal{C}$ has a finite cover.*

In this case, let $\Lambda \to C^{\Lambda}$ be a left minimal $\mathcal{C}$-approximation. Then $\mathcal{P}_0(\mathcal{C}) = \mathsf{add}\, C^{\Lambda}$ holds.

By using this, we obtain the following characterization of functorially finite wide subcategories.

Proposition 6.4.12. *Let Λ be an artin algebra and $\mathcal{W}$ a wide subcategory of $\mathsf{mod}\,\Lambda$. Then the following are equivalent.*

(1) *$\mathcal{W}$ is functorially finite.*
(2) *$\mathcal{W}$ is covariantly finite.*
(2)$'$ *$\mathcal{W}$ is contravariantly finite.*
(3) *$\mathcal{W}$ has an* Ext*-progenerator.*
(4) *$\mathcal{W}$ is equivalent to $\mathsf{mod}\,\Gamma$ for some artin algebra Γ.*

Proof. Note that $\mathcal{W}$ is closed under images, so we can apply Lemma 6.4.11.

(1) $\Rightarrow$ (2), (2)$'$: Trivial.

(2) $\Rightarrow$ (3): Take a left minimal $\mathcal{W}$-approximation $\Lambda \to P$ with P in $\mathcal{W}$. Then P is a minimal cover of Λ with $\mathsf{add}\, P = \mathcal{P}_0(\mathcal{W})$ by Lemma 6.4.11. We claim that P is an Ext-progenerator of $\mathcal{C}$. First, P is Ext-projective in $\mathcal{W}$ since it is split projective. Second, since P covers $\mathcal{W}$ and $\mathcal{W}$ is closed under kernels, for every object W in $\mathcal{W}$, there is a short exact sequence $0 \to W' \to P' \to W \to 0$ with $P' \in \mathsf{add}\, P$ and $W' \in \mathcal{W}$. This shows that $\mathcal{W}$ has enough Ext-projectives with $\mathcal{P}(\mathcal{C}) = \mathsf{add}\, P = \mathcal{P}_0(\mathcal{C})$.

(3) $\Rightarrow$ (4): Let P be an Ext-progenerator of $\mathcal{W}$. Define $\Gamma := \operatorname{End}_\Lambda(P)$ and consider the functor $\operatorname{Hom}_\Lambda(P, -)\colon \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\Gamma$. Then it easy to see that this induces an equivalence $\mathcal{W} \simeq \mathsf{mod}\,\Gamma$ by the standard argument in the Morita theory.

(4) $\Rightarrow$ (2): Let $F\colon \mathsf{mod}\,\Gamma \simeq \mathcal{W}$ be an equivalence. Then the composition $\mathsf{mod}\,\Gamma \xrightarrow{F} \mathcal{W} \hookrightarrow \mathsf{mod}\,\Lambda$ is exact since $\mathcal{W}$ is closed under kernels and cokernels in $\mathsf{mod}\,\Lambda$. Thus $\mathsf{add}(F\Gamma)$ is a cover of $\mathcal{W}$, since Γ is a cover of $\mathsf{mod}\,\Gamma$ and F preserves surjectivity. This implies that $\mathcal{W}$ is covariantly finite by Lemma 6.4.11.

By the dual argument, (4) implies (2)$'$. Thus (4) implies (1). $\square$

Now let us return to the hereditary setting. The key observation is the following.

Lemma 6.4.13. *Let Λ be a hereditary artin algebra and $\mathcal{C}$ an ICE-closed subcategory of $\mathsf{mod}\,\Lambda$ with enough* Ext*-projectives. Let P be an object in $\mathcal{P}_0(\mathcal{C})$ and Q a submodule of P satisfying $Q \in \mathcal{C}$. Then Q is also in $\mathcal{P}_0(\mathcal{C})$.*

Proof. Since there is an Ext-progenerator of $\mathcal{C}$ by Proposition 6.3.1, the category $\mathcal{C}$ has a finite cover, thus has a minimal cover $\mathcal{P}_0(\mathcal{C})$ by Proposition 6.4.8. Therefore there is a surjection $\pi\colon P_0 \twoheadrightarrow Q$ with $P_0 \in \mathcal{P}_0(\mathcal{C})$, hence we obtain a short exact sequence $0 \to \operatorname{Ker}\pi \to P_0 \xrightarrow{\pi} Q \to 0$.

Now since Λ is hereditary, the map $\operatorname{Ext}^1_\Lambda(P, \operatorname{Ker}\pi) \to \operatorname{Ext}^1_\Lambda(Q, \operatorname{Ker}\pi)$ induced by the inclusion $Q \hookrightarrow P$ is surjective. Thus we obtain the following exact commutative diagram.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Ker}\pi \arrow[r] \arrow[d, equal] & P_0 \arrow[r, "\pi"] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Ker}\pi \arrow[r] & E \arrow[r, "p"] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
 & & P/Q \arrow[r, equal] \arrow[d] & P/Q \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $\mathcal{C}$ is closed under cokernels, we have $P/Q \in \mathcal{C}$. Then the middle vertical exact sequence implies $E \in \mathcal{C}$ since $\mathcal{C}$ is extension-closed. Now p should split since P is split projective in $\mathcal{C}$, thus the middle horizontal short exact sequence splits. It follows that so does the top horizontal sequence, hence Q is a direct summand of P_0. Therefore $Q \in \mathcal{P}_0(\mathcal{C})$ holds. $\square$

By using this, we can show the following last result in this section on the relation between Fac-minimal rigid modules and wide subcategories. We denote by $\mathsf{rigid}_0\,\Lambda$ the set of isomorphism classes of basic rigid Λ-modules which are Fac-minimal.

Proposition 6.4.14. *Let Λ be a hereditary artin algebra. Then the following diagram commutes and the horizontal maps are bijections, where the map $(-)_0$ is given by taking the* Fac*-minimal version.*

$$\begin{tikzcd}
\mathsf{rigid}_0\,\Lambda \arrow[r, shift left, "\mathsf{cok}"] \arrow[d, hook] \arrow[dd, bend right=60, "1"'] & \mathsf{f\text{-}wide}\,\Lambda \arrow[l, shift left, "P"] \arrow[d, hook] \arrow[dd, bend left=60, "1"] \\
\mathsf{rigid}\,\Lambda \arrow[r, shift left, "\mathsf{cok}"] \arrow[d, two heads, "(-)_0"'] & \mathsf{ice_p}\,\Lambda \arrow[l, shift left, "P"] \arrow[d, two heads, "\mathsf{W}"] \\
\mathsf{rigid}_0\,\Lambda \arrow[r, shift left, "\mathsf{cok}"] & \mathsf{f\text{-}wide}\,\Lambda \arrow[l, shift left, "P"]
\end{tikzcd}$$

In particular, functorially finite wide subcategories are in bijection with basic Fac*-minimal rigid modules.*

Proof. It suffices to show the following for $U \in \mathsf{rigid}\,\Lambda$ by Theorem 6.2.3.

(1) $\mathsf{cok}\,U$ is wide if and only if U is Fac-minimal.
(2) U_0 is an Ext-progenerator of $\mathsf{W}(\mathsf{cok}\,U)$, where U_0 is the Fac-minimal version of U.

By Theorem 6.2.3, the category $\mathsf{cok}\, U$ is an ICE-closed subcategory of $\mathsf{mod}\,\Lambda$ with $\mathcal{P}(\mathsf{cok}\, U) = \mathsf{add}\, U$.

(1) It is easy to check that an ICE-subcategory of $\mathsf{mod}\,\Lambda$ is wide if and only if it is closed under epi-kernels. Thus the assertion follows from Proposition 6.4.10.

(2) Put $\mathcal{C} := \mathsf{cok}\, U$. Then $\mathcal{P}_0(\mathcal{C}) = \mathsf{add}\, U_0$ holds by Proposition 6.4.8. First we show $\mathcal{P}_0(\mathcal{C}) \subset \mathsf{W}(\mathcal{C})$. It suffices to show that every map $\varphi\colon C \to P_0$ with $C \in \mathcal{C}$ and $P_0 \in \mathcal{P}_0(\mathcal{C})$ satisfies $\operatorname{Ker}\varphi \in \mathcal{C}$. Since $\mathcal{C}$ is closed under images, we have $\operatorname{Im}\varphi \in \mathcal{C}$, and then Lemma 6.4.13 shows that $\operatorname{Im}\varphi$ is split projective in $\mathcal{C}$. Thus the induced surjection $C \twoheadrightarrow \operatorname{Im}\varphi$ splits, hence $\operatorname{Ker}\varphi$ is a direct summand of C. Thus $\operatorname{Ker}\varphi \in \mathcal{C}$ holds.

Now we have shown $U_0 \in \mathsf{W}(\mathcal{C})$. Moreover, U_0 is a cover of $\mathsf{W}(\mathcal{C})$ since it is a cover of $\mathcal{C}$, and U_0 is split projective in $\mathsf{W}(\mathcal{C})$ since it is so in $\mathcal{C}$. Then it is easy to see that U_0 is an Ext-progenerator of $\mathsf{W}(\mathcal{C})$ because $\mathsf{W}(\mathcal{C})$ is closed under kernels. □

It was shown in [IT, Corollary 2.17] that the maps $\mathsf{W}\colon \mathsf{f\text{-}tors}\,\Lambda \rightleftarrows \mathsf{f\text{-}wide}\,\Lambda\colon \mathsf{Fac}$ are mutually inverse bijections (although their definition of $\mathsf{f\text{-}wide}\,\Lambda$ is a bit different from ours). For the convenience of the reader, we give a short proof of this in our context.

Corollary 6.4.15. *Let Λ be a hereditary artin algebra. Then* $\mathsf{W}\colon \mathsf{f\text{-}tors}\,\Lambda \rightleftarrows \mathsf{f\text{-}wide}\,\Lambda\colon \mathsf{Fac}$ *are mutually inverse bijections between the sets of functorially finite torsion classes and functorially finite wide subcategories.*

Proof. Let U be a basic rigid Λ-module. According to Propositions 6.4.4 and 6.4.14, it suffices to show the following claims.

(1) If U is support tilting, then $U = P(\mathsf{Fac}\, U_0)$ holds, where U_0 is a Fac-minimal version of U.
(2) If U is Fac-minimal, then U is a Fac-minimal version of $P(\mathsf{Fac}\, U)$.

(1) By the definition of the Fac-minimal version, we have $\mathsf{Fac}\, U_0 = \mathsf{Fac}\, U$. Thus the claim follows from $U = P(\mathsf{Fac}\, U)$, which holds by Theorem 6.2.3.

(2) Since U is Fac-minimal, U is a minimal cover of $\mathsf{Fac}\, U$. Since $P(\mathsf{Fac}\, U)$ is a cover of $\mathsf{Fac}\, U$, its Fac-minimal version coincides with U. □

6.A. Enumerative results

Throughout this appendix, we denote by k a field. In this appendix, we give an explicit formula of the number $\#\,\mathsf{rigid}^i(kQ)$ of rigid kQ-modules with i non-isomorphic direct summands for a Dynkin quiver Q.

In [MRZ, Proposition 6.1], it was shown that this number does not depend on the orientation of Q, but the proof therein relies heavily on cluster combinatorics. In the first subsection, we give a short homological proof of this fact. In the second subsection, we give an explicit formula of $\#\,\mathsf{rigid}^i(kQ)$ by using the enumerative result on cluster complexes in [Krat].

Let us introduce the set which we want to enumerate in this appendix.

Definition 6.A.1. Let Λ be an artin algebra. For a non-negative integer i, we denote by $\mathsf{rigid}^i\,\Lambda$ the set of isomorphism classes of basic Λ-modules U satisfying $|U| = i$.

Note that if Λ is hereditary, then $\mathsf{rigid}^i\,\Lambda = \varnothing$ unless $0 \le i \le |\Lambda|$ by considering the Bongartz completion.

6.A.1. Invariance under sink mutation. Let us begin with recalling the *mutation* of a quiver at a sink.

Definition 6.A.2. Let Q be a quiver. A *sink* of Q is a vertex v such that there is no arrow starting at v. For a sink v of Q, the *sink mutation* $\mu_v Q$ is a new quiver obtained from Q by reversing all arrows which end at v.

Note that v becomes a source in $\mu_v Q$. It is well-known that if two quivers Q and Q' have the same underlying graph which is a tree, then there is a sequence of sink mutations which transforms Q into Q'.

Now the following is the main result in this subsection.

THEOREM 6.A.3. *Let Q be an acylic quiver with n vertices and v a sink of Q. Then for any $0 \leq i \leq n$, there is a bijection between $\mathsf{rigid}^i(kQ)$ and $\mathsf{rigid}^i(k(\mu_v Q))$.*

This result immediately yields the following corollary, since any two Dynkin quivers with the same underlying graph can be connected via a series of sink mutations.

COROLLARY 6.A.4 (c.f. [MRZ, Proposition 6.1]). *Let Q be a Dynkin quiver. Then $\# \mathsf{rigid}^i(kQ)$ only depends on the underlying Dynkin graph, not on the choice of an orientation.*

In the rest of this subsection, we will give a proof of Theorem 6.A.3.

DEFINITION 6.A.5. Let Q be an acylic quiver and v a sink or a source of Q.

(1) We denote by $\mathsf{mod}_v\, kQ$ the subcategory consisting of kQ-modules which do *not* contain $S(v)$ as a direct summand, where $S(v)$ is the simple module corresponding to v.
(2) For a kQ-module M, we denote by M_v a unique module in $\mathsf{mod}_v\, kQ$ such that there is a following decomposition for some $n \geq 0$.
$$M \cong M_v \oplus S(v)^{\oplus n}$$
(3) We define $\mathsf{rigid}^i_v(kQ) := \mathsf{rigid}^i(kQ) \cap \mathsf{mod}_v(kQ)$, that is, the set of basic rigid kQ-modules M with $|M| = i$ such that M *does not* contain $S(v)$ as an direct summand.
(4) We define $\mathsf{rigid}^i_{\langle v \rangle}(kQ) := \mathsf{rigid}^i(kQ) \setminus \mathsf{rigid}^i_v(kQ)$, that is, the set of basic rigid kQ-modules M with $|M| = i$ such that M contains $S(v)$ as an direct summand.

By definition, we have $\mathsf{rigid}^i(kQ) = \mathsf{rigid}^i_v(kQ) \sqcup \mathsf{rigid}^i_{\langle v \rangle}(kQ)$. Our strategy is to construct two bijections $\mathsf{rigid}^i_v(kQ) \cong \mathsf{rigid}^i_v(k(\mu_v Q))$ and $\mathsf{rigid}^i_{\langle v \rangle}(kQ) \cong \mathsf{rigid}^i_{\langle v \rangle}(k(\mu_v Q))$ separately. The first one is established by the *reflection functor*, and second one by the *perpendicular categories*.

First, we will use the following property of the classical BGP-reflection functor. For the proof, we refer the reader to standard textbooks on quiver representation theory such as [ASS].

PROPOSITION 6.A.6. *Let Q be an acylic quiver and v a sink of Q. There is a functor $R_v \colon \mathsf{mod}\, kQ \to \mathsf{mod}\, k(\mu_v Q)$ called the* reflection functor, *which induces an equivalence $\mathsf{mod}_v\, kQ \simeq \mathsf{mod}_v\, k(\mu_v Q)$. Moreover, this functor induces an isomorphism $\mathrm{Ext}^1_{kQ}(X, Y) \cong \mathrm{Ext}^1_{k(\mu_v Q)}(R_v X, R_v Y)$ for every $X, Y \in \mathsf{mod}_v\, kQ$.*

This immediately yields the following bijection.

COROLLARY 6.A.7. *Let Q be an acyclic quiver and v a sink of Q. Then we have an bijection $R_v \colon \mathsf{rigid}^i_v(kQ) \xrightarrow{\sim} \mathsf{rigid}^i_v(k(\mu_v Q))$ given by $U \mapsto R_v U$.*

Next we will construct a bijection $\mathsf{rigid}^i_{\langle v \rangle}(kQ) \cong \mathsf{rigid}^{i-1}(kQ_v)$, where Q_v denotes a quiver obtained by removing v from Q. We will use the following two subcategories determined by S_v.

DEFINITION 6.A.8. Let Λ be an artin algebra and $\mathcal{C}$ a collection of objects in $\mathsf{mod}\,\Lambda$. Then define two subcategories $\mathcal{C}^{\perp_{0,1}}$ and ${}^{\perp_{0,1}}\mathcal{C}$ of $\mathsf{mod}\,\Lambda$ as follows.

(1) $\mathcal{C}^{\perp_{0,1}}$ consists of modules X with $\mathrm{Hom}_{kQ}(\mathcal{C}, X) = 0 = \mathrm{Ext}^1_{kQ}(\mathcal{C}, X)$.
(2) ${}^{\perp_{0,1}}\mathcal{C}$ consists of modules X with $\mathrm{Hom}_{kQ}(X, \mathcal{C}) = 0 = \mathrm{Ext}^1_{kQ}(X, \mathcal{C})$.

These subcategories are called right and left *perpendicular categories* with respect to $\mathcal{C}$. By using this categories and their relation to (co)localizations, we can prove the key property of them in this case.

LEMMA 6.A.9. *Let Q be an acylic quiver and v a sink or a source of Q. Then both $S(v)^{\perp_{0,1}}$ and ${}^{\perp_{0,1}}S(v)$ are wide subcategories of $\mathsf{mod}\, kQ$, and equivalent to $\mathsf{mod}\, kQ_v$.*

PROOF. The fact that these categories are wide subcategories follows from [GL, Proposition 1.1] since kQ is hereditary, and its proof is quite straightforward, so we omit this.

For the rest, we will use the theory of a *recollement*. We refer the definitions and details to [Psa]. Let e_v be an idempotent corresponding to v, and put $e = 1 - e_v$. The we have the following recollement diagram, where I is the natural embedding functor.

$$\begin{tikzcd}
\mathsf{mod}\,\frac{kQ}{\langle e\rangle} \arrow[r, hook, "\mathsf{I}"] & \mathsf{mod}\,kQ \arrow[r, "\mathsf{E}"] \arrow[l, bend right] \arrow[l, bend left] & \mathsf{mod}\,e(kQ)e \arrow[l, bend right, "\mathsf{L}"'] \arrow[l, bend left, "\mathsf{R}"]
\end{tikzcd}$$

Here $\mathsf{E} := \mathrm{Hom}_{kQ}(e(kQ), -) = (-)e$, and L and R are left and right adjoint functors of E, and both are fully faithful. Consider the essential image $\mathrm{Im}\,\mathsf{I}$ of I. Since E is a localization and a colocaliztion with respect to the Serre subcategory $\mathrm{Im}\,\mathsf{I}$ [Psa, Remark 2.2], we have that $\mathrm{Im}\,\mathsf{R}$ and $\mathrm{Im}\,\mathsf{L}$ coincide with the perpendicular category $(\mathrm{Im}\,\mathsf{I})^{\perp_{0,1}}$ and ${}^{\perp_{0,1}}(\mathrm{Im}\,\mathsf{I})$ respectively [GL, Proposition 2.2]. On the other hand, $\mathrm{Im}\,\mathsf{I}$ consists of modules M with $Me = 0$, thus $\mathrm{Im}\,\mathsf{I} = \mathsf{add}\,S(v)$ holds. Therefore, we have $\mathrm{Im}\,\mathsf{R} = S(v)^{\perp_{0,1}}$ and $\mathrm{Im}\,\mathsf{L} = {}^{\perp_{0,1}}S(v)$. Since L and R are fully faithful, both $S(v)^{\perp_{0,1}}$ and ${}^{\perp_{0,1}}S(v)$ are equivalent to $\mathsf{mod}\,e(kQ)e$. Now the assertion holds since we clearly have an isomorphism of algebras $e(kQ)e \cong kQ_v$. $\square$

We remark that this lemma was proved in [CK, Lemma 3.2.5] in a more general setting.

By using this, we obtain the following bijection.

Proposition 6.A.10. *Let Q be an acylic quiver and v a sink or a source of Q. Then for $i \geq 1$, we have a bijection*

$$\mathsf{rigid}^i_{\langle v\rangle}(kQ) \cong \mathsf{rigid}^{i-1}(kQ_v).$$

The map is given by $U \mapsto (U_v)e \in \mathsf{mod}\,e(kQ)e$, where e is the same as in the proof of Lemma 6.A.9, and we identify $e(kQ)e$ with kQ_v.

Proof. We give a proof for the case v is a sink, and the same proof applies for the source case by using $S(v)^{\perp_{0,1}}$ instead of ${}^{\perp_{0,1}}S(v)$. First recall that these subcategories are wide subcategories of $\mathsf{mod}\,kQ$ by Lemma 6.A.9, thus are abelian categories. The key observation is the following claim.

(Claim): *For $X \in \mathsf{mod}_v(kQ)$, the following are equivalent:*

(1) *$X \oplus S(v)$ is rigid.*

(2) *$X \in {}^{\perp_{0,1}}S(v)$ holds, and X is rigid in the abelian category ${}^{\perp_{0,1}}S(v)$.*

Proof of (Claim).

(1) $\Rightarrow$ (2): Since $S(v) \oplus X$ is rigid, $\mathrm{Ext}^1_{kQ}(X, S(v)) = 0$ holds. Moreover, if we have a non-zero map $X \to S(v)$, then it must be surjective since $S(v)$ is simple, hence it splits since $S(v)$ is projective. This contradicts to $X \in \mathsf{mod}_v\,kQ$, therefore we have $\mathrm{Hom}_{kQ}(X, S(v)) = 0$. It follows that $X \in {}^{\perp_{0,1}}S(v)$ holds. Since ${}^{\perp_{0,1}}S(v)$ is a wide subcategory of $\mathsf{mod}\,kQ$, an Ext^1 inside ${}^{\perp_{0,1}}S(v)$ is the same as an Ext^1 inside $\mathsf{mod}\,kQ$. Thus X is rigid in the abelian category ${}^{\perp_{0,1}}S(v)$ since so is in $\mathsf{mod}\,kQ$.

(2) $\Rightarrow$ (1): By the above argument, X is a rigid kQ-module, and $\mathrm{Ext}^1_{kQ}(S(v), X \oplus S(v))$ vanishes since $S(v)$ is projective. Thus $X \oplus S(v)$ is rigid by $X \in {}^{\perp_{0,1}}S(v)$. $\blacksquare$

By (Claim), the map $U \mapsto U_v$ clearly induces the following bijection

$$\mathsf{rigid}^i_{\langle v\rangle}(kQ) \cong \mathsf{rigid}^{i-1}({}^{\perp_{0,1}}S(v)),$$

where the right hand side is the set of basic rigid objects X in the abelian category ${}^{\perp_{0,1}}S(v)$ satisfying $|X| = i - 1$. Now the assertion holds from Lemma 6.A.9, since ${}^{\perp_{0,1}}S(v)$ is equivalent to $\mathsf{mod}\,kQ_v$ as abelian categories. $\square$

Now we immediately obtain the second bijection.

Corollary 6.A.11. *Let Q be an acylic quiver and i a sink of Q. Then there is a bijection between $\mathsf{rigid}^i_{\langle v\rangle}(kQ)$ and $\mathsf{rigid}^i_{\langle v\rangle}\,k(\mu_v Q)$.*

Proof. Note that v is a source of $\mu_v Q$. Then by Proposition 6.A.10, we have two bijections between $\mathsf{rigid}^i_{\langle v\rangle}(kQ) \cong \mathsf{rigid}^{i-1}(kQ_v)$ and $\mathsf{rigid}^i_{\langle v\rangle}\,k(\mu_v Q) \cong \mathsf{rigid}^{i-1}\,k(\mu_v Q)_v$. Since $(\mu_v Q)_v = Q_v$ holds, we obtain a bijection between $\mathsf{rigid}^i_{\langle v\rangle}(kQ)$ and $\mathsf{rigid}^i_{\langle v\rangle}\,k(\mu_v Q)$ by composing the above two bijections. $\square$

Now we are ready to prove Theorem 6.A.3.

PROOF OF THEOREM 6.A.3. We have the following equalities by definition.

$$\mathsf{rigid}^i(kQ) = \mathsf{rigid}^i_v(kQ) \sqcup \mathsf{rigid}^i_{\langle v\rangle}(kQ)$$
$$\mathsf{rigid}^i(k(\mu_v Q)) = \mathsf{rigid}^i_v(k(\mu_v Q)) \sqcup \mathsf{rigid}^i_{\langle v\rangle}(k(\mu_v Q))$$

Now we have a bijection $\mathsf{rigid}^i_v(kQ) \cong \mathsf{rigid}^i_v(k(\mu_v Q))$ by Corollary 6.A.7, and a bijection $\mathsf{rigid}^i_{\langle v\rangle}(kQ) \cong \mathsf{rigid}^i_{\langle v\rangle}(k(\mu_v Q))$ by Corollary 6.A.11. Thus by combining these two, we obtain a bijection between $\mathsf{rigid}^i(kQ)$ and $\mathsf{rigid}^i(k(\mu_v Q))$. $\square$

6.A.2. Formula for the number of rigid modules. In this subsection, we give an explicit formula for $\#\,\mathsf{rigid}^i(kQ)$ for a Dynkin quiver Q. *From now on, we assume that Q is a Dynkin quiver of type $X_n \in \{A_n, D_n, E_6, E_7, E_8\}$ with n vertices.*

Let $\Delta(Q)$ be a simplicial complex defined as follows: the set of vertices is $\mathsf{rigid}(kQ)$, and an $(i-1)$-simplex consists of sets of rigid kQ-modules whose direct sum is rigid, or equivalently, belongs to $\mathsf{rigid}^i(kQ)$. This complex was introduced by Riedtmann and Schofield [RS]. By definition, $\#\,\mathsf{rigid}^i(kQ)$ is equal to the number of $(i+1)$-faces of $\Delta(Q)$, thus the calculation of $\#\,\mathsf{rigid}^i(kQ)$ is nothing but that of the face vector of $\Delta(Q)$. Although this complex is classical, there seems to be no papers which contain an explicit formula of $\#\,\mathsf{rigid}^i(kQ)$.

We give such an formula, by translating our problem to a combinatorial problem on a *cluster complex*. Let Φ be the root system of type X_n, and let $\Phi_{\geq -1}$ denote the set of *almost positive roots* of Φ, that is, positive roots together with negative simple roots. Then the *cluster complex* $\Delta(X_n)$ of type X_n, also known as the *generalized associahedron*, is a simplicial complex with the vertex set $\Phi_{\geq -1}$. We refer the reader to [FZ, MRZ] for the details. Then this complex contains $\Delta(Q)$ if Q is *bipirtite*, that is, every vertex is either a sink or a source. More precisely, the following holds.

PROPOSITION 6.A.12. *Let Q be a Dynkin quiver with a bipirtite orientation. Then taking dimension vectors, we have an embedding $\Delta(Q) \hookrightarrow \Delta(X_n)$, which induces an isomorphism between $\Delta(Q)$ and the full subcomplex of $\Delta(X_n)$ spanned by positive roots.*

We refer the reader to [MRZ, 4.12] for the proof, and to [BMRRT, Theorem 4.5] for the more theoretical explanation of this using the cluster category. As a corollary, we have the following equality.

COROLLARY 6.A.13. *Let Q be a Dynkin quiver. Then $\#\,\mathsf{rigid}^i(kQ)$ is equal to the number of $(i-1)$-faces of $\Delta(X_n)$ which contain no negative simple roots.*

PROOF. We can transform Q into a bipirtite Dynkin quiver by using sink mutations. Thus we may assume that Q is bipirtite by Corollary 6.A.4. Then the assertion is immediate from Proposition 6.A.12. $\square$

Now we are ready to show the formula of $\#\,\mathsf{rigid}^i(kQ)$ by using [Krat], which enumerates the number of faces of $\Delta(Q)$ satisfying various conditions.

THEOREM 6.A.14. *Let Q be a Dynkin quiver of type X_n. Then the number $\#\,\mathsf{rigid}^i(kQ)$ is equal to (X_n) in the following list, where $\binom{n}{i}$ denotes the binomial coefficient.*

(A_n) $$\frac{1}{i+1}\binom{n}{i}\binom{n+i}{i}$$

(D_n) $$\binom{n}{i}\binom{n+i-2}{i} + \binom{n-1}{i-1}\binom{n+i-3}{i-1} - \frac{1}{n-1}\binom{n-1}{i-1}\binom{n+i-2}{i}$$

(E_6)

i	0	1	2	3	4	5	6	total
	1	36	300	1035	1720	1368	418	4878

(E_7)

i	0	1	2	3	4	5	6	7	total
	1	63	777	3927	9933	13299	9009	2431	39440

(E_8)	i	0	1	2	3	4	5	6	7	8	total
		1	120	2135	15120	54327	108360	121555	71760	17342	390720

PROOF. The computation is achieved by specializing the results [Krat, Theorems FA, FD, Section 7] to $m = 1$ and $y = 0$. More precisely, in [Krat], the *m-generalization* of cluster complexes are studied, and $m = 1$ is the classical case. Then the number of faces of an m-cluster complex which consists of given numbers of positive roots and negative roots was computed, and $y = 0$ means that we exclude negative roots. $\square$

By our main result, $\#\, \mathsf{rigid}(kQ)$ is equal to the number of ICE-closed subcategories of $\mathsf{mod}\, kQ$. Since we have $\#\, \mathsf{rigid}(kQ) = \sum_{i=0}^{n} \#\, \mathsf{rigid}^i(kQ)$, we obtain the following enumeration.

COROLLARY 6.A.15. *Let Q be a Dynkin quiver of type X_n. Then the number of ICE-closed subcategories in $\mathsf{mod}\, kQ$ is equal to the sum of the numbers given in Theorem 6.A.14 over $i = 0, 1, \dots, n$. In particular, if Q is of type A_n, then the equality holds,*

$$\#\, \mathsf{ice}(kQ) = \sum_{i=0}^{n} \frac{1}{i+1} \binom{n}{i} \binom{n+i}{i},$$

where the right hand side is known as the n-th large Schröder number [OEIS, A006318].

REMARK 6.A.16. In [Eno6, Theorem 6.13], the author computes $\#\, \mathsf{ice}(kQ)$ for a linearly oriented A_n quiver by a different method. By combining this with Theorem 6.A.3, we can give another proof of the fact that $\#\, \mathsf{ice}\, kQ$ is equal to the n-th large Schröder number for a quiver of type A_n.

Bibliography

[AbBi] H. Abe, S. Billey, *Consequences of the Lakshmibai-Sandhya theorem: the ubiquity of permutation patterns in Schubert calculus and related geometry*, Adv. Stud. Pure Math., 71 (2016), Math. Soc. Japan, 1–52.

[AIR] T. Adachi, O. Iyama, I. Reiten, *τ-tilting theory*, Compos. Math. 150 (2014), no. 3, 415–452.

[AIRT] C. Amiot, O. Iyama, I. Reiten, G. Todorov, *Preprojective algebras and c-sortable words*, Proc. Lond. Math. Soc. (3) 104 (2012), no. 3, 513–539.

[Asa1] S. Asai, *Semibricks*, Int. Math. Res. Not. rny150, 2018.

[Asa2] S. Asai, *Bricks over preprojective algebras and join-irreducible elements in Coxeter groups*, arXiv:1712.08311.

[AP] S. Asai, C. Pfeifer, *Wide subcategories and lattices of torsion classes*, arXiv:1905.01148.

[ASS] I. Assem, D. Simson, A. Skowroński, *Elements of the representation theory of associative algebras Vol. 1*, London Mathematical Society Student Texts, 65, Cambridge University Press, Cambridge, 2006.

[Aus1] M. Auslander, *Coherent functors*, Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965) 189–231 Springer, New York.

[Aus2] M. Auslander, *Representation theory of Artin algebras II*, Comm. Algebra 1 (1974), 269–310.

[Aus3] M. Auslander, *Relations for Grothendieck groups of Artin algebras*, Proc. Amer. Math. Soc. 91 (1984), no. 3, 336–340.

[Aus4] M. Auslander, *Isolated singularities and existence of almost split sequences*, Representation theory, II (Ottawa, Ont., 1984), 194–242, Lecture Notes in Math., 1178, Springer, Berlin, 1986.

[AuBu] M. Auslander, R-O. Buchweitz, *The homological theory of maximal Cohen-Macaulay approximations*, Mém. Soc. Math. France (N.S.) No. 38 (1989), 5–37.

[AR1] M. Auslander, I. Reiten, *Grothendieck groups of algebras and orders*, J. Pure Appl. Algebra 39 (1986), no. 1–2, 1–51.

[AR2] M. Auslander, I. Reiten, *Applications of contravariantly finite subcategories*, Adv. Math. 86 (1991), no. 1, 111–152.

[ARS] M. Auslander, I. Reiten, S.O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, 36. Cambridge University Press, Cambridge, 1995.

[AS] M. Auslander, S.O. Smalø, *Preprojective modules over artin algebras*, J. Algebra 66 (1980), 61–122.

[BaGe] N.R. Baeth, A. Geroldinger, *Monoids of modules and arithmetic of direct-sum decompositions*, Pacific J. Math. 271 (2014), no. 2, 257–319.

[BeGr] A. Berenstein, J. Greenstein, *Primitively generated Hall algebras*, Pacific J. Math. 281 (2016), no. 2, 287–331.

[BB] A. Björner, F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, 231. Springer, New York, 2005.

[BMB] M. Bousquet-Mélou, S. Butler, *Forest-like permutations*, Ann. Comb. 11 (2007), no. 3-4, 335–354.

[Bro1] G. Brookfield, *Direct sum cancellation of Noetherian modules*, J. Algebra 200 (1998), no. 1, 207–224.

[Bro2] G. Brookfield, *The Grothendieck group and the extensional structure of Noetherian module categories*, Algebra and its applications (Athens, OH, 1999), 111–131, Contemp. Math., 259, Amer. Math. Soc., Providence, RI, 2000.

[Bro3] G. Brookfield, *The extensional structure of commutative Noetherian rings*, Comm. Algebra 31 (2003), no. 6, 2543–2571.

[Bru] R.A. Brualdi, Introductory Combinatorics, fifth ed., Pearson Prentice Hall, Upper Saddle River, NJ, 2010.

[BrGu] W. Bruns, J. Gubeladze, *Polytopes, rings, and K-theory*, Springer Monographs in Mathematics. Springer, Dordrecht, 2009.

[BHLR] T. Brüstle, S. Hassoun, D. Langford, S. Roy, *Reduction of exact structures*, J. Pure Appl. Algebra 224 (2020), no. 4, 106212, 29 pp.

[BST] T. Brüstle, D. Smith, H. Treffinger, *Wall and chamber structure for finite-dimensional algebras*, Adv. Math. 354 (2019), 106746, 31 pp.

[BIRS] A. B. Buan, O. Iyama, I. Reiten, J. Scott, *Cluster structures for 2-Calabi-Yau categories and unipotent groups*, Compos. Math. 145 (2009), no. 4, 1035–1079.

[BMRRT] A. Buan, R. Marsh, I. Reiten, M. Reineke, G. Todorov, *Tilting theory and cluster combinatorics*, Adv. in Math. 204 (2006), no. 2, 572–618.

[But] M. C. R. Butler, *Grothendieck groups and almost split sequences*, Lecture Notes in Math., 882, Springer, Berlin-New York, 1981.

[Büh] T. Bühler, *Exact categories*, Expo. Math. 28 (2010), no. 1, 1–69.

[CWY] W.Y.C. Chen, S.Y.J. Wu, C.H. Yan, *Linked partitions and linked cycles*, European J. Combin. 29 (2008), 1377–1520.

[CK] X-W. Chen, H. Krause, *Expansions of abelian categories*, J. Pure Appl. Algebra 215 (2011), 2873–2883.

[C-B] W. Crawley-Boevey, *On the exceptional fibres of Kleinian singularities*, Amer. J. Math. 122 (2000), no. 5, 1027–1037.

[DIJ] L. Demonet, O. Iyama, G. Jasso, *τ-tilting finite algebras, bricks, and g-vectors*, Int. Math. Res. Not. rnx135, 2017.

[DIRRT] L. Demonet, O. Iyama, N. Reading, I. Reiten, H. Thomas, Lattice theory of torsion classes, arXiv:1711.01785.

[DK] B. Keller, with an appendix by L. Demonet, *A survey on maximal green sequences*, arXiv:1904.09247.

[DR] V. Dlab, C. M. Ringel, *A module theoretical interpretation of properties of the root system*, Ring theory (Proc. Antwerp Conf. (NATO Adv. Study Inst.), Univ. Antwerp, Antwerp, 1978), pp. 435–451, Lecture Notes in Pure and Appl. Math., 51, Dekker, New York, 1979.

[Dye] M. J. Dyer, *Hecke algebras and shellings of Bruhat intervals*, Compositio Math. 89 (1993), no. 1, 91–115.

[Dyk] K. Dykema, *Multilinear function series and transforms in free probability theory*, Adv. Math. 208 (2007), no. 1, 351–407.

[Eno1] H. Enomoto, *Classifying exact categories via Wakamatsu tilting*, J. Algebra 485 (2017), 1–44.

[Eno2] H. Enomoto, *Classifications of exact structures and Cohen-Macaulay-finite algebras*, arXiv:1705.02163.

[Eno3] H. Enomoto, *Relations for Grothendieck groups and representation-finiteness*, J. Algebra 539 (2019), 152–176.

[Eno4] H. Enomoto, *The Jordan-Hölder property and Grothendieck monoids of exact categories*, arXiv:1908.05446.

[Eno5] H. Enomoto, *Bruhat inversions in Weyl groups and torsion-free classes over preprojective algebras*, arXiv:2002.09205.

[Eno6] H. Enomoto, *Monobrick, a uniform approach to torsion-free classes and wide subcategories*, arXiv:2005.01626.

[Eno7] H. Enomoto, *Rigid modules and ICE-closed subcategories over path algebras*, arXiv:2005.05536.

[Fac1] A. Facchini, *Direct sum decompositions of modules, semilocal endomorphism rings, and Krull monoids*, J. Algebra 256 (2002), no. 1, 280–307.

[Fac2] A. Facchini, *Krull monoids and their application in module theory*, Algebras, rings and their representations, 53–71, World Sci. Publ., Hackensack, NJ, 2006.

[FS] C. K. Fan, J. R. Stembridge, *Nilpotent orbits and commutative elements*, J. Algebra 196 (1997), no. 2, 490–498.

[Fio] L. Fiorot, *N-Quasi-Abelian Categories vs N-Tilting Torsion Pairs*, arXiv:1602.08253.

[FZ] S. Fomin, A. Zelevinsky, *Y-systems and generalized associahedra*, Ann. of Math. 158 (2003), 977–1018.

[GL] W. Geigle, H. Lenzing, *Perpendicular categories with applications to representations and sheaves*, J. Algebra 144 (1991), 339–389.

[GLS] C. Geiß, B. Leclerc, J. Schröer, *Kac-Moody groups and cluster algebras*, Adv. Math. 228 (2011), no. 1, 329–433.

[GH-K] A. Geroldinger, F. Halter-Koch, *Non-unique factorizations: Algebraic, combinatorial and analytic theory*, Pure and Applied Mathematics (Boca Raton), 278. Chapman & Hall/CRC, Boca Raton, FL, 2006.

[GL] R. M. Green, J. Losonczy, *Freely braided elements of Coxeter groups*, Ann. Comb. 6 (2002), no. 3-4, 337–348.

[Gri1] P. A. Grillet, *Semigroups: An introduction to the structure theory*, Monographs and Textbooks in Pure and Applied Mathematics, 193. Marcel Dekker, Inc., New York, 1995.

[Gri2] P. A. Grillet, *Commutative semigroups*, Advances in Mathematics (Dordrecht), 2. Kluwer Academic Publishers, Dordrecht, 2001.

[Gro] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2) 9 1957 119–221.

[Hap] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, 119. Cambridge University Press, Cambridge, 1988.

[HR] S. Hassoun, S. Roy, *Notes on Jordan-Hölder property for exact categories*, arXiv:1906.03246.

[Hir] N. Hiramatsu, *Relations for Grothendieck groups of Gorenstein rings*, Proc. Amer. Math. Soc. 145 (2017), no. 2, 559–562.

[Hum1] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Second printing, revised. Graduate Texts in Mathematics, 9. Springer-Verlag, New York-Berlin, 1978. xii+171 pp.

[Hum2] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, 29. Cambridge University Press, Cambridge, 1990. xii+204 pp.

[IT] C. Ingalls, H. Thomas, *Noncrossing partitions and representations of quivers*, Compos. Math. 145 (2009), no. 6, 1533–1562.

[Iya] O. Iyama, *The relationship between homological properties and representation theoretic realization of Artin algebras*, Trans. Amer. Math. Soc. 357 (2005), no. 2, 709–734.

[IRRT] O. Iyama, N. Reading, I. Reiten, H. Thomas, *Lattice structure of Weyl groups via representation theory of preprojective algebras*, Compos. Math. 154 (2018), no. 6, 1269–1305.

[KaSa] M. Kashiwara, Y. Saito, *Geometric construction of crystal bases*, Duke Math. J. 89 (1997), no. 1, 9–36.

[KaSc] M. Kashiwara, P. Schapira, *Categories and sheaves*, Grundlehren der Mathematischen Wissenschaften 332, Springer-Verlag, Berlin, 2006.

[Kob] T. Kobayashi, *Syzygies of Cohen-Macaulay modules and Grothendieck groups*, J. Algebra 490 (2017), 372–379.

[Krat] C. Krattenthaler, *The F-triangle of the generalised cluster complex*, Topics in Discrete Mathematics, dedicated to Jarik Nešetřil on the occasion of his 60th birthday (M. Klazar, J. Kratochvil, M. Loebl, J. Matoušek, R. Thomas and P. Valtr, ed.), Springer-Verlag, Berlin, New York, 2006, pp. 93–126.

[Krau] H. Krause, *Krull-Schmidt categories and projective covers*, Expo. Math. 33 (2015), no. 4, 535–549.

[Lus] G. Lusztig, *Canonical bases arising from quantized enveloping algebras II*, Common trends in mathematics and quantum field theories (Kyoto, 1990). Progr. Theoret. Phys. Suppl. No. 102 (1990), 175–201 (1991).

[MMP] E. N. Marcos, H. A. Merklen, M. I. Platzeck, *The Grothendieck group of the category of modules of finite projective dimension over certain weakly triangular algebras*, Comm. Algebra 28 (2000), no. 3, 1387–1404.

[MS] F. Marks, J. Šťovíček, *Torsion classes, wide subcategories and localisations*, Bull. Lond. Math. Soc. 49 (2017), no. 3, 405–416.

[MRZ] R. Marsh, M. Reineke, A. Zelevinsky, *Generalized associahedra via quiver representations*, Trans. Amer. Math. Soc. 355 (2003) 4171–4186.

[Miz] Y. Mizuno, *Classifying τ-tilting modules over preprojective algebras of Dynkin type*, Math. Z. 277 (2014), no. 3-4, 665–690.

[OEIS] *The On-Line Encyclopedia of Integer Sequences*, published electronically at https://oeis.org

[Pap] P. Papi, *A characterization of a special ordering in a root system*, Proc. Amer. Math. Soc. 120 (1994), no. 3, 661–665.

[PR] M. I. Platzeck, I. Reiten, *Modules of finite projective dimension for standardly stratified algebras*, Comm. Algebra 29 (2001), no. 3, 973–986.

[Psa] C. Psaroudakis, *Homological theory of recollements of abelian categories*, J. Algebra 398 (2014), 63–110.

[Qui] D. Quillen, *Higher algebraic K-theory: I*, Lecture Notes in Math., Vol. 341, Springer, Berlin 1973.

[Raĭ] D.A. Raĭkov, *Semiabelian categories*, Dokl. Akad. Nauk SSSR 188 1969 1006–1009.

[Rea1] N. Reading, *Clusters, Coxeter-sortable elements and noncrossing partitions*, Trans. Amer. Math. Soc. 359 (2007), no. 12, 5931–5958.

[Rea2] N. Reading, *Noncrossing arc diagrams and canonical join representations*, SIAM J. Discrete Math. 29 (2015), no. 2, 736–750.

[RS] C. Riedtmann, A. Schofield, *On a simplicial complex associated with tilting modules*, Comment. Math. Helv. 66, no. 1, 70–78 (1991).

[Rin1] C. M. Ringel, *Representations of K-species and bimodules.* J. Algebra 41 (1976), no. 2, 269–302.

[Rin2] C. M. Ringel, *The Catalan combinatorics of the hereditary artin algebras*, in Recent Developments in Representation Theory, Contemp. Math. 673, Amer. Math. Soc., Providence, RI, 2016.

[Rum1] W. Rump, *$*$-modules, tilting, and almost abelian categories*, Comm. Algebra 29 (2001), no. 8, 3293–3325.

[Rum2] W. Rump, *Almost abelian categories*, Cahiers Topologie Géom. Différentielle Catég. 42 (2001), no. 3, 163–225.

[Rum3] W. Rump, *A counterexample to Raikov's conjecture*, Bull. Lond. Math. Soc. 40 (2008), no. 6, 985–994.

[Sage] The Sage Developers, *SageMath, the Sage Mathematics Software System (Version 8.6)*, 2020, `https://www.sagemath.org`.

[ST] S. Schroll, H. Treffinger, *A τ-tilting approach to the first Brauer-Thrall conjecture*, arXiv:2004.14221.

[Sma] S. O. Smalø, *Torsion theories and tilting modules*, Bull. London Math. Soc. 16 (1984), no. 5, 518–522.

[Ste] B. Stenström, *Rings of quotients: An introduction to methods of ring theory*, Springer-Verlag, NewYork-Heidelberg, 1975.

[Tat] A. Tattar, *Torsion pairs and quasi-abelian categories*, arXiv:1907.10025.

[Tho] H. Thomas, *Coxeter groups and quiver representations*, In Surveys in representation theory of algebras, volume 716 of Contemp. Math., pages 173–186. Amer. Math. Soc., Providence, RI, 2018.

[Tre] H. Treffinger, *An algebraic approach to Harder-Narasimhan filtrations*, arXiv:1810.06322.

[Wei] C. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, 38, Cambridge University Press, Cambridge, 1994.

[WY] A. Woo, A. Yong, *When is a Schubert variety Gorenstein?*, Adv. Math. 207 (2006), no. 1, 205–220.

[Yos] Y. Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, London Mathematical Society Lecture Note Series 146, Cambridge University Press, Cambridge, 1990.