WELL-POSEDNESS FOR HIGHER ORDER NONLINEAR DISPERSIVE EQUATIONS
(高階の非線形分散型方程式に対する適切性)

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Contents

1. Introduction

In the present paper, we consider the Cauchy problem of higher order Benjamin-Ono type equations and Schrödinger type equations with constant coefficients. In particular, in Section 2, we consider third order Benjamin-Ono type equations (3BO), which reads

$$\partial_t u - \partial_x^3 u + u^2\partial_x u + c_1\partial_x(u\mathcal{H}\partial_x u) + c_2\mathcal{H}\partial_x(u\partial_x u) = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{T},$$

where the unknown function u is real valued and $c_1, c_2 \in \mathbb{R}$. $\mathcal{H}$ is the Hilbert transform, see Subsection 2.1 for its definition. In Section 3, we consider fourth order Benjamin-Ono type equations (4BO), which reads

$$\partial_t u = \partial_x K(u), \quad (t,x) \in \mathbb{R}\times\mathcal{M},$$

where $\mathcal{M} = \mathbb{R}$ (or $\mathbb{T}$),

$$\begin{aligned} K(u) := {} & \mathcal{H}\partial_x^3 u + c_1 u\partial_x^2 u + c_2(\partial_x u)^2 + c_3(\mathcal{H}\partial_x u)^2 + c_4\mathcal{H}(u\mathcal{H}\partial_x^2 u) \\ & + c_5\mathcal{H}(u^2\partial_x u) + c_6 u\mathcal{H}(u\partial_x u) + c_7 u^2\mathcal{H}\partial_x u - u^4 \end{aligned} \tag{1.1}$$

with $c_k \in \mathbb{R}$ for $k = 1, \dots, 7$ and the unknown function u is real valued. In Section 4, we consider higher order Schrödinger type equations (HS) with constant coefficients, which reads

$$D_t u = D_x^{2m} u + \sum_{j=1}^{2m} \left(a_j D_x^{2m-j} u + b_j D_x^{2m-j}\bar{u}\right), \quad (t,x) \in \mathbb{R}\times\mathcal{M} \tag{1.2}$$

where $D_t = -i\partial_t$, $D_x = -i\partial_x$, i is the imaginary unit and the unknown function u is complex valued. The constants $\{a_j\}, \{b_j\} \subset \mathbb{C}$ are given. Our main objects is to prove the local-wellposedness for (3BO) and (4BO) and to classify the Cauchy problem of (HS).

1.1. Introduction of (3BO).

In Section 2, we consider the Cauchy problem of (3BO):

$$\begin{cases} \partial_t u - \partial_x^3 u + u^2\partial_x u + c_1\partial_x(u\mathcal{H}\partial_x u) + c_2\mathcal{H}\partial_x(u\partial_x u) = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{T}, \\ u(0,x) = \varphi(x) \in H^s(\mathbb{T}). \end{cases} \tag{1.3}$$

Our first result is the following:

Theorem 1.1. *The Cauchy problem* (1.3) *is locally well-posed in* $H^s(\mathbb{T})$ *for* $s > 5/2$.

For more precise statement of Theorem 1.1, see Theorem 2.1 in Section 2. We make some comments on Theorem 1.1. Nonlinear terms $\partial_x(u\mathcal{H}\partial_x u)$ and $\mathcal{H}\partial_x(u\partial_x u)$ in (3BO) have two derivatives, and the energy estimate gives only the following:

$$\left|\frac{d}{dt}\|\partial_x^k u(t)\|_{L^2}^2 + c\int \partial_x u(\mathcal{H}\partial_x^{k+1}u)\partial_x^k u dx\right| \lesssim (1+\|\partial_x^2 u\|_{L^\infty})^2\|\partial_x^k u(t)\|_{L^2}^2, \qquad (1.4)$$

where c is a constant depending only on k. See (2.19) for details. It is difficult to handle the second term in the left hand side by $\|u\|_{H^k}$, which is the main difficulty in this problem. To overcome that difficulty, we add a correction term into the energy (see Definition 3 in Section 2):

$$E_*(u) := \|u\|_{L^2}^2 + \|D^s u\|_{L^2}^2 + a_s\|u\|_{L^2}^{4s+2} + b_s\int u(\mathcal{H}D^s u)D^{s-2}\partial_x u dx,$$

where $D := \mathcal{F}^{-1}|\xi|\mathcal{F}$, following the idea from Kwon [18], who studied the local well-posedness of the fifth order KdV equation (see also Segata [27], Kenig-Pilod [16] and Tsugawa [33]). The correction term allows us to cancel out the worst term in (1.4), which makes it possible to evaluate the H^s-norm of the solution by that of the initial data. It is worth pointing out that our proof refines the idea in [7]. Indeed, Feng introduced the following energy estimate in order to show the "weak" continuous dependence (see Subsection 2.1 for details):

$$\begin{aligned}&\frac{d}{dt}\left(\|\partial_x^{k-2}w\|_{L^2}^2 + \frac{2k-3}{4}\int_{\mathbb{R}}(u+v)\partial_x^{k-3}w\mathcal{H}\partial_x^{k-2}w dx\right)\\ &\le C(T,\|\varphi\|_{H^k},\|\psi\|_{H^k})\|w(t)\|_{H^{k-2}}^2,\end{aligned}$$

on $[0,T]$, where $w=u-v$ and $u,v\in C([0,T];H^k(\mathbb{R}))$ satisty (1.3) with $c_1=c_2=\sqrt{3}/2$ and initial data $\varphi,\psi\in H^k(\mathbb{R})$, respectively. Here, we would like to have the estimate for $\|w\|_{H^k}$. If we simply replace $k-2$ with k in the above estimate, the constant in the right hand side depends on $\|\varphi\|_{H^{k+2}}$ (resp. $\|\psi\|_{H^{k+2}}$), which cannot be handled by $\|\varphi\|_{H^k}$ (resp. $\|\psi\|_{H^k}$). Therefore, we need to find a different correction term (see Definition 3 in Section 2) and estimate the difference between two solutions in $H^k(\mathbb{T})$ more carefully (see the proof of Proposition 2.21) so as to complete the continuous dependence.

It is known that (3BO) with specific coefficients is completely integrable and has infinitely many conservation laws. In the integrable case, we can extend the solution obtained by Theorem 1.1 globally, using the conservation law corresponding to H^3-norm.

Corollary 1.2. *The Cauchy problem* (1.3) *with* $c_1=c_2=\sqrt{3}/2$ *is globally well-posed in* $H^s(\mathbb{T})$ *for* $s\ge 3$.

Corollary 1.2 can be shown by using the Gagliardo-Nirenberg inequality. On the real line case, there are some much better results with respect to the regularity of the initial data (see Subsection 2.1 for details). This is why we focus on our problem on $\mathbb{T}$. But Theorem 1.1 and Corollary 1.2 on (3BO) still hold on $\mathbb{R}$.

1.2. **Introduction of (4BO).** In Section 3, we consider the Cauchy problem of (4BO):

$$\begin{cases} \partial_t u = \partial_x K(u), \quad (t,x) \in \mathbb{R} \times \mathcal{M}, \\ u(0,x) = \varphi(x) \in H^s(\mathcal{M}). \end{cases} \tag{1.5}$$

Here, $K(u)$ is defined in (1.1). Our result is the following:

Theorem 1.3. *The Cauchy problem* (1.5) *is locally well-posed in* $H^s(\mathcal{M})$ *for* $s > 7/2$.

Now we mention the idea of the proof of Theorem 1.3. We may have the third order derivative loss since nonlinear terms in (4BO) have three derivatives at most. By the symmetry, it can be reduced to the second order derivative loss (see Lemma 3.25). Our proof is based on the energy method, and the standard energy estimate gives only the following:

$$\left| \frac{d}{dt} \|D^s u(t)\|_{L^2}^2 + L_1(u) + L_2(u) + L_3(u) \right| \leq C(1 + \|u\|_{H^{s_0}})^3 \|u(t)\|_{H^s}^2, \tag{1.6}$$

where $s_0 > 7/2$, $D = \mathcal{F}^{-1}|\xi|\mathcal{F}$ and

$$L_1(u) := \lambda_1(s) \int \partial_x u (D^s \partial_x u)^2 dx, \quad L_2(u) := \lambda_2(s) \int (\mathcal{H}\partial_x^2 u)(\mathcal{H} D^s \partial_x u) D^s u dx,$$
$$L_3(u) := \lambda_3(s) \int u \partial_x u (\mathcal{H} D^s \partial_x u) D^s u dx$$

(see Definition 6 in Subsection 3.2 for definitions of $\lambda_j(s)$). Here, we note that $L_1(u)$ is the second order derivative loss, and $L_2(u)$ and $L_3(u)$ are the first order derivative losses. It is impossible to handle $L_j(u)$ for $j = 1, 2, 3$ by $\|u\|_{H^s}$. In order to overcome this difficulty, we modify the energy by adding correction terms. Namely, we consider

$$E_s(u) := \frac{1}{2}\|u\|_{L^2}^2 (1 + C_s \|u\|_{L^2}^2 + C_s \|u\|_{L^2}^{4s}) + \frac{1}{2}\|D^s u\|_{L^2}^2 + \sum_{j=1}^{3} M_s^{(j)}(u),$$

with

$$M_s^{(1)}(u) := \frac{\lambda_1(s)}{4}\int u(\mathcal{H}D^s u)\mathcal{H}D^{s-1}u dx,$$
$$M_s^{(2)}(u) := \frac{\lambda_2(s)}{4}\int (\mathcal{H}\partial_x u)(D^{s-1}u)^2 dx,$$
$$M_s^{(3)}(u) := \frac{\lambda_1(s)\lambda_4(s)+4\lambda_3(s)}{32}\int u^2(D^{s-1}u)^2 dx$$

(see Definition 6 in Subsection 3.2). The first two terms correspond to $\|u\|_{H^s}$, and $M_s^{(1)}(u)$, $M_s^{(2)}(u)$ and $M_s^{(3)}(u)$ are correction terms. As defined in Definition 6, we note that $\lambda_j(s)$ for $j=1,2,3,4$ is a linear polynomial in s. The coefficient of $M_s^{(j)}(u)$ can be determined so that the time derivative of $M_s^{(j)}(u)$ cancels out $L_j(u)$ for $j=1,2$. On the other hand, the time derivative of $M_s^{(1)}(u)$ also yields $L_3(u)$, that is,

$$\frac{d}{dt}M_s^{(1)}(u) \sim L_1(u) + L_3(u)$$

since $L_1(u)$ is the second order derivative loss. Therefore, we need to collect coefficients of $L_3(u)$ resulting from both $\|D^s u\|$ and $M_s^{(1)}(u)$ when we determine the coefficient of $M_s^{(3)}(u)$. For this reason, the coefficient of $M_s^{(3)}(u)$ is a quadratic polynomial in s.

It is known that (4BO) with specific coefficients is completely integrable and has infinitely many conservation laws. As in the case of (3BO), we can extend the solution obtained by Theorem 1.3 globally.

Corollary 1.4. *The Cauchy problem* (1.5) *with* $c_1=3$, $-c_2=c_5=c_6=c_7=-2$ *and* $c_3=c_4=-1$ *is globally well-posed in* $H^s(\mathcal{M})$ *for* $s\geq 4$.

1.3. **Introduction of (HS).** In Section 4, we consider the Cauchy problem of (HS):

$$\begin{cases} D_t u = D_x^{2m}u + \sum_{j=1}^{2m}\left(a_j D_x^{2m-j}u + b_j D_x^{2m-j}\bar{u}\right), & (t,x)\in\mathbb{R}\times\mathcal{M}, \\ u(0,x)=\varphi(x)\in L^2(\mathcal{M}). \end{cases} \tag{1.7}$$

We introduce λ, which is used to classify (1.7) into three types.

Definition 1. $\gamma=\{\gamma_j\}_{j=1}^{m-1}$ and $\lambda=\{\lambda_j\}_{j=1}^{2m-1}$ are defined as

$$\gamma_j = b_{2j} - \sum_{k=1}^{j-1} \bar{a}_{2(j-k)}\gamma_k, \quad 1 \le j \le m-1,$$

$$\begin{cases} \lambda_{2j} = 2\operatorname{Im} a_{2j} - 2\displaystyle\sum_{k=1}^{j-1} \operatorname{Im} \bar{b}_{2(j-k)}\gamma_k, & 1 \le j \le m-1, \\ \lambda_{2j-1} = 2\operatorname{Im} a_{2j-1} + 2\displaystyle\sum_{k=1}^{j-1} \operatorname{Im} \bar{b}_{2(j-k)-1}\gamma_k, & 1 \le j \le m. \end{cases}$$

Our result is the following:

Theorem 1.5.
(Dispersive type, L^2 well-posedness) Assume that $\lambda_j=0$ for $1\le j\le 2m-1$. Then, for any $\varphi\in L^2(\mathcal{M})$, there exists a unique solution $u(t,x)$ of (4.1)–(4.2) *such that $u(t,x)\in C((-\infty,\infty);L^2(\mathcal{M}))$.*
(Parabolic type) Assume that there exists $j^*\in\mathbb{N}$ such that $\lambda_j=0$ for $1\le j<2j^*$ and $\lambda_{2j^*}>0$ (resp. $\lambda_{2j^*}<0$). Then, for any $\varphi\in L^2(\mathcal{M})$, there exist a unique solution $u(t,x)$ of (4.1)–(4.2) *on $[0,\infty)$ (resp. $(-\infty,0]$) such that $u(t,x)\in C([0,\infty);L^2(\mathcal{M}))\cap C^\infty((0,\infty)\times\mathcal{M})$ (resp. $C((-\infty,0];L^2(\mathcal{M}))\cap C^\infty((-\infty,0)\times\mathcal{M})$). For any $\varphi\in L^2(\mathcal{M})\backslash C^\infty(\mathcal{M})$ and $\delta>0$, no solution u of* (4.1)–(4.2) *exists on $(-\delta,0]$ (resp. $[0,\delta)$) such that $u(t,x)\in C((-\delta,0];L^2(\mathcal{M}))$ (resp. $C([0,\delta);L^2(\mathcal{M}))$).*
(Twisted parabolic type) Assume that there exists $j^*\in\mathbb{N}$ such that $\lambda_j=0$ for $1\le j<2j^*-1$ and $\lambda_{2j^*-1}>0$ (resp. $\lambda_{2j^*-1}<0$). Let $\varphi\in L^2(\mathcal{M})$ satisfy $P^+\varphi\notin H^{1/2}(\mathcal{M})$. Then, for any $\delta>0$, there exist no solution $u(t,x)$ of (4.1)–(4.2) *on $[-\delta,0]$ (resp. $[0,\delta]$) satisfying $u\in C([-\delta,0];L^2(\mathcal{M}))$ (resp. $u\in C([0,\delta];L^2(\mathcal{M}))$). Moreover, the same result as above holds even if we replace P^+, $[-\delta,0]$ and $[0,\delta]$ with P^-, $[0,\delta]$ and $[-\delta,0]$, respectively.*

Since the coefficients are constants, by the Fourier transform, the equation in (1.7) can be rewritten into the following:

$$D_t\widehat{u}(t,\xi) = \xi^{2m}\widehat{u}(t,\xi) + \sum_{j=1}^{2m}\left(a_j\xi^{2m-j}\widehat{u}(t,\xi) + b_j\xi^{2m-j}\overline{\widehat{u}}(t,-\xi)\right). \tag{1.8}$$

Here, we fix $\xi\in\mathbb{R}$ (or $\mathbb{Z}$) and put

$$U_\xi(t) = \begin{pmatrix} \widehat{u}(t,\xi) \\ \overline{\widehat{u}}(t,-\xi) \end{pmatrix}, \quad X_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_j = \begin{pmatrix} a_j & b_j \\ (-1)^{j+1}\overline{b_j} & (-1)^{j+1}\overline{a_j} \end{pmatrix},$$

for $1 \le j \le 2m$. Then, by (1.8) with $u(0,x) = \varphi(x)$, it follows that

$$D_t U_\xi(t) = \sum_{j=0}^{2m} \xi^{2m-j} X_j U_\xi(t), \quad U_\xi(0) = {}^t(\widehat{\varphi}(\xi), \overline{\widehat{\varphi}}(-\xi)), \tag{1.9}$$

which is a system of linear ordinary differential equations. We can easily obtain the unique solution

$$U_\xi(t) = U_\xi(0) \exp it \sum_{j=0}^{2m} \xi^{2m-j} X_j \tag{1.10}$$

on $t \in (-\infty, \infty)$ for each $\xi \in \mathbb{R}$ (or $\mathbb{T}$). Therefore, our interest in Theorem 1.5 is essentially on the regularity of the solution. Here, note that $X_j X_k = X_k X_j$ holds for any $0 \le j, k \le 2m$ if and only if $b_j = 0$ holds for any $1 \le j \le 2m$. If we assume this assumption, (1.9) is not a system but a single ordinary differential equation and

$$\widehat{u}(t,\xi) = \widehat{\varphi}(\xi) \exp it \Big(\xi^{2m} + \sum_{j=1}^{2m} \xi^{2m-j} a_j \Big) \tag{1.11}$$

for each $\xi \in \mathbb{R}$ (or $\mathbb{Z}$). Since $\gamma_j = 0$ and $\lambda_j = 2\mathrm{Im}\, a_j$, it follows that

$$|\widehat{u}(t,\xi)| = |\widehat{\varphi}(\xi)| \prod_{j=1}^{2m} \exp \frac{-t\xi^{2m-j}\lambda_j}{2},$$

by which we obtain Theorem 1.5 easily. On the other hand, it seems difficult to obtain Theorem 1.5 by (1.10) for general $\{b_j\}$ since $X_j X_k \ne X_k X_j$ for some j, k. To avoid this difficulty, we employ the energy estimate. Propositions 4.2 and 4.3 are main estimates in this paper. The first term of the left-hand side of (4.8) is the main part of the energy. The second term is the correction term. For "Dispersive type", the third and the fourth terms vanish. Thus, we easily obtain the L^2 a priori estimate. For "Parabolic type", the third term includes $\lambda_{2j^*} \| |\partial_x|^{m-j^*} u \|^2$. The parabolic smoothing is caused by the term. For "Twisted parabolic type", the fourth term includes $\lambda_{2j^*-1} \langle D_x^{2(m-j)+1} u, u \rangle$. We want to show the parabolic smoothing by making use of the term. However, the sign of the term is not definite. That is unfavorable in our argument. Therefore, we compute the energy inequalities of $P^+ u$ and $P^- u$ instead of u and obtain Proposition 4.4. Note that the sign of all terms except the correction terms in (4.11) and (4.12) are definite. Though (4.11) is the energy inequality for $\|P^+ u\|$, it includes $\lambda_j^- \| |\partial_x|^{m-j/2} P^- u \|^2$. This is because (4.1) is essentially coupled system of $P^+ u$ and $P^- u$ as (4.4). The term $\lambda_j^- \| |\partial_x|^{m-j/2} P^- u \|^2$ cannot be estimated by $\|u\|$. This is the main difficulty in the proof of "Twisted parabolic type" in Theorem 4.1. We analyse a property of $\{\lambda_j^-\}$ and use an additional correction term F_k^- to eliminate a bad effect caused by

$\lambda_j^- \||\partial_x|^{m-j/2} P^- u\|^2$ and obtain (4.9) (see also (4.10)). This is the key idea in this paper. This is a joint work with Professor Kotaro Tsugawa.

1.4. **Notations.** We denote the norm in $L^p(\mathcal{M})$ by $\|\cdot\|_p$. In particular, we simply write $\|\cdot\| := \|\cdot\|_2$. We write $D = |\partial_x| = \mathcal{F}^{-1}|\xi|\mathcal{F}$. We denote $\|f\|_{H^s} := 2^{-1/2}(\|f\|^2 + \|D^s f\|^2)^{1/2}$ for a function f and $s \geq 0$. Let $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2}$. We also use the same symbol for $\langle \cdot \rangle := (1 + |\cdot|^2)^{1/2}$. Let $[A, B] := AB - BA$, $P^+ f(x) := \mathcal{F}^{-1}(\chi(\xi \geq 1)\hat{f})(x)$, $P^- f(x) := \mathcal{F}^{-1}(\chi(\xi \leq -1)\hat{f})(x)$, $P_0 f(x) := \mathcal{F}^{-1}(\chi(|\xi| < 1)\hat{f})(x)$, $P_{\neq 0} f(x) := \mathcal{F}^{-1}(\chi(|\xi| \geq 1)\hat{f})(x)$. $\hat{f}$ is the Fourier transform of f:

$$\hat{f}(k) = \mathcal{F}f(k) = (2\pi)^{-1} \int_{\mathcal{M}} f(x) e^{-ixk} dx.$$

2. Local well-posedness for third order Benjamin-Ono type equations on the torus

2.1. **Introduction.** We consider the Cauthy problem of the following third order Benjamin-Ono type equations on the torus $\mathbb{T}(:= \mathbb{R}/2\pi\mathbb{Z})$:

$$\partial_t u - \partial_x^3 u + u^2 \partial_x u + c_1 \partial_x (u \mathcal{H} \partial_x u) + c_2 \mathcal{H} \partial_x (u \partial_x u) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{T}, \tag{2.1}$$

$$u(0,x) = \varphi(x), \tag{2.2}$$

where the initial data φ and the unknown function u are real valued, and $c_1, c_2 \in \mathbb{R}$. $\mathcal{H}$ is the Hilbert transform on the torus defined by

$$\widehat{\mathcal{H}f}(0) = 0 \quad \text{and} \quad \widehat{\mathcal{H}f}(k) = -i \operatorname{sgn}(k) \hat{f}(k), \quad k \in \mathbb{Z} \backslash \{0\}.$$

The well-known Benjamin-Ono equation

$$\partial_t u + \mathcal{H} \partial_x^2 u + 2u \partial_x u = 0 \tag{2.3}$$

describes the behavior of long internal waves in deep stratified fluids. The equation (2.3) also has infinitely many conservation laws, which generates a hierarchy of Hamiltonian equations of order j. The equation (2.1) with $c_1 = c_2 = \sqrt{3}/2$ is the second equation in the Benjamin-Ono hierarchy [20].

There are a lot of literature on the Cauchy problem on (2.3). On the real line case, Ionescu-Kenig [10] showed the local well-posedness in $H^s(\mathbb{R})$ for $s \geq 0$ (see also [23] for another proof and [11] for the local well-posedness with small complex valued data). On the periodic case, Molinet [21, 22] showed the local well-posedness in $H^s(\mathbb{T})$ for $s \geq 0$ and that this result was sharp. See [1, 2, 13, 15, 17, 25, 31] for former results.

On the Cauchy problem of (2.1) with $c_1 = c_2 = \sqrt{3}/2$ on the real line, Feng-Han [6] proved the unique existence in $H^s(\mathbb{R})$ for $4 \leq s \in \mathbb{N}$ by using the theory of complete integrability. They also used the energy method with a correction term in order to show the uniqueness. Feng [7] modified the energy method used in [6] and used an *a priori* bound of solutions in $H^s(\mathbb{R})$ to show the "weak" continuous dependence in the following sense:

$$\varphi_n \to \varphi \text{ in } H^{s-2}(\mathbb{R}) \text{ as } n \to \infty \Rightarrow u_n \to u \text{ in } C([0,T]; H^{s-2}(\mathbb{R})) \text{ as } n \to \infty, \tag{2.4}$$

for $\varphi, \varphi_n \in H^s(\mathbb{R})$ and $6 \leq s \in \mathbb{N}$. Here, u_n (resp. u) denotes the corresponding solution of (2.1) with $c_1 = c_2 = \sqrt{3}/2$ and the initial data φ_n for $n \in \mathbb{N}$ (resp. φ). Note that the topology of the convergence is weaker than H^s. Linares-Pilod-Ponce

[19] and Molinet-Pilod [24] succeed in proving the local well-posedness in $H^s(\mathbb{R})$ of the following equation

$$\partial_t u + d_1\partial_x^3 u - d_2\mathcal{H}\partial_x^2 u = d_3 u\partial_x u - d_4\partial_x(u\mathcal{H}\partial_x u + \mathcal{H}(u\partial_x u)), \tag{2.5}$$

for $s \geq 2$ and $s \geq 1$, respectively. Here, coefficients satisfy $d_1 \in \mathbb{R}$, $d_1 \neq 0$ and $d_j > 0$ for $j = 2, 3, 4$. Their proof involves the gauge transform and the Kato type smoothing estimate. Recently, Guo-Huo [9] showed the local well-posedness of (2.5) in $H^s(\mathbb{R})$ for $s \geq 3/4$ without the gauge transformation. They used so called the short-time $X^{s,b}$ method developed by Ionescu-Kenig-Tataru [12].

On the periodic case, as far as the author knows, there are no well-posedness results for the Cauchy problem of (2.1) available in the literature. Although proofs in Feng-Han [6] and Feng [7] above works, and we cannot obtain the local well-posedness, that is, the resultant continuous dependence (2.4) is weak. And their proofs heavily depend on the complete integrability. In particular, it is very important to have $c_1 = c_2$ in their proofs. It should also be pointed out that in the periodic case, we do not have the Kato type smoothing estimate, which implies that the local well-posedness is far from trivial.

Therefore, in this article, we are interested in establishing the local well-posedness of (2.1) in $H^s(\mathbb{T})$ for s less than 4 without using the theory of complete integrability. In particular, we improve the "weak" continuous dependence (2.4) shown in [7] in order to fulfill conditions of the local well-posedness. Moreover, thanks to Lemma 2.7, we can show the local well-posedness of the non-integrable case (2.1).

The main result is the following:

Theorem 2.1. *Let $s \geq s_0 > 5/2$. For any $\varphi \in H^s(\mathbb{T})$, there exist $T = T(\|\varphi\|_{H^{s_0}}) > 0$ and the unique solution $u \in C([-T,T]; H^s(\mathbb{T}))$ to the IVP (2.1)–(2.2) on $[-T,T]$. Moreover, for any $R > 0$, the solution map $\varphi \mapsto u(t)$ is continuous from the ball $\{\varphi \in H^s(\mathbb{T}); \|\varphi\|_{H^s} \leq R\}$ to $C([-T,T]; H^s(\mathbb{T}))$.*

Now, we mention the idea of the proof of Theorem 2.1. The standard energy method gives us the local well-posedness of (2.3) in $H^s(\mathbb{T})$ for $s > 3/2$. On the other hand, nonlinear terms $\partial_x(u\mathcal{H}\partial_x u)$ and $\mathcal{H}\partial_x(u\partial_x u)$ in (2.1) have two derivatives, and the energy estimate gives only the following:

$$\frac{d}{dt}\|\partial_x^k u(t)\|_{L^2}^2 \lesssim (1 + \|\partial_x^2 u\|_{L^\infty})^2\|\partial_x^k u(t)\|_{L^2}^2 + \left|\int \partial_x u(\mathcal{H}\partial_x^{k+1}u)\partial_x^k u dx\right|. \tag{2.6}$$

It is difficult to handle the last term in the right hand side by $\|u\|_{H^k}$, which is the main difficulty in this problem. To overcome that difficulty, we add a correction term into the energy (see Definition 3):

$$E_*(u) := \|u\|_{L^2}^2 + \|D^s u\|_{L^2}^2 + a_s\|u\|_{L^2}^{4s+2} + b_s \int u(\mathcal{H}D^s u)D^{s-2}\partial_x u dx,$$

where $D := \mathcal{F}^{-1}|\xi|\mathcal{F}$, following the idea from Kwon [18], who studied the local well-posedness of the fifth order KdV equation (see also Segata [27], Kenig-Pilod [16] and Tsugawa [33]). The correction term allows us to cancel out the worst term in (2.6), which makes it possible to evaluate the H^s-norm of the solution by that of the initial data. It is worth pointing out that our proof refines the idea in [7]. Indeed, Feng introduced the following energy estimate in order to show the "weak" continuous dependence (2.4):

$$\begin{aligned}&\frac{d}{dt}\left(\|\partial_x^{k-2}w\|_{L^2}^2 + \frac{2k-3}{4}\int_{\mathbb{R}}(u+v)\partial_x^{k-3}w\mathcal{H}\partial_x^{k-2}w dx\right)\\ &\le C(T, \|\varphi\|_{H^k}, \|\psi\|_{H^k})\|w(t)\|_{H^{k-2}}^2,\end{aligned}$$

on $[0,T]$, where $w = u - v$ and $u, v \in C([0,T]; H^k(\mathbb{R}))$ satisty (2.1) with $c_1 = c_2 = \sqrt{3}/2$ and initial data $\varphi, \psi \in H^k(\mathbb{R})$, respectively. Here, we would like to have the estimate for $\|w\|_{H^k}$. If we simply replace $k-2$ with k in the above estimate, the constant in the right hand side depends on $\|\varphi\|_{H^{k+2}}$ (resp. $\|\psi\|_{H^{k+2}}$), which cannot be handled by $\|\varphi\|_{H^k}$ (resp. $\|\psi\|_{H^k}$). Therefore, we need to find a different correction term (see Definition 3) and estimate the difference between two solutions in $H^k(\mathbb{T})$ more carefully (see the proof of Proposition 2.21) so as to complete the continuous dependence.

Another difficulty is the presence of the Hilbert transform $\mathcal{H}$, which restricts the possibility of using the integration by parts for some terms. Recall that for real valued functions f, g, we have

$$|\langle fD^s g, D^s\partial_x g\rangle_{L^2}| \le \frac{1}{2}\|\partial_x f\|_\infty \|D^s g\|_{L^2}^2.$$

However, in our problem we cannot apply the integration by parts to

$$\langle \partial_x f\mathcal{H}D^s\partial_x g, D^s g\rangle_{L^2},$$

which is nothing but the term which we cancel out by introducing a correction term.

We notice that the L^2-norm is conserved by solutions of equations (2.1) with $c_1 = c_2$ thanks to the following equality:

$$\langle \mathcal{H}\partial_x(u\partial_x u), u\rangle_{L^2} + \langle \partial_x(u\mathcal{H}\partial_x u), u\rangle_{L^2} = 0,$$

which helps us to handle nonlinear terms. In the case $c_1 \neq c_2$, we use Lemma 2.7 originally proved in [4].

Subsequently, using the conservation law corresponding to the H^3-norm of the solution, we can obtain the following result:

Corollary 2.2. *The Cauchy problem (2.1)–(2.2) with* $c_1 = c_2 = \sqrt{3}/2$ *is globally well-posed in* $H^s(\mathbb{T})$ *for* $s \geq 3$.

Although we focus on our problem on $\mathbb{T}$, our proof still holds on $\mathbb{R}$, i.e., we can obtain the local well-posedness for (2.1)–(2.2) on $\mathbb{R}$ in $H^s(\mathbb{R})$ for $s > 5/2$. Thus, we can improve the results shown in [7, 6]. There are two differences, and one is the following:

$$\mathcal{H}(\mathcal{H}f)(x) = \begin{cases} -f(x), & x \in \mathbb{R}, \\ -f(x) + \hat{f}(0), & x \in \mathbb{T}. \end{cases}$$

However, such a difference does not yield difficulties in our argument since we have $|\hat{f}(0)| \leq \|\hat{f}\|_{l^\infty(\mathbb{Z})} \leq \|f\|_{L^1(\mathbb{T})} \lesssim \|f\|_{L^2(\mathbb{T})}$. The other one is the Gagliardo-Nirenberg inequality (Lemma 2.3), that is, we do not need to add $\|f\|_{L^2(\mathbb{R})}$ on $\mathbb{R}$ when $l = 0$.

This section is organized as follows. In Subsection 2.2, we state a number of estimates. We also obtain a solution of the regularized equation associated to (2.1). In Subsection 2.3, we give an *a priori* estimate for the solution to (2.1). In Subsection 2.4, we show the existence of the solution, uniqueness, the persistence, and the continuous dependence.

2.2. **Preliminaries and parabolic regularization.** In this subsection, we collect a number of estimates which will be used throughout this paper. We use the following Gagliardo-Nirenberg inequality on the torus:

Lemma 2.3. *Assume that* $l \in \mathbb{N} \cup \{0\}$ *and* $s \geq 1$ *satisfy* $l \leq s - 1$ *and a real number* p *satisfies* $2 \leq p \leq \infty$. *Put* $\alpha = (l + 1/2 - 1/p)/s$. *Then, we have*

$$\|\partial_x^l f\|_p \lesssim \begin{cases} \|f\|^{1-\alpha}\|D^s f\|^\alpha \quad (\textit{when} \quad 1 \leq l \leq s-1), \\ \|f\|^{1-\alpha}\|D^s f\|^\alpha + \|f\| \quad (\textit{when} \quad l = 0), \end{cases}$$

for any $f \in H^s(\mathbb{T})$.

Proof. In the case s is an integer, see Section 2 in [26]. The general case follows from the integer case and the Hölder inequality. □

The following inequality is helpful when we estimate the difference between two solutions in L^2.

Lemma 2.4. *Let $k \in \mathbb{N} \cup \{0\}$. Then the following inequality holds true:*

$$\|\mathcal{H}\partial_x^k f + \langle D\rangle^{-1}\partial_x^{k+1} f\| \le 2\|f\|_{H^{k-2}}$$

for any $f \in H^{k-2}(\mathbb{T})$.

Proof. We claim that

$$|\operatorname{sgn}(\xi) - \xi\langle\xi\rangle^{-1}| \le 2\langle\xi\rangle^{-2} \tag{2.7}$$

for any $\xi \in \mathbb{Z}$. When $0 \le \xi \le 1$, we have $|1 - \xi\langle\xi\rangle^{-1}| \le 1 \le 2\langle\xi\rangle^{-2}$. Let $\xi > 1$. Set $g(x) := (x + \xi^2)^{1/2}$ for $x \ge 0$. The mean value theorem shows that there exists $\theta \in (0, 1)$ such that $g(1) - g(0) = g'(\theta)$. It then follows that

$$\frac{|\langle\xi\rangle - \xi|}{\langle\xi\rangle} \le \frac{1}{2\xi\langle\xi\rangle} \le \frac{1}{(1+\xi)\langle\xi\rangle} \le \langle\xi\rangle^{-2},$$

which shows (2.7) when $\xi \ge 0$. We omit the proof of (2.7) when $\xi < 0$ since it is similar. Therefore, using (2.7), we obtain

$$\|\mathcal{H}\partial_x^k f + \langle D\rangle^{-1}\partial_x^{k+1} f\| = \|(\operatorname{sgn}(\xi) - \xi\langle\xi\rangle^{-1})\xi^k \hat{f}(\xi)\|_{l^2} \le 2\|f\|_{H^{k-2}},$$

as desired. □

Definition 2. For $s \ge 0$ and functions f, g defined on $\mathbb{T}$, we define

$$\begin{aligned} P_s(f,g) &:= D^s\partial_x(f\partial_x g) - D^s\partial_x f\partial_x g - fD^s\partial_x^2 g - (s+1)\partial_x f D^s\partial_x g, \\ Q_s(f,g) &:= \mathcal{H}D^s\partial_x(f\partial_x g) - (\mathcal{H}D^s\partial_x f)\partial_x g - f\mathcal{H}D^s\partial_x^2 g \\ &\quad - (s+1)\partial_x f\mathcal{H}D^s\partial_x g. \end{aligned}$$

We introduce several commutator estimates. For general theory on the real line, see [8]. We shall use extensively the following commutator estimate.

Lemma 2.5. *Let $s \ge 1$ and $s_0 > 5/2$. Then there exists $C = C(s, s_0) > 0$ such that for any $f, g \in H^s(\mathbb{T}) \cap H^{s_0}(\mathbb{T})$,*

$$\|P_s(f,g)\|, \|Q_s(f,g)\| \le C(\|f\|_{H^{s_0}}\|g\|_{H^s} + \|f\|_{H^s}\|g\|_{H^{s_0}}).$$

Proof. We show only the inequality for $P_s(f, g)$ with $s > 1$. The case $s = 1$ follows from Lemma 2.7. The estimate for $Q_s(f, g)$ follows from a similar argument since $D = \mathcal{H}\partial_x$. It suffices to show that there exists $C = C(s)$ such that

$$\begin{aligned} &||\xi|^s\xi\eta - |\xi-\eta|^s(\xi-\eta)\eta - |\eta|^s\eta^2 - (s+1)(\xi-\eta)|\eta|^s\eta| \\ &\le C(|\xi-\eta|^s|\eta|^2 + |\xi-\eta|^2|\eta|^s) \end{aligned} \tag{2.8}$$

for any $\xi, \eta \in \mathbb{Z}$. We split the summation region into three regions: $R_1 = \{3|\eta| \le |\xi - \eta|\}$, $R_2 = \{|\eta| \ge 3|\xi - \eta|\}$ and $R_3 = \{|\xi - \eta|/4 \le |\eta| \le 4|\xi - \eta|\}$. On R_1, the mean value theorem shows that (2.8) holds. On R_2, note that $|\xi| \sim |\eta|$. It immediately follows that $|\xi-\eta|^s(\xi-\eta)\eta \lesssim |\xi-\eta|^s|\eta|^2$. Set $\sigma(x) = x|x|^s$ for $x \in \mathbb{R}$. Note that $\sigma \in C^2(\mathbb{R})$. The Taylor theorem shows that there exist $\tilde{\eta} \in (\xi,\eta)$ or $\tilde{\eta} \in (\eta,\xi)$ such that

$$\sigma(\xi) = \sigma(\eta) + \sigma'(\eta)(\xi-\eta) + \frac{\sigma''(\tilde{\eta})}{2}(\xi-\eta)^2.$$

This together with the fact that $|\tilde{\eta}| \sim |\xi| \sim |\eta|$ implies that (2.8) holds. On R_3, it is obvious. □

Lemma 2.6. *Let $s \geq 1$, $s_0 > 1/2$ and $\Lambda_s = D^s$ or $D^{s-1}\partial_x$. Then we have the following:*
(i) There exists $C(s,s_0) > 0$ such that for any $f,g \in H^{s_0+1}(\mathbb{T}) \cap H^s(\mathbb{T})$,

$$\|[\Lambda_s, f]\partial_x g\| \leq C(\|f\|_{H^{s_0+1}}\|g\|_{H^s} + \|f\|_{H^s}\|g\|_{H^{s_0+1}}).$$

(ii) There exists $C(s_0) > 0$ such that for any $f \in H^{s_0+1}(\mathbb{T})$ and $g \in L^2(\mathbb{T})$,

$$\|[\langle D\rangle^{-1}\Lambda_2, f]g\| \leq C\|f\|_{H^{s_0+1}}\|g\|.$$

Proof. We omit the proof of the (i) since it is identical with that of the previous lemma. We show the case (ii) with $\Lambda_2 = \partial_x^2$ only. The other case follows from a similar argument. It suffices to show that $|\xi^2\langle\xi\rangle^{-1} - \eta^2\langle\eta\rangle^{-1}| \lesssim |\xi-\eta|$ for any $\xi,\eta \in \mathbb{Z}$. Set $\sigma(x) = -x^2\langle x\rangle^{-1}$ for $x \in \mathbb{R}$. Note that $\sigma \in C^1(\mathbb{R})$ and that $\sigma'(x) = -(x^3+2x)\langle x\rangle^{-3}$. It then follows that there exists $C > 0$ such that $|\sigma'(x)| \leq C$ for any $x \in \mathbb{R}$. This together with the mean value theorem implies that we have

$$|\sigma(\xi) - \sigma(\eta)| \leq C|\xi-\eta|,$$

which completes the proof. □

The following estimate is essential for our analysis in the case $c_1 \neq c_2$ in (2.1). For L^p cases on the real line, see [4].

Lemma 2.7. *Let $s_0 > 1/2$ and $k \in \mathbb{N}$. Then, there exists $C = C(s_0) > 0$ such that for any $f \in H^{s_0}(\mathbb{T})$ and $g \in L^2(\mathbb{T})$*

$$\|[\mathcal{H}, f]\partial_x^k g\| \leq C\|f\|_{H^{s_0+k}}\|g\|.$$

Proof. It suffices to show that

$$|\operatorname{sgn}(\xi) - \operatorname{sgn}(\eta)||\eta|^k \lesssim |\xi-\eta|^k \tag{2.9}$$

for any $\xi,\eta \in \mathbb{Z}$. We split the summation region into three regions: $R_1 = \{3|\eta| \leq |\xi|\}$, $R_2 = \{|\eta| \geq 3|\xi|\}$ and $R_3 = \{|\xi|/4 \leq |\eta| \leq 4|\xi|\}$. It is clear that (2.9) holds on R_1 and R_2. It is also clear that (2.9) holds when $\xi\eta > 0$. Therefore, we consider the region $R_3 \cap \{\xi\eta \le 0\}$. We first assume that $\xi \ge 0$ and $\eta \le 0$. Note that $|\xi - \eta| \ge |\xi| \ge |\eta|/4$. Similarly, in the case $\xi \le 0$ and $\eta \ge 0$ we have $|\xi - \eta| \ge |\eta|$. Therefore, we have (2.9), which concludes the proof. □

Lemma 2.8. *Let $s_0 > 1/2$ and u, v be sufficiently smooth function defined on $\mathbb{T}$. Then there exists $C = C(s_0) > 0$ such that*

$$|\langle v\mathcal{H}\partial_x^2 u + \partial_x v\mathcal{H}\partial_x u, u\rangle| \le C\|v\|_{H^{s_0+2}}\|u\|^2.$$

Proof. This follows from the equality

$$2\langle v\mathcal{H}\partial_x^2 u + \partial_x v\mathcal{H}\partial_x u, u\rangle = -\langle[\mathcal{H}, v]\partial_x^2 u, u\rangle - \langle\partial_x^2 v\mathcal{H}u, u\rangle$$

together with Lemma 2.7. □

We shall also use extensively the following estimate.

Lemma 2.9. *Let $s_0 > 1/2$. Then, there exists $C = C(s_0) > 0$ such that for any $f \in H^{s_0+1}(\mathbb{T})$ and $g \in H^1(\mathbb{T})$*

$$|\langle f\partial_x g, g\rangle| \le C\|f\|_{H^{s_0+1}}\|g\|^2.$$

Proof. This follows from the density argument and the integration by parts. □

The following lemma helps us calculate a correction term.

Lemma 2.10. *For sufficiently smooth functions f, g and h defined on $\mathbb{T}$, it holds that*

$$\langle\partial_x^3 fg, h\rangle + \langle f\partial_x^3 g, h\rangle + \langle fg, \partial_x^3 h\rangle = 3\langle\partial_x f\partial_x g, \partial_x h\rangle.$$

Proof. See Lemma 2.2 in [16]. □

We shall repeatedly use estimates of the following type:

Lemma 2.11. *Let $s_0 > 5/2$.*
(i) Let $s \ge 1$. There exists $C(s, s_0) > 0$ such that for any $f_1 \in H^s(\mathbb{T}) \cap H^{s_0}(\mathbb{T})$ and $f_2 \in H^{s+1}(\mathbb{T}) \cap H^{s_0}(\mathbb{T})$,

$$\begin{aligned}&|\langle f_1\mathcal{H}D^s f_2, \mathcal{H}D^s(f_1\partial_x f_2)\rangle|\\ &\le C(\|f_1\|_{H^{s_0}}^2\|f_2\|_{H^s}^2 + \|f_1\|_{H^{s_0}}\|f_1\|_{H^s}\|f_2\|_{H^{s_0}}\|f_2\|_{H^s}).\end{aligned}$$

(ii) Let $s \geq 2$. There exists $C(s, s_0) > 0$ such that for any $f_1 \in H^{s+1}(\mathbb{T}) \cap H^{s_0}(\mathbb{T})$ and $f_2 \in H^{s+2}(\mathbb{T}) \cap H^{s_0}(\mathbb{T})$,

$$\begin{aligned}&|\langle f_1 \mathcal{H} D^s \partial_x (f_1 \mathcal{H} \partial_x f_2), D^{s-2} \partial_x f_2 \rangle| \\ &\leq C(\|f_1\|_{H^{s_0}}^2 \|f_2\|_{H^s}^2 + \|f_1\|_{H^{s_0}} \|f_1\|_{H^s} \|f_2\|_{H^{s_0}} \|f_2\|_{H^s}).\end{aligned}$$

Proof. First we show (i). Note that

$$\begin{aligned}&|\langle f_1 \mathcal{H} D^s f_2, \mathcal{H} D^s (f_1 \partial_x f_2) \rangle| \\ &\leq |\langle f_1 \mathcal{H} D^s f_2, [\mathcal{H} D^s, f_1] \partial_x f_2 \rangle| + |\langle f_1^2 \mathcal{H} D^s f_2, \mathcal{H} D^s \partial_x f_2 \rangle|.\end{aligned}$$

Lemma 2.9 together with (i) of Lemma 2.6 shows (i). Next we show (ii). Lemma 2.5 shows that

$$\begin{aligned}&|\langle D^{s+1}(f_1 \mathcal{H} \partial_x f_2), f_1 D^{s-2} \partial_x f_2 \rangle - R_1 - R_2 - R_3| \\ &\lesssim \|f_1\|_{H^{s_0}}^2 \|f_2\|_{H^s}^2 + \|f_1\|_{H^{s_0}} \|f_1\|_{H^s} \|f_2\|_{H^{s_0}} \|f_2\|_{H^s},\end{aligned}$$

where $R_1 = \langle D^{s+1} f_1 \mathcal{H} \partial_x f_2, f_1 D^{s-2} \partial_x f_2 \rangle$, $R_2 = \langle f_1 \mathcal{H} D^{s+1} \partial_x f_2, f_1 D^{s-2} \partial_x f_2 \rangle$ and $R_3 = (s+1) \langle \partial_x f_1 \mathcal{H} D^{s+1} f_2, f_1 D^{s-2} \partial_x f_2 \rangle$. It is easy to see that

$$|R_1| \lesssim \|f_1\|_{H^{s_0}} \|f_1\|_{H^s} \|f_2\|_{H^{s_0}} \|f_2\|_{H^s} \quad \text{and} \quad |R_3| \lesssim \|f_1\|_{H^{s_0}}^2 \|f_2\|_{H^s}^2.$$

For R_2, we have

$$\begin{aligned}R_2 &= -\langle f_1^2 D^s \partial_x^2 f_2, D^{s-2} \partial_x f_2 \rangle \\ &= 2 \langle f_1 \partial_x f_1 D^s \partial_x f_2, D^{s-2} \partial_x f_2 \rangle - \langle f_1^2 D^s \partial_x f_2, D^s f_2 \rangle \\ &= -2 \langle \partial_x (f_1 \partial_x f_1 D^{s-2} \partial_x f_2), D^s f_2 \rangle + \langle f_1 \partial_x f_1, (D^s f_2)^2 \rangle,\end{aligned}$$

which can be bounded by $\lesssim \|f_1\|_{H^{s_0}}^2 \|f_2\|_{H^s}^2$. This concludes the proof. □

Lemma 2.12. *For any $s \geq 1$ and $s_0 > 5/2$, there exists $C(s, s_0) > 0$ such that for any $u, v \in H^{s+2}(\mathbb{T}) \cap H^{s_0}(\mathbb{T})$,*

$$\begin{aligned}&|\langle D^s \partial_x (u \mathcal{H} \partial_x u - v \mathcal{H} \partial_x v), D^s w \rangle - s \langle \partial_x u \mathcal{H} D^s \partial_x w, D^s w \rangle| \\ &+ |\langle \mathcal{H} D^s \partial_x (u \partial_x u - v \partial_x v), D^s w \rangle - (s+1) \langle \partial_x u \mathcal{H} D^s \partial_x w, D^s w \rangle| \\ &\leq C \|w\|_{H^s} \{(\|u\|_{H^{s_0}} + \|v\|_{H^{s_0}}) \|w\|_{H^s} + (\|u\|_{H^s} + \|v\|_{H^s}) \|w\|_{H^{s_0}} \\ &\quad + \|w\|_{H^{s_0 - 2}} \|v\|_{H^{s+2}} + \|w\|_{H^{s_0 - 1}} \|v\|_{H^{s+1}}\},\end{aligned}$$

where $w = u - v$.

Proof. Adding and subtracting terms, we obtain

$$\begin{aligned}
&|\langle D^s\partial_x(u\mathcal{H}\partial_x w + w\mathcal{H}\partial_x v), D^s w\rangle - s\langle \partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle| \\
&\le |\langle P_s(u,\mathcal{H}w) + P_s(w,\mathcal{H}v), D^s w\rangle| + |\langle D^s\partial_x u\mathcal{H}\partial_x w, D^s w\rangle| \\
&\quad + |\langle u\mathcal{H}D^s\partial_x^2 w + \partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle| + \frac{1}{2}|\langle \mathcal{H}\partial_x^2 v, (D^s w)^2\rangle| \\
&\quad + |\langle w\mathcal{H}D^s\partial_x^2 v, D^s w\rangle| + (s+1)|\langle \partial_x w\mathcal{H}D^s\partial_x v, D^s w\rangle|, \\
&|\langle \mathcal{H}D^s\partial_x(u\partial_x w + w\partial_x v), D^s w\rangle - (s+1)\langle \partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle| \\
&\le |\langle Q_s(u,w) + Q_s(w,v), D^s w\rangle| + |\langle u\mathcal{H}D^s\partial_x^2 w + \partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle| \\
&\quad + |\langle w\mathcal{H}D^s\partial_x^2 v, D^s w\rangle| + (s+2)|\langle \partial_x w\mathcal{H}D^s\partial_x v, D^s w\rangle|
\end{aligned}$$

since we have

$$\langle \partial_x w\mathcal{H}D^s\partial_x u + \partial_x v\mathcal{H}D^s\partial_x w, D^s w\rangle = \langle \partial_x u\mathcal{H}D^s\partial_x w + \partial_x w\mathcal{H}D^s\partial_x v, D^s w\rangle.$$

Note that

$$\begin{aligned}
|\langle D^s\partial_x u\mathcal{H}\partial_x w, D^s w\rangle| &= |\langle D^s\partial_x w\mathcal{H}\partial_x w, D^s w\rangle + \langle D^s\partial_x v\mathcal{H}\partial_x w, D^s w\rangle| \\
&\lesssim \|w\|_{H^{s_0}}\|w\|_{H^s}^2 + \|w\|_{H^s}\|w\|_{H^{s_0-1}}\|v\|_{H^{s+1}}
\end{aligned}$$

by Lemma 2.9. This together with Lemma 2.5 and 2.8 gives the desired inequality, which completes the proof. □

Definition 3. Let $s \ge 2$ and $a, b, c \ge 0$. Set $\lambda(s') = -2((c_1+c_2)s' + c_2)/3$ for $s' \ge 0$. For $f, g \in H^s(\mathbb{T})$ we define

$$\begin{aligned}
E_s(f,g;a) &:= a\|f-g\|^2 + \|D^s(f-g)\|^2 \\
&\qquad + \lambda(s)\int_{\mathbb{T}} f(\mathcal{H}D^s(f-g))D^{s-2}\partial_x(f-g)dx, \\
E_s(f;b) &:= E_s(f,0;1) + b\|f\|^{4s+2}.
\end{aligned}$$

For $f, g \in L^2(\mathbb{T})$ we define

$$\tilde{E}(f,g;c) := c\|f-g\|_{H^{-1}}^2 + \|f-g\|^2 - \lambda(0)\int_{\mathbb{T}} f(\langle D\rangle^{-1}(f-g))(f-g)dx.$$

Lemma 2.13. *Let $s \ge s_0 > 5/2$ and $K > 0$. Then*
(i) If $f, g \in H^s(\mathbb{T})$ and f satisfies $\|f\| \le K$, then there exist $C = C(s,K)$ and $a = a(s,K)$ such that

$$\|f-g\|_{H^s}^2 \le E_s(f,g;a) \le C\|f-g\|_{H^s}^2. \tag{2.10}$$

(ii) If $f \in H^s(\mathbb{T})$, there exist $C = C(s)$ and $b = b(s)$ such that

$$\|f\|_{H^s}^2 \le E_s(f;b) \le C(1+\|f\|^{4s})\|f\|_{H^s}^2 \tag{2.11}$$

(iii) If $f, g \in L^2(\mathbb{T})$ and f satisfies $\|f\| \le K$, then there exist $c = c(K)$ and $C = C(K)$ such that

$$\frac{1}{2}\|f-g\|^2 \le \tilde{E}(f,g;c) \le C\|f-g\|^2. \tag{2.12}$$

Proof. We see from Lemma 2.3 and the Young inequality that

$$\begin{aligned}\int_{\mathbb{T}} |f(\mathcal{H}D^s(f-g))D^{s-2}\partial_x(f-g)|dx &\le \|f\|\|D^s(f-g)\|\|D^{s-2}\partial_x(f-g)\|_\infty \\ &\le C\|f-g\|^{1/2s}\|D^s(f-g)\|^{2-1/2s} \\ &\le C\|f-g\|^2 + \frac{1}{2}\|D^s(f-g)\|^2.\end{aligned}$$

Choosing $a > 0$ so that $a - C \ge 1/2$, we obtain the left hand side of (2.10). The right hand side of (2.10) follows immediately, which shows (i).

Next we prove (2.11). A similar argument to the proof of (2.10) yields that

$$\int_{\mathbb{T}} |f(\mathcal{H}D^s f)D^{s-2}\partial_x f|dx \le C\|f\|^{4s+2} + \frac{1}{2}\|D^s f\|^2.$$

Choosing $b > 0$ so that $b - C > 1/2$, we obtain (2.11). The proof of (iii) is identical with that of (i). □

In what follows, we simply write $E_s(f,g) := E_s(f,g;a)$, $E_s(f) := E_s(f;b)$ and $\tilde{E}_s(f,g) := \tilde{E}_s(f,g;c)$, where a, b and c are defined by Lemma 2.13.

Definition 4. Let $s \ge 0$, $f \in H^s(\mathbb{T})$ and $\gamma \in (0,1)$. And let $\rho \in C_0^\infty(\mathbb{R})$ satisfy $\operatorname{supp}\rho \subset [-2,2]$, $0 \le \rho \le 1$ on $\mathbb{R}$ and $\rho \equiv 1$ on $[-1,1]$. We put

$$\widehat{J_\gamma f}(k) := \rho(\gamma k)\hat{f}(k).$$

For the proof of the following lemma, see Remark 3.5 in [5].

Lemma 2.14. *Let $s \ge 0$, $\alpha \ge 0$, $\gamma \in (0,1)$ and $f \in H^s(\mathbb{T})$. Then, $J_\gamma f \in H^\infty(\mathbb{T})$ satisfies*

$$\|J_\gamma f - f\|_{H^s} \to 0 \quad (\gamma \to 0), \quad \|J_\gamma f - f\|_{H^{s-\alpha}} \lesssim \gamma^\alpha \|f\|_{H^s},$$
$$\|J_\gamma f\|_{H^{s-\alpha}} \le \|f\|_{H^{s-\alpha}}, \quad \|J_\gamma f\|_{H^{s+\alpha}} \lesssim \gamma^{-\alpha}\|f\|_{H^s}.$$

We employ the parabolic regularization on the problem (2.1)-(2.2):

$$\partial_t u - \partial_x^3 u + u^2\partial_x u + c_1\partial_x(u\mathcal{H}\partial_x u) + c_2\mathcal{H}\partial_x(u\partial_x u) = -\gamma D^{5/2}u, \tag{2.13}$$

$$u(0,x) = \varphi(x), \tag{2.14}$$

where $(t,x) \in [0,\infty) \times \mathbb{T}$ and $\gamma \in (0,1)$. In what follows, we only consider $t \geq 0$. In the case $t \leq 0$, we only need to replace $-\gamma D^{5/2}u$ with $\gamma D^{5/2}u$ in (2.13).

Proposition 2.15. *Let $s \geq 2$ and $\gamma \in (0,1)$. For any $\varphi \in H^s(\mathbb{T})$, there exist $T_\gamma \in (0,\infty]$ and the unique solution $u \in C([0,T_\gamma), H^s(\mathbb{T}))$ to the IVP (2.13)–(2.14) on $[0,T_\gamma)$ such that (i) $\liminf_{t\to T_\gamma} \|u(t)\|_{H^2} = \infty$ or (ii) $T_\gamma = \infty$ holds. Moreover, u satisfies*

$$u \in C((0,T_\gamma), H^{s+\alpha}(\mathbb{T})), \quad \forall \alpha > 0. \tag{2.15}$$

Proof. This follows from the standard argument, for example, see Proposition 2.8 in [33], but we reproduce the proof here for the sake of completeness. First we consider the case $s=2$. For simplicity, set $F(u) = -u^2\partial_x u - c_1\partial_x(u\mathcal{H}\partial_x u) - c_2\mathcal{H}\partial_x(u\partial_x u)$. Let $U_\gamma(t)$ be the linear propagator of the linear part of (2.13), i.e.,

$$U_\gamma(t)\varphi = \mathcal{F}^{-1}[e^{-i\xi^3 t - \gamma|\xi|^{5/2}t}\hat{\varphi}]$$

for a function φ. Note that

$$\|D^\alpha U_\gamma(t)\varphi\| \leq \frac{C(\alpha)}{(\gamma t)^{2\alpha/5}}\|\varphi\| \quad \text{and} \quad \|U_\gamma(t)\varphi\|_{H^\alpha} \leq C(\alpha)(1+(\gamma t)^{-2\alpha/5})\|\varphi\| \tag{2.16}$$

for $t>0$ and $\alpha>0$. We show the map

$$\Gamma(u(t)) = U_\gamma(t)\varphi + \int_0^t U_\gamma(t-\tau)F(u)d\tau$$

is a contraction on the ball

$$B_r = \left\{ u \in C([0,T]; H^2(\mathbb{T})); \|u\|_X := \sup_{t\in[0,T]} \|u(t)\|_{H^2} \leq r \right\},$$

where $r>0$ and T will be chosen later (which is sufficiently small and depends only on $\|\varphi\|_{H^2}$ and γ). Set $r = 2\|\varphi\|_{H^s}$. We show that Γ maps from B_r to B_r. Let $u \in B_r$. Obviously,

$$\|\Gamma(u(t))\|_{H^2} \leq \|\varphi\|_{H^2} + \int_0^t \|U_\gamma(t-t')F(u)\|_{H^2}dt'.$$

The Plancherel theorem implies that

$$\begin{aligned}\|U_\gamma(t-t')\partial_x u^3\|_{H^2} &= \|\langle\xi\rangle^2|\xi|e^{-\gamma(t-t')|\xi|^{5/2}}\mathcal{F}u^3\|_{l^2} \\ &\lesssim \gamma^{-2/5}(t-t')^{-2/5}\|u^3\|_{H^2} \lesssim \gamma^{-2/5}(t-t')^{-2/5}\|\varphi\|_{H^2}^3.\end{aligned}$$

Similarly, we have

$$\|U_\gamma(t-t')\mathcal{H}\partial_x(u\partial_x u)\|_{H^2} \lesssim \gamma^{-4/5}(t-t')^{-4/5}\|\varphi\|_{H^2}^2.$$

On the other hand,

$$\|U_\gamma(t-t')\partial_x(u\mathcal{H}\partial_x u)\|_{H^2} \lesssim (1+\gamma^{-4/5}(t-t')^{-4/5})\|\varphi\|_{H^2}^2.$$

It then follows that

$$\begin{aligned}&\sup_{t\in[0,T]}\|\Gamma(u(t))\|_{H^2}\\&\le \|\varphi\|_{H^2}+C\{\|\varphi\|_{H^2}^2\gamma^{-2/5}T^{3/5}+\|\varphi\|_{H^2}(T+\gamma^{-4/5}T^{1/5})\}\|\varphi\|_{H^2}\le 2\|\varphi\|_{H^2}\end{aligned}$$

for sufficiently small $T=T(\|\varphi\|_{H^2},\gamma)>0$ and any $u\in B_r$. By a similar argument, we can show that $\|\Gamma(u)-\Gamma(v)\|_X\le 2^{-1}\|u-v\|_X$ when $u,v\in B_r$. Therefore, Γ is a contraction map from B_r to B_r, which implies that there exists $u\in B_r$ such that $u=\Gamma(u)$ on $[0,T]$. Since $\|u(T)\|_{H^2}$ is finite, we can repeat the argument above with initial data $u(T)$ to obtain the solution on $[T,T+T']$. Iterating this process, we can extend the solution on $[0,T_\gamma)$ where $T_\gamma=\infty$ or $\liminf_{t\to T_\gamma}\|u(t)\|_{H^2}=\infty$ holds.

Next, we consider the case $s>2$. The solution obtained by the argument above satisfies

$$u(t)=U_\gamma(t)\varphi+\int_0^t U_\gamma(t-t')F(u)dt'. \tag{2.17}$$

Note that

$$\|U_\gamma(t-t')\partial_x u^3\|_{H^s}\lesssim \gamma^{-2/5}(t-t')^{-2/5}\|u^3\|_{H^s}\lesssim \gamma^{-2/5}(t-t')^{-2/5}\|\varphi\|_{H^2}^2\|\varphi\|_{H^s}.$$

We can estimate the other nonlinear terms in the same manner as above. It then follows that

$$\begin{aligned}&\sup_{t\in[0,T]}\|u(t)\|_{H^s}\\&\le \|\varphi\|_{H^s}+C\{\|\varphi\|_{H^2}^2\gamma^{-2/5}T^{3/5}+\|\varphi\|_{H^2}(T+\gamma^{-4/5}T^{1/5})\}\|\varphi\|_{H^s}\\&\le 2\|\varphi\|_{H^s}\end{aligned}$$

for sufficiently small $T=T(\|\varphi\|_{H^2},\gamma)>0$. By using (2.17), we also obtain $u\in C([0,T];H^s(\mathbb{T}))$. Since $\|u(T)\|_{H^s}$ is finite, we can repeat the argument above with initial data $u(T)$ to obtain $u\in C([T,T+T'];H^s(\mathbb{T}))$. We can iterate this process as far as $\|u(t)\|_{H^2}<\infty$. Therefore, we obtain $u\in C([0,T_\gamma);H^s(\mathbb{T}))$. We omit the proof of the uniqueness since it follows from a standard argument. Let $0<\delta<T_\gamma/2$. We see from (2.16) and (2.17) that $u\in C([\delta,T_\gamma);H^{s+1/4}(\mathbb{T}))$. The same argument as above with the initial data $u(\delta)\in H^{s+1/4}(\mathbb{T})$ shows that $u\in C([\delta+\delta/2,T_\gamma);H^{s+1/2}(\mathbb{T}))$. Iterating this procedure, we obtain (2.15) since δ is arbitrary, which completes the proof. □

2.3. **Energy estimate.** In this subsection, we obtain an *a priori* estimate of the solution of (2.1), which is important to have the time T independent of γ.

Proposition 2.16. *Let $s \geq s_0 > 5/2$, $\gamma \in (0,1)$, $\varphi \in H^s(\mathbb{T})$. Let $T_\gamma > 0$ and let $u \in C([0,T_\gamma), H^s(\mathbb{T})) \cap C((0,T_\gamma); H^{s+3}(\mathbb{T}))$ be the solution to (2.13)–(2.14), both of which are obtained by Proposition 2.15. Then, there exist $T = T(s_0, \|\varphi\|_{H^{s_0}}) > 0$ and $C = C(s, s_0, \|\varphi\|_{H^{s_0}}) > 0$ such that*

$$T_\gamma \geq T, \quad \sup_{t\in[0,T]} E_s(u(t)) \leq CE_s(\varphi), \quad \frac{d}{dt}E_s(u(t)) \leq CE_s(u(t)) \tag{2.18}$$

on $[0,T]$, where T (resp. C) is monotone decreasing (resp. increasing) with $\|\varphi\|_{H^{s_0}}$.

Before proving Proposition 2.16, we give the following lemma.

Lemma 2.17. *Let $s \geq s_0 > 5/2$, $\gamma \in [0,1)$, $T > 0$, $u \in C([0,T], H^s(\mathbb{T})) \cap C((0,T]; H^{s+3}(\mathbb{T}))$ satisfy (2.13) on $[0,T] \times \mathbb{T}$ and $\sup_{t\in[0,T]} E_{s_0}(u(t)) \leq K$ for $K > 0$. Then, there exists $C = C(s, s_0, K) > 0$ such that*

$$\frac{d}{dt}E_s(u(t)) \leq CE_s(u(t))$$

on $[0,T]$.

Proof. First observe that

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2 &= 2\langle \partial_x^3 u - u^2\partial_x u - c_1\partial_x(u\mathcal{H}\partial_x u) - c_2\mathcal{H}\partial_x(u\partial_x u) - \gamma D^{5/2}u, u\rangle \\ &\lesssim \|u(t)\|_{H^1}^2 \leq \|u(t)\|_{H^s}^2.\end{aligned}$$

We can estimate the time derivative of $\|u(t)\|^{4s+2}$ in a similar manner. Next we consider

$$\begin{aligned}&\frac{d}{dt}\|D^s u\|^2 \\ &= 2\langle D^s\partial_x^3 u, D^s u\rangle - 2\langle D^s(u^2\partial_x u), D^s u\rangle - 2c_1\langle D^s\partial_x(u\mathcal{H}\partial_x u), D^s u\rangle \\ &\quad - 2c_2\langle \mathcal{H}D^s\partial_x(u\partial_x u), D^s u\rangle - 2\gamma\langle D^{s+5/2}u, D^s u\rangle \\ &=: R_1 + R_2 + R_3 + R_4 + R_5.\end{aligned}$$

It is clear that $R_1 = 0$. We have

$$|R_2| \leq 2|\langle [D^s, u^2]\partial_x u, D^s u\rangle| + 2|\langle u^2 D^s\partial_x u, D^s u\rangle| \lesssim \|u\|_{H^s}^2$$

by (i) of Lemma 2.6 and Lemma 2.9. Lemma 2.12 with $v = 0$ shows that

$$\begin{aligned}&|R_3 + 2c_1 s\langle \partial_x u\mathcal{H}D^s\partial_x u, D^s u\rangle| + |R_4 + 2c_2(s+1)\langle \partial_x u\mathcal{H}D^s\partial_x u, D^s u\rangle| \\ &\lesssim \|u\|_{H^s}^2\end{aligned}$$

Finally, we have $R_5 = -2\gamma\|D^{s+5/4}u\|^2$. Therefore, we have

$$\frac{d}{dt}\|D^s u\|^2 \le C\|u\|_{H^s}^2 + 3\lambda(s)\int_{\mathbb{T}} \partial_x u(\mathcal{H}D^s\partial_x u)D^s u dx - 2\gamma\|D^{s+5/4}u\|^2, \qquad (2.19)$$

where $\lambda(s)$ is defined in Definition 3. Next we evaluate the correction term. We put

$$\begin{aligned}
&\frac{d}{dt}\langle u\mathcal{H}D^s u, D^{s-2}\partial_x u\rangle \\
&= \langle \partial_t u\mathcal{H}D^s u, D^{s-2}\partial_x u\rangle + \langle u\mathcal{H}D^s\partial_t u, D^{s-2}\partial_x u\rangle + \langle u\mathcal{H}D^s u, D^{s-2}\partial_x\partial_t u\rangle \\
&=: R_6 + R_7 + R_8.
\end{aligned}$$

Moreover, we set

$$\begin{aligned}
R_6 =& \langle \partial_x^3 u\mathcal{H}D^s u, D^{s-2}\partial_x u\rangle - \langle u^2\partial_x u\mathcal{H}D^s u, D^{s-2}\partial_x u\rangle \\
&- c_1\langle \partial_x(u\mathcal{H}\partial_x u)\mathcal{H}D^s u, D^{s-2}\partial_x u\rangle - c_2\langle (\mathcal{H}\partial_x(u\partial_x u))\mathcal{H}D^s u, D^{s-2}\partial_x u\rangle \\
&- \gamma\langle D^{5/2}u\mathcal{H}D^s u, D^{s-2}\partial_x u\rangle =: R_{61} + R_{62} + R_{63} + R_{64} + R_{65}.
\end{aligned}$$

And we set

$$\begin{aligned}
R_7 =& \langle u\mathcal{H}D^s\partial_x^3 u, D^{s-2}\partial_x u\rangle - \langle u\mathcal{H}D^s(u^2\partial_x u), D^{s-2}\partial_x u\rangle \\
&- c_1\langle u\mathcal{H}D^s\partial_x(u\mathcal{H}\partial_x u), D^{s-2}\partial_x u\rangle + c_2\langle uD^s\partial_x(u\partial_x u), D^{s-2}\partial_x u\rangle \\
&- \gamma\langle u\mathcal{H}D^{s+5/2}u, D^{s-2}\partial_x u\rangle =: R_{71} + R_{72} + R_{73} + R_{74} + R_{75}.
\end{aligned}$$

Finally, we set

$$\begin{aligned}
R_8 =& \langle u\mathcal{H}D^s u, D^{s-2}\partial_x^4 u\rangle - \langle u\mathcal{H}D^s u, D^{s-2}\partial_x(u^2\partial_x u)\rangle \\
&+ c_1\langle u\mathcal{H}D^s u, D^s(u\mathcal{H}\partial_x u)\rangle + c_2\langle u\mathcal{H}D^s u, \mathcal{H}D^s(u\partial_x u)\rangle \\
&- \gamma\langle u\mathcal{H}D^s u, D^{s+1/2}\partial_x u\rangle =: R_{81} + R_{82} + R_{83} + R_{84} + R_{85}.
\end{aligned}$$

Lemma 2.10 shows that

$$R_{61} + R_{71} + R_{81} = 3\langle \partial_x u\mathcal{H}D^s\partial_x u, D^{s-2}\partial_x^2 u\rangle = -3\langle \partial_x u\mathcal{H}D^s\partial_x u, D^s u\rangle,$$

which cancels out the second term in the right hand side in (2.19) by multiplying $\lambda(s)$. It is easy to see that $|R_{62}| + |R_{63}| + |R_{64}| \lesssim \|u\|_{H^s}^2$. By (i) of Lemma 2.6, we have $|R_{72}| + |R_{82}| \lesssim \|u\|_{H^s}^2$. We see from (ii) of Lemma 2.11 that $|R_{73}| \lesssim \|u\|_{H^s}^2$. Lemma 2.9 and (i) of Lemma 2.6 give $|R_{74}| + |R_{83}| \lesssim \|u\|_{H^s}^2$. For R_{84}, it follows from (i) of Lemma 2.11 that $|R_{84}| \lesssim \|u\|_{H^s}^2$. Finally, we estimate R_{65}, R_{75} and R_{85}. Lemma 2.3 implies that

$$\begin{aligned}
\|D^{s-2}\partial_x u\|_\infty &\le C\|D^{s-2}u\|^{1/4}\|D^s u\|^{3/4} \\
&\le C\|u\|^{1-(4s-2)/(4s+5)}\|D^{s+5/4}u\|^{(4s-2)/(4s+5)}.
\end{aligned}$$

Then we have

$$\begin{aligned}|R_{65}| &\leq \gamma\|D^{5/2}u\|\|D^s u\|\|D^{s-2}\partial_x u\|_\infty\\ &\leq \gamma C\|u\|^{1+2/(4s+5)}\|D^{s+5/4}u\|^{2-2/(4s+5)}\\ &\leq C\|u\|^{4s+7}+\frac{\gamma^{1+1/4(s+1)}}{3}\|D^{s+5/4}u\|^2.\end{aligned}$$

A similar argument yields

$$|R_{75}|+|R_{85}| \leq C\|u\|^{4s+7}+C\|u\|^{2s+9/2}+\frac{2\gamma^{1+1/4(s+1)}}{3}\|D^{s+5/4}u\|^2.$$

Therefore, the fact that $\gamma\in[0,1)$ shows that

$$\frac{d}{dt}E_s(u(t)) \leq C\|u(t)\|_{H^s}^2 \leq CE_s(u(t))$$

on $[0,T]$. Note that the implicit constant does not depend on γ. This completes the proof. □

Now, we are ready to prove Proposition 2.16.

Proof of Proposition 2.16. Assume that the set $F=\{t\geq 0; E_{s_0}(u(t))>2E_{s_0}(\varphi)\}$ is not empty. Set $T_\gamma^*=\inf F$. Note that $0<T_\gamma^*\leq T_\gamma$ and $E_{s_0}(u(t))\leq 2E_{s_0}(\varphi)$ on $[0,T_\gamma^*]$. Assume that there exists $t'\in[0,T_\gamma^*]$ such that $E_{s_0}(u(t'))>2E_{s_0}(\varphi)$. This implies that $t'\geq T_\gamma^*$ by the definition of T_γ^*. Then we have $t'=T_\gamma^*$. Thus, $\sup_{t\in[0,T_\gamma^*]}E_{s_0}(u(t))\leq C(\|\varphi\|_{H^{s_0}})$ by (ii) of Lemma 2.13. By Proposition 2.17, there exists $C_s'=C(s,s_0,\|\varphi\|_{H^{s_0}})$ such that

$$\frac{d}{dt}E_s(u(t)) \leq C_s'E(u(t))$$

on $[0,T_\gamma^*]$. The Gronwall inequality gives that

$$E_s(u(t)) \leq E_s(\varphi)\exp(C_s't) \tag{2.20}$$

on $[0,T_\gamma^*]$. Here, we put $T=\min\{(2C_{s_0}')^{-1},T_\gamma^*\}$. Then (2.20) with $s=s_0$ shows that

$$E_{s_0}(u(t)) \leq E_{s_0}(\varphi)\exp(2^{-1}) < 2E_{s_0}(\varphi),$$

on $[0,T]$. By the definition of T_γ^* and the continuity of $E_{s_0}(u(t))$, we obtain $0<T=(2C_{s_0}')^{-1}<T_\gamma^*\leq T_\gamma$. If F is empty, then we have $T_\gamma^*=T_\gamma=\infty$. In particular, we can take $T=(2C_{s_0}')^{-1}<\infty$, which concludes the proof. □

2.4. **Uniqueness, persistence and continuous dependence.** In this subsection, we prove Theorem 2.1. We first show the existence of the solution of (2.1) by the limiting procedure. We also prove the uniqueness and the persistence property $u \in C([0,T];H^s(\mathbb{T}))$. Then we estimate difference between two solutions of (2.23)–(2.24) in $H^s(\mathbb{T})$, which is essential to show the continuous dependence.

Lemma 2.18. *Let $s \geq s_0 > 5/2$, $\gamma_j \in (0,1)$, $T > 0$. Assume that $u_j \in C([0,T];H^s(\mathbb{T})) \cap C((0,T];H^{s+1}(\mathbb{T}))$ satisfies (2.13) with $\gamma = \gamma_j$ on $[0,T]\times\mathbb{T}$ and $\sup_{t\in[0,T]} \|u_j(t)\|_{H^{s_0}} \leq K$ for $K > 0$, $j = 1, 2$. Then there exists $C = C(K,s)$ such that*

$$\frac{d}{dt}\tilde{E}(u_1,u_2) \leq C(\tilde{E}(u_1,u_2) + \max\{\gamma_1^2,\gamma_2^2\}) \tag{2.21}$$

on $[0,T]$.

Proof. Set $w := u_1 - u_2$ so that w satisfies the following equation:

$$\begin{aligned}
&\partial_t w - \partial_x^3 w + \frac{1}{3}\partial_x\{(u_1^2 + u_1u_2 + u_2^2)w\} \\
&+ \frac{c_1}{2}\partial_x(w\mathcal{H}\partial_x z) + \frac{c_1}{2}\partial_x(z\mathcal{H}\partial_x w) + \frac{c_2}{2}\mathcal{H}\partial_x(w\partial_x z) + \frac{c_2}{2}\mathcal{H}\partial_x(z\partial_x w) \\
&= -\gamma_1 D^{5/2}w - (\gamma_1 - \gamma_2)D^{5/2}u_2,
\end{aligned} \tag{2.22}$$

where $z = u_1 + u_2$. By the presence of the operator $\langle D\rangle^{-1}$, we can easily obtain

$$\frac{d}{dt}\|\langle D\rangle^{-1}w\|^2 \lesssim \|w\|^2 + \max\{\gamma_1^2,\gamma_2^2\}.$$

Indeed, note that $\partial_x(z\mathcal{H}\partial_x w) = \partial_x^2(z\mathcal{H}w) - \partial_x(\partial_x z\mathcal{H}w)$. Then we have

$$\begin{aligned}
&|\langle\langle D\rangle^{-1}\partial_x(z\mathcal{H}\partial_x w), \langle D\rangle^{-1}w\rangle| \\
&\leq |\langle\langle D\rangle^{-1}\partial_x^2(z\mathcal{H}w), \langle D\rangle^{-1}w\rangle| + |\langle\langle D\rangle^{-1}\partial_x(\partial_x z\mathcal{H}w), \langle D\rangle^{-1}w\rangle| \lesssim \|w\|^2.
\end{aligned}$$

Other terms can be estimated in a similar manner. Next, we estimate the L^2-norm of w. Set

$$\begin{aligned}
\frac{d}{dt}\|w\|^2 &= 2\langle\partial_x^3 w, w\rangle - \frac{2}{3}\langle\partial_x\{(u_1^2 + u_1u_2 + u_2^2)w\}, w\rangle - c_1\langle\partial_x(w\mathcal{H}\partial_x z), w\rangle \\
&\quad - c_1\langle\partial_x(z\mathcal{H}\partial_x w), w\rangle - c_2\langle\mathcal{H}\partial_x(w\partial_x z), w\rangle - c_2\langle\mathcal{H}\partial_x(z\partial_x w), w\rangle \\
&\quad - 2\gamma_1\langle D^{5/2}w, w\rangle - 2(\gamma_1 - \gamma_2)\langle D^{5/2}u_2, w\rangle \\
&=: R_9 + R_{10} + R_{11} + R_{12} + R_{13} + R_{14} + R_{15} + R_{16}.
\end{aligned}$$

Again, it is clear that $R_9 = 0$. By Lemma 2.9, we have $|R_{10}| + |R_{11}| \lesssim \|w\|^2$. Note that

$$\begin{aligned}
&\langle [\mathcal{H}, \partial_x z]\partial_x w, w\rangle + \langle [\mathcal{H}, z]\partial_x^2 w, w\rangle \\
&= \langle \mathcal{H}(\partial_x z \partial_x w), w\rangle - \langle \partial_x z \mathcal{H}\partial_x w, w\rangle + \langle \mathcal{H}(z\partial_x^2 w), w\rangle - \langle z\mathcal{H}\partial_x^2 w, w\rangle \\
&= \langle \partial_x(\partial_x z \mathcal{H} w), w\rangle - \langle \partial_x z \mathcal{H}\partial_x w, w\rangle - \langle \partial_x^2(z\mathcal{H}w), w\rangle - \langle z\mathcal{H}\partial_x^2 w, w\rangle \\
&= -2\langle \partial_x(z\mathcal{H}\partial_x w), w\rangle.
\end{aligned}$$

Then Lemma 2.7 shows that $|R_{12}| + |R_{14}| \lesssim \|w\|^2$. We can reduce R_{13} to

$$R_{13} = -2c_2\langle \partial_x u_1 \mathcal{H}\partial_x w, w\rangle - c_2\langle \partial_x w \mathcal{H}\partial_x w, w\rangle$$

since $z = 2u_1 - w$. The last term in the right hand side can be bounded by $\lesssim \|w\|^2$ by using Lemma 2.9. Observe that $R_{15} = -\gamma_1\|D^{5/4}w\|^2 \le 0$ and that $|R_{16}| \lesssim \|w\|^2 + \max\{\gamma_1^2, \gamma_2^2\}$. Therefore, we have

$$\frac{d}{dt}\|w\|^2 \le C\|w\|^2 + 3\lambda(0)\int_{\mathbb{T}} \partial_x u_1(\mathcal{H}\partial_x w)w dx + \max\{\gamma_1^2, \gamma_2^2\}.$$

The correction term in $\widetilde{E}$ cannot exactly cancel out the second term, but Lemma 2.4 shows that the difference is harmless. Set

$$\begin{aligned}
\frac{d}{dt}\langle u_1\langle D\rangle^{-1}w, w\rangle &= \langle \partial_t u_1\langle D\rangle^{-1}w, w\rangle + \langle u_1\langle D\rangle^{-1}\partial_t w, w\rangle + \langle u_1\langle D\rangle^{-1}w, \partial_t w\rangle \\
&=: R_{17} + R_{18} + R_{19}.
\end{aligned}$$

Moreover, we set $R_{171} = \langle \partial_x^3 u_1\langle D\rangle^{-1}w, w\rangle$ and set

$$\begin{aligned}
R_{18} &= \langle u_1\langle D\rangle^{-1}\partial_x^3 w, w\rangle - \frac{1}{3}\langle u_1\langle D\rangle^{-1}\partial_x\{(u_1^2 + u_1u_2 + u_2^2)w\}, w\rangle \\
&\quad - \frac{c_1}{2}\langle u_1\langle D\rangle^{-1}\partial_x(w\mathcal{H}\partial_x z), w\rangle - \frac{c_1}{2}\langle u_1\langle D\rangle^{-1}\partial_x(z\mathcal{H}\partial_x w), w\rangle \\
&\quad - \frac{c_2}{2}\langle u_1\langle D\rangle^{-1}\mathcal{H}\partial_x(w\partial_x z), w\rangle - \frac{c_2}{2}\langle u_1\langle D\rangle^{-1}\mathcal{H}\partial_x(z\partial_x w), w\rangle \\
&\quad - \gamma_1\langle u_1\langle D\rangle^{-1}D^{5/2}w, w\rangle - (\gamma_1 - \gamma_2)\langle u_1\langle D\rangle^{-1}D^{5/2}u_2, w\rangle \\
&=: R_{181} + R_{182} + R_{183} + R_{184} + R_{185} + R_{186} + R_{187} + R_{188}
\end{aligned}$$

We set R_{19k} for $k = 1, \dots, 8$ in the same manner as above. Lemma 2.10 shows that

$$R_{171} + R_{181} + R_{191} = -3\langle \partial_x u_1\langle D\rangle^{-1}\partial_x^2 w, w\rangle - 3\langle \partial_x^2 u_1\langle D\rangle^{-1}\partial_x w, w\rangle,$$

which together with Lemma 2.4 shows that $|R_{13} - \lambda(0)(R_{171} + R_{181} + R_{191})| \lesssim \|w\|^2$. It is easy to see that

$$\begin{aligned}
&|\langle (u_1^2\partial_x u_1 + c_1\partial_x(u_1\mathcal{H}\partial_x u_1) + c_2\mathcal{H}\partial_x(u_1\partial_x u_1) + \gamma_1 D^{5/2}u_1)\langle D\rangle^{-1}w, w\rangle| \\
&\lesssim \|w\|^2.
\end{aligned}$$

We have $|R_{182}| + |R_{183}| + |R_{185}| + |R_{192}| + |R_{193}| + |R_{195}| \lesssim \|w\|^2$ because of the presence of the operator $\langle D\rangle^{-1}$. In order to handle $R_{184}, R_{186}, R_{194}$ and R_{196}, we see from Lemma 2.4 and (i) of Lemma 2.6 that

$$\begin{aligned}|R_{196}| &= \left|-\frac{c_2}{2}\langle u_1\langle D\rangle^{-1}w, \mathcal{H}\partial_x^2(zw)\rangle + \frac{c_2}{2}\langle u_1\langle D\rangle^{-1}w, \mathcal{H}\partial_x(\partial_x zw)\rangle\right| \\ &\lesssim |\langle u_1\langle D\rangle^{-1}\partial_x w, (\mathcal{H}\partial_x + \langle D\rangle^{-1}\partial_x^2)(zw)\rangle| + |\langle u_1\langle D\rangle^{-1}\partial_x w, \langle D\rangle^{-1}\partial_x^2(zw)\rangle| \\ &\quad + \|w\|^2 \\ &\lesssim |\langle u_1\langle D\rangle^{-1}\partial_x w, [\langle D\rangle^{-1}\partial_x^2, z]w\rangle| + |\langle u_1 z\langle D\rangle^{-1}\partial_x w, \langle D\rangle^{-1}\partial_x^2 w\rangle| + \|w\|^2 \\ &\lesssim \|w\|^2.\end{aligned}$$

We can obtain $|R_{184}| + |R_{186}| + |R_{194}| \lesssim \|w\|^2$ from a similar argument. Finally, it is easy to see that $|R_{187}| + |R_{188}| + |R_{197}| + |R_{198}| \lesssim \|w\|^2 + \max\{\gamma_1^2, \gamma_2^2\}$. Summing these estimates above and applying (iii) of Lemma 2.13, we obtain (2.21), which concludes the proof. □

Now we obtain the solution to (2.1)–(2.2). Let $\varphi \in H^s(\mathbb{T})$ and let $\gamma_1, \gamma_2 \in (0,1)$. Let u_{γ_j} be the solution to (2.13)–(2.14) with $\gamma = \gamma_j$ for $j = 1, 2$, obtained by Proposition 2.15. Note that $\tilde{E}(u_{\gamma_1}(0), u_{\gamma_2}(0)) = \tilde{E}(\varphi, \varphi) = 0$. Proposition 2.16 shows that there exists $T = T(s_0, \|\varphi\|_{H^{s_0}})$ such that (2.18) holds. We see from (iii) of Lemma 2.13 and Lemma 2.18 that

$$\sup_{t\in[0,T]} \|u_{\gamma_1}(t) - u_{\gamma_2}(t)\|^2 \le \sup_{t\in[0,T]} \tilde{E}(u_{\gamma_1}(t), u_{\gamma_2}(t)) \le C\max\{\gamma_1^2, \gamma_2^2\} \to 0$$

as $\gamma_1, \gamma_2 \to 0$. This implies that there exists $u \in C([0,T]; L^2(\mathbb{T}))$ such that

$$u_\gamma \to u \quad \text{in} \quad C([0,T]; L^2(\mathbb{T})) \quad \text{as} \quad \gamma \to 0.$$

The above convergence can be verified in $C([0,T]; H^r(\mathbb{T}))$ for any $r < s$ by interpolating with $L^\infty([0,T]; H^s(\mathbb{T}))$. It is clear that u satisfies (2.1)–(2.2) on $[0,T]$.

For the proof of the following uniqueness result, see Thorem 6.22 in [14].

Lemma 2.19 (Uniqueness). *Let $\delta > 0$ and $\varepsilon > 0$, $u_j \in L^\infty([0,\delta]; H^{5/2+\varepsilon}(\mathbb{T}))$ satisfy (2.1) on $[0,\delta]$ with $u_1(0) = u_2(0)$ and satisfy*

$$u_j \in C([0,\delta]; H^2(\mathbb{T})) \cap C^1([0,\delta]; H^{-1}(\mathbb{T}))$$

for $j = 1, 2$. Then $u_1 \equiv u_2$ on $[0,\delta]$.

It remains to show the persistent property, i.e., $u \in C([0,T]; H^s(\mathbb{T}))$ and the continuous dependence. In what follows, we employ the Bona-Smith approximation argument. We consider the following initial value problem:

$$\partial_t u - \partial_x^3 u + u^2\partial_x u + c_1\partial_x(u\mathcal{H}\partial_x u) + c_2\mathcal{H}\partial_x(u\partial_x u) = 0, \quad x \in \mathbb{T}, \tag{2.23}$$

$$u(0,x) = J_\gamma\varphi(x), \tag{2.24}$$

where $J_\gamma\varphi$ is defined in Definition 4. Let $s \geq s_0 > 5/2$, $\varphi \in H^s(\mathbb{T})$ and $\epsilon > 0$. Lemma 2.14 shows that $J_\gamma\varphi \in H^\infty(\mathbb{T})$. Let $u_\gamma \in C([0,T_\gamma); H^{s+3+\epsilon}(\mathbb{T}))$ be the solution (2.13) with the initial data $J_\gamma\varphi$ obtained by Proposition 2.15. Lemma 2.14 and Proposition 2.16 imply that there exists $T = T(s_0, \|\varphi\|_{H^{s_0}})$ such that (2.18) holds for $s+3+\epsilon$. Lemma 2.18 and the above argument show that there exists $\tilde{u} \in C([0,T]; H^{s+3}(\mathbb{T}))$ such that $\tilde{u}$ solves (2.23)–(2.24). Therefore, we have the following corollary:

Corollary 2.20. *Let $s \geq s_0 > 5/2$, $T > 0$, $u_j \in C([0,T]; H^{s+1}(\mathbb{T}))$ satisfy (2.23) on $[0,T] \times \mathbb{T}$ and $\sup_{t\in[0,T]}\|u_j(t)\|_{H^s} \leq K$ for $K > 0$, $j = 1,2$. Then there exists $C = C(K, s_0, s)$ such that*

$$\frac{d}{dt}\tilde{E}(u_1(t), u_2(t)) \leq C\tilde{E}(u_1(t), u_2(t)) \tag{2.25}$$

on $[0,T]$.

Proposition 2.21. *Let $s \geq s_0 > 5/2$, $T > 0$, $u_j \in C([0,T]; H^{s+3}(\mathbb{T}))$ satisfy (2.23) on $[0,T] \times \mathbb{T}$ and $\sup_{t\in[0,T]}\|u_j(t)\|_{H^s} \leq K$ for $K > 0$, $j = 1,2$. Then there exists $C = C(s, s_0, K)$ such that*

$$\begin{aligned}\frac{d}{dt}E_s(u_1(t), u_2(t)) \leq & C(\|u_1(t)-u_2(t)\|_{H^s}^2 + \|u_1(t)-u_2(t)\|_{H^{s_0-1}}^2\|u_2\|_{H^{s+1}}^2 \\ & + \|u_1(t)-u_2(t)\|_{H^{s_0-2}}^2\|u_2\|_{H^{s+2}}^2)\end{aligned} \tag{2.26}$$

on $[0,T]$.

Proof. Set $w = u_1 - u_2$ and $z = u_1 + u_2$. It is easy to see that

$$\frac{d}{dt}\|w\|^2 \lesssim \|w\|_{H^1}^2 \leq \|w\|_{H^s}^2.$$

Set

$$\begin{aligned}\frac{d}{dt}\|D^s w\|^2 = & 2\langle D^s\partial_x^3 w, D^s w\rangle - 2\langle D^s(u_1^2\partial_x w), D^s w\rangle - 2\langle D^s(zw\partial_x u_2), D^s w\rangle \\ & - 2c_1\langle D^s\partial_x(u_1\mathcal{H}\partial_x u_1 - u_2\mathcal{H}\partial_x u_2), D^s w\rangle \\ & - 2c_2\langle \mathcal{H}D^s\partial_x(u_1\partial_x u_1 - u_2\partial_x u_2), D^s w\rangle \\ =: & R_1 + R_2 + R_3 + R_4 + R_5.\end{aligned}$$

It is easy to see that $R_1 = 0$ and $|R_2| \lesssim \|w\|_{H^s}^2$ by (i) of Lemma 2.6. For R_3, we have $|R_3| \lesssim \|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2 \|u_2\|_{H^{s+1}}^2$. Lemma 2.12 shows that

$$\begin{aligned} &|R_4 + R_5 - 3\lambda(s)\langle \partial_x u_1 \mathcal{H} D^s \partial_x w, D^s w\rangle| \\ &\lesssim \|w\|_{H^s}^2 + \|w\|_{H^{s_0-2}}^2 \|u_2\|_{H^{s+2}}^2 + \|w\|_{H^{s_0-1}}^2 \|u_2\|_{H^{s+1}}^2. \end{aligned}$$

Therefore, the time derivative of $\|D^s w\|^2$ yields

$$\begin{aligned} \frac{d}{dt}\|D^s w\|^2 \le C\|w\|_{H^s}^2 + C\|w\|_{H^{s_0-1}}^2 \|u_2\|_{H^{s+1}}^2 + C\|w\|_{H^{s_0-2}}^2 \|u_2\|_{H^{s+2}}^2 \\ + 3\lambda(s)\int_{\mathbb{T}} \partial_x u_1 (\mathcal{H} D^s \partial_x w) D^s w dx. \end{aligned} \tag{2.27}$$

Next, we evaluate the time derivative of the correction term. Lemma 2.10 with $f = u_1$, $g = \mathcal{H} D^s w$ and $h = D^{s-2}\partial_x w$ shows that

$$\begin{aligned} &\langle \partial_x^3 u_1 \mathcal{H} D^s w, D^{s-2}\partial_x w\rangle + \langle u_1 \mathcal{H} D^s \partial_x^3 w, D^{s-2}\partial_x w\rangle + \langle u_1 \mathcal{H} D^s w, D^{s-2}\partial_x^4 w\rangle \\ &= 3\langle \partial_x u_1 \mathcal{H} D^s \partial_x w, D^{s-2}\partial_x^2 w\rangle = -3\langle \partial_x u_1 \mathcal{H} D^s \partial_x w, D^s w\rangle. \end{aligned}$$

Multiplying by $\lambda(s)$, we can cancel out the last term in the right hand side in (2.27). On the other hand, it is easy to see that

$$\langle (\partial_t u_1 - \partial_x^3 u_1)\mathcal{H} D^s w, D^{s-2}\partial_x w\rangle \lesssim \|w\|_{H^s}^2.$$

We set

$$\begin{aligned} &\langle u_1 \mathcal{H} D^s(\partial_t w - \partial_x^3 w), D^{s-2}\partial_x w\rangle \\ &= -\frac{1}{3}\langle u_1 \mathcal{H} D^s \partial_x\{(u_1^2 + u_1 u_2 + u_2^2)w\}, D^{s-2}\partial_x w\rangle \\ &\quad - c_1\langle u_1 \mathcal{H} D^s \partial_x(u_1 \mathcal{H}\partial_x w), D^{s-2}\partial_x w\rangle + c_2\langle u_1 D^s \partial_x(u_1 \partial_x w), D^{s-2}\partial_x w\rangle \\ &\quad - c_1\langle u_1 \mathcal{H} D^s \partial_x(w \mathcal{H}\partial_x u_2), D^{s-2}\partial_x w\rangle + c_2\langle u_1 D^s \partial_x(w \partial_x u_2), D^{s-2}\partial_x w\rangle \\ &=: R_9 + R_{10} + R_{11} + R_{12} + R_{13} \end{aligned}$$

and

$$\begin{aligned} &\langle u_1 \mathcal{H} D^s w, D^{s-2}\partial_x(\partial_t w - \partial_x^3 w)\rangle \\ &= \frac{1}{3}\langle u_1 \mathcal{H} D^s w, D^s\{(u_1^2 + u_1 u_2 + u_2^2)w\}\rangle + c_1\langle u_1 \mathcal{H} D^s w, D^s(u_1 \mathcal{H}\partial_x w)\rangle \\ &\quad + c_2\langle u_1 \mathcal{H} D^s w, \mathcal{H} D^s(u_1 \partial_x w)\rangle + c_1\langle u_1 \mathcal{H} D^s w, D^s(w \mathcal{H}\partial_x u_2)\rangle \\ &\quad + c_2\langle u_1 \mathcal{H} D^s w, \mathcal{H} D^s(w \partial_x u_2)\rangle =: R_{14} + R_{15} + R_{16} + R_{17} + R_{18}. \end{aligned}$$

By (i) of Lemma 2.6, we have $|R_9| + |R_{14}| \lesssim \|w\|_{H^s}^2$. We see from (ii) of Lemma 2.11 that $|R_{10}| \lesssim \|w\|_{H^s}^2$. We also have $|R_{16}| \lesssim \|w\|_{H^s}^2$ by (i) of Lemma 2.11. Similarly, we can obtain $|R_{11}| + |R_{15}| \lesssim \|w\|_{H^s}^2$. On the other hand, by (i) of Lemma 2.6 we have $|R_{12}| + |R_{13}| + |R_{17}| + |R_{18}| \lesssim \|w\|_{H^s}^2 + \|w\|_{H^{s_0-2}}^2 \|u_2\|_{H^{s+1}}^2$. Summing these estimates above, we obtain (2.26) on $[0, T]$, which concludes the proof. $\square$

Now, we can show the persistence property and the continuous dependence.

Proof of Theorem 2.1. In what follows, without loss of generality, we may assume that s_0 is strictly smaller than s since the assumption $\|\varphi\|_{H^{s_0}} \leq K$ is weaker than $\|\varphi\|_{H^{s_0'}} \leq K$ when $s_0 < s_0'$. First we prove the persistence property. Let $0 < \gamma_1 < \gamma_2 < 1$. Let $u_{\gamma_j} \in C([0, T]; H^{s+3}(\mathbb{T}))$ be the solution to (2.23)–(2.24) with the initial data $J_\gamma \varphi$ for $\varphi \in H^s(\mathbb{T})$ and $j = 1, 2$. Corollary 2.20 with the Gronwall inequality shows that

$$\sup_{t\in[0,T]} \|u_{\gamma_1}(t) - u_{\gamma_2}(t)\|^2 \leq C\tilde{E}(u_{\gamma_1}(0), u_{\gamma_2}(0)) \leq C\|J_{\gamma_1}\varphi - J_{\gamma_2}\varphi\|^2 \leq C\gamma_2^{2s}$$

since $\gamma_1 < \gamma_2$. This together with the interpolation implies that

$$\sup_{t\in[0,T]} \|u_{\gamma_1}(t) - u_{\gamma_2}(t)\|_{H^\alpha}^2 \leq C\gamma_2^{2(s-\alpha)}$$

for any $0 \leq \alpha < s$. On the other hand, Lemma 2.14 and 2.17 show that

$$\sup_{t\in[0,t]} \|u_{\gamma_2}(t)\|_{H^{s+\alpha}}^2 \leq C\|J_{\gamma_2}\varphi\|_{H^{s+\alpha}}^2 \leq C\gamma_2^{-2\alpha}\|\varphi\|_{H^s}^2$$

for $\alpha \geq 0$. This together with the Gronwall inequality and Proposition 2.21 implies that

$$\sup_{t\in[0,T]} \|u_{\gamma_1}(t) - u_{\gamma_2}(t)\|_{H^s}^2 \lesssim \|J_{\gamma_1}\varphi - J_{\gamma_2}\varphi\|_{H^s}^2 + \gamma_2^{2(s-s_0)} \to 0$$

as $\gamma_2, \gamma_1 \to 0$ since $\|J_{\gamma_1}\varphi - J_{\gamma_2}\varphi\|_{H^s} \to 0$ as $\gamma_1, \gamma_2 \to 0$. Then, there exists $\tilde{u} \in C([0, T]; H^s(\mathbb{T}))$ such that

$$u_\gamma \to \tilde{u} \ \text{ in } \ C([0, T]; H^s(\mathbb{T})) \quad \text{as} \quad \gamma \to 0.$$

It is clear that the function $\tilde{u}$ coincides with our solution $u \in C([0, T]; H^r(\mathbb{T}))$ for $r < s$ to (2.1)–(2.2), which shows the persistence property.

Finally, we prove the continuous dependence, which is the only thing left to prove. We will claim that

$$\begin{gathered}\forall\varphi \in H^s(\mathbb{T}), \forall\epsilon > 0, \exists\delta > 0, \forall\psi \in H^s(\mathbb{T}) : \\ \left[\|\varphi - \psi\|_{H^s} < \delta \Rightarrow \sup_{t\in[0,T/2]} \|u(t) - v(t)\|_{H^s} < \epsilon\right],\end{gathered} \tag{2.28}$$

where u, v represent the solution to (2.1) with initial data $\varphi, \psi \in H^s(\mathbb{T})$, respectively, which are obtained by the above argument. In (2.28) we take the interval $[0, T/2]$ with T as defined by Proposition 2.16 to guarantee that if $\|\varphi - \psi\|_{H^s} < \delta$, then the solution $v(t)$ is defined in the time interval $[0,T/2]$. Fix $\varphi \in H^s(\mathbb{T})$ and $\epsilon > 0$. Let $0 < \gamma_1 < \gamma_2 < 1$. Assume that $\|\varphi - \psi\|_{H^s} < \delta$, where $\delta > 0$ will be chosen later. Note that by the triangle inequality we have

$$\begin{aligned} &\sup_{t\in[0,T/2]} \|u(t) - v(t)\|_{H^s} \\ &\le \sup_{t\in[0,T/2]} \|u(t) - u^{\gamma_2}(t)\|_{H^s} + \sup_{t\in[0,T/2]} \|u^{\gamma_2}(t) - v^{\gamma_1}(t)\|_{H^s} \\ &\quad + \sup_{t\in[0,T/2]} \|v^{\gamma_1}(t) - v(t)\|_{H^s}, \end{aligned} \tag{2.29}$$

where u^{γ_2} and v^{γ_1} represent the solution to the IVP (2.1) with the initial data $J_{\gamma_2}\varphi$ and $J_{\gamma_1}\psi$, respectively. First we handle the second term in the right hand side in (2.29). Again, the triangle inequality shows that

$$\|J_{\gamma_2}\varphi - J_{\gamma_1}\psi\|_{H^r} \le \|J_{\gamma_2}\varphi - \varphi\|_{H^r} + \|\varphi - \psi\|_{H^r} + \|\psi - J_{\gamma_1}\psi\|_{H^r}$$

for $r \le s$. Proposition 2.21 with $u_1 = v^{\gamma_1}$ and $u_2 = u^{\gamma_2}$ gives that

$$\begin{aligned} &\sup_{t\in[0,T/2]} \|u^{\gamma_2}(t) - v^{\gamma_1}(t)\|_{H^s} \\ &\le C\|J_{\gamma_2}\varphi - \varphi\|_{H^s} + C\delta + C\|\psi - J_{\gamma_1}\psi\|_{H^s} + C\gamma_2^{s-s_0} + C\gamma_2^{-1}\delta^{1+1/s-s_0/s} \\ &\quad + C\gamma_2^{-1}\|\psi - J_{\gamma_1}\psi\|^{1+1/s-s_0/s} + C\gamma_2^{s-s_0} + C\gamma_2^{-2}\delta^{1+2/s-s_0/s} \\ &\quad + C\gamma_2^{-2}\|\psi - J_{\gamma_1}\psi\|^{1+2/s-s_0/s}. \end{aligned}$$

Therefore, we choose $\gamma_2 > 0$ so that

$$\sup_{t\in[0,T/2]} \|u(t) - u^{\gamma_2}(t)\|_{H^s} + C\|J_{\gamma_2}\varphi - \varphi\|_{H^s} + C\gamma_2^{s-s_0} < \frac{\epsilon}{3},$$

Then we take $\delta > 0$ such that

$$C(\delta + \gamma_2^{-1}\delta^{1+1/s-s_0/s} + \gamma_2^{-2}\delta^{1+2/s-s_0/s}) < \frac{\epsilon}{3}$$

and finally for each $\psi \in H^s(\mathbb{T})$ satisfying $\|\varphi - \psi\|_{H^s} < \delta$ we take $\gamma_1 \in (0, \gamma_2)$ such that

$$\begin{aligned} &\sup_{t\in[0,T/2]} \|v^{\gamma_1}(t) - v(t)\|_{H^s} + C\|\psi - J_{\gamma_1}\psi\|_{H^s} \\ &+ C\gamma_2^{-1}\|\psi - J_{\gamma_1}\psi\|^{1+1/s-s_0/s} + C\gamma_2^{-2}\|\psi - J_{\gamma_1}\psi\|^{1+2/s-s_0/s} < \frac{\epsilon}{3}. \end{aligned}$$

which completes the proof of (2.28). □

3. Local well-posedness for fourth order Benjamin-Ono type equations

3.1. **Introduction.** We consider the Cauthy problem of the following fourth order Benjamin-Ono type equations:

$$\partial_t u = \partial_x K(u), \tag{3.1}$$

$$u(0,x) = \varphi(x), \tag{3.2}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}$ or $\mathbb{T}(=: \mathbb{R}/2\pi\mathbb{Z})$, $u = u(t,x), \varphi = \varphi(x) \in \mathbb{R}$,

$$\begin{aligned} K(u) := \mathcal{H}\partial_x^3 u + c_1 u \partial_x^2 u + c_2(\partial_x u)^2 + c_3(\mathcal{H}\partial_x u)^2 + c_4\mathcal{H}(u\mathcal{H}\partial_x^2 u) \\ + c_5\mathcal{H}(u^2\partial_x u) + c_6 u\mathcal{H}(u\partial_x u) + c_7 u^2\mathcal{H}\partial_x u - u^4 \end{aligned} \tag{3.3}$$

and $c_j \in \mathbb{R}$ for $j = 1, \dots, 7$. $\mathcal{H}$ is the Hilbert transform defined by

$$\widehat{\mathcal{H}f}(0) = 0 \quad \text{and} \quad \widehat{\mathcal{H}f}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi)$$

for $\xi \in \mathbb{R}\backslash\{0\}$ or $\mathbb{Z}\backslash\{0\}$. The well-known Benjamin-Ono equation

$$\partial_t u + \mathcal{H}\partial_x^2 u + 2u\partial_x u = 0 \tag{3.4}$$

describes the behavior of long internal waves in deep stratified fluids. The equation (3.4) also has infinitely many conservation laws, which generates a hierarchy of Hamiltonian equations of order j. The equation (3.1) with $c_1 = 3$, $-c_2 = c_5 = c_6 = c_7 = -2$ and $c_3 = c_4 = -1$ is integrable and the third equation in the Benjamin-Ono hierarchy [20].

There are a lot of literature on the Cauchy problem on (3.4). On the real line case, Ionescu-Kenig [10] showed the local well-posedness in $H^s(\mathbb{R})$ for $s \geq 0$ (see also [23] for another proof and [11] for the local well-posedness with small complex valued data). On the periodic case, Molinet [21, 22] showed the local well-posedness in $H^s(\mathbb{T})$ for $s \geq 0$ and that this result was sharp. See [1, 2, 13, 15, 17, 25, 31] for former results.

In [29], we studied the local well-posedness for the equation

$$\partial_t u = \partial_x(\partial_x^2 u + d_1 u\mathcal{H}\partial_x u + d_2\mathcal{H}(u\partial_x u) - u^3), \quad x \in \mathbb{T}, \tag{3.5}$$

where $d_1, d_2 \in \mathbb{R}$. The equation (3.5) with $d_1 = d_2 = 3/2$ is integrable and the second equation in the Benjamin-Ono hierarchy. The local well-posedness for (3.5) is based on the energy method with a correction term. Namely, we employ the energy method to

$$E_*(u) := \|u\|_{L^2}^2 + \|D^s u\|_{L^2}^2 + a_s\|u\|_{L^2}^{4s+2} + b_s\int u(\mathcal{H}D^s u)D^{s-2}\partial_x u dx$$

(see Definition 2 in [29]) in order to eliminate the first order derivative loss. In fact, we have the second order derivative loss resulting from nonlinear terms in the energy inequality, but it can be reduced to the first order derivative loss because of the symmetry (see Lemma 2.6 in [29]). For related results such as the local well-posedness on the real line, see [6, 7, 19, 24].

On the other hand, as far as we know, there are no well-posedness results for (3.1) either on the real line or on the torus. In particular, some of nonlinear terms in (3.1) have three derivatives, which implies that the local well-posedness for (3.1) is far from trivial. The main result is the following:

Theorem 3.1. *We write $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$. Let $s \geq s_0 > 7/2$. For any $\varphi \in H^s(\mathcal{M})$, there exist $T = T(\|\varphi\|_{H^{s_0}}) > 0$ and the unique solution $u \in C([-T,T]; H^s(\mathcal{M}))$ to the IVP (3.1)–(3.2) on $[-T,T]$. Moreover, for any $R > 0$, the solution map $\varphi \mapsto u(t)$ is continuous from the ball $\{\varphi \in H^s(\mathcal{M}); \|\varphi\|_{H^s} \leq R\}$ to $C([-T,T]; H^s(\mathcal{M}))$.*

Now we mention the idea of the proof of Theorem 3.1. We may have the third order derivative loss since nonlinear terms in (3.1) have three derivatives at most. By the symmetry, it can be reduced to the second order derivative loss (see Lemma 3.25). Our proof is based on the energy method, and the standard energy estimate gives only the following:

$$\frac{d}{dt}\|D^s u(t)\|_{L^2}^2 \leq C(1 + \|u\|_{H^{s_0}})^3 \|u(t)\|_{H^s}^2 + |L_1(u)| + |L_2(u)| + |L_3(u)|, \tag{3.6}$$

where $s_0 > 7/2$, $D = \mathcal{F}^{-1}|\xi|\mathcal{F}$ and

$$L_1(u) := \lambda_1(s) \int \partial_x u (D^s \partial_x u)^2 dx, \quad L_2(u) := \lambda_2(s) \int (\mathcal{H}\partial_x^2 u)(\mathcal{H}D^s \partial_x u) D^s u dx,$$
$$L_3(u) := \lambda_3(s) \int u \partial_x u (\mathcal{H}D^s \partial_x u) D^s u dx$$

(see Definition 6 for definitions of $\lambda_j(s)$). Here, we note that $L_1(u)$ is the second order derivative loss, and $L_2(u)$ and $L_3(u)$ are the first order derivative losses. We need to to handle $L_j(u)$ for $j = 1, 2, 3$ by $\|u\|_{H^s}$ if we use the standard argument. However, it is impossible to do that. In order to overcome this difficulty, we modify the energy by adding correction terms, following the idea from Kwon [18] who studied the local well-posedness for the fifth order KdV equation (see also Segata [27], Kenig-Pilod [16] and Tsugawa [33]). Namely, we consider

$$E_s(u) := \frac{1}{2}\|u\|_{L^2}^2 (1 + C_s \|u\|_{L^2}^2 + C_s \|u\|_{L^2}^{4s}) + \frac{1}{2}\|D^s u\|_{L^2}^2 + \sum_{j=1}^{3} M_s^{(j)}(u),$$

with

$$M_s^{(1)}(u) := \frac{\lambda_1(s)}{4}\int u(\mathcal{H}D^s u)\mathcal{H}D^{s-1}u dx,$$
$$M_s^{(2)}(u) := \frac{\lambda_2(s)}{4}\int (\mathcal{H}\partial_x u)(D^{s-1}u)^2 dx,$$
$$M_s^{(3)}(u) := \frac{\lambda_1(s)\lambda_4(s)+4\lambda_3(s)}{32}\int u^2(D^{s-1}u)^2 dx$$

(see Definition 6). The first two terms correspond to $\|u\|_{H^s}$, and $M_s^{(1)}(u)$, $M_s^{(2)}(u)$ and $M_s^{(3)}(u)$ are correction terms. As defined in Definition 6, we note that $\lambda_j(s)$ for $j=1,2,3,4$ is a linear polynomial in s. The coefficient of $M_s^{(j)}(u)$ can be determined so that the time derivative of $M_s^{(j)}(u)$ cancels out $L_j(u)$ for $j=1,2$. On the other hand, the time derivative of $M_s^{(1)}(u)$ also yields $L_3(u)$, that is,

$$\frac{d}{dt}M_s^{(1)}(u) \sim L_1(u) + L_3(u)$$

since $L_1(u)$ is the second order derivative loss. Therefore, we need to collect coefficients of $L_3(u)$ resulting from both $\|D^s u\|$ and $M_s^{(1)}(u)$ when we determine the coefficient of $M_s^{(3)}(u)$. For this reason, the coefficient of $M_s^{(3)}(u)$ is a quadratic polynomial in s.

Subsequently, using the conservation law corresponding to the H^4-norm of the solution, we can obtain an *a priori* estimate of solutions in H^4. Therefore, we can easily extend the solution obtained in Theorem 3.1 globally. Namely, we obtain the following result:

Corollary 3.2. *We write* $\mathcal{M} = \mathbb{R}$ *or* $\mathbb{T}$. *The Cauchy problem* (3.1)–(3.2) *with* $c_1 = 3$, $-c_2 = c_5 = c_6 = c_7 = -2$ *and* $c_3 = c_4 = -1$ *is globally well-posed in* $H^s(\mathcal{M})$ *for* $s \geq 4$.

In what follows, we consider our problem only on $\mathcal{M} = \mathbb{T}$, and the proof on $\mathbb{R}$ is alomst same as that on $\mathbb{T}$. There are two differences, and one is the following:

$$\mathcal{H}(\mathcal{H}f)(x) = \begin{cases} -f(x), & x \in \mathbb{R}, \\ -f(x) + \hat{f}(0), & x \in \mathbb{T}. \end{cases}$$

However, such a difference does not yield difficulties in our argument since we have $|\hat{f}(0)| \leq \|\hat{f}\|_{l^\infty(\mathbb{Z})} \leq \|f\|_{L^1(\mathbb{T})} \lesssim \|f\|_{L^2(\mathbb{T})}$. The other one is the Gagliardo-Nirenberg inequality (Lemma 3.3), that is, we do not need to add $\|f\|_{L^2(\mathbb{R})}$ on $\mathbb{R}$ when $l = 0$.

This section is organized as follows. In Subsection 3.2, we prove the main result, admitting two Propositions 3.8 and 3.9. In Subsection 3.3, we show the main estimate which is Proposition 3.8, that is, the energy inequality between two solutions in H^s. In Subsection 3.4, we give a proof of the energy estimate in L^2 which is Proposition 3.9.

3.2. **Proof of Theorem 3.1.** In this subsection we prove Theorem 3.1, admitting two propositions.

Definition 5. For a function u, we define

$$F_1(u) := \mathcal{H}\partial_x^3 u, \quad F_2(u) := c_1 u\partial_x^2 u + c_2(\partial_x u)^2 + c_3(\mathcal{H}\partial_x u)^2 + c_4\mathcal{H}(u\mathcal{H}\partial_x^2 u),$$
$$F_3(u) := c_5\mathcal{H}(u^2\partial_x u) + c_6 u\mathcal{H}(u\partial_x u) + c_7 u^2\mathcal{H}\partial_x u, \quad F_4(u) := -u^4.$$

Recall that $K(u) = F_1(u) + F_2(u) + F_3(u) + F_4(u)$.

Lemma 3.3. *Assume that $l \in \mathbb{N}\cup\{0\}$ and $s \geq 1$ satisfy $l \leq s-1$ and a real number p satisfies $2 \leq p \leq \infty$. Put $\alpha = (l + 1/2 - 1/p)/s$. Then, we have*

$$\|\partial_x^l f\|_p \lesssim \begin{cases} \|f\|^{1-\alpha}\|D^s f\|^{\alpha} \quad (when \quad 1 \leq l \leq s-1), \\ \|f\|^{1-\alpha}\|D^s f\|^{\alpha} + \|f\| \quad (when \quad l = 0), \end{cases}$$

for any $f \in H^s(\mathbb{T})$.

Proof. See Section 2 in [26] and Lemma 2.1 in [29]. □

We employ the parabolic regularization:

$$\partial_t u = \partial_x K(u) - \varepsilon\partial_x^4 u, \tag{3.7}$$
$$u(0,x) = \varphi(x), \tag{3.8}$$

where $t \geq 0$ and $\varepsilon > 0$. In what follows, we consider only $t \geq 0$. In the case $t \leq 0$, we only need to replace $-\varepsilon\partial_x^4 u$ with $\varepsilon\partial_x^4 u$ in (3.7). By the standard argument, we can establish the local well-posedness for (3.7)–(3.8) as follows.

Proposition 3.4. *Let $s \geq 3$ and $\varepsilon \in (0,1)$. For any $\varphi \in H^s(\mathbb{T})$, there exist $T_\varepsilon \in (0,\infty]$ and the unique solution $u \in C([0,T_\varepsilon), H^s(\mathbb{T}))$ to the IVP (3.7)–(3.8) on $[0,T_\varepsilon)$ such that (i) $\liminf_{t\to T_\varepsilon}\|u(t)\|_{H^3} = \infty$ or (ii) $T_\varepsilon = \infty$ holds. Moreover, we assume $\varphi^{(j)}, \varphi^{(\infty)} \in H^s(\mathbb{T})$ satisfies $\|\varphi^{(j)} - \varphi^{(\infty)}\|_{H^s} \to 0$ as $j \to \infty$. Let $u^{(j)}$ (resp. $u^{(\infty)}$) $\in C([0,T_\varepsilon); H^s(\mathbb{T}))$ be the solution to (3.7)–(3.8) with initial data $\varphi = \varphi^{(j)}$ (resp. $\varphi = \varphi^{(\infty)}$). Then, for any $T \in (0,T_\varepsilon)$, we have $\sup_{t\in[0,T]}\|u^{(j)}(t) - u^{(\infty)}(t)\|_{H^s} \to 0$ as $j \to \infty$.*

Proof. See Proposition 2.8 in [33] or Proposition 2.13 in [29]. □

We construct a solution to (3.1)–(3.2) by a limiting procedure for solutions obtained by Proposition 3.4. In this argument, it is important to establish the time T independent of ε, which is proved in Proposition 3.10. For that purpose, we define the energy with correction terms in $H^s(\mathbb{T})$. As stated in Section 1, we note that the coefficient of $M_s^{(3)}$ is a quadratic polynomial in s.

Definition 6. Let $s \geq 1$. We define

$$\lambda_1(s) := (c_1 - c_4)s - \frac{c_1}{2} + 2c_2 + \frac{c_4}{2}, \qquad \lambda_2(s) := -2c_3 s - c_4,$$
$$\lambda_3(s) := -2(c_5 + c_6 + c_7)s - 2c_5 - c_6, \quad \lambda_4(s) := 2(c_1 - c_4)s - 5c_1 + 4c_2 + 5c_4.$$

For functions $f, g \in H^s(\mathbb{T})$, we also define

$$E_s(f,g) := \frac{1}{2}\|f-g\|^2(1 + C_s\|f\|^2 + C_s\|f\|^{4s}) + \frac{1}{2}\|D^s(f-g)\|^2 + \sum_{j=1}^{3} M_s^{(j)}(f,g),$$

where

$$M_s^{(1)}(f,g) := \frac{\lambda_1(s)}{4}\int_{\mathbb{T}} f(\mathcal{H}D^s(f-g))\mathcal{H}D^{s-1}(f-g)dx,$$
$$M_s^{(2)}(f,g) := \frac{\lambda_2(s)}{4}\int_{\mathbb{T}} (\mathcal{H}\partial_x f)(D^{s-1}(f-g))^2 dx,$$
$$M_s^{(3)}(f,g) := \frac{\lambda_1(s)\lambda_4(s) + 4\lambda_3(s)}{32}\int_{\mathbb{T}} f^2(D^{s-1}(f-g))^2 dx$$

and C_s is sufficiently large constant such that Lemma 3.6 holds. For simplicity, we write $E_s(f) := E_s(f,0)$ and $M_s^{(j)}(u) := M_s^{(j)}(u,0)$ for $j = 1,2,3$.

We define the energy with correction terms in $L^2(\mathbb{T})$ since there is a problem to define D^{-1} at very low frequency in $E_0(f,g)$. For that purpose, we introduce the following.

Definition 7. Let $\psi \in C^\infty(\mathbb{R})$ be a function satisfying $0 \leq \psi \leq 1$ on $\mathbb{R}$ and

$$\psi(\xi) = \begin{cases} 1, & |\xi| \geq 2, \\ 0, & |\xi| \leq 1. \end{cases}$$

We also define the operator

$$Jf(x) := \mathcal{F}^{-1}\left(\frac{\psi(\xi)}{|\xi|}\hat{f}(\xi)\right)(x)$$

for a function f.

Lemma 3.5. *It holds that*

$$\|Jf\| \le 2\|f\|_{H^{-1}}$$

for any $f \in H^{-1}(\mathbb{T})$.

Proof. This follows from the fact that $\langle \xi \rangle \le 2|\xi|$ for $|\xi| \ge 1$. □

Definition 8. For functions $f, g \in H^1(\mathbb{T})$, we define

$$E(f,g) := \frac{1}{2}\|f-g\|^2 + \frac{1}{2}\|f-g\|_{H^{-1}}^2(1 + C\|f\|^2 + C\|f\|^4) + \sum_{j=1}^{3} M^{(j)}(f,g),$$

where

$$M^{(1)}(f,g) := \frac{\lambda_1(0)}{4}\int_{\mathbb{T}} f(\mathcal{H}(f-g))\mathcal{H}J(f-g)dx,$$
$$M^{(2)}(f,g) := \frac{\lambda_2(0)}{4}\int_{\mathbb{T}} (\mathcal{H}\partial_x f)(J(f-g))^2 dx,$$
$$M^{(3)}(f,g) := \frac{\lambda_1(0)\lambda_4(0) + 4\lambda_3(0)}{32}\int_{\mathbb{T}} f^2(J(f-g))^2 dx$$

and C is sufficiently large constant such that Lemma 3.7 holds.

Lemma 3.6. *Let* $s \ge 1$ *and let* $C_s > 0$ *be sufficiently large. Then for any* $f, g \in H^s(\mathbb{T})$, *it follows that*

$$E_s(f,g) \le \|f-g\|^2(1 + C_s\|f\|^2 + C_s\|f\|^{4s}) + \|D^s(f-g)\|^2 \le 4E_s(f,g).$$

Proof. Lemma 3.3 shows that

$$\begin{aligned}
|M_s^{(2)}(f,g)| &= \left|\frac{\lambda_2(s)}{4}\int_{\mathbb{T}} (\mathcal{H}f)(D^{s-1}(f-g))D^{s-1}\partial_x(f-g)dx\right| \\
&\le C\|f\|\|D^s(f-g)\|\|\mathcal{H}D^{s-1}(f-g)\|_\infty \\
&\le C\|f\|\|f-g\|^{1/2s}\|D^s(f-g)\|^{2-1/2s} + C\|f\|\|f-g\|\|D^s(f-g)\| \\
&\le C\|f-g\|^2(\|f\|^2 + \|f\|^{4s}) + \frac{1}{12}\|D^s(f-g)\|^2.
\end{aligned}$$

Similarly, we can estimate $M_s^{(1)}(f,g)$ and $M_s^{(3)}(f,g)$ as follows:

$$|M_s^{(1)}(f,g)|, |M_s^{(3)}(f,g)| \le C\|f-g\|^2(\|f\|^2 + \|f\|^{4s}) + \frac{1}{12}\|D^s(f-g)\|^2,$$

which completes the proof. □

A similar argument of the previous lemma together with Lemma 3.5 yields the following.

Lemma 3.7. *Let $C>0$ be sufficiently large. Then for any $f,g\in H^1(\mathbb{T})$, it follows that*

$$E(f,g)\le \|f-g\|_{H^{-1}}^2(1+C\|f\|^2+C\|f\|^4)+\|f-g\|^2\le 4E(f,g).$$

Definition 9. Let $s\ge 0$. For f,g, we define

$$I_s(f,g):=1+\|f\|_{H^s}+\|g\|_{H^s}.$$

The main estimate in this section is the following.

Proposition 3.8. *Let $s\ge s_0>7/2$, $1\le s'\le s$, $\varepsilon_j\in(0,1)$, $\varphi_j\in H^{s+4}(\mathbb{T})$ and $u_j\in C([0,T_{\varepsilon_j});H^{s+4}(\mathbb{T}))$ be the solution to* (3.7)–(3.8) *obtained by Proposition 3.4 with $\varepsilon=\varepsilon_j$ and $\varphi=\varphi_j$ for $j=1,2$. Then there exists $C=C(s',s_0)>0$ such that*

$$\begin{aligned}&\frac{d}{dt}E_{s'}(u_1(t),u_2(t))\\ &\le CI_{s_0}(u_1,u_2)^{2(s'+2)}\{\|w\|_{H^{s'}}^2+\|w\|_{H^{s_0-3}}^2\|u_2\|_{H^{s'+3}}^2\\ &\quad+\|w\|_{H^{s_0}}^2(\|u_1\|_{H^{s'}}^2+\|u_2\|_{H^{s'}}^2)\}+\max\{\varepsilon_1^2,\varepsilon_2^2\}\|u_2\|_{H^{s'+4}}^2\end{aligned}\tag{3.9}$$

on $[0,\min\{T_{\varepsilon_1},T_{\varepsilon_2}\})$, where $w=u_1-u_2$.

Proposition 3.9. *Let $s_0>7/2$, $T>0$ and $\varepsilon_j\in(0,1)$. Let $u_j\in C([0,T];H^{s_0}(\mathbb{T}))\cap C((0,T);H^{s_0+1}(\mathbb{T}))$ satisfy* (3.7) *with $\varepsilon=\varepsilon_j$ on $[0,T]$ for $j=1,2$. Then there esists $C=C(s_0)>0$ such that*

$$\frac{d}{dt}E(u_1(t),u_2(t))\le CI_{s_0}(u_1,u_2)^7E(u_1(t),u_2(t))+\max\{\varepsilon_1^2,\varepsilon_2^2\}\|u_2\|_{H^{s_0+1}}^2\tag{3.10}$$

on $[0,T]$, where $w=u_1-u_2$.

If we admit Propositions 3.8 and 3.9, we can show the main result. We prove Proposition 3.8 (resp. Proposition 3.9) in Section 3 (resp. Section 4).

Proposition 3.10. *Let $s\ge s_0>7/2$, $\varepsilon\in(0,1)$, $\varphi\in H^s(\mathbb{T})$. Let $T_\varepsilon>0$ and let $u\in C([0,T_\varepsilon),H^s(\mathbb{T}))\cap C((0,T_\varepsilon);H^{s+4}(\mathbb{T}))$ be the solution to* (3.7)–(3.8)*, both of which are obtained by Proposition 3.4. Then, there exist $T=T(s_0,\|\varphi\|_{H^{s_0}})>0$ and $C=C(s,s_0,\|\varphi\|_{H^{s_0}})>0$ such that*

$$T_\varepsilon\ge T,\quad \sup_{t\in[0,T]}E_s(u(t))\le CE_s(\varphi),\quad \frac{d}{dt}E_s(u(t))\le CE_s(u(t))\tag{3.11}$$

on $[0,T]$, where T (resp. C) is monotone decreasing (resp. increasing) with $\|\varphi\|_{H^{s_0}}$.

Proof. Assume that the set $F=\{t\ge 0;E_{s_0}(u(t))>2E_{s_0}(\varphi)\}$ is not empty. Set $T_\varepsilon^*=\inf F$. Note that $0<T_\varepsilon^*\le T_\varepsilon$ and $E_{s_0}(u(t))\le 2E_{s_0}(\varphi)$ on $[0,T_\varepsilon^*]$. Assume that there exists $t'\in[0,T_\varepsilon^*]$ such that $E_{s_0}(u(t'))>2E_{s_0}(\varphi)$. This implies that

$t' \geq T_\varepsilon^*$ by the definition of T_ε^*. Then we have $t' = T_\varepsilon^*$. Thus, $\sup_{t\in[0,T_\varepsilon^*]} E_{s_0}(u(t)) \leq C(\|\varphi\|_{H^{s_0}})$ by (ii) of Lemma 3.6. By Proposition 3.8 with $\varphi_2 = 0$, there exists $C_s' = C(s, s_0, \|\varphi\|_{H^{s_0}})$ such that

$$\frac{d}{dt}E_s(u(t)) \leq C_s' E(u(t))$$

on $[0, T_\varepsilon^*]$. The Gronwall inequality gives that

$$E_s(u(t)) \leq E_s(\varphi)\exp(C_s' t) \tag{3.12}$$

on $[0, T_\varepsilon^*]$. Here, we put $T = \min\{(2C_{s_0}')^{-1}, T_\varepsilon^*\}$. Then (3.12) with $s = s_0$ shows that

$$E_{s_0}(u(t)) \leq E_{s_0}(\varphi)\exp(2^{-1}) < 2E_{s_0}(\varphi),$$

on $[0, T]$. By the definition of T_ε^* and the continuity of $E_{s_0}(u(t))$, we obtain $0 < T = (2C_{s_0}')^{-1} < T_\varepsilon^* \leq T_\varepsilon$. If F is empty, then we have $T_\varepsilon^* = T_\varepsilon = \infty$. In particular, we can take $T = (2C_{s_0}')^{-1} < \infty$, which concludes the proof. □

For the proof of the following uniqueness result, see Thorem 6.22 in [14].

Lemma 3.11 (Uniqueness)**.** *Let* $s_0 > 7/2$, $\delta > 0$ *and* $u, v \in L^\infty([0,\delta]; H^{s_0}(\mathbb{T}))$ *satisfy* (3.1) *on* $[0,\delta]$ *with* $u(0) = v(0)$ *and satisfy*

$$u, v \in C([0,\delta]; H^3(\mathbb{T})) \cap C^1([0,\delta]; H^{-1}(\mathbb{T})).$$

Then $u \equiv v$ *on* $[0,\delta]$.

It is important to employ the Bona-Smith type argument in the energy inequality for two solutions in H^s. For that purpose, we introduce the following.

Definition 10. Let $s \geq 0$, $f \in H^s(\mathbb{T})$ and $\eta \in (0,1)$. And let $\rho \in C_0^\infty(\mathbb{R})$ be $\rho(x) := 1 - \psi(x)$ for $x \in \mathbb{R}$. We put

$$\widehat{L_\eta f}(k) := \rho(\eta k)\hat{f}(k).$$

For the proof of the following lemma, see Remark 3.5 in [5].

Lemma 3.12. *Let* $s \geq 0$, $\alpha \geq 0$, $\eta \in (0,1)$ *and* $f \in H^s(\mathbb{T})$. *Then,* $L_\eta f \in H^\infty(\mathbb{T})$ *satisfies*

$$\|L_\eta f - f\|_{H^s} \to 0 \quad (\eta \to 0), \quad \|L_\eta f - f\|_{H^{s-\alpha}} \lesssim \gamma^\alpha \|f\|_{H^s},$$
$$\|L_\eta f\|_{H^{s-\alpha}} \leq \|f\|_{H^{s-\alpha}}, \quad \|L_\eta f\|_{H^{s+\alpha}} \lesssim \gamma^{-\alpha}\|f\|_{H^s}.$$

Proof of Theorem 3.1. We only need to prove Theorem 3.1 for $t \geq 0$ thanks to the transform $t \to -t$. In what follows, without loss of generality, we may assume that s_0 is strictly smaller than s since the assumption $\|\varphi\|_{H^{s_0}} \leq K$ is weaker than $\|\varphi\|_{H^{s_0'}} \leq K$ when $s_0 < s_0'$. First we prove the existence of the solution. For $\varphi \in H^s(\mathbb{T})$, we put $\varphi_\eta := L_\eta \varphi \in H^\infty(\mathbb{T})$ for $\eta \in (0,1)$. By Proposition 3.4, there exists the unique solution $u_{\epsilon,\eta} \in C([0,T_\varepsilon); H^s(\mathbb{T}))$ to (3.7) with the initial data φ_η on $[0,T_\varepsilon)$. We see from Lemma 3.12 that

$$\|\varphi_\eta\|_{H^s} \leq \|\varphi\|_{H^s}, \quad \|\varphi_\eta\|_{H^{s_0}} \leq \|\varphi\|_{H^{s_0}}.$$

Then, Proposition 3.10 with Lemma 3.6 shows that there exists $T = T(s, s_0, \|\varphi\|_{H^{s_0}}) > 0$ such that

$$\sup_{t\in[0,T]} \|u_{\varepsilon,\eta}(t)\|_{H^s} \lesssim \sup_{t\in[0,T]} E_s(u_{\varepsilon,\eta}(t))^{1/2} \lesssim E_s(u_{\varepsilon,\eta}(0))^{1/2} \lesssim \|\varphi_\eta\|_{H^s},$$

which implies that

$$\sup_{t\in[0,T]} \|u_{\varepsilon,\eta}(t)\|_{H^{s+3}} \lesssim \eta^{-3}\|\varphi\|_{H^s}. \tag{3.13}$$

Let $0 < \varepsilon_1 \leq \varepsilon_2 < 1$ and $\eta_j = \varepsilon_j^{1/2s}$ for $j = 1, 2$. Proposition 3.9 with $s' = s$ shows that there exists $C = C(s, s_0, T, \|\varphi\|_{H^{s_0}}) > 0$ such that

$$\begin{aligned}\sup_{t\in[0,T]} \|u_{\varepsilon_1,\eta_1}(t) - u_{\varepsilon_2,\eta_2}(t)\| &\leq CE(u_{\varepsilon_1,\eta_1}(0), u_{\varepsilon_2,\eta_2}(0)) \\ &\leq C(\|\varphi_{\eta_1} - \varphi_{\eta_2}\|^2 + \varepsilon_2^{2-2/s})^{1/2} \leq C\varepsilon_2^{1/2}.\end{aligned}$$

By interpolation, it holds that for $\alpha \in [0, s]$,

$$\sup_{t\in[0,T]} \|u_{\varepsilon_1,\eta_1}(t) - u_{\varepsilon_2,\eta_2}(t)\|_{H^{s-\alpha}} \lesssim \varepsilon_2^{\alpha/2s}. \tag{3.14}$$

Therefore, Proposition 3.8 together with (3.13) and (3.14) shows that

$$\sup_{t\in[0,T]} \|u_{\varepsilon_1,\eta_1}(t) - u_{\varepsilon_2,\eta_2}(t)\|_{H^s} \lesssim \|\varphi_{\eta_1} - \varphi_{\eta_2}\|_{H^s} + \varepsilon_2^{(1-s_0/s)/2} \tag{3.15}$$

since $0 < (1 - s_0/s)/2 < 1 - s_0/s < 1 - 2/s$. Then, $\{u_{\varepsilon,\eta}\}_{\varepsilon=\eta^{2s}}$ is a Cauchy sequence in $C([0,T]; H^s(\mathbb{T}))$ as $\varepsilon \to 0$ and there exists the limit $u \in C([0,T]; H^s(\mathbb{T}))$. It is clear that u satisfy (3.1)–(3.2) on $[0,T]$. We also note that letting $\varepsilon_1 \to 0$ in (3.15),

$$\sup_{t\in[0,T]} \|u(t) - u_\varepsilon(t)\|_{H^s} \lesssim \|\varphi - \varphi_{\varepsilon^{1/2s}}\|_{H^s} + \varepsilon^{(1-s_0/s)/2} \tag{3.16}$$

for $\varepsilon \in (0,1)$, where $u_\varepsilon := u_{\varepsilon,\varepsilon^{1/2s}}$

Finally, we show the continuous dependence. We claim that if $\|\varphi^{(j)} - \varphi\|_{H^s} \to 0$ as $j \to \infty$, then $\sup_{t\in[0,T]} \|u^{(j)}(t) - u(t)\|_{H^s} \to 0$ as $j \to \infty$, where $u^{(j)}$ (resp. u) is the solution to (3.1) with the initial data $\varphi^{(j)}$ (resp. φ) for $j \in \mathbb{N}$. First note that the triangle inequality with Lemma 3.12 gives that

$$\begin{aligned}\|\varphi^{(j)} - \varphi^{(j)}_{\varepsilon^{1/2s}}\|_{H^s} &\le \|\varphi^{(j)} - \varphi\|_{H^s} + \|\varphi - \varphi_{\varepsilon^{1/2s}}\|_{H^s} + \|\varphi_{\varepsilon^{1/2s}} - \varphi^{(j)}_{\varepsilon^{1/2s}}\|_{H^s} \\ &\lesssim \|\varphi^{(j)} - \varphi\|_{H^s} + \|\varphi - \varphi_{\varepsilon^{1/2s}}\|_{H^s}.\end{aligned}$$

This together with (3.16) implies that

$$\begin{aligned}&\sup_{t\in[0,T]} \|u^{(j)}(t) - u(t)\|_{H^s} \\ &\le \sup_{t\in[0,T]} \|u^{(j)}(t) - u^{(j)}_\varepsilon(t)\|_{H^s} + \sup_{t\in[0,T]} \|u^{(j)}_\varepsilon(t) - u_\varepsilon(t)\|_{H^s} + \sup_{t\in[0,T]} \|u_\varepsilon(t) - u(t)\|_{H^s} \\ &\le C\left(\|\varphi^{(j)} - \varphi\|_{H^s} + \|\varphi - \varphi_{\varepsilon^{1/2s}}\|_{H^s} + \sup_{t\in[0,T]} \|u^{(j)}_\varepsilon(t) - u_\varepsilon(t)\|_{H^s} + \varepsilon^{(1-s_0/s)/2}\right).\end{aligned}$$

Let $\delta > 0$. Then, there exists $\varepsilon_0 \in (0,1)$ such that for any $\varepsilon \in (0,\varepsilon_0)$

$$C(\|\varphi - \varphi_{\varepsilon^{1/2s}}\|_{H^s} + \varepsilon^{(1-s_0/s)/2}) < \frac{\delta}{2}.$$

For each $\varepsilon \in (0,\varepsilon_0)$, we see from Proposition 3.4 that there exists $N_0 \in \mathbb{N}$ such that if $j > N_0$, then

$$C\|\varphi^{(j)} - \varphi\|_{H^s} + C\sup_{t\in[0,T]} \|u^{(j)}_\varepsilon(t) - u_\varepsilon(t)\|_{H^s} < \frac{\delta}{2},$$

which completes the proof of Theorem 3.1. □

3.3. **The energy estimate in H^s.** In this subsection, we prove Proposition 3.8, which is the main estimate in this section. Before proving Proposition 3.8, we introduce some commutator estimates which are useful in evaluating nonlinear terms.

Definition 11. For $s \geq 0$ and functions f, g, h we define

$$\begin{aligned}
P_s^{(1)}(f,g) &:= D^s\partial_x(f\partial_x^2 g) - D^s\partial_x f\partial_x^2 g - fD^s\partial_x^3 g - (s+1)\partial_x f D^s\partial_x^2 g \\
&\quad - \frac{s(s+1)}{2}\partial_x^2 f D^s\partial_x g, \\
P_s^{(2)}(f,g) &:= \mathcal{H}D^s\partial_x(f\partial_x^2 g) - (\mathcal{H}D^s\partial_x f)\partial_x^2 g - f\mathcal{H}D^s\partial_x^3 g \\
&\quad - (s+1)\partial_x f\mathcal{H}D^s\partial_x^2 g - \frac{s(s+1)}{2}\partial_x^2 f\mathcal{H}D^s\partial_x g, \\
P_s^{(3)}(f,g) &:= D^s\partial_x(\partial_x f\partial_x g) - D^s\partial_x^2 f\partial_x g - (s+1)D^s\partial_x f\partial_x^2 g \\
&\quad - \partial_x f D^s\partial_x^2 g - (s+1)\partial_x^2 f D^s\partial_x g, \\
P_s^{(4)}(f,g) &:= \mathcal{H}D^s\partial_x(\partial_x f\partial_x g) - (\mathcal{H}D^s\partial_x^2 f)\partial_x g - (s+1)(\mathcal{H}D^s\partial_x f)\partial_x^2 g \\
&\quad - \partial_x f(\mathcal{H}D^s\partial_x^2 g) - (s+1)\partial_x^2 f(\mathcal{H}D^s\partial_x g), \\
P_s^{(5)}(f,g,h) &:= D^s\partial_x(fg\partial_x h) - D^s\partial_x fg\partial_x h - fD^s\partial_x g\partial_x h - fgD^s\partial_x^2 h \\
&\quad - (s+1)\partial_x fgD^s\partial_x h - (s+1)f\partial_x gD^s\partial_x h, \\
P_s^{(6)}(f,g,h) &:= \mathcal{H}D^s\partial_x(fg\partial_x h) - (\mathcal{H}D^s\partial_x f)g\partial_x h - f(\mathcal{H}D^s\partial_x g)\partial_x h - fg\mathcal{H}D^s\partial_x^2 h \\
&\quad - (s+1)\partial_x fg\mathcal{H}D^s\partial_x h - (s+1)f\partial_x g\mathcal{H}D^s\partial_x h, \\
P_s^{(7)}(f,g,h) &:= D^s\partial_x(f\mathcal{H}(g\partial_x h)) - D^s\partial_x f\mathcal{H}(g\partial_x h) - f(\mathcal{H}D^s\partial_x g)\partial_x h - fg\mathcal{H}D^s\partial_x^2 h \\
&\quad - (s+1)\partial_x fg\mathcal{H}D^s\partial_x h - (s+1)f\partial_x g\mathcal{H}D^s\partial_x h, \\
P_s^{(8)}(f,g) &:= D^s\partial_x(f\partial_x g) - D^s\partial_x f\partial_x g - fD^s\partial_x^2 g - (s+1)\partial_x fD^s\partial_x g, \\
P_s^{(9)}(f,g) &:= \mathcal{H}D^s\partial_x(f\partial_x g) - (\mathcal{H}D^s\partial_x f)\partial_x g - f\mathcal{H}D^s\partial_x^2 g - (s+1)\partial_x f\mathcal{H}D^s\partial_x g.
\end{aligned}$$

Lemma 3.13. *Let $s_0 > 1/2$ and $s \geq 0$. Then*

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^{s_0}} + \|f\|_{H^{s_0}}\|g\|_{H^s}$$

for any $f, g \in H^{\max\{s_0,s\}}(\mathbb{T})$.

Proof. This follows from the fact that $\langle\xi\rangle^s \lesssim \langle\xi-\eta\rangle^s + \langle\eta\rangle^s$ for any $\xi, \eta \in \mathbb{Z}$. □

For the proofs of the following three lemmas, see [29].

Lemma 3.14. *Let $s_0 > 3/2$ and $s \geq 1$. Then*

$$\|[D^s, f]\partial_x g\| \lesssim \|f\|_{H^s}\|g\|_{H^{s_0}} + \|f\|_{H^{s_0}}\|g\|_{H^s}$$

for any $f, g \in H^{\max\{s_0,s\}}(\mathbb{T})$.

Lemma 3.15. *Let $s_0 > 1/2$, $k \in \mathbb{N}$ and $s_1, s_2 \geq 0$. Suppose that $s_1 + s_2 = k$. Then, there exists $C = C(s_0) > 0$ such that for any $f \in H^{s_0+s_1}(\mathbb{T})$ and $g \in H^{s_2}(\mathbb{T})$*

$$\|[\mathcal{H}, f]\partial_x^k g\| \leq C\|f\|_{H^{s_0+s_1}}\|g\|_{H^{s_2}}.$$

Lemma 3.16. *Let $s_0 > 5/2$ and $s \geq 0$. Then there exists $C = C(s, s_0) > 0$ such that*

$$\|P_s^{(8)}(u,v)\|, \|P_s^{(9)}(u,v)\| \leq C(\|u\|_{H^{s_0}}\|v\|_{H^s} + \|u\|_{H^s}\|v\|_{H^{s_0}})$$

for any $u, v \in H^{\max\{s,s_0\}}(\mathbb{T})$.

Lemma 3.17. *Let $s_0 > 7/2$ and $s \geq 0$. Then there exists $C = C(s, s_0) > 0$ such that*

$$\|P_s^{(1)}(u,v)\|, \|P_s^{(2)}(u,v)\| \leq C(\|u\|_{H^s}\|v\|_{H^{s_0}} + \|u\|_{H^{s_0}}\|v\|_{H^s})$$

for any $u, v \in H^{\max\{s,s_0\}}(\mathbb{T})$.

Proof. We show only the inequality for $P_s^{(1)}$. The other one follows from a similar argument. It suffices to show that

$$\begin{aligned}&\left||\xi|^s\xi\eta^2 - |\xi-\eta|^s(\xi-\eta)\eta^2 - |\eta|^s\eta^3 - (s+1)(\xi-\eta)|\eta|^s\eta^2 - \frac{s(s+1)}{2}(\xi-\eta)^2|\eta|^s\eta\right| \\ &\lesssim |\eta|^3|\xi-\eta|^s + |\eta|^s|\xi-\eta|^3,\end{aligned} \tag{3.17}$$

for any $\xi, \eta \in \mathbb{Z}$. We split the summation region into three regions: $R_1 = \{3|\eta| \leq |\xi-\eta|\}$, $R_2 = \{|\xi-\eta|/4 \leq |\eta| \leq 4|\xi-\eta|\}$ and $R_3 = \{|\eta| \geq 3|\xi-\eta|\}$. On R_1, the mean value theorem shows that (3.25) holds. On R_2, it is obvious. On R_3, it immediately follows that $|\xi-\eta|^{s+1}|\eta|^2 \lesssim |\xi-\eta|^s|\eta|^3$. Set $\sigma(x) = x|x|^s$ for $x \in \mathbb{R}$. Note that $\sigma \in C^3(\mathbb{R})$ when $s > 2$. The Taylor theorem shows that there exist $\tilde{\eta} \in (\xi, \eta)$ or $\tilde{\eta} \in (\eta, \xi)$ such that

$$\sigma(\xi) = \sigma(\eta) + \sigma'(\eta)(\xi-\eta) + \frac{\sigma''(\eta)}{2}(\xi-\eta)^2 + \frac{\sigma'''(\widetilde{\eta})}{6}(\xi-\eta)^3.$$

This together with the fact that $|\tilde{\eta}| \sim |\xi| \sim |\eta|$ implies that (3.25) holds. When $1 < s \leq 2$, (3.25) holds since $|\xi-\eta|^2|\eta|^{s+1} = |\xi-\eta|^{2-s}|\xi-\eta|^s|\eta|^{s+1} \lesssim |\xi-\eta|^s|\eta|^3$ on R_3. Similarly, when $0 \leq s \leq 1$, (3.25) holds by the above inequality with $|\xi-\eta||\eta|^{s+2} \lesssim |\xi-\eta|^s|\eta|^3$ on R_3, which concludes the proof. □

Lemma 3.18. *Let $s_0 > 7/2$ and $s \geq 0$. Then there exists $C = C(s, s_0) > 0$ such that*

$$\|P_s^{(3)}(u,v)\|, \|P_s^{(4)}(u,v)\| \leq C(\|u\|_{H^s}\|v\|_{H^{s_0}} + \|u\|_{H^{s_0}}\|v\|_{H^s}),$$

for any $u, v \in H^{\max\{s,s_0\}}(\mathbb{T})$.

Proof. This follows from a similar argument of the previous lemma. □

Lemma 3.19. *Let $s_0 > 3/2$ and $s \geq 0$. Then there exists $C = C(s, s_0) > 0$ such that*

$$\|\Lambda_{s+1}(uv) - \Lambda_{s+1}uv - u\Lambda_{s+1}v\| \leq C(\|u\|_{H^s}\|v\|_{H^{s_0}} + \|u\|_{H^{s_0}}\|v\|_{H^s})$$

for any $u, v \in H^{\max\{s,s_0\}}(\mathbb{T})$, where $\Lambda_{s+1} = D^s\partial_x$ or D^{s+1}.

Proof. It suffices to show that for any $\xi, \eta \in \mathbb{Z}$

$$||\xi|^{s+1} - |\xi-\eta|^{s+1} - |\eta|^{s+1}|, ||\xi|^s\xi - |\xi-\eta|^s(\xi-\eta) - |\eta|^s\eta| \lesssim |\eta||\xi-\eta|^s + |\xi-\eta||\eta|^s.$$

If $s = 0$, then it is obvious by the triangle inequality. In the case $s > 0$, this follows from a similar argument of the proof of Lemma 3.17. □

Lemma 3.20. *Let $s_0 > 5/2$ and $s \geq 0$. Then there exists $C = C(s, s_0) > 0$ such that*

$$\begin{aligned}&\|P_s^{(5)}(u_1,u_2,u_3)\|, \|P_s^{(6)}(u_1,u_2,u_3)\|\\ &\leq C(\|u_1\|_{H^s}\|u_2\|_{H^{s_0}}\|u_3\|_{H^{s_0}} + \|u_1\|_{H^{s_0}}\|u_2\|_{H^s}\|u_3\|_{H^{s_0}} + \|u_1\|_{H^{s_0}}\|u_2\|_{H^{s_0}}\|u_3\|_{H^s})\end{aligned}$$

for any $u_j \in H^{\max\{s,s_0\}}(\mathbb{T})$ for $j = 1, 2, 3$.

Proof. We show only the inequality for $P_s^{(5)}$. The other one follows from a similar argument. Aplying Lemma 3.13, 3.16 and 3.19, we have

$$\begin{aligned}&\|P_s^{(5)}(u_1,u_2,u_3)\|\\ &\leq \|P_s^{(8)}(u_1u_2,u_3)\| + \|D^s\partial_x(u_1u_2) - D^s\partial_x u_1 u_2 - u_1 D^s\partial_x u_2\|\|\partial_x u_3\|_\infty,\\ &\lesssim \|u_1\|_{H^s}\|u_2\|_{H^{s_0}}\|u_3\|_{H^{s_0}} + \|u_1\|_{H^{s_0}}\|u_2\|_{H^s}\|u_3\|_{H^{s_0}} + \|u_1\|_{H^{s_0}}\|u_2\|_{H^{s_0}}\|u_3\|_{H^s},\end{aligned}$$

which completes the proof. □

Lemma 3.21. *Let $s_0 > 5/2$ and $s \geq 0$. Then there exists $C = C(s, s_0) > 0$ such that*

$$\begin{aligned}&\|P_s^{(7)}(u_1,u_2,u_3)\|\\ &\leq C(\|u_1\|_{H^s}\|u_2\|_{H^{s_0}}\|u_3\|_{H^{s_0}} + \|u_1\|_{H^{s_0}}\|u_2\|_{H^s}\|u_3\|_{H^{s_0}} + \|u_1\|_{H^{s_0}}\|u_2\|_{H^{s_0}}\|u_3\|_{H^s})\end{aligned}$$

for any $u_j \in H^{\max\{s,s_0\}}(\mathbb{T})$ for $j = 1, 2, 3$.

Proof. We see from the proof of Lemma 3.16 and 3.19 that for any $\xi, \xi_1, \xi_2 \in \mathbb{Z}$

$$\begin{aligned}&||\xi|^s\xi\xi_2 - |\xi-\xi_1-\xi_2|^s(\xi-\xi_1-\xi_2)\xi_2 - |\xi_1|^s\xi_1\xi_2\\ &\quad - |\xi_2|^s\xi_2^2 - (s+1)(\xi-\xi_1-\xi_2)|\xi_2|^s\xi_2 - (s+1)\xi_1|\xi_2|^s\xi_2|\\ &\leq ||\xi|^s\xi\xi_2 - |\xi-\xi_2|^s(\xi-\xi_2)\xi_2 - |\xi_2|^s\xi_2^2 - (s+1)(\xi-\xi_2)|\xi_2|^s\xi_2|\\ &\quad + ||\xi-\xi_2|^s(\xi-\xi_2) - |\xi_1|^s\xi_1 - |\xi-\xi_1-\xi_2|^s(\xi-\xi_1-\xi_2)||\xi_2| \lesssim \Xi(\xi,\xi_1,\xi_2),\end{aligned}$$

where

$$\Xi(\xi,\xi_1,\xi_2) := |\xi-\xi_2|^2|\xi_2|^s + |\xi-\xi_2|^s|\xi_2|^2 + (|\xi-\xi_1-\xi_2|^s|\xi_1| + |\xi-\xi_1-\xi_2||\xi_1|^s)|\xi_2|.$$

We see from Lemma 3.15 that

$$\begin{aligned}
&\|P_s^{(7)}(u_1,u_2,u_3)\| \\
&\le \|D^s\partial_x(u_1\mathcal{H}(u_2\partial_x u_3)) - D^s\partial_x u_1\mathcal{H}(u_2\partial_x u_3) - u_1\mathcal{H}(D^s\partial_x u_2\partial_x u_3) \\
&\quad - u_1\mathcal{H}(u_2 D^s\partial_x^2 u_3) - (s+1)\partial_x u_1\mathcal{H}(u_2 D^s\partial_x u_3) - (s+1)u_1\mathcal{H}(\partial_x u_2 D^s\partial_x u_3)\| \\
&\quad + \|u_1\|_\infty(\|[\mathcal{H},\partial_x u_3]D^s\partial_x u_2\| + \|[\mathcal{H},u_2]D^s\partial_x^2 u_3\|) \\
&\quad + (s+1)(\|\partial_x u_1\|_\infty\|[\mathcal{H},u_2]D^s\partial_x u_3\| + \|u_1\|_\infty\|[\mathcal{H},\partial_x u_2]D^s\partial_x u_3\|) \\
&\lesssim \left\|\sum_{\xi_1,\xi_2}\Xi(\xi,\xi_1,\xi_2)|\hat{u_1}(\xi-\xi_1-\xi_2)||\hat{u_2}(\xi_1)||\hat{u_3}(\xi_2)|\right\|_{l^2_\xi} \\
&\quad + \|u_1\|_{H^{s_0}}\|u_2\|_{H^{s_0}}\|u_3\|_{H^s} + \|u_1\|_{H^{s_0}}\|u_2\|_{H^s}\|u_3\|_{H^{s_0}} \\
&\lesssim \|u_1\|_{H^s}\|u_2\|_{H^{s_0}}\|u_3\|_{H^{s_0}} + \|u_1\|_{H^{s_0}}\|u_2\|_{H^s}\|u_3\|_{H^{s_0}} + \|u_1\|_{H^{s_0}}\|u_2\|_{H^{s_0}}\|u_3\|_{H^s},
\end{aligned}$$

which completes the proof. □

Lemma 3.22. *Let $s_0 > 5/2$ and $s \ge 0$. Then there exists $C = C(s,s_0) > 0$ such that*

$$\|\Lambda_s(u\partial_x^2 v) - u\Lambda_s\partial_x^2 v - s\partial_x u\Lambda_s\partial_x v\| \le C(\|u\|_{H^{s_0}}\|v\|_{H^s} + \|u\|_{H^s}\|v\|_{H^{s_0}})$$

for any $u, v \in H^{\max\{s,s_0\}}(\mathbb{T})$, where $\Lambda_s = D^s$ or $\mathcal{H}D^s$.

Proof. The proof is very similar to that of Lemma 3.17. □

Lemma 3.23. *Let $s_0 > 3/2$ and $s \ge 0$. Then there exists $C = C(s,s_0) > 0$ such that*

$$\begin{aligned}
&\|\Lambda_s(u_1\mathcal{H}(u_2\partial_x u_3)) - u_1u_2\mathcal{H}\Lambda_s\partial_x u_3\| \\
&\le C(\|u_1\|_{H^s}\|u_2\|_{H^{s_0}}\|u_3\|_{H^{s_0}} + \|u_1\|_{H^{s_0}}\|u_2\|_{H^s}\|u_3\|_{H^{s_0}} + \|u_1\|_{H^{s_0}}\|u_2\|_{H^{s_0}}\|u_3\|_{H^s})
\end{aligned}$$

for any $u_j \in H^{\max\{s,s_0\}}(\mathbb{T})$ for $j = 1,2,3$, where $\Lambda_s = D^s$ or $\mathcal{H}D^s$.

Proof. It suffices to show that

$$\begin{aligned}
&\|\Lambda_s(u_1\mathcal{H}(u_2\partial_x u_3)) - u_1\mathcal{H}(u_2\Lambda_s\partial_x u_3)\| \\
&\lesssim \|u_1\|_{H^s}\|u_2\|_{H^{s_0}}\|u_3\|_{H^{s_0}} + \|u_1\|_{H^{s_0}}\|u_2\|_{H^s}\|u_3\|_{H^{s_0}} + \|u_1\|_{H^{s_0}}\|u_2\|_{H^{s_0}}\|u_3\|_{H^s}.
\end{aligned}$$

Indeed, Lemma 3.15 shows that

$$\|u_1[\mathcal{H},u_2]\Lambda_s\partial_x u_3\| \lesssim \|u_1\|_{H^{s_0-1}}\|u_2\|_{H^{s_0}}\|u_3\|_{H^s}.$$

The standard argument implies that

$$||\xi|^s\xi_2 - |\xi_2|^s\xi_2|, |\operatorname{sgn}(\xi)|\xi|^s\xi_2 - \operatorname{sgn}(\xi_2)|\xi_2|^s\xi_2| \lesssim |\xi-\xi_2|^s|\xi_2| + |\xi-\xi_2||\xi_s|^s,$$

which completes the proof by the triangle inequality. □

Lemma 3.24. *Let $s \geq 0$ and $s_0 > 5/2$. Let $u \in H^{s_0}(\mathbb{T})$ and $w \in H^{s+2}(\mathbb{T})$. Then*

$$|\langle uD^s\partial_x^2 w, D^s w\rangle + \langle u, (D^s\partial_x w)^2\rangle| \lesssim \|u\|_{H^{s_0}}\|w\|_{H^s}^2$$

Proof. Note that

$$\langle uD^s\partial_x^2 w, D^s w\rangle = \frac{1}{2}\langle \partial_x^2 u, (D^s w)^2\rangle - \langle u, (D^s\partial_x w)^2\rangle,$$

which shows the claim. □

As stated in Section 1, by the integration by parts, the third order derivative loss can be reduced to the second order one.

Lemma 3.25. *Let $s \geq 0$ and $s_0 > 7/2$. Let $u \in H^{s_0}(\mathbb{T})$ and $w \in H^{s+3}(\mathbb{T})$. Then*

$$\left|\langle uD^s\partial_x^3 w, D^s w\rangle - \frac{3}{2}\langle \partial_x u, (D^s\partial_x w)^2\rangle\right| \lesssim \|u\|_{H^{s_0}}\|w\|_{H^s}^2$$

Proof. Note that

$$\langle uD^s\partial_x^3 w, D^s w\rangle = -\langle \partial_x u D^s\partial_x^2 w, D^s w\rangle + \frac{1}{2}\langle \partial_x u, (D^s\partial_x w)^2\rangle,$$

which together with Lemma 3.24 shows the claim. □

Lemma 3.26. *Let $s_0 > 1/2$ and u, v be sufficiently smooth function defined on $\mathbb{T}$. Then there exists $C = C(s_0) > 0$ such that*

$$|\langle \partial_x(v\mathcal{H}\partial_x u), u\rangle| \leq C\|v\|_{H^{s_0+2}}\|u\|^2.$$

Proof. See Lemma 2.6 in [29]. □

Lemma 3.27. *Let $s \geq 0$ and $s_0 > 5/2$. Let $u \in H^{s_0}(\mathbb{T})$ and $w \in H^{s+2}(\mathbb{T})$. Then*

$$|\langle u, (\mathcal{H}D^s\partial_x w)^2\rangle - \langle u, (D^s\partial_x w)^2\rangle| \lesssim \|u\|_{H^{s_0}}\|w\|_{H^s}^2.$$

Proof. We have

$$\langle u, (\mathcal{H}D^s\partial_x w)^2\rangle = \langle \mathcal{H}(\partial_x u\mathcal{H}D^s\partial_x w), D^s w\rangle + \langle \mathcal{H}(u\mathcal{H}D^s\partial_x^2 w), D^s w\rangle =: A + B.$$

For A, note that

$$|A| \leq |\langle [\mathcal{H}, \partial_x u]\mathcal{H}D^s\partial_x w, D^s w\rangle| + |\langle \partial_x u D^s\partial_x w, D^s w\rangle| \lesssim \|u\|_{H^{s_0}}\|w\|_{H^s}^2.$$

For B, we have

$$B = \langle [\mathcal{H}, u]\mathcal{H}D^s\partial_x^2 w, D^s w\rangle - \langle uD^s\partial_x^2 w, D^s w\rangle$$
$$= \langle [\mathcal{H}, u]\mathcal{H}D^s\partial_x^2 w, D^s w\rangle - \frac{1}{2}\langle \partial_x^2 u, (D^s w)^2\rangle + \langle u, (D^s\partial_x w)^2\rangle,$$

which concludes the proof. □

We are ready to evaluate nonlinear terms. First, we estimate terms in $F_2(u)$.

Lemma 3.28. *Let $s_0 > 7/2$ and $s \geq 0$. Let $u, v \in H^{\max\{s+3, s_0\}}(\mathbb{T})$. Then*

$$|\langle D^s\partial_x(u\partial_x^2 u - v\partial_x^2 v), D^s w\rangle + (s-1/2)\langle \partial_x u, (D^s\partial_x w)^2\rangle|$$
$$\lesssim I_{s_0}(u, v)\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-3}}^2\|v\|_{H^{s+3}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},$$

where $w = u - v$.

Proof. Set

$$\langle D^s\partial_x(u\partial_x^2 u - v\partial_x^2 v), D^s w\rangle = \langle D^s\partial_x(u\partial_x^2 w), D^s w\rangle + \langle D^s\partial_x(w\partial_x^2 v), D^s w\rangle =: R_1 + R_2.$$

Lemma 3.17, 3.24 and 3.25 show that

$$|R_1 + (s-1/2)\langle \partial_x u, (D^s\partial_x w)^2\rangle|$$
$$\leq |\langle P_s^{(1)}(u, w), D^s w\rangle| + |\langle D^s\partial_x(w+v)\partial_x^2 w, D^s w\rangle| + |\langle uD^s\partial_x^3 w, D^s w\rangle - (3/2)\langle \partial_x u, (D^s\partial_x w)^2\rangle|$$
$$+ (s+1)|\langle \partial_x u D^s\partial_x^2 w, D^s w\rangle + \langle \partial_x u, (D^s\partial_x w)^2\rangle| + s(s+1)|\langle \partial_x^3 u, (D^s w)^2\rangle|/4$$
$$\lesssim \|w\|_{H^s}(\|w\|_{H^{s_0-1}}\|v\|_{H^{s+1}} + \|u\|_{H^{s_0}}\|w\|_{H^s} + \|w\|_{H^{s_0}}\|u\|_{H^s} + \|w\|_{H^{s_0}}\|w\|_{H^s}).$$

We see from a similar argument that

$$|R_2| \lesssim I_{s_0}(u, v)\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-3}}^2\|v\|_{H^{s+3}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},$$

which completes the proof. □

Lemma 3.29. *Let $s_0 > 7/2$ and $s \geq 0$. Let $u, v \in H^{\max\{s+2, s_0\}}(\mathbb{T})$. Then*

$$|\langle D^s\partial_x((\partial_x u)^2 - (\partial_x v)^2), D^s w\rangle + 2\langle \partial_x u, (D^s\partial_x w)^2\rangle|$$
$$\lesssim I_{s_0}(u, v)\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-2}}^2\|v\|_{H^{s+2}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},$$

where $w = u - v$.

Proof. Set $z = u + v$. Note that

$$\langle D^s\partial_x^2 z\partial_x w, D^s w\rangle + \langle \partial_x z D^s\partial_x^2 w, D^s w\rangle$$
$$= \langle \partial_x w D^s\partial_x^2 w, D^s w\rangle + 2\langle \partial_x w D^s\partial_x^2 v, D^s w\rangle + \langle \partial_x z D^s\partial_x^2 w, D^s w\rangle$$
$$= 2\langle \partial_x u D^s\partial_x^2 w, D^s w\rangle + 2\langle \partial_x w D^s\partial_x^2 v, D^s w\rangle.$$

Lemma 3.18 and 3.24 show that

$$\begin{aligned}
&|\langle D^s\partial_x(\partial_x z\partial_x w), D^s w\rangle + 2\langle \partial_x u, (D^s\partial_x w)^2\rangle| \\
&\le |\langle P_s^{(3)}(z,w), D^s w\rangle| + (s+1)|\langle \partial_x^3 u, (D^s w)^2\rangle| + 2(s+1)|\langle \partial_x w D^s\partial_x v, D^s w\rangle| \\
&\quad + 2|\langle \partial_x u D^s\partial_x^2 w, D^s w\rangle + \langle \partial_x u, (D^s\partial_x w)^2\rangle| + 2|\langle \partial_x w D^s\partial_x^2 v, D^s w\rangle| \\
&\lesssim I_{s_0}(u,v)\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-2}}^2\|v\|_{H^{s+2}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

which completes the proof. □

Lemma 3.30. *Let $s_0 > 7/2$ and $s \ge 0$. Let $u, v \in H^{\max\{s+2,s_0\}}(\mathbb{T})$. Then*

$$\begin{aligned}
&|\langle D^s\partial_x((\mathcal{H}\partial_x u)^2 - (\mathcal{H}\partial_x v)^2), D^s w\rangle - 2s\langle (\mathcal{H}\partial_x^2 u)\mathcal{H}D^s\partial_x w, D^s w\rangle| \\
&\lesssim I_{s_0}(u,v)\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-2}}^2\|v\|_{H^{s+2}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

where $w = u - v$.

Proof. Set $z = u + v$. As in the proof of Lemma 3.29, we have

$$\begin{aligned}
&\langle (\mathcal{H}\partial_x w)\mathcal{H}D^s\partial_x^2 z, D^s w\rangle + \langle (\mathcal{H}\partial_x z)\mathcal{H}D^s\partial_x^2 w, D^s w\rangle \\
&= 2\langle (\mathcal{H}\partial_x u)\mathcal{H}D^s\partial_x^2 w, D^s w\rangle + 2\langle (\mathcal{H}\partial_x w)\mathcal{H}D^s\partial_x^2 v, D^s w\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\langle (\mathcal{H}\partial_x^2 w)\mathcal{H}D^s\partial_x z, D^s w\rangle + \langle (\mathcal{H}\partial_x^2 z)\mathcal{H}D^s\partial_x w, D^s w\rangle \\
&= 2\langle (\mathcal{H}\partial_x^2 u)\mathcal{H}D^s\partial_x w, D^s w\rangle + 2\langle (\mathcal{H}\partial_x^2 w)\mathcal{H}D^s\partial_x v, D^s w\rangle.
\end{aligned}$$

Then Lemma 3.18, 3.24 and 3.26 show that

$$\begin{aligned}
&|\langle D^s\partial_x((\mathcal{H}\partial_x z)\mathcal{H}\partial_x w), D^s w\rangle - 2s\langle (\mathcal{H}\partial_x^2 u)\mathcal{H}D^s\partial_x w, D^s w\rangle| \\
&\le |\langle P_s^{(3)}(\mathcal{H}z, \mathcal{H}w), D^s w\rangle| + 2|\langle \partial_x((\mathcal{H}\partial_x u)\mathcal{H}D^s\partial_x w), D^s w\rangle| \\
&\quad + 2|\langle (\mathcal{H}\partial_x w)\mathcal{H}D^s\partial_x^2 v, D^s w\rangle| + 2(s+1)|\langle (\mathcal{H}\partial_x w)\mathcal{H}D^s\partial_x v, D^s w\rangle| \\
&\lesssim I_{s_0}(u,v)\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-2}}^2\|v\|_{H^{s+2}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

which completes the proof. □

Lemma 3.31. *Let $s_0 > 7/2$ and $s \ge 0$. Let $u, v \in H^{\max\{s+3,s_0\}}(\mathbb{T})$. Then*

$$\begin{aligned}
&|\langle \mathcal{H}D^s\partial_x(u\mathcal{H}\partial_x^2 u - v\mathcal{H}\partial_x^2 v), D^s w\rangle - \langle (\mathcal{H}\partial_x^2 u)\mathcal{H}D^s\partial_x w, D^s w\rangle - (s-1/2)\langle \partial_x u, (D^s\partial_x w)^2\rangle| \\
&\lesssim I_{s_0}(u,v)\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-3}}^2\|v\|_{H^{s+3}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

where $w = u - v$.

Proof. Set $z = u + v$. Note that

$$\begin{aligned}
&\langle (\mathcal{H}D^s\partial_x u)\mathcal{H}\partial_x^2 w, D^s w\rangle + \langle (\mathcal{H}D^s\partial_x w)\mathcal{H}\partial_x^2 v, D^s w\rangle \\
&= \langle (\mathcal{H}D^s\partial_x w)\mathcal{H}\partial_x^2 w, D^s w\rangle + \langle (\mathcal{H}D^s\partial_x w)\mathcal{H}\partial_x^2 v, D^s w\rangle + \langle (\mathcal{H}D^s\partial_x v)\mathcal{H}\partial_x^2 w, D^s w\rangle \\
&= \langle (\mathcal{H}\partial_x^2 u)\mathcal{H}D^s\partial_x w, D^s w\rangle + \langle (\mathcal{H}\partial_x^2 w)\mathcal{H}D^s\partial_x v, D^s w\rangle.
\end{aligned}$$

Lemma 3.17, 3.24 and 3.25 show that

$$\begin{aligned}
&|\langle \mathcal{H}D^s\partial_x(u\mathcal{H}\partial_x^2 u - v\mathcal{H}\partial_x^2 v), D^s w\rangle - \langle (\mathcal{H}\partial_x^2 u)\mathcal{H}D^s\partial_x w, D^s w\rangle - (s-1/2)\langle \partial_x u, (D^s\partial_x w)^2\rangle| \\
&\le |\langle P_s^{(2)}(u, \mathcal{H}w) + P_s^{(2)}(w, \mathcal{H}v), D^s w\rangle| + |\langle uD^s\partial_x^3 w, D^s w\rangle - (3/2)\langle \partial_x u, (D^s\partial_x w)^2\rangle| \\
&\quad + (s+1)|\langle \partial_x u D^s\partial_x^2 w, D^s w\rangle + \langle \partial_x u, (D^s\partial_x w)^2\rangle| + |\langle wD^s\partial_x^3 v, D^s w\rangle| \\
&\quad + s(s+1)|\langle \partial_x^3 u, (D^s w)^2\rangle|/4 + (s+1)|\langle \partial_x w D^s\partial_x^2 v, D^s w\rangle| + s(s+1)|\langle \partial_x^2 w D^s\partial_x v, D^s w\rangle|/2 \\
&\quad + |\langle (\mathcal{H}\partial_x^2 w)\mathcal{H}D^s\partial_x v, D^s w\rangle| \\
&\lesssim I_{s_0}(u,v)\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-3}}^2\|v\|_{H^{s+3}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

which completes the proof. □

Next, we estimate nonlinear terms in $F_3(u)$ and $F_4(u)$.

Lemma 3.32. *Let $s_0 > 7/2$ and $s \ge 0$. Let $u, v \in H^{\max\{s+2,s_0\}}(\mathbb{T})$. Then*

$$\begin{aligned}
&|\langle \mathcal{H}D^s\partial_x(u^2\partial_x u - v^2\partial_x v), D^s w\rangle - 2(s+1)\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle| \\
&\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-2}}^2\|v\|_{H^{s+2}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

where $w = u - v$.

Proof. Set $z = u + v$. Note that

$$\begin{aligned}
&2\langle u\partial_x w\mathcal{H}D^s\partial_x u, D^s w\rangle + \langle w\partial_x v\mathcal{H}D^s\partial_x z, D^s w\rangle + \langle z\partial_x v\mathcal{H}D^s\partial_x w, D^s w\rangle \\
&= 2\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle + 2\langle u\partial_x w\mathcal{H}D^s\partial_x v, D^s w\rangle + 2\langle w\partial_x v\mathcal{H}D^s\partial_x v, D^s w\rangle
\end{aligned}$$

and that $u^2\partial_x u - v^2\partial_x v = u^2\partial_x w + zw\partial_x v$. Lemma 3.20, 3.15 and 3.26 show that

$$\begin{aligned}
&|\langle \mathcal{H}D^s\partial_x(u^2\partial_x w + zw\partial_x v), D^s w\rangle - 2(s+1)\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle| \\
&\le |\langle P_s^{(6)}(u,u,w) + P_s^{(6)}(z,w,v), D^s w\rangle| + 2|\langle u\partial_x w\mathcal{H}D^s\partial_x v, D^s w\rangle| \\
&\quad + 2|\langle w\partial_x v\mathcal{H}D^s\partial_x v, D^s w\rangle| + |\langle \partial_x(u^2\mathcal{H}D^s\partial_x w), D^s w\rangle| \\
&\quad + |\langle zw\mathcal{H}D^s\partial_x^2 v, D^s w\rangle| + (s+1)|\langle \partial_x(zw)\mathcal{H}D^s\partial_x v, D^s w\rangle| \\
&\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-2}}^2\|v\|_{H^{s+2}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

which completes the proof. □

Lemma 3.33. *Let $s_0 > 7/2$ and $s \geq 0$. Let $u, v \in H^{\max\{s+2,s_0\}}(\mathbb{T})$. Then*

$$\begin{aligned}&|\langle D^s\partial_x(u\mathcal{H}(u\partial_x u) - v\mathcal{H}(v\partial_x v)), D^s w\rangle - (2s+1)\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle| \\ &\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-2}}^2\|v\|_{H^{s+2}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},\end{aligned}$$

where $w = u - v$.

Proof. Note that

$$\begin{aligned}&\langle u\partial_x w\mathcal{H}D^s\partial_x u, D^s w\rangle + \langle u\partial_x v\mathcal{H}D^s\partial_x w, D^s w\rangle \\ &= \langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle + \langle u\partial_x w\mathcal{H}D^s\partial_x v, D^s w\rangle.\end{aligned}$$

Then Lemma 3.21 and 3.26 show that

$$\begin{aligned}&|\langle D^s\partial_x(u\mathcal{H}(u\partial_x w) + u\mathcal{H}(w\partial_x v) + w\mathcal{H}(v\partial_x v)), D^s w\rangle - (2s+1)\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle| \\ &\leq |\langle P_s^{(7)}(u,u,w) + P_s^{(7)}(u,w,v) + P_s^{(7)}(w,v,v), D^s w\rangle| + |\langle \mathcal{H}\partial_x(u\partial_x u), (D^s w)^2\rangle|/2 \\ &\quad + |\langle D^s\partial_x v\mathcal{H}(u\partial_x w), D^s w\rangle| + |\langle D^s\partial_x v\mathcal{H}(w\partial_x v), D^s w\rangle| + |\langle \partial_x(u^2\mathcal{H}D^s\partial_x w), D^s w\rangle| \\ &\quad + |\langle uw\mathcal{H}D^s\partial_x^2 v, D^s w\rangle| + |\langle vw\mathcal{H}D^s\partial_x^2 v, D^s w\rangle| + (2s+3)|\langle \partial_x(zw)\mathcal{H}D^s\partial_x v, D^s w\rangle|/2 \\ &\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-2}}^2\|v\|_{H^{s+2}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},\end{aligned}$$

which completes the proof. □

Lemma 3.34. *Let $s_0 > 7/2$ and $s \geq 0$. Let $u, v \in H^{\max\{s+2,s_0\}}(\mathbb{T})$. Then*

$$\begin{aligned}&|\langle D^s\partial_x(u^2\mathcal{H}\partial_x u - v^2\mathcal{H}\partial_x v), D^s w\rangle - 2s\langle u\partial_x u\mathcal{H}D^s w, D^s w\rangle| \\ &\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-2}}^2\|v\|_{H^{s+2}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},\end{aligned}$$

where $w = u - v$.

Proof. Set $z = u + v$. Lemma 3.26 and 3.21 show that

$$\begin{aligned}&|\langle D^s\partial_x(u^2\mathcal{H}\partial_x w + zw\mathcal{H}\partial_x v), D^s w\rangle - 2s\langle u\partial_x u\mathcal{H}D^s w, D^s w\rangle| \\ &\leq |\langle P_s^{(6)}(u,u,\mathcal{H}w) + P_s^{(6)}(z,w,\mathcal{H}v), D^s w\rangle + |\langle \partial_x(u\mathcal{H}\partial_x u), (D^s w)^2\rangle| \\ &\quad + 2|\langle u(\mathcal{H}\partial_x w)D^s\partial_x v, D^s w\rangle| + |\langle \partial_x(u^2\mathcal{H}D^s\partial_x w), D^s w\rangle| \\ &\quad + |\langle \partial_x(w\mathcal{H}\partial_x v), (D^s w)^2\rangle|/2 + 2|\langle w(\mathcal{H}\partial_x v)D^s\partial_x v, D^s w\rangle| \\ &\quad + |\langle zw\mathcal{H}D^s\partial_x^2 v, D^s w\rangle| + (s+1)|\langle \partial_x(zw)\mathcal{H}D^s\partial_x v, D^s w\rangle| \\ &\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-2}}^2\|v\|_{H^{s+2}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},\end{aligned}$$

which completes the proof. □

Lemma 3.35. *Let* $s_0 > 7/2$ *and* $s \geq 0$. *Let* $u, v \in H^{\max\{s+1,s_0\}}(\mathbb{T})$. *Then*

$$\begin{aligned}&|\langle D^s\partial_x(u^4 - v^4), D^s w\rangle| \\ &\lesssim I_{s_0}(u,v)^3\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},\end{aligned}$$

where $w = u - v$.

Proof. Lemma 3.13 and 3.14 show that

$$\begin{aligned}&|\langle D^s(u^3\partial_x w + w(u^2 + uv + v^2)\partial_x v), D^s w\rangle| \\ &\leq |\langle [D^s, u^3]\partial_x w, D^s w\rangle| + |\langle [D^s, w(u^2 + uv + v^2)]\partial_x v, D^s w\rangle| \\ &\quad + |\langle \partial_x(u^3), (D^s w)^2\rangle|/2 + |\langle w(u^2 + uv + v^2)D^s\partial_x v, D^s w\rangle| \\ &\lesssim I_{s_0}(u,v)^3\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},\end{aligned}$$

which completes the proof. □

Remark 3.1. In Lemma 3.28, 3.29, 3.30, 3.31, 3.32, 3.33, 3.34 and 3.35 with $s = 0$, we do not have terms such as $\|w\|_{H^{s_0-j}}^2\|v\|_{H^{s+j}}^2$ for $j = 1, 2, 3$ and $(\|u\|_{H^s}^2+\|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2$ in the right hand side. This can be verified by a simple caluculation. Indeed, for example, on Lemma 3.28 with $s = 0$, we have

$$\begin{aligned}\langle \partial_x(u\partial_x^2 u - v\partial_x^2 v), w\rangle &= -\langle u\partial_x^2 w, \partial_x w\rangle - \langle w\partial_x^2 v, \partial_x w\rangle \\ &= \frac{1}{2}\langle \partial_x u, (\partial_x w)^2\rangle + \frac{1}{2}\langle \partial_x^3 v, w^2\rangle.\end{aligned}$$

The second term in the right hand side can be estimated by $\lesssim \|v\|_{H^{s_0}}\|w\|^2$. For this reason, we obtain the following.

Lemma 3.36. *Let* $s_0 > 7/2$ *and* $u, v \in H^{s_0}(\mathbb{T})$. *Then*

$$\begin{aligned}&\Bigg|\sum_{j=2}^{4}\langle \partial_x(F_j(u) - F_j(v)), w\rangle + \lambda_1(0)\langle \partial_x u, (\partial_x w)^2\rangle + \lambda_2(0)\langle (\mathcal{H}\partial_x^2 u)\mathcal{H}\partial_x w, w\rangle \\ &\quad + \lambda_3(0)\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle\Big| \\ &\lesssim I_{s_0}(u,v)^3\|w\|^2,\end{aligned}$$

where $w = u - v$.

Now, we estimate the time derivatives of $M_s^{(1)}$, $M_s^{(2)}$ and $M_s^{(3)}$. The following lemma helps us to calculate correction terms. Note that Lemma 3.37 is more complicated than Lemma 2.8 in [29] because of the presence of $\mathcal{H}$.

Lemma 3.37. *Let f, g, h be sufficiently smooth real-valued functions defined on $\mathbb{T}$. Then,*

$$\begin{aligned}&\langle \mathcal{H}\partial_x^4 f, gh\rangle + \langle f\mathcal{H}\partial_x^4 g, h\rangle + \langle fg, \mathcal{H}\partial_x^4 h\rangle \\ &= -\langle [\mathcal{H}, h]\partial_x^4 f, g\rangle - \langle [\mathcal{H}, f]\partial_x^4 h, g\rangle + 4\langle \partial_x^3 f\mathcal{H}g, \partial_x h\rangle - 4\langle \partial_x f\mathcal{H}\partial_x g, \partial_x^2 h\rangle + 2\langle \partial_x^2 f\mathcal{H}g, \partial_x^2 h\rangle.\end{aligned}$$

Proof. Observe that

$$\begin{aligned}&\langle fg, \mathcal{H}\partial_x^4 h\rangle \\ &= -\langle [\mathcal{H}, f]\partial_x^4 h, g\rangle - \langle \partial_x^4 f\mathcal{H}g + 4\partial_x^3 f\mathcal{H}\partial_x g + 6\partial_x^2 f\mathcal{H}\partial_x^2 g + 4\partial_x f\mathcal{H}\partial_x^3 g + f\mathcal{H}\partial_x^4 g, h\rangle \\ &= -\langle [\mathcal{H}, f]\partial_x^4 h, g\rangle - \langle \partial_x^4 f\mathcal{H}g, h\rangle - 4\langle \partial_x f\mathcal{H}\partial_x g, \partial_x^2 h\rangle + 2\langle \partial_x^2 f\mathcal{H}\partial_x^2 g, h\rangle - \langle f\mathcal{H}\partial_x^4 g, h\rangle \\ &= -\langle [\mathcal{H}, f]\partial_x^4 h, g\rangle + \langle \partial_x^4 f\mathcal{H}g, h\rangle + 4\langle \partial_x^3 f\mathcal{H}g, \partial_x h\rangle - 4\langle \partial_x f\mathcal{H}\partial_x g, \partial_x^2 h\rangle \\ &\quad + 2\langle \partial_x^2 f\mathcal{H}g, \partial_x^2 h\rangle - \langle f\mathcal{H}\partial_x^4 g, h\rangle.\end{aligned}$$

Note that

$$\langle \mathcal{H}\partial_x^4 f, gh\rangle + \langle \partial_x^4 f\mathcal{H}g, h\rangle = -\langle [\mathcal{H}, h]\partial_x^4 f, g\rangle,$$

which completes the proof. □

Lemma 3.38. *Let $s_0 > 7/2$ and $s \geq 1$. Let $u, w \in H^{\max\{s+4, s_0\}}(\mathbb{T})$. Then*

$$\begin{aligned}&|\langle (\mathcal{H}\partial_x^4 u)\mathcal{H}D^s w, \mathcal{H}D^{s-1}w\rangle - \langle uD^s\partial_x^4 w, \mathcal{H}D^{s-1}w\rangle + \langle u\mathcal{H}D^s w, \mathcal{H}D^s\partial_x^3 w\rangle - 4\langle \partial_x u, (D^s\partial_x w)^2\rangle| \\ &\lesssim \|u\|_{H^{s_0}}\|w\|_{H^s}^2.\end{aligned}$$

Proof. We use Lemma 3.37 with $f = u$, $g = \mathcal{H}D^s w$ and $h = \mathcal{H}D^{s-1}w$. Then Lemma 3.15 shows that

$$|\langle [\mathcal{H}, h]\partial_x^4 f, g\rangle| + |\langle [\mathcal{H}, f]\partial_x^4 h, g\rangle| + |\langle \partial_x^3 f\mathcal{H}g, \partial_x h\rangle| \lesssim \|u\|_{H^{s_0}}\|w\|_{H^s}^2. \tag{3.18}$$

Note that $-4\langle \partial_x f\mathcal{H}\partial_x g, \partial_x^2 h\rangle = 4\langle \partial_x u, (D^s\partial_x w)^2\rangle$. And finally, we see from the integration by parts that

$$2|\langle \partial_x^2 f\mathcal{H}g, \partial_x^2 h\rangle| = |\langle \partial_x^3 u, (D^s w)^2\rangle| \lesssim \|u\|_{H^{s_0}}\|w\|_{H^s}^2$$

which concludes the proof. □

Lemma 3.39. *Let $s_0 > 7/2$ and $s \geq 1$. Let $u, w \in H^{\max\{s+4, s_0\}}(\mathbb{T})$. Then*

$$\begin{aligned}&|\langle \partial_x^5 u, (D^{s-1}w)^2\rangle - 2\langle (\mathcal{H}\partial_x u)D^s\partial_x^3 w, D^{s-1}w\rangle + 4\langle (\mathcal{H}\partial_x^2 u)\mathcal{H}D^s\partial_x w, D^s w\rangle| \\ &\lesssim \|u\|_{H^{s_0}}\|w\|_{H^s}^2.\end{aligned}$$

Proof. The integration by parts shows that

$$\begin{aligned}
&|\langle \partial_x^5 u, (D^{s-1}w)^2\rangle - 2\langle (\mathcal{H}\partial_x u)D^s\partial_x^3 w, D^{s-1}w\rangle + 4\langle (\mathcal{H}\partial_x^2 u)\mathcal{H}D^s\partial_x w, D^s w\rangle| \\
&= |\langle \partial_x^4 u D^{s-1}\partial_x w, D^{s-1}w\rangle - \langle (\mathcal{H}u)D^s\partial_x^4 w, D^{s-1}w\rangle + \langle (\mathcal{H}u)D^s\partial_x^3 w, D^{s-1}\partial_x w\rangle \\
&\quad - 2\langle (\mathcal{H}\partial_x^2 u)\mathcal{H}D^s\partial_x w, D^s w\rangle|,
\end{aligned}$$

which allows us to use Lemma 3.37 with $f = \mathcal{H}u$, $g = D^{s-1}\partial_x w$ and $h = D^{s-1}w$. It is cleat that (3.18) holds in this case. Lemma 3.26 implies that

$$|\langle \partial_x f \mathcal{H}\partial_x g, \partial_x^2 h\rangle| = |\langle \partial_x((\mathcal{H}\partial_x u)\mathcal{H}D^s\partial_x w), D^s w\rangle| \lesssim \|u\|_{H^{s_0}}\|w\|_{H^s}^2.$$

On the other hand, we have $2\langle \partial_x^2 f \mathcal{H}g, \partial_x^2 h\rangle = -2\langle (\mathcal{H}\partial_x^2 u)\mathcal{H}D^s\partial_x w, D^s w\rangle$, which completes the proof. □

Lemma 3.40. *Let $s_0 > 7/2$ and $s \geq 1$. Let $u, w \in H^{\max\{s+3,s_0\}}(\mathbb{T})$. Then*

$$|\langle u\mathcal{H}\partial_x^4 u, (D^{s-1}w)^2\rangle + \langle u^2 D^{s-1}w, D^s\partial_x^3 w\rangle - 4\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle| \lesssim \|u\|_{H^{s_0}}^2\|w\|_{H^s}^2.$$

Proof. Adding and subtraction a term, we have

$$\begin{aligned}
&2|\langle u\mathcal{H}\partial_x^4 u, (D^{s-1}w)^2\rangle + \langle u^2 D^{s-1}w, D^s\partial_x^3 w\rangle - 4\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle| \\
&\leq |\langle \mathcal{H}\partial_x^4(u^2), (D^{s-1}w)^2\rangle + 2\langle u^2 D^{s-1}w, D^s\partial_x^3 w\rangle - 8\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle| \\
&\quad + |\langle 2u\mathcal{H}\partial_x^4 u - \mathcal{H}\partial_x^4(u^2), (D^{s-1}w)^2\rangle|.
\end{aligned}$$

Lemma 3.15 shows that the second term in the right hand side can be estimated by $\lesssim \|u\|_{H^{s_0}}^2\|w\|_{H^s}^2$. We use Lemma 3.37 with $f = u^2$, $g = h = D^{s-1}w$. It is clear that (3.18) holds in this case. Note that $-4\langle \partial_x f\mathcal{H}\partial_x g, \partial_x^2 h\rangle = 8\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle$. Finally, we have

$$|\langle \partial_x^2 f\mathcal{H}g, \partial_x^2 h\rangle| = |\langle \partial_x(\partial_x^2(u^2)\mathcal{H}D^{s-1}w), \mathcal{H}D^s w\rangle| \lesssim \|u\|_{H^{s_0}}^2\|w\|_{H^s}^2,$$

which completes the proof. □

We observe the first order derivative loss resulting from $M_s^{(1)}$.

Lemma 3.41. *Let $s_0 > 7/2$ and $s \geq 1$. Let $u, v \in H^{\max\{s+3,s_0\}}(\mathbb{T})$. Then*

$$\begin{aligned}
&|\langle u\mathcal{H}D^s\partial_x(u\partial_x^2 u - v\partial_x^2 v), \mathcal{H}D^{s-1}w\rangle + (s-3)\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle| \\
&\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-2}}^2\|v\|_{H^{s+2}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

where $w = u - v$.

Proof. Lemma 3.17 and 3.26 show that

$$\begin{aligned}
&|\langle u\mathcal{H}D^s\partial_x(u\partial_x^2 w + w\partial_x^2 v), \mathcal{H}D^{s-1}w\rangle + (s-3)\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle| \\
&\le |\langle u(P_s^{(2)}(u,w) + P_s^{(2)}(w,v)), \mathcal{H}D^{s-1}w\rangle| + |\langle \partial_x(u\partial_x^2 w\mathcal{H}D^{s-1}w), \mathcal{H}D^s u\rangle| \\
&\quad + |\langle \partial_x^3(u^2)\mathcal{H}D^s w, \mathcal{H}D^{s-1}w\rangle| + |\langle \partial_x^2(u^2)\mathcal{H}D^s w, D^s w\rangle| + |\langle \partial_x(u^2\mathcal{H}D^s\partial_x w), D^s w\rangle| \\
&\quad + (s+1)|\langle \partial_x(\partial_x^2(u^2)\mathcal{H}D^{s-1}w), \mathcal{H}D^s w\rangle|/2 + s(s+1)|\langle \partial_x(u\partial_x^2 u\mathcal{H}D^{s-1}w), \mathcal{H}D^s w\rangle|/2 \\
&\quad + |\langle \partial_x(u\partial_x^2 v\mathcal{H}D^{s-1}w), \mathcal{H}D^s w\rangle| + |\langle \partial_x(uw\mathcal{H}D^{s-1}w), \mathcal{H}D^s\partial_x^2 v\rangle| \\
&\quad + (s+1)|\langle \partial_x(u\partial_x w\mathcal{H}D^{s-1}w), \mathcal{H}D^s\partial_x v\rangle| + s(s+1)|\langle \partial_x(u\partial_x^2 w\mathcal{H}D^{s-1}w), \mathcal{H}D^s v\rangle|/2 \\
&\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-2}}^2\|v\|_{H^{s+2}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

which completes the proof. □

Lemma 3.42. *Let $s_0 > 7/2$ and $s \ge 1$. Let $u, v \in H^{\max\{s+2,s_0\}}(\mathbb{T})$. Then*

$$\begin{aligned}
&|\langle u\mathcal{H}D^s w, D^s(u\partial_x^2 u - v\partial_x^2 v)\rangle + (s-2)\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle| \\
&\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-2}}^2\|v\|_{H^{s+2}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

where $w = u - v$.

Proof. Lemma 3.22 together with Lemma 3.26 shows that

$$\begin{aligned}
&|\langle u\mathcal{H}D^s w, D^s(u\partial_x^2 w + w\partial_x^2 v)\rangle + (s-2)\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle| \\
&\le |\langle u\mathcal{H}D^s w, D^s(u\partial_x^2 w) - uD^s\partial_x^2 w - s\partial_x uD^s\partial_x w\rangle| \\
&\quad + |\langle u\mathcal{H}D^s w, D^s(w\partial_x^2 v) - wD^s\partial_x^2 v - s\partial_x wD^s\partial_x v\rangle| + |\langle \partial_x^2(u^2)\mathcal{H}D^s w, D^s w\rangle| \\
&\quad + |\langle \partial_x(u^2\mathcal{H}D^s\partial_x w), D^s w\rangle| + s|\langle \partial_x(u\partial_x u)\mathcal{H}D^s w, D^s w\rangle| \\
&\quad + |\langle u\mathcal{H}D^s w, wD^s\partial_x^2 v\rangle| + s|\langle u\mathcal{H}D^s w, \partial_x wD^s\partial_x v\rangle| \\
&\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-2}}^2\|v\|_{H^{s+2}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

which completes the proof. □

Lemma 3.43. *Let $s_0 > 7/2$ and $s \ge 1$. Let $u, v \in H^{\max\{s+2,s_0\}}(\mathbb{T})$. Then*

$$\begin{aligned}
&|\langle u\mathcal{H}D^s\partial_x((\partial_x u)^2 - (\partial_x v)^2), \mathcal{H}D^{s-1}w\rangle + 2\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle| \\
&\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

where $w = u - v$.

Proof. Set $z = u + v$. Lemma 3.22 shows that

$$\begin{aligned}
&|\langle u\mathcal{H}D^s\partial_x(\partial_x z\partial_x w), \mathcal{H}D^{s-1}w\rangle + 2\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle| \\
&\le |\langle uP_s^{(4)}(z,w), \mathcal{H}D^{s-1}w\rangle| + 2|\langle \partial_x(u\partial_x w\mathcal{H}D^{s-1}w), \mathcal{H}D^s\partial_x v\rangle| \\
&\quad + (s+1)|\langle \partial_x(u\partial_x^2 w\mathcal{H}D^{s-1}w), \mathcal{H}D^s z\rangle| + (s+1)|\langle \partial_x(u\partial_x^2 z\mathcal{H}D^{s-1}w), \mathcal{H}D^s w\rangle| \\
&\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

which completes the proof. □

Lemma 3.44. *Let $s_0 > 7/2$ and $s \ge 1$. Let $u, v \in H^{\max\{s+1,s_0\}}(\mathbb{T})$. Then*

$$\begin{aligned}
&|\langle u\mathcal{H}D^s w, D^s((\partial_x u)^2 - (\partial_x v)^2)\rangle + 2\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle| \\
&\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

where $w = u - v$.

Proof. Lemma 3.19 shows that

$$\begin{aligned}
&|\langle u\mathcal{H}D^s w, D^s(\partial_x z\partial_x w)\rangle + 2\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle| \\
&\le |\langle u\mathcal{H}D^s w, D^s(\partial_x z\partial_x w) - D^s\partial_x z\partial_x w - \partial_x z D^s\partial_x w\rangle| + 2|\langle \partial_x(u\partial_x u)\mathcal{H}D^s w, D^s w\rangle| \\
&\quad + 2|\langle u\partial_x w\mathcal{H}D^s w, D^s\partial_x v\rangle| \\
&\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

which completes the proof. □

Lemma 3.45. *Let $s_0 > 7/2$ and $s \ge 1$. Let $u, v \in H^{\max\{s+2,s_0\}}(\mathbb{T})$. Then*

$$\begin{aligned}
&|\langle u\mathcal{H}D^s\partial_x((\mathcal{H}\partial_x u)^2 - (\mathcal{H}\partial_x v)^2), \mathcal{H}D^{s-1}w\rangle| \\
&\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

where $w = u - v$.

Proof. Set $z = u + v$. Lemma 3.18 shows that

$$\begin{aligned}
&|\langle u\mathcal{H}D^s\partial_x((\mathcal{H}\partial_x z)(\mathcal{H}\partial_x w)), \mathcal{H}D^{s-1}w\rangle| \\
&\le |\langle uP_s^{(4)}(\mathcal{H}z, \mathcal{H}w), \mathcal{H}D^{s-1}w\rangle + 3|\langle \partial_x(u\mathcal{H}\partial_x u), (D^s w)^2\rangle| \\
&\quad + 2|\langle \partial_x^2(u\mathcal{H}\partial_x u)D^s w, \mathcal{H}D^{s-1}w\rangle| + 2|\langle \partial_x(u(\mathcal{H}\partial_x w)\mathcal{H}D^{s-1}w), D^s\partial_x v\rangle| \\
&\quad + (s+1)|\langle \partial_x(u(\mathcal{H}\partial_x^2 w)\mathcal{H}D^{s-1}w), D^s z)\rangle| + (s+1)|\langle \partial_x(u(\mathcal{H}\partial_x^2 z)\mathcal{H}D^{s-1}w), D^s w)\rangle| \\
&\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

which completes the proof. □

Lemma 3.46. *Let* $s_0 > 7/2$ *and* $s \geq 1$. *Let* $u, v \in H^{\max\{s+1,s_0\}}(\mathbb{T})$. *Then*

$$\begin{aligned}&|\langle u\mathcal{H}D^s w, D^s((\mathcal{H}\partial_x u)^2 - (\mathcal{H}\partial_x v)^2)\rangle|\\ &\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},\end{aligned}$$

where $w = u - v$.

Proof. Set $z = u + v$. Lemma 3.19 shows that

$$\begin{aligned}&|\langle u\mathcal{H}D^s w, D^s((\mathcal{H}\partial_x z)\mathcal{H}\partial_x w)\rangle|\\ &\leq |\langle u\mathcal{H}D^s w, D^s((\mathcal{H}\partial_x z)\mathcal{H}\partial_x w) - (\mathcal{H}\partial_x w)\mathcal{H}D^s\partial_x z - (\mathcal{H}\partial_x z)\mathcal{H}D^s\partial_x w\rangle|\\ &\quad + |\langle \partial_x(u\mathcal{H}\partial_x^2 u), (\mathcal{H}D^s w)^2\rangle| + 2|\langle u(\mathcal{H}\partial_x w)\mathcal{H}D^s w, \mathcal{H}D^s\partial_x v\rangle|\\ &\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},\end{aligned}$$

which completes the proof. □

Lemma 3.47. *Let* $s_0 > 7/2$ *and* $s \geq 1$. *Let* $u, v \in H^{\max\{s+3,s_0\}}(\mathbb{T})$. *Then*

$$\begin{aligned}&|\langle uD^s\partial_x(u\mathcal{H}\partial_x^2 u - v\mathcal{H}\partial_x^2 v), \mathcal{H}D^{s-1}w\rangle + (s-3)\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle|\\ &\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-2}}^2\|v\|_{H^{s+2}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},\end{aligned}$$

where $w = u - v$.

Proof. We see from Lemma 3.17 and 3.26 that

$$\begin{aligned}&|\langle uD^s\partial_x(u\mathcal{H}\partial_x^2 w + w\mathcal{H}\partial_x^2 v), \mathcal{H}D^{s-1}w\rangle + (s-3)\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle|\\ &\leq |\langle u(P_s^{(1)}(u,\mathcal{H}w) + P_s^{(1)}(w,\mathcal{H}v)), \mathcal{H}D^{s-1}w\rangle| + |\langle \partial_x(u(\mathcal{H}\partial_x^2 w)\mathcal{H}D^{s-1}w), D^s u\rangle|\\ &\quad + |\langle \partial_x(\partial_x^2(u^2)\mathcal{H}D^{s-1}w), \mathcal{H}D^s w\rangle| + |\langle \partial_x(u^2\mathcal{H}D^s\partial_x w), D^s w\rangle|\\ &\quad + (s+1)|\langle \partial_x(\partial_x(u\partial_x u)\mathcal{H}D^{s-1}w), \mathcal{H}D^s w\rangle| + s(s+1)|\langle \partial_x(u\partial_x^2 u\mathcal{H}D^{s-1}w), \mathcal{H}D^s w\rangle|/2\\ &\quad + |\langle \partial_x(u(\mathcal{H}\partial_x^2 v)\mathcal{H}D^{s-1}w), D^s w\rangle| + |\langle \partial_x(uw\mathcal{H}D^{s-1}w), \mathcal{H}D^s\partial_x^2 v\rangle|\\ &\quad + (s+1)|\langle \partial_x(u\partial_x w\mathcal{H}D^{s-1}w), \mathcal{H}D^s\partial_x v\rangle| + s(s+1)|\langle \partial_x(u\partial_x^2 w\mathcal{H}D^{s-1}w), \mathcal{H}D^s v\rangle|/2\\ &\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-2}}^2\|v\|_{H^{s+2}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},\end{aligned}$$

which completes the proof. □

Lemma 3.48. *Let* $s_0 > 7/2$ *and* $s \geq 1$. *Let* $u, v \in H^{\max\{s+2,s_0\}}(\mathbb{T})$. *Then*

$$\begin{aligned}&|\langle u\mathcal{H}D^s w, D^{s-1}\partial_x(u\mathcal{H}\partial_x^2 u - v\mathcal{H}\partial_x^2 v)\rangle + (s-2)\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle|\\ &\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},\end{aligned}$$

where $w = u - v$.

Proof. Lemma 3.22 and 3.26 shows that

$$
\begin{aligned}
&|\langle u\mathcal{H}D^s w, D^{s-1}\partial_x(u\mathcal{H}\partial_x^2 w + w\mathcal{H}\partial_x^2 v)\rangle + (s-2)\langle u\partial_x u\mathcal{H}D^s\partial_x w, D^s w\rangle| \\
&\le |\langle u\mathcal{H}D^s w, D^{s-1}\partial_x(u\mathcal{H}\partial_x^2 w) - u\mathcal{H}D^{s-1}\partial_x^3 w - s\partial_x u\mathcal{H}D^{s-1}\partial_x^2 w\rangle| \\
&\quad + |\langle u\mathcal{H}D^s w, D^{s-1}\partial_x(w\mathcal{H}\partial_x^2 v) - w\mathcal{H}D^{s-1}\partial_x^3 v - s\partial_x w\mathcal{H}D^{s-1}\partial_x^2 v\rangle| \\
&\quad + |\langle [\mathcal{H}, u^2]D^s\partial_x^2 w, D^s w\rangle| + |\langle \partial_x(u^2\mathcal{H}D^s\partial_x w), D^s w\rangle| \\
&\quad + s|\langle [\mathcal{H}, u\partial_x u]D^s\partial_x w, D^s w\rangle| + |\langle u\mathcal{H}D^s w, wD^{s+2}v\rangle| + s|\langle u\mathcal{H}D^s w, \partial_x w D^s\partial_x v\rangle| \\
&\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}
$$

which completes the proof. □

Lemma 3.49. *Let $s_0 > 7/2$ and $s \ge 1$. Let $u, v \in H^{\max\{s+2,s_0\}}(\mathbb{T})$. Then*

$$
\begin{aligned}
&|\langle (\mathcal{H}\partial_x u)D^{s-1}w, D^{s-1}\partial_x(u\partial_x^2 u - v\partial_x^2 v)\rangle| \\
&\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}
$$

where $w = u - v$.

Proof. Note that

$$
\begin{aligned}
&\langle (\mathcal{H}\partial_x u)D^{s-1}w, uD^{s-1}\partial_x^3 w\rangle \\
&= -\langle \partial_x(u\mathcal{H}\partial_x u)D^{s-1}w, D^{s-1}\partial_x^2 w\rangle - \langle u(\mathcal{H}\partial_x u)D^{s-1}\partial_x w, D^{s-1}\partial_x^2 w\rangle \\
&= \langle \partial_x(\partial_x(u\mathcal{H}\partial_x u)D^{s-1}w), D^{s-1}\partial_x w\rangle + \frac{1}{2}\langle \partial_x(u\mathcal{H}\partial_x u), (D^{s-1}\partial_x w)^2\rangle.
\end{aligned}
$$

Lemma 3.22 shows that

$$
\begin{aligned}
&|\langle (\mathcal{H}\partial_x u)D^{s-1}w, D^{s-1}\partial_x(u\partial_x^2 w + w\partial_x^2 v)\rangle| \\
&\le |\langle (\mathcal{H}\partial_x u)D^{s-1}w, D^{s-1}\partial_x(u\partial_x^2 w) - uD^s\partial_x^3 w - s\partial_x uD^{s-1}\partial_x^2 w\rangle| \\
&\quad + |\langle (\mathcal{H}\partial_x u)D^{s-1}w, D^{s-1}\partial_x(w\partial_x^2 v) - wD^{s-1}\partial_x^3 v - s\partial_x wD^{s-1}\partial_x^2 v\rangle| \\
&\quad + |\langle (\mathcal{H}\partial_x u)D^{s-1}w, uD^{s-1}\partial_x^3 w\rangle| + s|\langle \partial_x(\partial_x u(\mathcal{H}\partial_x u)D^{s-1}w), D^{s-1}\partial_x w\rangle| \\
&\quad + |\langle \partial_x(w(\mathcal{H}\partial_x u)D^{s-1}w), D^{s-1}\partial_x^2 v\rangle| + s|\langle \partial_x(\partial_x w(\mathcal{H}\partial_x u)D^{s-1}w), D^{s-1}\partial_x v\rangle| \\
&\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}
$$

which completes the proof. □

Lemma 3.50. *Let $s_0 > 7/2$ and $s \ge 1$. Let $u, v \in H^{\max\{s+1,s_0\}}(\mathbb{T})$. Then*

$$
\begin{aligned}
&|\langle (\mathcal{H}\partial_x u)D^{s-1}w, D^{s-1}\partial_x((\partial_x u)^2 - (\partial_x v)^2)\rangle| \\
&\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}
$$

where $w = u - v$.

Proof. Lemma 3.19 shows that

$$\begin{aligned}
&|\langle (\mathcal{H}\partial_x u) D^{s-1}w, D^{s-1}\partial_x(\partial_x z \partial_x w)\rangle| \\
&\le |\langle (\mathcal{H}\partial_x u) D^{s-1}w, D^{s-1}\partial_x(\partial_x z \partial_x w) - D^{s-1}\partial_x^2 z \partial_x w - \partial_x z D^{s-1}\partial_x^2 w\rangle| \\
&\quad + |\langle \partial_x(\partial_x w (\mathcal{H}\partial_x u) D^{s-1}w), D^{s-1}\partial_x z\rangle| + |\langle \partial_x(\partial_x z (\mathcal{H}\partial_x u) D^{s-1}w), D^{s-1}\partial_x w\rangle| \\
&\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

which completes the proof. □

Lemma 3.51. *Let $s_0 > 7/2$ and $s \ge 1$. Let $u, v \in H^{\max\{s+2,s_0\}}(\mathbb{T})$. Then*

$$\begin{aligned}
&|\langle (\mathcal{H}\partial_x u) D^{s-1}w, D^s(u\mathcal{H}\partial_x^2 u - v\mathcal{H}\partial_x^2 v)\rangle| \\
&\lesssim I_{s_0}(u,v)^2\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

where $w = u - v$.

Proof. Note that

$$\begin{aligned}
&\langle (\mathcal{H}\partial_x u) D^{s-1}w, u\mathcal{H}D^s\partial_x^2 w\rangle \\
&= -\langle \partial_x(u\mathcal{H}\partial_x u) D^{s-1}w, \mathcal{H}D^s\partial_x w\rangle + \langle u(\mathcal{H}\partial_x u)\mathcal{H}D^s w, \mathcal{H}D^s\partial_x w\rangle \\
&= \langle \partial_x(\partial_x(u\mathcal{H}\partial_x u) D^{s-1}w), \mathcal{H}D^s w\rangle - \frac{1}{2}\langle \partial_x(u\mathcal{H}\partial_x u), (\mathcal{H}D^s w)^2\rangle.
\end{aligned}$$

We have

$$\begin{aligned}
&|\langle (\mathcal{H}\partial_x u) D^{s-1}w, D^s(u\mathcal{H}\partial_x^2 w + w\mathcal{H}\partial_x^2 v)\rangle| \\
&\le |\langle (\mathcal{H}\partial_x u) D^{s-1}w, D^s(u\mathcal{H}\partial_x^2 w) - u\mathcal{H}D^s\partial_x^2 w - s\partial_x u\mathcal{H}D^s\partial_x w\rangle| \\
&\quad + |\langle (\mathcal{H}\partial_x u) D^{s-1}w, D^s(w\mathcal{H}\partial_x^2 v) - w\mathcal{H}D^s\partial_x^2 v - s\partial_x w\mathcal{H}D^s\partial_x v\rangle| \\
&\quad + |\langle (\mathcal{H}\partial_x u) D^{s-1}w, u\mathcal{H}D^s\partial_x^2 w\rangle| + s|\langle (\mathcal{H}\partial_x u) D^{s-1}w, \partial_x u\mathcal{H}D^s\partial_x w\rangle| \\
&\quad + |\langle (\mathcal{H}\partial_x u) D^{s-1}w, w\mathcal{H}D^s\partial_x^2 v\rangle| + s|\langle (\mathcal{H}\partial_x u) D^{s-1}w, \partial_x w\mathcal{H}D^s\partial_x v\rangle|
\end{aligned}$$

which completes the proof by Lemma 3.22. □

Lemma 3.52. *Let $s_0 > 7/2$ and $s \ge 1$. Let $u, v \in H^{\max\{s+2,s_0\}}(\mathbb{T})$. Then*

$$\begin{aligned}
&|\langle uD^s\partial_x(u^2\partial_x u - v^2\partial_x v), \mathcal{H}D^{s-1}w\rangle| \\
&\lesssim I_{s_0}(u,v)^3\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

where $w = u - v$.

Proof. Set $z=u+v$. Lemma 3.14 shows that

$$\begin{aligned}
&|\langle uD^s\partial_x(u^2\partial_x w+zw\partial_x v),\mathcal{H}D^{s-1}w\rangle|\\
&\le |\langle \partial_x uD^s(u^2\partial_x w+zw\partial_x v),\mathcal{H}D^{s-1}w\rangle|+|\langle uD^s(u^2\partial_x w+zw\partial_x v),D^s w\rangle|\\
&\le |\langle D(\partial_x u\mathcal{H}D^{s-1}w),D^{s-1}(u^2\partial_x w+zw\partial_x v)\rangle|+|\langle u[D^s,u^2]\partial_x w,D^s w\rangle|\\
&\quad+\frac{3}{2}|\langle u^2\partial_x u,(D^s w)^2\rangle|+|\langle uD^s(zw\partial_x v),D^s w\rangle|,
\end{aligned}$$

which completes the proof. □

Lemma 3.53. *Let $s_0>7/2$ and $s\ge 1$. Let $u,v\in H^{\max\{s+2,s_0\}}(\mathbb{T})$. Then*

$$\begin{aligned}
&|\langle u\mathcal{H}D^s\partial_x(u\mathcal{H}(u\partial_x u)-v\mathcal{H}(v\partial_x v)),\mathcal{H}D^{s-1}w\rangle|\\
&\lesssim I_{s_0}(u,v)^3\{\|w\|_{H^s}^2+\|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2+(\|u\|_{H^s}^2+\|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

where $w=u-v$.

Proof. Note that

$$\begin{aligned}
&|\langle u\mathcal{H}D^s\partial_x(u\mathcal{H}(u\partial_x w)+u\mathcal{H}(w\partial_x v)+w\mathcal{H}(v\partial_x v)),\mathcal{H}D^{s-1}w\rangle|\\
&\le |\langle D(\partial_x u\mathcal{H}D^{s-1}w),\mathcal{H}D^{s-1}(u\mathcal{H}(u\partial_x w))\rangle|+|\langle u\mathcal{H}D^s(u\mathcal{H}(u\partial_x w)),D^s w\rangle|\\
&\quad+|\langle \partial_x(u\mathcal{H}D^{s-1}w),\mathcal{H}D^s(u\mathcal{H}(w\partial_x v)+w\mathcal{H}(v\partial_x v))\rangle|.
\end{aligned}$$

The second term in the right hand side can be estimated as follows:

$$\begin{aligned}
&|\langle u\mathcal{H}D^s(u\mathcal{H}(u\partial_x w)),D^s w\rangle|\\
&\le |\langle \mathcal{H}D^s(u\mathcal{H}(u\partial_x w))+u^2D^s\partial_x w,uD^s w\rangle|+\frac{3}{2}|\langle u^2\partial_x u,(D^s w)^2\rangle|.
\end{aligned}$$

Then Lemma 3.23 completes the proof. □

Lemma 3.54. *Let $s_0>7/2$ and $s\ge 1$. Let $u,v\in H^{\max\{s+2,s_0\}}(\mathbb{T})$. Then*

$$\begin{aligned}
&|\langle u\mathcal{H}D^s\partial_x(u^2\mathcal{H}\partial_x u-v^2\mathcal{H}\partial_x v),\mathcal{H}D^{s-1}w\rangle|\\
&\lesssim I_{s_0}(u,v)^3\{\|w\|_{H^s}^2+\|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2+(\|u\|_{H^s}^2+\|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},
\end{aligned}$$

where $w=u-v$.

Proof. Set $z=u+v$. Note that

$$\begin{aligned}
&|\langle u\mathcal{H}D^s\partial_x(u^2\mathcal{H}\partial_x w+zw\mathcal{H}\partial_x v),\mathcal{H}D^{s-1}w\rangle|\\
&\le |\langle D(\partial_x u\mathcal{H}D^{s-1}w),\mathcal{H}D^{s-1}(u^2\mathcal{H}\partial_x w)\rangle|+|\langle u[\mathcal{H}D^s,u^2]\mathcal{H}\partial_x w,D^s w\rangle|\\
&\quad+\frac{3}{2}|\langle u^2\partial_x u,(D^s w)^2\rangle|+|\langle \partial_x(u\mathcal{H}D^{s-1}w),\mathcal{H}D^s(zw\mathcal{H}\partial_x v)\rangle|.
\end{aligned}$$

Then Lemma 3.14 completes the proof. □

Lemma 3.55. *Let $s_0 > 7/2$ and $s \geq 1$. Let $u, v \in H^{\max\{s+2,s_0\}}(\mathbb{T})$. Then*

$$\begin{aligned}&|\langle u\mathcal{H}D^s w, D^{s-1}\partial_x(u^2\partial_x u - v^2\partial_x v)\rangle|\\&\lesssim I_{s_0}(u,v)^3\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},\end{aligned}$$

where $w = u - v$.

Proof. Note that

$$\begin{aligned}&|\langle u\mathcal{H}D^s w, D^{s-1}\partial_x(u^2\partial_x w + zw\partial_x v)\rangle|\\&\leq |\langle u\mathcal{H}D^s w, [D^{s-1}\partial_x, u^2]\partial_x w\rangle| + \frac{3}{2}|\langle u^2\partial_x u, (\mathcal{H}D^s w)^2\rangle|\\&\quad + |\langle u\mathcal{H}D^s w, D^{s-1}\partial_x(zw\partial_x v)\rangle|.\end{aligned}$$

Then, Lemma 3.14 completes the proof. □

Lemma 3.56. *Let $s_0 > 7/2$ and $s \geq 1$. Let $u, v \in H^{\max\{s+1,s_0\}}(\mathbb{T})$. Then*

$$\begin{aligned}&|\langle u\mathcal{H}D^s w, D^s(u\mathcal{H}(u\partial_x u) - v\mathcal{H}(v\partial_x v))\rangle|\\&\lesssim I_{s_0}(u,v)^3\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},\end{aligned}$$

where $w = u - v$.

Proof. Note that

$$\begin{aligned}&|\langle u\mathcal{H}D^s w, D^s(u\mathcal{H}(u\partial_x w) + u\mathcal{H}(w\partial_x v) + w\mathcal{H}(v\partial_x v))\rangle|\\&\leq |\langle u\mathcal{H}D^s w, D^s(u\mathcal{H}(u\partial_x w)) - u^2\mathcal{H}D^s\partial_x w\rangle| + \frac{3}{2}|\langle u^2\partial_x u, (\mathcal{H}D^s w)^2\rangle|\\&\quad + |\langle u\mathcal{H}D^s w, D^s(u\mathcal{H}(w\partial_x v) + w\mathcal{H}(v\partial_x v))\rangle|.\end{aligned}$$

We see that Lemma 3.23 completes the proof. □

Lemma 3.57. *Let $s_0 > 7/2$ and $s \geq 1$. Let $u, v \in H^{\max\{s+1,s_0\}}(\mathbb{T})$. Then*

$$\begin{aligned}&|\langle u\mathcal{H}D^s w, D^s(u^2\mathcal{H}\partial_x u - v^2\mathcal{H}\partial_x v)\rangle|\\&\lesssim I_{s_0}(u,v)^3\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},\end{aligned}$$

where $w = u - v$.

Proof. Set $z = u + v$. Note that

$$\begin{aligned}&|\langle u\mathcal{H}D^s w, D^s(u^2\mathcal{H}\partial_x w + zw\mathcal{H}\partial_x v)\rangle|\\&\leq |\langle u\mathcal{H}D^s w, [D^s, u^2]\mathcal{H}\partial_x w\rangle| + \frac{3}{2}|\langle u^2\partial_x u, (\mathcal{H}D^s w)^2\rangle| + |\langle u\mathcal{H}D^s w, D^s(zw\mathcal{H}\partial_x v)\rangle|.\end{aligned}$$

We see that Lemma 3.14 completes the proof. □

By the presence of $\mathcal{H}D^{s-1}$, the following lemma is clear:

Lemma 3.58. *Let $s_0 > 7/2$ and $s \geq 1$. Let $u, v \in H^{\max\{s+1,s_0\}}(\mathbb{T})$. Then*

$$|\langle u\mathcal{H}D^s\partial_x(u^4 - v^4), \mathcal{H}D^{s-1}w\rangle| + |\langle u\mathcal{H}D^s w, D^s(u^4 - v^4)\rangle|$$
$$\lesssim I_{s_0}(u,v)^3\{\|w\|_{H^s}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},$$

where $w = u - v$.

Lemma 3.59. *Let $s_0 > 7/2$ and $s \geq 1$. Let $u, v \in H^{\max\{s+1,s_0\}}(\mathbb{T})$. Then*

$$|\langle(\mathcal{H}\partial_x u)D^{s-1}w, D^{s-1}\partial_x(F_3(u) - F_3(v))\rangle| + |\langle(\mathcal{H}\partial_x u)D^{s-1}w, D^{s-1}\partial_x(u^4 - v^4)\rangle|$$
$$\lesssim I_{s_0}(u,v)^4\{\|w\|_{H^s}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},$$

where $w = u - v$.

Proof. This follows from Lemma 3.13 because of the presence of D^{s-1}. □

Lemma 3.60. *Let $s_0 > 7/2$ and $s \geq 1$. Let $u, v \in H^{\max\{s+2,s_0\}}(\mathbb{T})$. Then*

$$|\langle u^2D^{s-1}w, D^{s-1}\partial_x(u\partial_x^2u - v\partial_x^2v)\rangle|$$
$$\lesssim I_{s_0}(u,v)^3\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},$$

where $w = u - v$.

Proof. Note that

$$|\langle u^2D^{s-1}w, D^{s-1}\partial_x(u\partial_x^2w + w\partial_x^2v)\rangle|$$
$$\leq |\langle u^2D^{s-1}w, D^{s-1}\partial_x(u\partial_x^2w) - uD^{s-1}\partial_x^3w - s\partial_xuD^{s-1}\partial_x^2w\rangle|$$
$$+ |\langle\partial_x(u^2D^{s-1}w), D^{s-1}(w\partial_x^2v)\rangle| + 3|\langle\partial_x(u^2\partial_xuD^{s-1}w), D^{s-1}\partial_xw\rangle|$$
$$+ \frac{3}{2}|\langle u^2\partial_xu, (\mathcal{H}D^sw)^2\rangle| + s|\langle\partial_x(u^2\partial_xuD^{s-1}w), D^{s-1}\partial_xw\rangle|,$$

which completes the proof. □

Lemma 3.61. *Let $s_0 > 7/2$ and $s \geq 1$. Let $u, v \in H^{\max\{s+2,s_0\}}(\mathbb{T})$. Then*

$$|\langle u^2D^{s-1}w, \mathcal{H}D^{s-1}\partial_x(u\mathcal{H}\partial_x^2u - v\mathcal{H}\partial_x^2v)\rangle|$$
$$\lesssim I_{s_0}(u,v)^3\{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2\|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},$$

where $w = u - v$.

Proof. This follows from a similar argument to Lemma 3.60. □

Cobmining Lemma 3.60 and 3.61, we obtain the following:

Lemma 3.62. *Let $s_0 > 7/2$ and $s \geq 1$. Let $u, v \in H^{\max\{s+2,s_0\}}(\mathbb{T})$. Then*

$$\begin{aligned}&\sum_{j=2}^{4} |\langle u^2 D^{s-1} w, D^{s-1}\partial_x (F_j(u) - F_j(v))\rangle| \\ &\lesssim I_{s_0}(u,v)^5 \{\|w\|_{H^s}^2 + \|w\|_{H^{s_0-1}}^2 \|v\|_{H^{s+1}}^2 + (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|w\|_{H^{s_0}}^2\},\end{aligned}$$

where $w = u - v$.

Definition 12. Let $s \geq 0$ and $k \in \mathbb{N}$ satisfy $2(s+2) > k$. We define

$$p(k) := \frac{2(s+2)}{2(s+2)-k}, \quad q(k) := \frac{2(s+2)}{k}.$$

Note that $p(k) > 1$ and $1/p(k) + 1/q(k) = 1$.

The following five lemmas are estimates for viscous terms $-\varepsilon_1 \partial_x^4 u + \varepsilon_2 \partial_x^4 v$ in $M_s^{(1)}(u,v)$.

Lemma 3.63. *Let $s \geq 1$, $s_0 > 7/2$ and $\varepsilon_1 \in [0,1]$. Then there exists $C = C(s_0, s) > 0$ such that for any $u, v \in H^{\max\{s+2,s_0\}}(\mathbb{T})$,*

$$\begin{aligned}&\left|\varepsilon_1 \int_{\mathbb{T}} \partial_x^4 u (\mathcal{H} D^s w) \mathcal{H} D^{s-1} w dx\right| \\ &\leq \frac{\varepsilon_1^{p(4)}}{100} \|D^{s+2} w\|^2 + C\|u\|_{H^{s_0}}^{q(4)} \|w\|^2 + C\|u\|_{H^{s_0}} \|w\|_{H^s}^2,\end{aligned}$$

where $w = u - v$.

Proof. We set

$$\begin{aligned}\int_{\mathbb{T}} \partial_x^4 u (\mathcal{H} D^s w) \mathcal{H} D^{s-1} w dx &= -\int_{\mathbb{T}} \partial_x^3 u D^{s+1} w \mathcal{H} D^{s-1} w dx - \int_{\mathbb{T}} \partial_x^3 u (\mathcal{H} D^s w) D^s w dx \\ &=: A + B.\end{aligned}$$

It is clear that $|B| \lesssim \|u\|_{H^{s_0}} \|w\|_{H^s}^2$ Interpolation and the Young inequality show that

$$\begin{aligned}\varepsilon_1 |A| &\leq \varepsilon_1 C \|u\|_{H^{s_0}} \|D^{s+1} w\| \|D^{s-1} w\| \\ &\leq \varepsilon_1 C \|u\|_{H^{s_0}} \|w\|^{4/(s+2)} \|D^{s+2} w\|^{2-4/(s+2)} \leq \frac{\varepsilon_1^{p(4)}}{100} \|D^{s+2} w\|^2 + C\|u\|_{H^{s_0}}^{q(4)} \|w\|^2,\end{aligned}$$

as desired. □

By a similar argument to the proof of Lemma 3.63, we can show the rest of estimates for viscous terms in $M_s^{(1)}(u,v)$.

Lemma 3.64. *Let $s \geq 1$, $s_0 > 7/2$ and $\varepsilon_1 \in [0,1]$. Then there exists $C = C(s_0, s) > 0$ such that for any $u, v \in H^{\max\{s+4,s_0\}}(\mathbb{T})$,*

$$\left|\varepsilon_1 \int_{\mathbb{T}} u(\mathcal{H}D^s\partial_x^4 w)\mathcal{H}D^{s-1}w dx\right| \leq \frac{\sum_{j=1}^3 \varepsilon_1^{p(j)}}{100}\|D^{s+2}w\|^2 + C\sum_{j=1}^3 \|u\|_{H^{s_0}}^{q(j)}\|w\|^2,$$

where $w = u - v$.

Lemma 3.65. *Let $s \geq 1$, $s_0 > 7/2$ and $\varepsilon_1 \in [0,1]$. Then there exists $C = C(s_0, s) > 0$ such that for any $u, v \in H^{\max\{s+3,s_0\}}(\mathbb{T})$,*

$$\left|\varepsilon_1 \int_{\mathbb{T}} u(\mathcal{H}D^s w)D^s\partial_x^3 w dx\right| \leq \frac{\sum_{j=1}^2 \varepsilon_1^{p(j)}}{100}\|D^{s+2}w\|^2 + C\sum_{j=1}^2 \|u\|_{H^{s_0}}^{q(j)}\|w\|^2,$$

where $w = u - v$.

Lemma 3.66. *Let $s \geq 1$, $s_0 > 7/2$ and $\varepsilon_1, \varepsilon_2 \in [0,1]$. Then there exists $C = C(s_0, s) > 0$ such that for any $u, v \in H^{\max\{s+4,s_0\}}(\mathbb{T})$,*

$$\left|(\varepsilon_1 - \varepsilon_2) \int_{\mathbb{T}} u(\mathcal{H}D^s\partial_x^4 v)\mathcal{H}D^{s-1}w dx\right| \leq C\max\{\varepsilon_1^2, \varepsilon_2^2\}\|v\|_{H^{s+4}}^2 + C\|u\|_{H^{s_0}}^2\|w\|_{H^s}^2,$$

where $w = u - v$.

Lemma 3.67. *Let $s \geq 1$, $s_0 > 7/2$ and $\varepsilon_1, \varepsilon_2 \in [0,1]$. Then there exists $C = C(s_0, s) > 0$ such that for any $u, v \in H^{\max\{s+4,s_0\}}(\mathbb{T})$,*

$$\left|(\varepsilon_1 - \varepsilon_2) \int_{\mathbb{T}} u(\mathcal{H}D^s w)D^s\partial_x^3 v dx\right| \leq C\max\{\varepsilon_1^2, \varepsilon_2^2\}\|v\|_{H^{s+3}}^2 + C\|u\|_{H^{s_0}}^2\|w\|_{H^s}^2,$$

where $w = u - v$.

The following three lemmas are estimates for viscous terms $-\varepsilon_1\partial_x^4 u + \varepsilon_2\partial_x^4 v$ in $M_s^{(2)}(u, v)$. We omit the proofs of these lemmas since they are similar to that of Lemma 3.63.

Lemma 3.68. *Let $s \geq 1$, $s_0 > 7/2$ and $\varepsilon_1 \in [0,1]$. Then there exists $C = C(s_0, s) > 0$ such that for any $u, v \in H^{\max\{s+4,s_0\}}(\mathbb{T})$,*

$$\left|\varepsilon_1 \int_{\mathbb{T}} (\mathcal{H}\partial_x^5 u)(D^{s-1}w)^2 dx\right| \leq \frac{\varepsilon_1^{p(4)}}{100}\|D^{s+2}w\|^2 + C\|u\|_{H^{s_0}}^{q(4)}\|w\|^2 + C\|u\|_{H^{s_0}}\|w\|_{H^s}^2,$$

where $w = u - v$.

Lemma 3.69. *Let $s \geq 1$, $s_0 > 7/2$ and $\varepsilon_1 \in [0,1]$. Then there exists $C = C(s_0, s) > 0$ such that for any $u, v \in H^{\max\{s+3,s_0\}}(\mathbb{T})$,*

$$\left|\varepsilon_1 \int_{\mathbb{T}} (\mathcal{H}\partial_x u)D^{s-1}wD^{s-1}\partial_x^4 w dx\right| \leq \frac{\sum_{j=2}^3 \varepsilon_1^{p(j)}}{100}\|D^{s+2}w\|^2 + C\sum_{j=2}^3 \|u\|_{H^{s_0}}^{q(j)}\|w\|^2,$$

where $w = u - v$.

Lemma 3.70. *Let* $s \geq 1$, $s_0 > 7/2$ *and* $\varepsilon_1, \varepsilon_2 \in [0,1]$. *Then there exists* $C = C(s_0, s) > 0$ *such that for any* $u, v \in H^{\max\{s+3,s_0\}}(\mathbb{T})$,

$$\left|(\varepsilon_1 - \varepsilon_2)\int_{\mathbb{T}}(\mathcal{H}\partial_x u)D^{s-1}wD^{s-1}\partial_x^4 v dx\right| \leq C\max\{\varepsilon_1^2, \varepsilon_2^2\}\|v\|_{H^{s+3}}^2 + C\|u\|_{H^{s_0}}^2\|w\|_{H^s}^2,$$

where $w = u - v$.

The following three lemmas are estimates for viscous terms $-\varepsilon_1\partial_x^4 u + \varepsilon_2\partial_x^4 v$ in $M_s^{(3)}(u,v)$.

Lemma 3.71. *Let* $s \geq 1$, $s_0 > 7/2$ *and* $\varepsilon_1 \in [0,1]$. *Then there exists* $C = C(s_0, s) > 0$ *such that for any* $u, v \in H^{\max\{s+3,s_0\}}(\mathbb{T})$,

$$\left|\varepsilon_1\int_{\mathbb{T}} u\partial_x^4 u(D^{s-1}w)^2 dx\right| \leq C\|u\|_{H^{s_0}}^2\|w\|_{H^s}^2,$$

where $w = u - v$.

Proof. This is obvious thanks to the integration by parts. □

Lemma 3.72. *Let* $s \geq 1$, $s_0 > 7/2$ *and* $\varepsilon_1 \in [0,1]$. *Then there exists* $C = C(s_0, s) > 0$ *such that for any* $u, v \in H^{\max\{s+3,s_0\}}(\mathbb{T})$,

$$\left|\varepsilon_1\int_{\mathbb{T}} u^2 D^{s-1}wD^{s-1}\partial_x^4 w dx\right| \leq \frac{\sum_{j=2}^3 \varepsilon_1^{p(j)}}{100}\|D^{s+2}w\|^2 + C\sum_{j=2}^3\|u\|_{H^{s_0}}^{2q(j)}\|w\|^2,$$

where $w = u - v$.

Proof. First we set

$$\begin{aligned}
&\int_{\mathbb{T}} u^2 D^{s-1}wD^{s-1}\partial_x^4 w dx \\
&= -2\int_{\mathbb{T}} u\partial_x u D^{s-1}wD^{s-1}\partial_x^3 w dx - \int_{\mathbb{T}} u^2 D^{s-1}\partial_x wD^{s-1}\partial_x^3 w dx =: A + B.
\end{aligned}$$

The same argument as before implies that

$$\begin{aligned}
|A| &\leq C\|u\|_{H^{s_0}}^2\|w\|^{2/q(3)}\|D^{s+2}w\|^{2/p(3)}, \\
|B| &\leq C\|u\|_{H^{s_0}}^2\|w\|^{2/q(2)}\|D^{s+2}w\|^{2/p(2)},
\end{aligned}$$

which completes the proof. □

Lemma 3.73. *Let* $s \geq 1$, $s_0 > 7/2$ *and* $\varepsilon_1 \in [0,1]$. *Then there exists* $C = C(s_0, s) > 0$ *such that for any* $u, v \in H^{\max\{s+3,s_0\}}(\mathbb{T})$,

$$\left|(\varepsilon_1 - \varepsilon_2)\int_{\mathbb{T}} u^2 D^{s-1}wD^{s-1}\partial_x^4 v dx\right| \leq C\max\{\varepsilon_1^2, \varepsilon_2^2\}\|v\|_{H^{s+3}}^2 + C\|u\|_{H^{s_0}}^4\|w\|_{H^s}^2,$$

where $w = u - v$.

Proof. This follows from the Hölder inequality. □

Finally, we are ready to show the main inequality in this section.

Proof of Proposition 3.8. Let $s' \in [1, s]$. Put $w := u_1 - u_2$. Note that w satisfies

$$\partial_t w = \partial_x(K(u_1) - K(u_2)) - \varepsilon_1 \partial_x^4 w + (\varepsilon_1 - \varepsilon_2)\partial_x^4 u_2 \tag{3.19}$$

on $[0, \min\{T_{\varepsilon_1}, T_{\varepsilon_2}\})$. By Lemma 3.28, 3.29, 3.30, 3.31, 3.32, 3.33, 3.34 and 3.35, we have

$$\begin{aligned} &\Big|\frac{1}{2}\frac{d}{dt}\|D^{s'}w\|^2 + \lambda_1(s')\langle \partial_x u_1, (D^{s'}\partial_x w)^2\rangle + \lambda_2(s')\langle (\mathcal{H}\partial_x^2 u_1)\mathcal{H}D^{s'}\partial_x w, D^{s'}w\rangle \\ &\quad + \lambda_3(s')\langle u_1\partial_x u_1 \mathcal{H}D^{s'}\partial_x w, D^{s'}w\rangle + 2\varepsilon_1\|D^{s'+2}w\|^2\Big| \\ &= |\langle D^{s'}\{\partial_x(K(u_1) - K(u_2)) - \varepsilon_1\partial_x^4 w + (\varepsilon_1 - \varepsilon_2)\partial_x^4 u_2\}, D^{s'}w\rangle \\ &\quad + \lambda_1(s')\langle \partial_x u_1, (D^{s'}\partial_x w)^2\rangle + \lambda_2(s')\langle (\mathcal{H}\partial_x^2 u_1)\mathcal{H}D^{s'}\partial_x w, D^{s'}w\rangle + 2\varepsilon_1\|D^{s'+2}w\|^2| \\ &\le CI_{s_0}(u_1, u_2)^3\{\|w\|_{H^{s'}}^2 + \|w\|_{H^{s_0-3}}^2\|u_2\|_{H^{s'+3}}^2 + \|w\|_{H^{s_0}}^2(\|u_1\|_{H^{s'}}^2 + \|u_2\|_{H^{s'}}^2)\} \\ &\quad + \max\{\varepsilon_1^2, \varepsilon_2^2\}\|u_2\|_{H^{s'+4}}^2. \end{aligned} \tag{3.20}$$

By Lemma 3.38, 3.41, 3.42, 3.43, 3.44, 3.45, 3.46, 3.47, 3.48, 3.52, 3.53, 3.54, 3.55, 3.56, 3.57, 3.58, 3.63, 3.64, 3.65, 3.66 and 3.67, we also have

$$\begin{aligned} &\left|\frac{d}{dt}M_{s'}^{(1)}(u_1, u_2) - \lambda_1(s')\langle \partial_x u_1, (D^{s'}\partial_x w)^2\rangle + \frac{\lambda_1(s')\lambda_4(s')}{4}\langle (\mathcal{H}\partial_x^2 u_1)\mathcal{H}D^{s'}\partial_x w, D^{s'}w\rangle\right| \\ &= \Big|\frac{\lambda_1(s')}{4}(\langle \partial_t u\mathcal{H}D^{s'}w, \mathcal{H}D^{s'-1}w\rangle + \langle u\mathcal{H}D^{s'}\partial_t w, \mathcal{H}D^{s'-1}w\rangle + \langle u\mathcal{H}D^{s'}w, \mathcal{H}D^{s'-1}\partial_t w\rangle) \\ &\quad -\lambda_1(s')\langle \partial_x u_1, (D^{s'}\partial_x w)^2\rangle + \frac{\lambda_1(s')\lambda_4(s')}{4}\langle (\mathcal{H}\partial_x^2 u_1)\mathcal{H}D^{s'}\partial_x w, D^{s'}w\rangle\Big| \\ &\le CI_{s_0}(u_1, u_2)^{2(s'+2)}\{\|w\|_{H^{s'}}^2 + \|w\|_{H^{s_0-3}}^2\|u_2\|_{H^{s'+3}}^2 + \|w\|_{H^{s_0}}^2(\|u_1\|_{H^{s'}}^2 + \|u_2\|_{H^{s'}}^2)\} \\ &\quad + \frac{\varepsilon_1}{10}\|D^{s'+2}w\|^2 + \max\{\varepsilon_1^2, \varepsilon_2^2\}\|u_2\|_{H^{s'+4}}^2. \end{aligned} \tag{3.21}$$

Similarly, by Lemma 3.39, 3.49, 3.50, 3.51, 3.59, 3.68, 3.69, 3.70 and 3.71, we have

$$\begin{aligned} &\left|\frac{d}{dt}M_{s'}^{(2)}(u_1, u_2) - \lambda_2(s')\langle (\mathcal{H}\partial_x^2 u_1)\mathcal{H}D^{s'}\partial_x w, D^{s'}w\rangle\right| \\ &\le CI_{s_0}(u_1, u_2)^{2(s'+2)}\{\|w\|_{H^{s'}}^2 + \|w\|_{H^{s_0-3}}^2\|u_2\|_{H^{s'+3}}^2 \\ &\quad + \|w\|_{H^{s_0}}^2(\|u_1\|_{H^{s'}}^2 + \|u_2\|_{H^{s'}}^2)\} + \frac{\varepsilon_1}{10}\|D^{s'+2}w\|^2 + \max\{\varepsilon_1^2, \varepsilon_2^2\}\|u_2\|_{H^{s'+4}}^2. \end{aligned} \tag{3.22}$$

Moreover, by Lemma 3.40, 3.62, 3.72 and 3.73, we obtain

$$\begin{aligned}
&\left|\frac{d}{dt}M_{s'}^{(3)}(u_1,u_2)-\frac{\lambda_1(s')\lambda_4(s')+4\lambda_3(s')}{4}\langle u\partial_x u\mathcal{H}D^{s'}\partial_x w, D^{s'}w\rangle\right|\\
&\le CI_{s_0}(u_1,u_2)^{2(s'+2)}\{\|w\|_{H^{s'}}^2+\|w\|_{H^{s_0-3}}^2\|u_2\|_{H^{s'+3}}^2 \qquad (3.23)\\
&\quad+\|w\|_{H^{s_0}}^2(\|u_1\|_{H^{s'}}^2+\|u_2\|_{H^{s'}}^2)\}+\frac{\varepsilon_1}{10}\|D^{s'+2}w\|^2+\max\{\varepsilon_1^2,\varepsilon_2^2\}\|u_2\|_{H^{s'+4}}^2.
\end{aligned}$$

It is easy to see that

$$\frac{d}{dt}\{\|w\|^2(1+C\|u_1\|^2+C\|u_1\|^{4s'})\}\le CI_{s_0}(u_1,u_2)^{4s'+3}\|w\|_{H^{s'}}^2. \qquad (3.24)$$

Therefore, collecting (3.20), (3.21), (3.22), (3.23) and (3.24), we obtain (3.9). □

3.4. **The energy estimate in L^2.** In this subsection, we prove Proposition 3.9, which is the only thing left to prove. We introduce some estimates for the operator J.

Lemma 3.74. *Let $k\in\mathbb{N}\cup\{0\}$. There exists $C=C(k)>0$ such that*

$$\|\mathcal{H}J\partial_x^{k+1}f-\partial_x^k f\|\le C\|f\|$$

for any $f\in L^2(\mathbb{T})$.

Proof. It suffices to show that there exists $C=C(k)>0$ such that

$$\left|-i\,\mathrm{sgn}(\xi)\frac{\psi(\xi)}{|\xi|}(i\xi)^{k+1}-(i\xi)^k\right|\le C$$

for any $\xi\in\mathbb{Z}$. But this is clear since the left hand side is equal to $|\xi|^k|\psi(\xi)-1|$ and $\mathrm{supp}(1-\psi)\subset\{|\xi|\le 2\}$. □

Lemma 3.75. *Let $s_0>5/2$. Let $u\in H^{s_0}(\mathbb{T})$ and $v\in L^2(\mathbb{T})$. Then*

$$\|J\partial_x(u\partial_x^2 v)+u\mathcal{H}\partial_x^2 v\|\lesssim\|u\|_{H^{s_0}}\|v\|.$$

Proof. Note that

$$\begin{aligned}
&J\partial_x(u\partial_x^2 v)+u\mathcal{H}\partial_x^2 v\\
&=(J\partial_x^3+\mathcal{H}\partial_x^2)(uv)-2(J\partial_x^2+\mathcal{H}\partial_x)(\partial_x uv)+J\partial_x(\partial_x^2uv)-[\mathcal{H},u]\partial_x^2v+\mathcal{H}(\partial_x^2uv).
\end{aligned}$$

Lemma 3.74 shows the desired inequality. □

As a corollary, we have the following.

Corollary 3.76. *Let $s_0>5/2$. Let $u\in H^{s_0}(\mathbb{T})$ and $v\in L^2(\mathbb{T})$. Then*

$$\|\mathcal{H}J\partial_x(u\mathcal{H}\partial_x^2 v)-u\mathcal{H}\partial_x^2 v\|\lesssim\|u\|_{H^{s_0}}\|v\|.$$

Lemma 3.77. *Let $s_0 > 5/2$. Let $u \in H^{s_0}(\mathbb{T})$ and $v \in L^2(\mathbb{T})$. Then*

$$\|J\partial_x(\partial_x u\partial_x v) + \partial_x u\mathcal{H}\partial_x v\| \lesssim \|u\|_{H^{s_0}}\|v\|.$$

Proof. This follows from the follwing equality

$$\begin{aligned}&J\partial_x(\partial_x u\partial_x v) + \partial_x u\mathcal{H}\partial_x v\\&= (J\partial_x^2 + \mathcal{H}\partial_x)(\partial_x uv) - J\partial_x(\partial_x^2 uv) - \mathcal{H}(\partial_x^2 uv) - [\mathcal{H}, \partial_x u]\partial_x v\end{aligned}$$

and Lemma 3.74. □

As a corollary, we have the following.

Corollary 3.78. *Let $s_0 > 5/2$. Let $u \in H^{s_0}(\mathbb{T})$ and $v \in L^2(\mathbb{T})$. Then*

$$\|\mathcal{H}J\partial_x((\mathcal{H}\partial_x u)\mathcal{H}\partial_x v) - (\mathcal{H}\partial_x u)\mathcal{H}\partial_x v\| \lesssim \|u\|_{H^{s_0}}\|v\|.$$

Lemma 3.79. *Let $s_0 > 1/2$ and $\Lambda = D^2$ or $D\partial_x$. There exists $C(s_0) > 0$ such that for any $f \in H^{s_0+1}(\mathbb{T})$ and $g \in L^2(\mathbb{T})$,*

$$\|[\langle D\rangle^{-1}\Lambda, f]g\| \le C\|f\|_{H^{s_0+1}}\|g\|.$$

Proof. See (ii) of Lemma 2.4 in [29]. □

We estimate the time derivative of $M^{(j)}(u, v)$ for $j = 1, 2, 3$.

Lemma 3.80. *Let $s_0 > 7/2$. Let $u, w \in H^{s_0+1}(\mathbb{T})$. Then*

$$|\langle(\mathcal{H}\partial_x^4 u)\mathcal{H}w, \mathcal{H}Jw\rangle - \langle u\partial_x^4 w, \mathcal{H}Jw\rangle - \langle u\mathcal{H}w, J\partial_x^4 w\rangle - 4\langle\partial_x u, (\partial_x w)^2\rangle| \lesssim \|u\|_{H^{s_0}}\|w\|^2.$$

Proof. We use Lemma 3.37 with $f = u$, $g = \mathcal{H}w$ and $h = \mathcal{H}Jw$. It is clear that (3.18) with $s = 0$ holds. Set

$$-4\langle\partial_x f\mathcal{H}\partial_x g, \partial_x^2 h\rangle + 2\langle\partial_x^2 f\mathcal{H}g, \partial_x^2 h\rangle = 6\langle\partial_x^2 f\mathcal{H}g, \partial_x^2 h\rangle + 4\langle\partial_x f\mathcal{H}g, \partial_x^3 h\rangle =: A + B.$$

For A, Lemma 3.74 shows that

$$\begin{aligned}|A| &\le 6|\langle w\partial_x^2 u, (\mathcal{H}J\partial_x^2 - \partial_x)w\rangle| + 3|\langle\partial_x^3 u, w^2\rangle| + 6|\hat{w}(0)\langle\partial_x^3 u, \mathcal{H}J\partial_x w\rangle|\\&\lesssim \|u\|_{H^{s_0}}\|w\|^2.\end{aligned}$$

Similarly, Lemma 3.25 and 3.74 show that

$$\begin{aligned}&|B - 4\langle\partial_x u, (\partial_x w)^2\rangle|\\&\le 4|\langle w\partial_x u, (\mathcal{H}J\partial_x^3 - \partial_x^2)w\rangle| + 4|\langle\partial_x uw, \partial_x^2 w\rangle + \langle\partial_x u, (\partial_x w)^2\rangle| \lesssim \|u\|_{H^{s_0}}\|w\|^2,\end{aligned}$$

which completes the proof. □

Lemma 3.81. *Let* $s_0 > 7/2$. *Let* $u, w \in H^{s_0+1}(\mathbb{T})$. *Then*

$$|\langle \partial_x^5 u, (Jw)^2\rangle - 2\langle (\mathcal{H}\partial_x u)Jw, \mathcal{H}J\partial_x^4 w\rangle + 4\langle (\mathcal{H}\partial_x^2 u)\mathcal{H}\partial_x w, w\rangle| \lesssim \|u\|_{H^{s_0}}\|w\|^2.$$

Proof. The integration by parts shows that

$$\begin{aligned}
&|\langle \partial_x^5 u, (Jw)^2\rangle - 2\langle (\mathcal{H}\partial_x u)Jw, \mathcal{H}J\partial_x^4 w\rangle + 4\langle (\mathcal{H}\partial_x^2 u)\mathcal{H}\partial_x w, w\rangle| \\
&= |\langle \partial_x^4 u Jw, J\partial_x w\rangle - \langle (\mathcal{H}u)J\partial_x w, \mathcal{H}J\partial_x^4 w\rangle - \langle (\mathcal{H}u)Jw, \mathcal{H}J\partial_x^5 w\rangle \\
&\quad - 2\langle (\mathcal{H}\partial_x^2 u)\mathcal{H}\partial_x w, w\rangle|,
\end{aligned}$$

which allows us to use Lemma 3.37 with $f = \mathcal{H}u$, $g = J\partial_x w$ and $h = Jw$. Then (3.18) with $s = 0$ holds. Lemma 3.26 shows that

$$|\langle \partial_x f \mathcal{H}\partial_x g, \partial_x^2 h\rangle| = |\langle \partial_x((\mathcal{H}\partial_x u)\mathcal{H}J\partial_x^2 w), J\partial_x w\rangle| \lesssim \|u\|_{H^{s_0}}\|w\|^2.$$

Finally, we have

$$\begin{aligned}
&2\langle \partial_x^2 f \mathcal{H}g, \partial_x^2 h\rangle \\
&= 2\langle (\mathcal{H}\partial_x^2 u)\mathcal{H}J\partial_x w, J\partial_x^2 w\rangle \\
&= 2\langle (\mathcal{H}\partial_x^2 u)\mathcal{H}J\partial_x w, (J\partial_x^2 + \mathcal{H}\partial_x)w\rangle + 2\langle (\mathcal{H}\partial_x^3 u)(\mathcal{H}J\partial_x - 1)w, \mathcal{H}w\rangle \\
&\quad + 2\langle (\mathcal{H}\partial_x^2 u)(\mathcal{H}J\partial_x^2 - \partial_x)w, \mathcal{H}w\rangle - 2\langle (\mathcal{H}\partial_x^2 u)\mathcal{H}\partial_x w, w\rangle,
\end{aligned}$$

which completes the proof. □

Lemma 3.82. *Let* $s_0 > 7/2$. *Let* $u, w \in H^{s_0+1}(\mathbb{T})$. *Then*

$$|\langle u\mathcal{H}\partial_x^4 u, (Jw)^2\rangle + \langle u^2 Jw, \mathcal{H}J\partial_x^4 w\rangle - 4\langle u\partial_x u \mathcal{H}\partial_x w, w\rangle| \lesssim \|u\|_{H^{s_0}}^2\|w\|^2.$$

Proof. Adding and subtraction a term, we have

$$\begin{aligned}
&2|\langle u\mathcal{H}\partial_x^4 u, (Jw)^2\rangle + \langle u^2 Jw, \mathcal{H}J\partial_x^4 w\rangle - 4\langle u\partial_x u \mathcal{H}\partial_x w, w\rangle| \\
&\le |\langle \mathcal{H}\partial_x^4(u^2), (Jw)^2\rangle + 2\langle u^2 Jw, \mathcal{H}J\partial_x^4 w\rangle - 8\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle| \\
&\quad + |\langle 2u\mathcal{H}\partial_x^4 u - \mathcal{H}\partial_x^4(u^2), (Jw)^2\rangle|.
\end{aligned}$$

Lemma 3.5 and 3.15 show that the second term in the right hand side can be estimated by $\lesssim \|u\|_{H^{s_0}}^2\|w\|^2$. We use Lemma 3.37 with $f = u^2$ and $g = h = Jw$. Note that (3.18) with $s = 0$ holds. It is easy to see that

$$|\langle \partial_x^2 f \mathcal{H}g, \partial_x^2 h\rangle| = |\langle \partial_x(\partial_x^2(u^2)\mathcal{H}Jw), J\partial_x w\rangle| \lesssim \|u\|_{H^{s_0}}^2\|w\|^2.$$

Finally, Lemma 3.74 shows that

$$\begin{aligned}&-4\langle \partial_x f\mathcal{H}\partial_x g, \partial_x^2 h\rangle\\&=-8\langle u\partial_x u\mathcal{H}J\partial_x w, (J\partial_x^2+\mathcal{H}\partial_x)w\rangle-8\langle \partial_x(u\partial_x u)(\mathcal{H}J\partial_x-1)w, \mathcal{H}w\rangle\\&\quad-8\langle u\partial_x u(\mathcal{H}J\partial_x^2-\partial_x)w, \mathcal{H}w\rangle+8\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle,\end{aligned}$$

which completes the proof. □

Lemma 3.83. *Let $s_0>7/2$. Let $u, v\in H^{s_0}(\mathbb{T})$. Then*

$$|\langle u\mathcal{H}\partial_x(u\partial_x^2 u-v\partial_x^2 v), \mathcal{H}Jw\rangle-3\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle|\lesssim I_{s_0}(u,v)^2\|w\|^2,$$

where $w=u-v$.

Proof. First we set

$$A:=\langle u\mathcal{H}\partial_x(u\partial_x^2 w), \mathcal{H}Jw\rangle, \quad B:=\langle u\mathcal{H}\partial_x(w\partial_x^2 v), \mathcal{H}Jw\rangle.$$

It is clear that $|B|\lesssim\|u\|_{H^{s_0}}\|v\|_{H^{s_0}}\|w\|^2$. Note that

$$\partial_x(u\partial_x^2 w)=\partial_x^3(uw)-2\partial_x^2(\partial_x uw)+\partial_x(\partial_x^2 uw).$$

Then we set

$$\begin{aligned}A&=\langle u\mathcal{H}\partial_x^3(uw), \mathcal{H}Jw\rangle-2\langle u\mathcal{H}\partial_x^2(\partial_x uw), \mathcal{H}Jw\rangle+\langle u\mathcal{H}\partial_x(\partial_x^2 uw), \mathcal{H}Jw\rangle\\&=: A_1+A_2+A_3.\end{aligned}$$

It is clear that $|A_3|\lesssim\|u\|_{H^{s_0}}^2\|w\|^2$. For A_1, we have

$$\begin{aligned}A_1&=\langle uw, \mathcal{H}\partial_x^3(u\mathcal{H}Jw)\rangle\\&=\langle uw, \mathcal{H}(\partial_x^3 u\mathcal{H}Jw)\rangle+3\langle uw, \mathcal{H}(\partial_x^2 u\mathcal{H}J\partial_x w)\rangle+3\langle uw, \mathcal{H}(\partial_x u\mathcal{H}J\partial_x^2 w)\rangle\\&\quad+\langle uw, \mathcal{H}(u\mathcal{H}J\partial_x^3 w)\rangle=: A_{11}+\cdots+A_{14}.\end{aligned}$$

It is clear that $|A_{11}|+|A_{12}|\lesssim\|u\|_{H^{s_0}}^2\|w\|^2$. For A_{13}, we have

$$\begin{aligned}A_{13}&=3\langle uw, [\mathcal{H}, \partial_x u]\mathcal{H}J\partial_x^2 w\rangle-3\langle u\partial_x uw, J\partial_x^2 w\rangle\\&=3\langle uw, [\mathcal{H}, \partial_x u]\mathcal{H}J\partial_x^2 w\rangle-3\langle u\partial_x uw, J\partial_x^2 w+\mathcal{H}\partial_x w\rangle+3\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle.\end{aligned}$$

Similarly, we have

$$\begin{aligned}A_{14}&=\langle uw, [\mathcal{H}, u]\mathcal{H}J\partial_x^3 w\rangle-\langle u^2 w, J\partial_x^3+\mathcal{H}\partial_x^2 w\rangle\\&\quad+\langle \partial_x(u^2\mathcal{H}\partial_x w), w\rangle-2\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle.\end{aligned}$$

Finally, we have

$$\begin{aligned} A_2 &= 2\langle \partial_x uw, \mathcal{H}(\partial_x^2 u\mathcal{H}Jw)\rangle + 4\langle \partial_x uw, \mathcal{H}(\partial_x u\mathcal{H}J\partial_x w)\rangle + 2\langle \partial_x uw, \mathcal{H}(u\mathcal{H}J\partial_x^2 w)\rangle \\ &=: A_{21} + A_{22} + A_{23}. \end{aligned}$$

Obviously, $|A_{21}| + |A_{22}| \lesssim \|u\|_{H^{s_0}}^2 \|w\|^2$. Observe that

$$A_{23} = 2\langle \partial_x uw, [\mathcal{H}, u]\mathcal{H}J\partial_x^2 w\rangle - 2\langle u\partial_x u(J\partial_x^2 w + \mathcal{H}\partial_x w), w\rangle + 2\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle.$$

Therefore, we have

$$\begin{aligned} &|A + B - 3\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle| \\ &\le |A_1 - \langle u\partial_x u\mathcal{H}\partial_x w, w\rangle| + |A_2 - 2\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle| + |A_3| + |B| \lesssim I_{s_0}(u,v)^2\|w\|^2, \end{aligned}$$

which completes the proof. □

Lemma 3.84. *Let $s_0 > 7/2$. Let $u, v \in H^{s_0+1}(\mathbb{T})$. Then*

$$|\langle u\mathcal{H}w, \mathcal{H}J\partial_x(u\partial_x^2 u - v\partial_x^2 v)\rangle - 2\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle| \lesssim I_{s_0}(u,v)^2\|w\|^2,$$

where $w = u - v$.

Proof. First we set $A := \langle u\mathcal{H}w, \mathcal{H}J\partial_x(u\partial_x^2 w)\rangle$ and $B := \langle u\mathcal{H}w, \mathcal{H}J\partial_x(w\partial_x^2 v)\rangle$. It is easy to see that $|B| \lesssim I(u,v)^2\|w\|^2$. We have

$$\begin{aligned} A = &\langle u\mathcal{H}w, \mathcal{H}\{J\partial_x(u\partial_x^2 w) + u\mathcal{H}\partial_x^2 w\}\rangle - \langle u\mathcal{H}w, [\mathcal{H}, u]\partial_x^2 w\rangle \\ &+ \langle \partial_x^2(u^2)\mathcal{H}w, w\rangle + \langle \partial_x(u^2\mathcal{H}\partial_x w), w\rangle + 2\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle. \end{aligned}$$

Lemma 3.15, 3.26 and 3.75 show that

$$|A - 2\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle| \lesssim I_{s_0}(u,v)^2\|w\|^2,$$

which completes the proof. □

We modify Lemma 3.43 in $L^2(\mathbb{T})$.

Lemma 3.85. *Let $s_0 > 7/2$. Let $u, v \in H^{s_0}(\mathbb{T})$. Then*

$$|\langle u\mathcal{H}\partial_x((\partial_x u)^2 - (\partial_x v)^2), \mathcal{H}Jw\rangle + 2\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle| \lesssim I_{s_0}(u,v)^2\|w\|^2,$$

where $w = u - v$.

Proof. Set $z = u + v$. First we set

$$\begin{aligned} \langle u\mathcal{H}\partial_x(\partial_x z\partial_x w), \mathcal{H}Jw\rangle &= \langle u\mathcal{H}\partial_x^2(\partial_x zw), \mathcal{H}Jw\rangle - \langle u\mathcal{H}\partial_x(\partial_x^2 zw), \mathcal{H}Jw\rangle \\ &=: A + B. \end{aligned}$$

It is clear that $|B| \lesssim I(u,v)^2\|w\|^2$. Moreover, we set

$$\begin{aligned} A &= -\langle \partial_x z w, \mathcal{H}\partial_x^2(u\mathcal{H}Jw)\rangle \\ &= -\langle \partial_x z w, \mathcal{H}(\partial_x^2 u\mathcal{H}Jw)\rangle - 2\langle \partial_x z w, \mathcal{H}(\partial_x u\mathcal{H}J\partial_x w)\rangle - \langle \partial_x z w, \mathcal{H}(u\mathcal{H}J\partial_x^2 w)\rangle \\ &=: A_1 + A_2 + A_3. \end{aligned}$$

It is also clear that $|A_1| + |A_2| \lesssim I(u,v)^2\|w\|^2$. For A_3, we have

$$\begin{aligned} A_3 = &-\langle \partial_x z w, [\mathcal{H}, u]\mathcal{H}J\partial_x^2 w\rangle + \langle \partial_x z w, uJ\partial_x^2 w + \mathcal{H}\partial_x w\rangle - 2\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle \\ &- \frac{1}{2}\langle \partial_x(u\mathcal{H}\partial_x w), w^2\rangle, \end{aligned}$$

from which follows that

$$\begin{aligned} &|A + B + 2\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle| \\ &\le |A_1| + |A_2| + |A_3 + 2\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle| + |B| \lesssim I_{s_0}(u,v)^2\|w\|^2, \end{aligned}$$

which completes the proof. □

Lemma 3.86. *Let $s_0 > 7/2$. Let $u, v \in H^{s_0}(\mathbb{T})$. Then*

$$|\langle u\mathcal{H}w, \mathcal{H}J\partial_x((\partial_x u)^2 - (\partial_x v)^2)\rangle + 2\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle| \lesssim I_{s_0}(u,v)^2\|w\|^2,$$

where $w = u - v$.

Proof. Set $z = u + v$. Note that

$$\begin{aligned} &\langle u\mathcal{H}w, \mathcal{H}J\partial_x(\partial_x z\partial_x w)\rangle \\ &= \langle u\mathcal{H}w, \mathcal{H}\{J\partial_x(\partial_x z\partial_x w) + \partial_x z\mathcal{H}\partial_x w\}\rangle - \langle u\mathcal{H}w, [\mathcal{H}, \partial_x z]\mathcal{H}\partial_x w\rangle \\ &\quad - \langle \partial_x(u\partial_x z)\mathcal{H}w, w\rangle - \langle \partial_x(u\mathcal{H}\partial_x w), w^2\rangle/2 - 2\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle. \end{aligned}$$

Then, Lemma 3.15 together with Lemma 3.77 completes the proof. □

Lemma 3.87. *Let $s_0 > 7/2$. Let $u, v \in H^{s_0}(\mathbb{T})$. Then*

$$|\langle u\mathcal{H}\partial_x((\mathcal{H}\partial_x u)^2 - (\mathcal{H}\partial_x v)^2), \mathcal{H}Jw\rangle| \lesssim I_{s_0}(u,v)^2\|w\|^2,$$

where $w = u - v$.

Proof. Set $z = u + v$ and set

$$\begin{aligned} &\langle u\mathcal{H}\partial_x((\mathcal{H}\partial_x z)\mathcal{H}\partial_x w), \mathcal{H}Jw\rangle \\ &= \langle u\mathcal{H}\partial_x^2((\mathcal{H}\partial_x z)\mathcal{H}w), \mathcal{H}Jw\rangle - \langle u\mathcal{H}\partial_x((\mathcal{H}\partial_x^2 z)\mathcal{H}w), \mathcal{H}Jw\rangle =: A + B. \end{aligned}$$

It is clear that $|B| \lesssim I_{s_0}(u,v)^2\|w\|^2$. Observe that

$$\begin{aligned}A &= -\langle (\mathcal{H}\partial_x z)\mathcal{H}w, \mathcal{H}\partial_x^2(u\mathcal{H}Jw)\rangle \\ &= -\langle (\mathcal{H}\partial_x z)\mathcal{H}w, \mathcal{H}(\partial_x^2 u\mathcal{H}Jw)\rangle - 2\langle (\mathcal{H}\partial_x z)\mathcal{H}w, \mathcal{H}(\partial_x u\mathcal{H}J\partial_x w)\rangle \\ &\quad - \langle (\mathcal{H}\partial_x z)\mathcal{H}w, \mathcal{H}(u\mathcal{H}J\partial_x^2 w)\rangle =: A_1 + A_2 + A_3.\end{aligned}$$

Again, $|A_1| + |A_2| \lesssim I_{s_0}(u,v)^2\|w\|^2$. And we note that

$$\begin{aligned}A_3 &= -\langle (\mathcal{H}\partial_x z)\mathcal{H}w, [\mathcal{H},u]\mathcal{H}J\partial_x^2 w\rangle + \langle (\mathcal{H}\partial_x z)\mathcal{H}w, uJ\partial_x^2 w\rangle \\ &= -\langle (\mathcal{H}\partial_x z)\mathcal{H}w, [\mathcal{H},u]\mathcal{H}J\partial_x^2 w\rangle + \langle u(\mathcal{H}\partial_x z)\mathcal{H}w, J\partial_x^2 w + \mathcal{H}\partial_x w\rangle \\ &\quad + \frac{1}{2}\langle \partial_x(u\mathcal{H}\partial_x z), (\mathcal{H}w)^2\rangle,\end{aligned}$$

which completes the proof. □

Lemma 3.88. *Let $s_0 > 7/2$. Let $u, v \in H^{s_0}(\mathbb{T})$. Then*

$$|\langle u\mathcal{H}w, \mathcal{H}J\partial_x((\mathcal{H}\partial_x u)^2 - (\mathcal{H}\partial_x v)^2)\rangle| \lesssim I_{s_0}(u,v)^2\|w\|^2,$$

where $w = u - v$.

Proof. Set $z = u + v$. Note that

$$\begin{aligned}&\langle u\mathcal{H}w, \mathcal{H}J\partial_x((\mathcal{H}\partial_x z)\mathcal{H}\partial_x w)\rangle \\ &= \langle u\mathcal{H}w, \mathcal{H}J\partial_x((\mathcal{H}\partial_x z)\mathcal{H}\partial_x w) - (\mathcal{H}\partial_x z)\mathcal{H}\partial_x w\rangle - \frac{1}{2}\langle \partial_x(u\mathcal{H}\partial_x z), (\mathcal{H}w)^2\rangle.\end{aligned}$$

Corollary 3.78 completes the proof. □

Lemma 3.89. *Let $s_0 > 7/2$. Let $u, v \in H^{s_0}(\mathbb{T})$. Then*

$$|\langle u\partial_x(u\mathcal{H}\partial_x^2 u - v\mathcal{H}\partial_x^2 v), \mathcal{H}Jw\rangle - 3\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle| \lesssim I_{s_0}(u,v)^2\|w\|^2,$$

where $w = u - v$.

Proof. First we set

$$A := \langle u\partial_x(u\mathcal{H}\partial_x^2 w), \mathcal{H}Jw\rangle, \quad B := \langle u\partial_x(w\mathcal{H}\partial_x^2 v), \mathcal{H}Jw\rangle.$$

It is clear that $|B| \lesssim I_{s_0}(u,v)^2\|w\|^2$. We also set

$$\begin{aligned}A &= \langle u\partial_x^3(u\mathcal{H}w), \mathcal{H}Jw\rangle - 2\langle u\partial_x^2(\partial_x u\mathcal{H}w), \mathcal{H}Jw\rangle + \langle u\partial_x(\partial_x^2 u\mathcal{H}w), \mathcal{H}Jw\rangle \\ &=: A_1 + A_2 + A_3.\end{aligned}$$

Again, it is clear that $|A_3| \lesssim I_{s_0}(u,v)^2\|w\|^2$. Note that

$$\begin{aligned}A_1 &= -\langle u\mathcal{H}w, \partial_x^3(u\mathcal{H}Jw)\rangle \\ &= -\langle u\mathcal{H}w, \partial_x^3 u\mathcal{H}Jw\rangle - 3\langle u\mathcal{H}w, \partial_x^2 u\mathcal{H}J\partial_x w\rangle - 3\langle u\mathcal{H}w, \partial_x u\mathcal{H}J\partial_x^2 w\rangle \\ &\quad - \langle u^2\mathcal{H}w, \mathcal{H}J\partial_x^3 w\rangle =: A_{11} + A_{12} + A_{13} + A_{14}.\end{aligned}$$

It is easy to see that $|A_{11}| + |A_{12}| \lesssim I_{s_0}(u,v)^2\|w\|^2$. We have

$$A_{13} = -3\langle u\partial_x u\mathcal{H}w, (\mathcal{H}J\partial_x^2 - \partial_x)w\rangle + 3\langle \partial_x(u\partial_x u)\mathcal{H}w, w\rangle + 3\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle$$

and

$$\begin{aligned}A_{14} &= -\langle u^2\mathcal{H}w, (\mathcal{H}J\partial_x^3 - \partial_x^2)w\rangle - \langle \partial_x^2(u^2)\mathcal{H}w, w\rangle - \langle \partial_x(u^2\mathcal{H}\partial_x w), w\rangle \\ &\quad - 2\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle.\end{aligned}$$

Similarly, we have

$$\begin{aligned}A_2 &= -2\langle \partial_x u\mathcal{H}w, \partial_x^2 u\mathcal{H}Jw + 2\partial_x u\mathcal{H}J\partial_x w\rangle - 2\langle u\partial_x u\mathcal{H}w, \mathcal{H}J\partial_x^2 w\rangle \\ &= -2\langle \partial_x u\mathcal{H}w, \partial_x^2 u\mathcal{H}Jw + 2\partial_x u\mathcal{H}J\partial_x w\rangle - 2\langle u\partial_x u\mathcal{H}w, (\mathcal{H}J\partial_x^2 - \partial_x)w\rangle \\ &\quad + 2\langle \partial_x(u\partial_x u)\mathcal{H}w, w\rangle + 2\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle.\end{aligned}$$

Therefore, we see that

$$\begin{aligned}&|A + B - 3\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle| \\ &\le |A_{11}| + |A_{12}| + |A_{13} - 3\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle| + |A_{14} + 2\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle| \\ &\quad + |A_2 - 2\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle| + |A_3| + |B| \lesssim I_{s_0}(u,v)^2\|w\|^2,\end{aligned}$$

which completes the proof. □

Lemma 3.90. *Let $s_0 > 7/2$. Let $u, v \in H^{s_0}(\mathbb{T})$. Then*

$$|\langle u\mathcal{H}w, J\partial_x(u\mathcal{H}\partial_x^2 u - v\mathcal{H}\partial_x^2 v)\rangle - 2\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle| \lesssim I_{s_0}(u,v)^2\|w\|^2,$$

where $w = u - v$.

Proof. First we set $A = \langle u\mathcal{H}w, J\partial_x(u\mathcal{H}\partial_x^2 w)\rangle$ and $B := \langle u\mathcal{H}w, J\partial_x(w\partial_x^2 v)\rangle$. It is clear that $|B| \lesssim I(u,v)^2\|w\|^2$. On the other hand, we have

$$\begin{aligned}A &= \langle u\mathcal{H}w, J\partial_x(u\mathcal{H}\partial_x^2 w) - u\partial_x^2 w\rangle + \langle \partial_x^2(u^2)\mathcal{H}w, w\rangle + \langle \partial_x(u^2\mathcal{H}\partial_x w), w\rangle \\ &\quad + 2\langle u\partial_x u\mathcal{H}\partial_x w, w\rangle.\end{aligned}$$

Then, Lemma 3.75 completes the proof. □

Lemma 3.91. *Let* $s_0 > 7/2$. *Let* $u, v \in H^{s_0}(\mathbb{T})$. *Then*

$$|\langle (\mathcal{H}\partial_x u)Jw, J\partial_x(u\partial_x^2 u - v\partial_x^2 v)\rangle| \lesssim I_{s_0}(u,v)^2\|w\|^2,$$

where $w = u - v$.

Proof. First we set $A = \langle (\mathcal{H}\partial_x u)Jw, J\partial_x(u\partial_x^2 w)\rangle$ and $B = \langle (\mathcal{H}\partial_x u)Jw, J\partial_x(w\partial_x^2 v)\rangle$. It is easy to see that $|B| \lesssim I_{s_0}(u,v)^2\|w\|^2$. We also set $A' = \langle (\mathcal{H}\partial_x u)Jw, u\mathcal{H}\partial_x^2 w\rangle$. Lemma 3.75 shows that $|A + A'| \lesssim I_{s_0}(u,v)^2\|w\|^2$. So we consider A'. Note that

$$\begin{aligned} A' &= \langle \partial_x^2(u\mathcal{H}\partial_x u)Jw, \mathcal{H}w\rangle + 2\langle \partial_x(u\mathcal{H}\partial_x u)J\partial_x w, \mathcal{H}w\rangle + \langle u(\mathcal{H}\partial_x u)J\partial_x^2 w, \mathcal{H}w\rangle \\ &=: A_1' + A_2' + A_3'. \end{aligned}$$

It is clear that $|A_1'| + |A_2'| \lesssim I_{s_0}(u,v)^2\|w\|^2$. Lemma 3.74 shows that

$$A_3' = \langle u(\mathcal{H}\partial_x u)(J\partial_x^2 + \mathcal{H}\partial_x)w, \mathcal{H}w\rangle + \frac{1}{2}\langle \partial_x(u\mathcal{H}\partial_x u), (\mathcal{H}u)^2\rangle,$$

which completes the proof. □

Lemma 3.92. *Let* $s_0 > 7/2$. *Let* $u, v \in H^{s_0}(\mathbb{T})$. *Then*

$$|\langle (\mathcal{H}\partial_x u)Jw, J\partial_x((\partial_x u)^2 - (\partial_x v)^2)\rangle| \lesssim I_{s_0}(u,v)^2\|w\|^2,$$

where $w = u - v$.

Proof. Set $z = u + v$. Note that

$$\langle (\mathcal{H}\partial_x u)Jw, J\partial_x((\partial_x u)^2 - (\partial_x v)^2)\rangle = -\langle \partial_x(\partial_x z J\partial_x((\mathcal{H}\partial_x u)Jw)), w\rangle,$$

which shows the desired inequality. □

Lemma 3.93. *Let* $s_0 > 7/2$. *Let* $u, v \in H^{s+2}(\mathbb{T})$. *Then*

$$|\langle (\mathcal{H}\partial_x u)Jw, J\partial_x((\mathcal{H}\partial_x u)^2 - (\mathcal{H}\partial_x v)^2)\rangle| \lesssim I_{s_0}(u,v)^2\|w\|^2,$$

where $w = u - v$.

Proof. The proof is identical with that of the previous lemma. □

A similar argument to the proof of Lemma 3.91 with using Corollary 3.76, we can show the following:

Lemma 3.94. *Let* $s_0 > 7/2$. *Let* $u, v \in H^{s_0}(\mathbb{T})$. *Then*

$$|\langle (\mathcal{H}\partial_x u)Jw, \mathcal{H}J\partial_x(u\mathcal{H}\partial_x^2 u - v\mathcal{H}\partial_x^2 v)\rangle| \lesssim I_{s_0}(u,v)^2\|w\|^2,$$

where $w = u - v$.

By the integration by parts with Lemma 3.15 and 3.74, we obtain the following:

Lemma 3.95. *Let $s_0 > 7/2$. Let $u, v \in H^{s_0}(\mathbb{T})$. Then*

$$\begin{aligned}&\sum_{j=3}^{4}(|\langle u\mathcal{H}\partial_x(F_j(u)-F_j(v)), \mathcal{H}Jw\rangle| + |\langle u\mathcal{H}w, \mathcal{H}J\partial_x(F_j(u)-F_j(v))\rangle|)\\ &\lesssim I_{s_0}(u,v)^4\|w\|^2,\end{aligned}$$

where $w = u - v$.

By the presence of J, we can easily obtain the following two lemmas:

Lemma 3.96. *Let $s_0 > 7/2$. Let $u, v \in H^{s_0}(\mathbb{T})$. Then*

$$\sum_{j=3}^{4}|\langle(\mathcal{H}\partial_x u)Jw, J\partial_x(F_j(u)-F_j(v))\rangle| \lesssim I_{s_0}(u,v)^4\|w\|^2,$$

where $w = u - v$.

Lemma 3.97. *Let $s_0 > 7/2$. Let $u, v \in H^{s_0}(\mathbb{T})$. Then*

$$\sum_{j=2}^{4}|\langle u^2 Jw, J\partial_x(F_j(u)-F_j(v))\rangle| \lesssim I_{s_0}(u,v)^5\|w\|^2,$$

where $w = u - v$.

Proof of Proposition 3.9. The proof is similar to that of Proposition 3.10. Put $w := u_1 - u_2$. Then w satisfies (3.19) on $[0, T]$. By Lemma 3.36, we have

$$\begin{aligned}&\left|\frac{1}{2}\frac{d}{dt}\|w\|^2 + \lambda_1(0)\langle\partial_x u_1, (\partial_x w)^2\rangle + \lambda_2(0)\langle(\mathcal{H}\partial_x^2 u_1)\mathcal{H}\partial_x w, w\rangle + \lambda_3(0)\langle u_1\partial_x u_1\mathcal{H}\partial_x w, w\rangle\right|\\ &\le CI_{s_0}(u_1,u_2)^3\|w\|_{H^s}^2 + \max\{\varepsilon_1^2, \varepsilon_2^2\}\|u_2\|_{H^{s_0+1}}^2.\end{aligned} \tag{3.25}$$

By Lemma 3.80, 3.83, 3.84, 3.85, 3.86, 3.87, 3.88, 3.89, 3.90, 3.95, we also have

$$\begin{aligned}&\left|\frac{d}{dt}M^{(1)}(u_1,u_2) - \lambda_1(0)\langle\partial_x u_1, (\partial_x w)^2\rangle + \frac{\lambda_1(0)\lambda_4(0)}{4}\langle(\mathcal{H}\partial_x^2 u_1)\mathcal{H}\partial_x w, w\rangle\right|\\ &\le CI_{s_0}(u_1,u_2)^5\|w\|_{H^s}^2 + \max\{\varepsilon_1^2, \varepsilon_2^2\}\|u_2\|_{H^{s_0+1}}^2\end{aligned} \tag{3.26}$$

Similarly, by Lemma 3.81, 3.91, 3.92, 3.93, 3.94, 3.96, we have

$$\begin{aligned}&\left|\frac{d}{dt}M^{(2)}(u_1,u_2) - \lambda_2(0)\langle(\mathcal{H}\partial_x^2 u_1)\mathcal{H}\partial_x w, w\rangle\right|\\ &\le CI_{s_0}(u_1,u_2)^5\|w\|_{H^s}^2 + \max\{\varepsilon_1^2, \varepsilon_2^2\}\|u_2\|_{H^{s_0+1}}^2.\end{aligned} \tag{3.27}$$

Moreover, by Lemma 3.82 and 3.97, we obtain

$$\begin{aligned}&\left|\frac{d}{dt}M^{(3)}(u_1,u_2)-\frac{\lambda_1(0)\lambda_4(0)+4\lambda_3(0)}{4}\langle u\partial_x u\mathcal{H}\partial_x w,w\rangle\right|\\&\le CI_{s_0}(u_1,u_2)^5\|w\|_{H^s}^2+\max\{\varepsilon_1^2,\varepsilon_2^2\}\|u_2\|_{H^{s_0+1}}^2.\end{aligned}\tag{3.28}$$

It is easy to see that

$$\frac{d}{dt}\{\|w\|_{H^{-1}}^2(1+C\|u_1\|^2+C\|u_1\|^4)\}\le CI_{s_0}(u_1,u_2)^7\|w\|^2.\tag{3.29}$$

Indeed, we have

$$\begin{aligned}&\langle\langle D\rangle^{-1}\partial_x(u_1\partial_x^2u_1-u_2\partial_x^2u_2),\langle D\rangle^{-1}w\rangle\\&=-\langle[\langle D\rangle^{-1}\partial_x^2,u_1]w,\langle D\rangle^{-1}\partial_x w\rangle-\frac{1}{2}\langle\partial_x u_1,(\langle D\rangle^{-1}\partial_x w)^2\rangle\\&\quad+2\langle\langle D\rangle^{-1}\partial_x(\partial_x u_1 w),\langle D\rangle^{-1}\partial_x w\rangle-\langle\langle D\rangle^{-1}(\partial_x^2u_1 w),\langle D\rangle^{-1}\partial_x w\rangle\\&\quad+\langle\langle D\rangle^{-1}\partial_x(w\partial_x^2u_2),\langle D\rangle^{-1}w\rangle,\end{aligned}$$

which together with Lemma 3.79 implies that

$$|\langle\langle D\rangle^{-1}\partial_x(u_1\partial_x^2u_1-u_2\partial_x^2u_2),\langle D\rangle^{-1}w\rangle|\lesssim I_{s_0}(u_1,u_2)\|w\|^2.$$

Other terms can be estimated in a simiar way, and then we obtain (3.29). Therefore, collecting (3.25), (3.26), (3.27), (3.28) and (3.29), we obtain (3.10). □

4. Well-posedness and parabolic smoothing effect for higher order Schrödinger type equations with constant coefficients

4.1. **Introduction.** We consider the Cauchy problem of the following:

$$D_t u(t,x) = D_x^{2m} u(t,x) + \sum_{j=1}^{2m} \big(a_j D_x^{2m-j} u(t,x) + b_j D_x^{2m-j} \bar{u}(t,x)\big), \tag{4.1}$$

$$u(0,x) = \varphi(x), \tag{4.2}$$

where $1 \le m \in \mathbb{N}$, $\mathcal{M} = \mathbb{R}\,(\text{or}\,\mathbb{T})$, $(t,x) \in (-\infty,\infty) \times \mathcal{M}$, $D_t = -i\partial_t$, $D_x = -i\partial_x$ and i is the imaginary unit. The constants $\{a_j\}, \{b_j\} \subset \mathbb{C}$ and the initial data $\varphi(x) : \mathcal{M} \to \mathbb{C}$ are given and $u(t,x) : (-\infty,\infty) \times \mathcal{M} \to \mathbb{C}$ is unknown. We are interested in the Cauchy problem of the following higher order nonlinear Schrödinger type equations:

$$i\partial_t u(t,x) - \partial_x^{2m} u(t,x) = F(\partial_x^{2m-1} u, \partial_x^{2m-1}\overline{u}, \partial_x^{2m-2} u, \partial_x^{2m-2}\overline{u}, \dots u, \overline{u}), \tag{4.3}$$

with (4.2), where F is a polynomial. As important examples, this class of equations includes the nonlinear Schrödinger hierarchy and the derivative nonlinear Schrödinger hierarchy, which are integrable systems appearing in the soliton theory. In [3], Chihara studied the well-posedness and the ill-posedness of (4.3) for $m = 1$ with (4.2) on $\mathbb{T}$. Recently, in [32], Tsugawa has studied similar problem and shown a non-existence result of solutions of (4.3) for some nonlinearity and $m = 1$ with (4.2) on $\mathbb{T}$ by employing the twisted parabolic smoothing. In their proofs, the so called "energy inequality" of (4.1) with variable coefficients $\{a_j(t,x)\}$ and $\{b_j(t,x)\}$ plays an important role. Our plan is to extend this result to $m \ge 2$. However, the energy inequality for higher m is much complicated. Therefore, we assume $\{a_j\}$ and $\{b_j\}$ are constants to make the problem simple in the present paper and will study the variable coefficients case in the forthcoming paper. λ defined below is used to classify (4.1) into three types.

Definition 13. $\gamma = \{\gamma_j\}_{j=1}^{m-1}$ and $\lambda = \{\lambda_j\}_{j=1}^{2m-1}$ are defined as

$$\gamma_j = b_{2j} - \sum_{k=1}^{j-1} \bar{a}_{2(j-k)}\gamma_k, \quad 1 \le j \le m-1,$$

$$\begin{cases} \lambda_{2j} = 2\operatorname{Im} a_{2j} - 2\displaystyle\sum_{k=1}^{j-1} \operatorname{Im} \bar{b}_{2(j-k)}\gamma_k, \quad 1 \le j \le m-1, \\ \lambda_{2j-1} = 2\operatorname{Im} a_{2j-1} + 2\displaystyle\sum_{k=1}^{j-1} \operatorname{Im} \bar{b}_{2(j-k)-1}\gamma_k, \quad 1 \le j \le m. \end{cases}$$

Our main result is the following.

Theorem 4.1.
(Dispersive type, L^2 well-posedness) Assume that $\lambda_j = 0$ for $1 \le j \le 2m-1$. Then, for any $\varphi \in L^2(\mathcal{M})$, there exists a unique solution $u(t,x)$ of (4.1)–(4.2) such that $u(t,x) \in C((-\infty,\infty); L^2(\mathcal{M}))$.
(Parabolic type) Assume that there exists $j^* \in \mathbb{N}$ such that $\lambda_j = 0$ for $1 \le j < 2j^*$ and $\lambda_{2j^*} > 0$ (resp. $\lambda_{2j^*} < 0$). Then, for any $\varphi \in L^2(\mathcal{M})$, there exist a unique solution $u(t,x)$ of (4.1)–(4.2) on $[0,\infty)$ (resp. $(-\infty,0]$) such that $u(t,x) \in C([0,\infty); L^2(\mathcal{M})) \cap C^\infty((0,\infty)\times\mathcal{M})$ (resp. $C((-\infty,0]; L^2(\mathcal{M})) \cap C^\infty((-\infty,0)\times \mathcal{M})$). For any $\varphi \in L^2(\mathcal{M}) \backslash C^\infty(\mathcal{M})$ and $\delta > 0$, no solution u of (4.1)–(4.2) exists on $(-\delta,0]$ (resp. $[0,\delta)$) such that $u(t,x) \in C((-\delta,0]; L^2(\mathcal{M}))$ (resp. $C([0,\delta); L^2(\mathcal{M}))$).
(Twisted parabolic type) Assume that there exists $j^* \in \mathbb{N}$ such that $\lambda_j = 0$ for $1 \le j < 2j^*-1$ and $\lambda_{2j^*-1} > 0$ (resp. $\lambda_{2j^*-1} < 0$). Let $\varphi \in L^2(\mathcal{M})$ satisfy $P^+\varphi \notin H^{1/2}(\mathcal{M})$. Then, for any $\delta > 0$, there exist no solution $u(t,x)$ of (4.1)–(4.2) on $[-\delta,0]$ (resp. $[0,\delta]$) satisfying $u \in C([-\delta,0]; L^2(\mathcal{M}))$ (resp. $u \in C([0,\delta]; L^2(\mathcal{M}))$). Moreover, the same result as above holds even if we replace P^+, $[-\delta,0]$ and $[0,\delta]$ with P^-, $[0,\delta]$ and $[-\delta,0]$, respectively.

Remark 4.1. Put $v(t) = \langle \partial_x \rangle^{-s} u(t)$. Then v satisfies (4.1) if u is the solution of (4.1) and $u(t) \in L^2(\mathcal{M}) \Leftrightarrow v(t) \in H^s(\mathcal{M})$. Therefore, Theorem 4.1 holds even if we replace $L^2(\mathcal{M})$ with $H^s(\mathcal{M})$ and $H^{1/2}(\mathcal{M})$ with $H^{s+1/2}(\mathcal{M})$ for any $s \in \mathbb{R}$.

Remark 4.2. In "Dispersive type", the persistence of regularity holds on both $(-\infty,0]$ and $[0,\infty)$. In "Parabolic type", the equations have the parabolic smoothing effect on either $(-\infty,0]$ or $[0,\infty)$, which means the persistence of regularity breaks down on either $[0,\infty)$ or $(-\infty,0]$. Non-existence results in "Parabolic type" and "Twisted parabolic type" is by the breakdown of the persistence of regularity.

Since the coefficients are constants, by the Fourier transform, (4.1) can be rewritten into the following:

$$D_t \widehat{u}(t,\xi) = \xi^{2m}\widehat{u}(t,\xi) + \sum_{j=1}^{2m} \left(a_j \xi^{2m-j}\widehat{u}(t,\xi) + b_j \xi^{2m-j}\overline{\widehat{u}}(t,-\xi)\right). \tag{4.4}$$

Here, we fix $\xi \in \mathbb{R}$ (or $\mathbb{Z}$) and put

$$U_\xi(t) = \begin{pmatrix} \widehat{u}(t,\xi) \\ \overline{\widehat{u}}(t,-\xi) \end{pmatrix}, \quad X_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_j = \begin{pmatrix} a_j & b_j \\ (-1)^{j+1}\overline{b_j} & (-1)^{j+1}\overline{a_j} \end{pmatrix},$$

for $1 \le j \le 2m$. Then, by (4.4) with (4.2), it follows that

$$D_t U_\xi(t) = \sum_{j=0}^{2m} \xi^{2m-j} X_j U_\xi(t), \quad U_\xi(0) = {}^t(\widehat{\varphi}(\xi), \overline{\widehat{\varphi}}(-\xi)), \tag{4.5}$$

which is a system of linear ordinary differential equations. We can easily obtain the unique solution

$$U_\xi(t) = U_\xi(0) \exp it \sum_{j=0}^{2m} \xi^{2m-j} X_j \tag{4.6}$$

on $t \in (-\infty, \infty)$ for each $\xi \in \mathbb{R}$ (or $\mathbb{T}$). Therefore, our interest in Theorem 4.1 is essentially on the regularity of the solution. Here, note that $X_j X_k = X_k X_j$ holds for any $0 \le j, k \le 2m$ if and only if $b_j = 0$ holds for any $1 \le j \le 2m$. If we assume this assumption, (4.5) is not a system but a single ordinary differential equation and

$$\widehat{u}(t,\xi) = \widehat{\varphi}(\xi) \exp it \Big(\xi^{2m} + \sum_{j=1}^{2m} \xi^{2m-j} a_j \Big) \tag{4.7}$$

for each $\xi \in \mathbb{R}$ (or $\mathbb{Z}$). Since $\gamma_j = 0$ and $\lambda_j = 2 \mathrm{Im}\, a_j$, it follows that

$$|\widehat{u}(t,\xi)| = |\widehat{\varphi}(\xi)| \prod_{j=1}^{2m} \exp \frac{-t\xi^{2m-j}\lambda_j}{2},$$

by which we obtain Theorem 4.1 easily. On the other hand, it seems difficult to obtain Theorem 4.1 by (4.6) for general $\{b_j\}$ since $X_j X_k \ne X_k X_j$ for some j, k. To avoid this difficulty, we employ the energy estimate. Propositions 4.2 and 4.3 are main estimates in this paper. The first term of the left-hand side of (4.8) is the main part of the energy. The second term is the correction term. For "Dispersive type", the third and the fourth terms vanish. Thus, we easily obtain the L^2 a priori estimate. For "Parabolic type", the third term includes $\lambda_{2j^*} \| |\partial_x|^{m-j^*} u\|^2$. The parabolic smoothing is caused by the term. For "Twisted parabolic type", the fourth term includes $\lambda_{2j^*-1} \langle D_x^{2(m-j)+1} u, u\rangle$. We want to show the parabolic smoothing by making use of the term. However, the sign of the term is not definite. That is unfavorable in our argument. Therefore, we compute the energy inequalities of P^+u and P^-u instead of u and obtain Proposition 4.4. Note that the sign of all terms except the correction terms in (4.11) and (4.12) are definite. Though (4.11) is the energy inequality for $\|P^+u\|$, it includes $\lambda_j^- \| |\partial_x|^{m-j/2} P^- u\|^2$. This is because (4.1) is essentially coupled system of P^+u and P^-u as (4.4). The term $\lambda_j^- \| |\partial_x|^{m-j/2} P^- u\|^2$ cannot be estimated by $\|u\|$. This is the main difficulty in the proof of "Twisted parabolic type" in Theorem 4.1. We analyse a property of $\{\lambda_j^-\}$ and use an additional correction term F_k^- to eliminate a bad effect caused by

$\lambda_j^- \||\partial_x|^{m-j/2}P^- u\|^2$ and obtain (4.9) (see also (4.10)). This is the key idea in this paper.

4.2. **The energy estimates.** Our purpose in this subsection is to show Propositions 4.2 and 4.3. Proposition 4.2 below is used to show "Dispersive type" and "Parabolic type" in Theorem 4.1.

Proposition 4.2. *Let u satisfy* (4.1). *Then, there exists $C = C(\{a_j\}, \{b_j\}) > 0$ such that*

$$\begin{aligned}\Bigg| \frac{d}{dt}\left(\|u\|^2 + \sum_{j=1}^{m-1} \operatorname{Re} \gamma_j \langle D_x^{-2j} P_{\neq 0}\bar{u}, P_{\neq 0} u\rangle \right) \\ + \sum_{j=1}^{m-1} \lambda_{2j} \||\partial_x|^{m-j} u\|^2 + \sum_{j=1}^{m} \lambda_{2j-1} \langle D_x^{2(m-j)+1} u, u\rangle \Bigg| \le C\|u\|^2. \end{aligned} \tag{4.8}$$

Definition 14. $\alpha = \{\alpha_j\}_{j=1}^{2m-1}, \lambda^+ = \{\lambda_j^+\}_{j=1}^{2m-1}, \lambda^- = \{\lambda_j^-\}_{j=1}^{2m-1}$ are defined as

$$\alpha_j = b_j - \frac{1}{2}\sum_{k=1}^{j-1} (1 + (-1)^{j-k}) \bar{a}_{j-k}\alpha_k, \quad 1 \le j \le 2m-1,$$

$$\lambda_j^+ = 2 \operatorname{Im} a_j + \sum_{k=1}^{j-1} (-1)^{j-k+1} \operatorname{Im} \bar{b}_{j-k}\alpha_k, \quad 1 \le j \le 2m-1,$$

$$\lambda_j^- = -\sum_{k=1}^{j-1} \operatorname{Im} \bar{b}_{j-k}\alpha_k, \quad 1 \le j \le 2m-1,$$

and $\beta^+ = \{\beta_k^+\}_{j=1}^{2(m-j^*-1)}$ and $\beta^- = \{\beta_k^-\}_{j=1}^{2(m-j^*-1)}$ for $1 \le j^* \le m-2$ are defined as

$$\lambda_{2j^*+k+1}^- = \sum_{j=1}^{k} (-1)^{k-j} \lambda_{2j^*+k-j-1}^+ \beta_k^+, \quad 1 \le k \le 2(m-j^*-1),$$

$$\lambda_{2j^*+k+1}^- = \sum_{j=1}^{k} (-1)^{k-1} \lambda_{2j^*+k-j-1}^+ \beta_k^-, \quad 1 \le k \le 2(m-j^*-1).$$

Remark 4.3. It is easy to see that $\gamma_j = \alpha_{2j}$ for $1 \le j \le m-1$. Then, we have

$$\lambda_{2j} = \lambda_{2j}^+ + \lambda_{2j}^-, \quad \lambda_{2k-1} = \lambda_{2k-1}^+ - \lambda_{2k-1}^-$$

for $1 \le j \le m-1$ and $1 \le k \le m$.

Proposition 4.3 below is used to show "Twisted parabolic type" in Theorem 4.1.

Proposition 4.3. *Let u satisfy* (4.1). *Assume that there exists $j^* \in \mathbb{N}$ such that $\lambda_j = 0$ for $1 \le j \le 2(j^*-1)$ and $\lambda_{2j^*-1} \neq 0$. Put*

$$F_k^-(u) = \||\partial_x|^{-(k+2)/2} P^- u\|^2 + \sum_{j=1}^{2m-1} \operatorname{Re} \alpha_j \langle D_x^{-j} |\partial_x|^{-k-2} \overline{P^+ u}, P^- u\rangle,$$

$$F_k^+(u) = \||\partial_x|^{-(k+2)/2} P^+ u\|^2 + \sum_{j=1}^{2m-1} \operatorname{Re} \alpha_j \langle D_x^{-j} |\partial_x|^{-k-2} \overline{P^- u}, P^+ u\rangle.$$

Then, there exists $C = C(\{a_j\}, \{b_j\}) > 0$ such that

$$\begin{aligned} &\Bigg| \frac{d}{dt}\Bigg(\|P^+u\|^2 + \sum_{j=1}^{2m-1} \operatorname{Re} \alpha_j \langle D_x^{-j} \overline{P^- u}, P^+ u\rangle + \sum_{k=1}^{2(m-j^*-1)} \beta_k^+ F_k^-(u) \Bigg) \\ &+ \lambda_{2j^*-1}^+ \||\partial_x|^{m-j^*+1/2} P^+ u\|^2 \Bigg| \le C\|u\|^2 + C\||\partial_x|^{m-j^*} P^+ u\|^2, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &\Bigg| \frac{d}{dt}\Bigg(\|P^-u\|^2 + \sum_{j=1}^{2m-1} \operatorname{Re} \alpha_j \langle D_x^{-j} \overline{P^+ u}, P^- u\rangle + \sum_{k=1}^{2(m-j^*-1)} \beta_k^- F_k^+(u) \Bigg) \\ &- \lambda_{2j^*-1}^+ \||\partial_x|^{m-j^*+1/2} P^- u\|^2 \Bigg| \le C\|u\|^2 + C\||\partial_x|^{m-j^*} P^- u\|^2. \end{aligned} \tag{4.10}$$

To prove Propositions 4.2 and 4.3, we use the following lemma.

Lemma 4.4. *Let u satisfy* (4.1). *Then, there exists $C = C(\{a_j\}, \{b_j\}) > 0$ such that*

$$\begin{aligned} &\Bigg| \frac{d}{dt}\Bigg(\|P^+u\|^2 + \sum_{j=1}^{2m-1} \operatorname{Re} \alpha_j \langle D_x^{-j} \overline{P^- u}, P^+ u\rangle \Bigg) \\ &+ \sum_{j=1}^{2m-1} (\lambda_j^+ \||\partial_x|^{m-j/2} P^+ u\|^2 + \lambda_j^- \||\partial_x|^{m-j/2} P^- u\|^2) \Bigg| \le C\|u\|^2 \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} &\Bigg| \frac{d}{dt}\Bigg(\|P^-u\|^2 + \sum_{j=1}^{2m-1} \operatorname{Re} \alpha_j \langle D_x^{-j} \overline{P^+ u}, P^- u\rangle \Bigg) \\ &+ \sum_{j=1}^{2m-1} (-1)^j (\lambda_j^+ \||\partial_x|^{m-j/2} P^- u\|^2 + \lambda_j^- \||\partial_x|^{m-j/2} P^+ u\|^2) \Bigg| \le C\|u\|^2. \end{aligned} \tag{4.12}$$

Proof of Lemma 4.4. First, we show (4.11). For simplicity, we set $v^+ := P^+u$ and $v^- := P^-u$. Note that $P^+\bar{u} = \overline{P^-u} = \overline{v^-}$ and $P^-\bar{u} = \overline{P^+u} = \overline{v^+}$. Then, v^+ and v^-

satisfy

$$D_t v^+ = D_x^{2m} v^+ + \sum_{k=1}^{2m} (a_k D_x^{2m-k} v^+ + b_k D_x^{2m-k} \overline{v^-}) \tag{4.13}$$

and

$$D_t \overline{v^-} = -D_x^{2m} \overline{v^-} - \sum_{k=1}^{2m} (-1)^k (\bar{a}_k D_x^{2m-k} \overline{v^-} + \bar{b}_k D_x^{2m-k} v^+). \tag{4.14}$$

By (4.13), we have

$$\begin{aligned}
\frac{d}{dt}\|v^+\|^2 &= 2\,\mathrm{Re}\,\langle \partial_t v^+, v^+\rangle = -2\,\mathrm{Im}\,\langle D_t v^+, v^+\rangle \\
&= -2\sum_{j=1}^{2m} (\mathrm{Im}\, a_j \langle D_x^{2m-j} v^+, v^+\rangle + \mathrm{Im}\, b_j \langle D_x^{2m-j} \overline{v^-}, v^+\rangle) \\
&= -2\sum_{j=1}^{2m} (\mathrm{Im}\, a_j \| |\partial_x|^{m-j/2} P^+ u\|^2 + \mathrm{Im}\, b_j \langle D_x^{2m-j} \overline{v^-}, v^+\rangle).
\end{aligned}$$

Here, we consider the time derivative of correction terms to cancel out the second term. Fix $1 \le j \le 2m-1$. We see from (4.13) and (4.14) that

$$\begin{aligned}
&\frac{d}{dt}\,\mathrm{Re}\,\alpha_j \langle D_x^{-j} \overline{P^- u}, v^+\rangle = -\,\mathrm{Im}\,\alpha_j \langle D_x^{-j} D_t \overline{v^-}, v^+\rangle + \mathrm{Im}\,\alpha_j \langle D_x^{-j} \overline{v^-}, D_t v^+\rangle \\
&= \mathrm{Im}\,\alpha_j \langle D_x^{-j} (D_x^{2m} \overline{v^-}), v^+\rangle + \mathrm{Im}\,\alpha_j \langle D_x^{-j} \overline{v^-}, D_x^{2m} v^+\rangle \\
&\quad + \sum_{k=1}^{2m} ((-1)^k\,\mathrm{Im}\,\alpha_j \bar{a}_k \langle D_x^{2m-k-j} \overline{v^-}, v^+\rangle + (-1)^k\,\mathrm{Im}\,\alpha_j \bar{b}_k \langle D_x^{2m-k-j} v^+, v^+\rangle \\
&\quad + \mathrm{Im}\,\alpha_j \bar{a}_k \langle D_x^{2m-k-j} \overline{v^-}, v^+\rangle + \mathrm{Im}\,\alpha_j \bar{b}_k \langle D_x^{2m-k-j} \overline{v^-}, \overline{v^-}\rangle) \\
&=: A_1^j + B_1^j + \sum_{k=1}^{2m} (A_{2,k}^j + A_{3,k}^j + B_{2,k}^j + B_{3,k}^j).
\end{aligned}$$

Observe that

$$\begin{aligned}
A_1^j + B_1^j &= 2\,\mathrm{Im}\,\alpha_j \langle D_x^{2m-j} \overline{v^-}, v^+\rangle, \\
A_{2,k}^j + B_{2,k}^j &= (1 + (-1)^k)\,\mathrm{Im}\,\alpha_j \bar{a}_k \langle D_x^{2m-k-j} \overline{v^-}, v^+\rangle, \\
A_{3,k}^j &= (-1)^k\,\mathrm{Im}\,\alpha_j \bar{b}_k \| |\partial_x|^{m-(k+j)/2} P^+ u\|^2, \\
B_{3,k}^j &= \mathrm{Im}\,\alpha_j \bar{b}_k \| |\partial_x|^{m-(k+j)/2} P^- u\|^2.
\end{aligned}$$

We collect coefficients of derivative losses with rearranging the summation order. Note that for any sequences $c_{j,k}$, it holds that

$$\sum_{j=1}^{p} \sum_{k=1}^{p-j} c_{j,k} = \sum_{j=1}^{p-1} \sum_{k=0}^{p-1-j} c_{j,k+1} = \sum_{j=1}^{p-1} \sum_{k=1}^{j} c_{k,j-k+1}. \tag{4.15}$$

It is easy to see that

$$\left|\sum_{j=1}^{2m}\sum_{k=2m-j}^{2m}(A_{2,k}^j+A_{3,k}^j+B_{2,k}^j+B_{3,k}^j)\right|\lesssim\|u\|^2.$$

Then, by (4.15), we have

$$\begin{aligned}\sum_{j=1}^{2m-1}\sum_{k=1}^{2m-1-j}(A_{2,k}^j+B_{2,k}^j)&=\sum_{j=1}^{2(m-1)}\sum_{k=1}^{j}(A_{2,j-k+1}^k+B_{2,j-k+1}^k)\\&=\sum_{j=1}^{2(m-1)}\sum_{k=1}^{j}(1+(-1)^{j-k+1})\operatorname{Im}\alpha_k\bar{a}_{j-k+1}\langle D_x^{2m-1-j}\overline{v^-},v^+\rangle.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\sum_{j=1}^{2m-1}\sum_{k=1}^{2m-1-j}A_{3,k}^j&=\sum_{j=1}^{2(m-1)}\sum_{k=1}^{j}(-1)^{j-k+1}\operatorname{Im}\alpha_k\bar{b}_{j-k+1}\||\partial_x|^{m-(j+1)/2}P^+u\|^2,\\\sum_{j=1}^{2m-1}\sum_{k=1}^{2m-1-j}B_{3,k}^j&=\sum_{j=1}^{2(m-1)}\sum_{k=1}^{j}\operatorname{Im}\alpha_k\bar{b}_{j-k+1}\||\partial_x|^{m-(j+1)/2}P^-u\|^2.\end{aligned}$$

This concludes the proof of (4.11). For the proof of (4.12), we set $v^+:=P^-u$ and $v^-:=P^+u$. Then, they satisfy (4.13) and (4.14). Therefore, the exactly same proof works. □

Now we are ready to prove Proposition 4.2. Though we can prove it directly without using Lemma 4.4, we give the proof of it by the lemma.

Proof of Proposition 4.2. Note that $\langle P^+f,P^-g\rangle=\langle P^-f,P^+g\rangle=0$ for any functions f,g. This implies that $\langle P_{\neq0}\bar{f},P_{\neq0}g\rangle=\langle\overline{P^-f},P^+g\rangle+\langle\overline{P^+f},P^-g\rangle$. So, collecting (4.11) and (4.12), we obtain

$$\begin{aligned}&\left|\frac{d}{dt}\left(\|P_{\neq0}u\|^2+\sum_{j=1}^{m-1}\operatorname{Re}\alpha_{2j}\langle D_x^{-2j}P_{\neq0}\bar{u},P_{\neq0}u\rangle\right)\right.\\&\left.+\sum_{j=1}^{m-1}\lambda_{2j}\||\partial_x|^{m-j}P_{\neq0}u\|^2+\sum_{j=1}^{m}\lambda_{2j-1}\langle D_x^{2(m-j)+1}P_{\neq0}u,P_{\neq0}u\rangle\right|\le C\|u\|^2.\end{aligned}$$

We also note that $\gamma_k=\alpha_{2k}$. Finally, it is easy to see that

$$\left|\frac{d}{dt}\|P_0u\|^2+\sum_{j=1}^{m-1}\lambda_{2j}\||\partial_x|^{m-j}P_0u\|^2+\sum_{j=1}^{m}\lambda_{2j-1}\langle D_x^{2(m-j)+1}P_0u,P_0u\rangle\right|\le C\|u\|^2.$$

Therefore, we have (4.8). □

The terms $\lambda_j^- \| |\partial_x|^{m-j/2} P^- u \|^2$ with $1 \le j \le 2j^* - 1$ in Lemma 4.4 are unfavorable in our argument to prove Proposition 4.3. Therefore, we analyse the coefficients λ^- below.

Lemma 4.5. *It holds that*

$$\lambda_{j+1}^- = -\frac{1}{2}\sum_{l=1}^{j-1}(1+(-1)^l)(\operatorname{Re} a_l)\lambda_{j+1-l}^- + \frac{1}{2}\sum_{l=1}^{j-1}\sum_{k=1}^{j-l}(1+(-1)^l)(\operatorname{Im} a_l)\operatorname{Re}\bar{b}_{j-l-k+1}\alpha_k$$

for $1 \le j \le 2(m-1)$.

Proof. By the definitions of λ_j^- and α_k, we have

$$\lambda_{j+1}^- = -\sum_{l=1}^{j}\operatorname{Im} b_l \bar{b}_{j-l+1} + \frac{1}{2}\sum_{l=1}^{j}\sum_{k=1}^{l-1}(1+(-1)^{l-k})\operatorname{Im}\bar{b}_{j-l+1}\bar{a}_{l-k}\alpha_k =: A + B.$$

It is easy to see that $A = 0$. Observe that

$$\sum_{l=1}^{p}\sum_{k=1}^{l-1} c_{l-k} d_l e_k = \sum_{l=1}^{p-1}\sum_{k=1}^{p-l} c_l d_{l+k} e_k \tag{4.16}$$

for any sequences $\{c_j\}, \{d_j\}$ and $\{e_j\}$. This implies that

$$\begin{aligned} B &= \frac{1}{2}\sum_{l=1}^{j-1}\sum_{k=1}^{j-l}(1+(-1)^l)\operatorname{Im}\bar{b}_{j-l-k+1}\bar{a}_l\alpha_k \\ &= \frac{1}{2}\sum_{l=1}^{j-1}\sum_{k=1}^{j-l}(1+(-1)^l)((\operatorname{Re} a_l)\operatorname{Im}\bar{b}_{j-l-k+1}\alpha_k - (\operatorname{Im} a_l)\operatorname{Re}\bar{b}_{j-l-k+1}\alpha_k). \end{aligned}$$

Here we used the fact that $\operatorname{Im} cd = (\operatorname{Re} c)\operatorname{Im} d + (\operatorname{Im} c)\operatorname{Re} d$ for any $c, d \in \mathbb{C}$. This completes the proof. □

Lemma 4.6. *Assume that there exists* $j^* \in \mathbb{N}$ *such that* $\lambda_{2j} = 0$ *for* $1 \le j \le j^*$. *Then, it holds that* $\operatorname{Im} a_{2j} = \lambda_{2j}^+ = 0$ *for* $1 \le j \le j^*$ *and* $\lambda_j^- = 0$ *for* $1 \le j \le 2j^* + 3$.

Proof. The proof proceeds by the induction on j. We prove the following: assume that there exists $j^* \in \mathbb{N}$ such that $\lambda_{2j} = 0$ for $1 \le j \le j^*$. Then it holds that

$$\operatorname{Im} a_{2j} = \lambda_{2j}^+ = \lambda_{2j}^- = \lambda_{2j^*+1}^- = \lambda_{2j^*+2}^- = \lambda_{2j^*+3}^- = \sum_{k=1}^{j-1}\operatorname{Im}\bar{b}_{2(j-k)}\alpha_{2k} = 0$$

for $1 \le j \le j^*$. It is easy to see that the claim above with $j^* = 1$ follows. Assume that $\lambda_{2j^*+2} = 0$. By the hypothesis, it holds that $\lambda_{2j^*+n}^- = 0$ for $1 \le n \le 3$. Then we have $\lambda^+_{2j^*+2} = 0$. Now, we claim that $\operatorname{Im} a_{2j^*+2} = 0$. Since $\lambda_{2j^*+2} = \lambda^+_{2j^*+2} + \lambda^-_{2j^*+2}$, we have

$$\operatorname{Im} a_{2j^*+2} - \sum_{l=1}^{j^*} \operatorname{Im} \bar{b}_{2(j^*-l+1)}\alpha_{2l} = 0.$$

Since $\gamma_l = \alpha_{2l}$, we see from the definition of γ that

$$\sum_{l=1}^{j^*} \operatorname{Im} \bar{b}_{2(j^*-l+1)}\gamma_l = \sum_{l=1}^{j^*} \operatorname{Im} \bar{b}_{2(j^*-l+1)} b_{2l} - \sum_{l=1}^{j^*}\sum_{k=1}^{l-1} \operatorname{Im} \bar{b}_{2(j^*-l+1)}\bar{a}_{2(l-k)}\gamma_k =: A + B.$$

It is easy to see that $A = 0$. We have

$$\begin{aligned} B &= -\sum_{l=1}^{j^*-1}\sum_{k=1}^{j^*-l} \operatorname{Im} \bar{b}_{2(j^*-l-k+1)}\bar{a}_{2l}\gamma_k \\ &= -\sum_{l=1}^{j^*-1}(\operatorname{Re} a_{2l})\sum_{k=1}^{j^*-l} \operatorname{Im} \bar{b}_{2(j^*-l-k+1)}\gamma_k + \sum_{l=1}^{j^*-1}(\operatorname{Im} a_{2l})\sum_{k=1}^{j^*-l} \operatorname{Im} \bar{b}_{2(j^*-l-k+1)}\gamma_k = 0 \end{aligned}$$

by (4.16) and the hypothesis, which shows that $\operatorname{Im} a_{2j^*+2} = 0$. Using Lemma 4.5 again, we obtain $\lambda^-_{2j^*+4} = \lambda^-_{2j^*+5} = 0$. Then we obtain $\lambda^+_{2j^*+2} = 0$, which completes the proof. □

Remark 4.4. From the proof of the above lemma, we also see that

$$\lambda_{2j^*+2} = 2 \operatorname{Im} a_{2j^*+2}, \quad \lambda_{2j^*+4} = 2 \operatorname{Im} a_{2j^*+4}$$

when $\lambda_{2j} = 0$ for $1 \le j \le j^*$.

Now, we prove Proposition 4.3.

Proof of Proposition 4.3. We give only the proof of (4.9) since we can show (4.10) in the same manner. When $j^* = 1$, we see from the definition that $\lambda^-_n = 0$ for $n = 1, 2, 3$. When $j^* \ge 2$, Lemma 4.6 implies that $\lambda^+_j = \lambda^-_j = 0$ for $1 \le j \le 2(j^*-1)$. Moreover, it holds that $\lambda^-_{2j^*+n} = 0$ for $n = -1, 0, 1$ and $\lambda^+_{2j^*-1} \ne 0$. By (4.11), we have

$$\begin{aligned} &\Bigg| \frac{d}{dt}\left(\|P^+u\|^2 + \sum_{j=1}^{2m-1} \operatorname{Re} \alpha_j \langle D_x^{-j}\overline{P^-u}, P^+u\rangle \right) + \sum_{j=2j^*+2}^{2m-1} \lambda^-_j \||\partial_x|^{m-j/2}P^-u\|^2 \\ &+ \lambda^+_{2j^*-1}\||\partial_x|^{m-j^*+1/2}P^+u\|^2 \Bigg| \lesssim \|u\|^2 + \||\partial_x|^{m-j^*}P^+u\|^2 \end{aligned}$$

Thus, we only need to show

$$\begin{aligned}&\left|\frac{d}{dt}\sum_{k=1}^{2(m-j^*-1)}\beta_k^+F_k^-(u)-\sum_{j=2j^*+2}^{2m-1}\lambda_j^-\||\partial_x|^{m-j/2}P^-u\|^2\right|\\&\lesssim\|u\|^2+\||\partial_x|^{m-j^*}P^+u\|^2.\end{aligned}\tag{4.17}$$

Put $v=|\partial_x|^{-(k+2)/2}P_{\neq 0}u$. Since v satisfies (4.1), by (4.12), we have

$$\begin{aligned}&\left|\frac{d}{dt}\left(\|P^-v\|^2+\sum_{j=1}^{2m-1}\operatorname{Re}\alpha_j\langle D_x^{-j}\overline{P^+v},P^-v\rangle\right)+\sum_{j=2j^*-1}^{2m-1}(-1)^j\lambda_j^+\||\partial_x|^{m-j/2}P^-v\|^2\right|\\&\lesssim\|v\|^2+\||\partial_x|^{m-j^*}P^+v\|^2.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\left|\sum_{k=1}^{2(m-j^*-1)}\beta_k^+\Big(\frac{d}{dt}F_k^-(u)+\sum_{j=2j^*-1}^{2m-k-3}(-1)^j\lambda_j^+\||\partial_x|^{m-(j+k+2)/2}P^-u\|^2\Big)\right|\\&\lesssim\|u\|^2+\||\partial_x|^{m-j^*}P^+u\|^2.\end{aligned}$$

By (4.15), we have

$$\begin{aligned}&\sum_{k=1}^{2(m-j^*-1)}\sum_{j=2j^*-1}^{2m-k-3}(-1)^j\beta_k^+\lambda_j^+\||\partial_x|^{m-(j+k+2)/2}P^-u\|^2\\&=\sum_{k=1}^{2(m-j^*-1)}\sum_{j=1}^{k}(-1)^{k-j+1}\beta_j^+\lambda_{2j^*+k-j-1}^+\||\partial_x|^{m-j^*-(k+1)/2}P^-u\|^2.\end{aligned}$$

Therefore, by the definition of β_k^+, we conclude (4.17). □

4.3. **Proof of main theorem.** In this subsection, we show Theorem 4.1.

Definition 15. For $f\in L^2(\mathcal{M})$ and $N>0$, we define

$$E(f;N):=\|f\|^2+N\|\partial_x^{-m}P_{\neq 0}f\|^2+\sum_{j=1}^{m-1}\operatorname{Re}\gamma_j\langle D_x^{-2j}P_{\neq 0}\bar{f},P_{\neq 0}f\rangle.$$

We choose N sufficiently large so that Lemma 4.7 holds. If there is no confusion, we write $E(f):=E(f;N)$.

Lemma 4.7. *Let $N>0$ be sufficiently large. Then, for any $f\in L^2(\mathcal{M})$ it holds that*

$$\frac{1}{2}E(f)\le\|f\|^2+N\|\partial_x^{-m}P_{\neq 0}f\|^2\le 2E(f).$$

Proof. The Gagliardo-Nirenberg inequality and the Young inequality show that

$$\sum_{j=1}^{m-1} |\operatorname{Re}\gamma_j \langle D_x^{-2j} P_{\neq 0}\bar{f}, P_{\neq 0} f\rangle| \leq \frac{1}{2}\|f\|^2 + C\|\partial_x^{-m} P_{\neq 0} f\|^2. \tag{4.18}$$

So, it suffices to choose $N = 2C$. □

We prove the first part of Theorem 4.1.

Proof of "Dispersive type" in Theorem 4.1. We consider our problem only on $[0, \infty)$ since the result on $(-\infty, 0]$ follows from the same argument. Let $T > 0$, which can be arbitrary large. We first show the a priori estimate $\sup_{t\in[0,T]} \|u(t)\| \leq C\|\varphi\|$. We assume that u satisfies (4.1) and (4.2). Then, it is easy to see that $\frac{d}{dt}\|\partial_x^{-m} P_{\neq 0} u\|^2 \leq 2|\langle D_t \partial_x^{-2m} P_{\neq 0} u, P_{\neq 0} u\rangle| \leq C\|u\|^2$. This together with (4.8), Lemma 4.7 and $\lambda_j = 0$ for $1 \leq j \leq 2m-1$ implies that $\frac{d}{dt}E(u(t)) \leq CE(u(t))$ on $[0, T]$. Thus, by the Gronwall inequality and Lemma 4.7, we obtain the a priori estimate. Next, we show the existence. Let $\varphi_n = \mathcal{F}^{-1}\chi(|\xi| < n)\mathcal{F}\varphi$ for $n \in \mathbb{N}$. Then, we have the solution u_n of (4.1) with $u_n(0) = \varphi_n$ by (4.6). Moreover, $u_n \in C([0, T]; L^2(\mathcal{M}))$ since $|\sum_{j=0}^{2m} \xi^{2m-j} X_j| \leq C(\{a_j\}, \{b_j\}, n)$ for $|\xi| < n$. Since $\{\varphi_n\}$ is a Cauchy sequence in $L^2(\mathcal{M})$, by the a priori estimate, we conclude $\{u_n\}$ is also a Cauchy sequence in $C([0, T]; L^2(\mathcal{M}))$. Thus, we obtain the solution $u \in C([0, T]; L^2(\mathcal{M}))$ of (4.1)–(4.2) as the limit of u_n. Finally, the uniqueness easily follows from the a priori estimate. □

Proof of "Parabolic type" in Theorem 4.1. We use the argument from the proof of Theorem 1.2 in [33]. We consider only the case $\lambda_{2j^*} > 0$ since the other case follows from the same argument. Let $T > 0$, which can be arbitrary large. By the Gagliardo-Nirenberg inequality and the Young inequality, we have

$$\left| \sum_{j=j_*+1}^{m-1} \lambda_{2j} \||\partial_x|^{m-j} u\|^2 + \sum_{j=j_*+1}^{m} \lambda_{2j-1} \langle D_x^{2(m-j)+1} u, u\rangle \right| \leq \frac{1}{2}\lambda_{2j^*} \||\partial_x|^{m-j^*} u\|^2 + C\|u\|^2.$$

Recall that $\lambda_j = 0$ for $1 \leq j \leq 2j^* - 1$. Therefore, in the same manner as the proof of "Dispersive type", we obtain the a priori estimate:

$$\sup_{t\in[0,T]} \left(\|u(t)\|^2 + \frac{\lambda_{2j^*}}{2} \int_0^t \||\partial_x|^{m-j^*} u(\tau)\|^2 d\tau \right) \leq C\|\varphi\|^2.$$

It then follows that we have the unique existence of the solution $u \in C([0, T]; L^2(\mathcal{M})) \cap L^2([0, T]; H^{m-j^*}(\mathcal{M}))$, which implies that $u(t) \in H^{m-j^*}(\mathcal{M})$ for a.e. $t \in [0, T]$. Let $0 < \varepsilon < T$. Then there exists $t_0 \in (0, \varepsilon/2)$ such that $u(t_0) \in H^{m-j^*}(\mathcal{M})$. Since $\langle\partial_x\rangle^{m-j^*} u$ satisfies (4.1)–(4.2) with initial data $\varphi := \langle\partial_x\rangle^{m-j^*} u(t_0) \in L^2(\mathcal{M})$, applying the same argument as above, we conclude $\langle\partial_x\rangle^{m-j^*} u \in C([t_0, T]; L^2(\mathcal{M})) \cap$

$L^2([t_0,T];H^{m-j^*}(\mathcal{M}))$. That is, $u\in C([t_0,T];H^{m-j^*}(\mathcal{M}))\cap L^2([t_0,T];H^{2(m-j^*)}(\mathcal{M}))$. We can choose t_1 so that $\varepsilon/2<t_1<\varepsilon/2+\varepsilon/4$ and $u(t_1)\in H^{2(m-j^*)}(\mathcal{M})$. Again, applying the same argument as above with the initial data $\varphi:=\langle\partial_x\rangle^{2(m-j^*)}u(t_0)\in L^2(\mathcal{M})$, we conclude $u\in C([t_1,T];H^{2(m-j^*)}(\mathcal{M}))\cap L^2([t_1,T];H^{3(m-j^*)}(\mathcal{M}))$. By repeating this process, we conclude $u\in C([\varepsilon,T];H^{k(m-j^*)}(\mathcal{M}))$ for any $k\in\mathbb{N}$, which implies $u\in C^\ell([\varepsilon,T];H^{k(m-j^*)-2m\ell}(\mathcal{M}))$ for any $k,\ell\in\mathbb{N}$ by (4.1). By the Sobolev embedding, we obtain $u\in C^\infty([\varepsilon,T]\times\mathcal{M})$. Since we can take $\varepsilon>0$ arbitrary small and $T>0$ arbitrary large, we conclude $u\in C^\infty((0,\infty)\times\mathcal{M})$. Finally, we show the nonexistence result by contradiction. Assume that there exists a solution $u\in C((-\delta,0];L^2(\mathcal{M}))$ of (4.1)–(4.2) with $\varphi\in L^2(\mathcal{M})\setminus C^\infty(\mathcal{M})$. We take t_0 such that $-\delta<t_0<0$. Then, as we proved above, we have $u\in C^\infty((t_0,0]\times\mathcal{M})$, which contradicts to the assumption $\varphi=u(0)\notin C^\infty(\mathcal{M})$. □

The following proposition is the main tool to show the result for "Twisted parabolic type" in Theorem 4.1.

Proposition 4.8 (the twisted parabolic smoothing). *Let $u\in C([t_0,t_1];L^2(\mathcal{M}))$ satisfy* (4.1). *Assume that there exists $j^*\in\mathbb{N}$ such that $\lambda_j=0$ for $1\le j<2j^*-1$ and $\lambda_{2j^*-1}>0$ (resp. <0). Then, it follows that*

$$P^+u\ (\textit{resp.}\ P^-u)\in C((t_0,t_1];H^{1/2}(\mathcal{M}))\quad(\textit{forward smoothing}),\tag{4.19}$$

$$P^-u\ (\textit{resp.}\ P^+u)\in C([t_0,t_1);H^{1/2}(\mathcal{M}))\quad(\textit{backward smoothing}).\tag{4.20}$$

In particular, it holds that $u\in C^\infty((t_0,t_1)\times\mathcal{M})$.

Proof. We consider only the case $\lambda^+_{2j^*-1}>0$ since the same proof works for the case $\lambda^+_{2j^*-1}<0$. For simplicity, set

$$G^+(u):=\sum_{j=1}^{2m-1}\operatorname{Re}\alpha_j\langle D_x^{-j}\overline{P^-u},P^+u\rangle+\sum_{k=1}^{2(m-j^*-1)}\beta_k^+F_k^-(u),$$

where F_k^- is defined in Proposition 4.3 and $\{\alpha_j\}$ and $\{\beta_k\}$ are defined in Definiton 14. Set $M:=\sup_{t\in[t_0,t_1]}\|u(t)\|$. Note that $\sup_{t\in[t_0,t_1]}(|G^+(u(t))|+|\,G^+(|\partial_x|^{1/2}u(t))|)\le CM$ and $G^+(|\partial_x|^{1/2}u(t))$ is continuous on $[t_0,t_1]$ by the presence of D_x^{-j} in the definition of $G^+(u)$ above. By the Gagliardo-Nirenberg inequality and the Young inequality, we have

$$\||\partial_x|^{m-j^*}Qu\|^2\le\delta\||\partial_x|^{m-j^*+1/2}Qu\|^2+C\delta^{-1}\|u\|^2$$

for $\delta > 0$, $Q = P^+$ or P^-. Take $\delta > 0$ sufficiently small. Then, this together with (4.9) and (4.10) yields

$$\lambda^+_{2j^*-1} \int_{t_0}^{t_1} \||\partial_x|^{m-j^*+1/2} Q u(\tau)\|^2 d\tau \le C(M)(1 + |t_1 - t_0|),$$

for $Q = P^+$ or P^-. By the interpolation, we also have

$$\begin{aligned} &\int_{t_0}^{t_1} \||\partial_x|^s u(\tau)\|^2 d\tau \\ &= \int_{t_0}^{t_1} (\||\partial_x|^s P^- u(\tau)\|^2 + \||\partial_x|^s P_0 u(\tau)\|^2 + \||\partial_x|^s P^+ u(\tau)\|^2) d\tau \\ &\le C(M, \lambda^+_{2j^*-1})(1 + |t_1 - t_0|) \end{aligned} \tag{4.21}$$

for $0 \le s \le m + j^* - 1/2$. It then follows that $\||\partial_x|^{m-j^*+1/2} u(t)\| < \infty$ for a.e. $t \in [t_0, t_1]$. Then, for any $\varepsilon > 0$ there exists $t_* \in (t_0, t_0 + \varepsilon)$ such that $\||\partial_x|^{m-j^*+1/2} u(t_*)\| < \infty$. Note that (4.9) holds even if we replace u with $|\partial_x|^{1/2} u$ since $|\partial_x|^{1/2} u$ satisfies (4.1). Thus,

$$\begin{aligned} &\left| \frac{d}{dt} \big(\||\partial_x|^{1/2} P^+ u\|^2 + G^+(|\partial_x|^{1/2} u) \big) + \lambda^+_{2j^*-1} \||\partial_x|^{m-j^*+1} P^+ u\|^2 \right| \\ &\le C \||\partial_x|^{1/2} u\|^2 + C \||\partial_x|^{m-j^*+1/2} P^+ u\|^2, \end{aligned} \tag{4.22}$$

By the Gagliardo-Nirenberg inequality and the Young inequality, we have

$$\||\partial_x|^{m-j^*+1/2} P^+ u\|^2 \le \delta \||\partial_x|^{m-j^*+1} P^+ u\|^2 + C\delta^{-1} \||\partial_x|^{1/2} u\|^2$$

for $\delta > 0$. Taking $\delta > 0$ sufficiently small and integrating (4.22) on $[t_*, t] (\subset [t_0, t_1])$ with (4.21), we obtain

$$\begin{aligned} &\||\partial_x|^{1/2} P^+ u(t)\|^2 + \frac{\lambda^+_{2j^*-1}}{2} \int_{t_*}^{t} \||\partial_x|^{m-j^*+1} P^+ u(\tau)\|^2 d\tau \\ &\le C(M, \lambda^+_{2j^*-1}, |t_1 - t_0|) + \||\partial_x|^{1/2} P^+ u(t_*)\|^2 < \infty \end{aligned} \tag{4.23}$$

since $u(t_*) \in H^{m-j^*+1/2}(\mathcal{M})$. Therefore, by (4.22) again, it follows that for any $t_* \le t' \le t \le t_1$

$$\begin{aligned}
&\Big| \||\partial_x|^{1/2} P^+ u(t)\|^2 - \||\partial_x|^{1/2} P^+ u(t')\|^2 \Big| \\
&\le \left| \Big[|\partial_x|^{1/2} P^+ u(\tau)\|^2 + G^+(|\partial_x|^{1/2} u) \Big]_{\tau=t'}^{\tau=t} + \lambda^+_{2j^*-1} \int_{t'}^{t} \||\partial_x|^{m-j^*+1} P^+ u(\tau)\|^2 d\tau \right| \\
&\quad + \left| \Big[G^+(|\partial_x|^{1/2} u) \Big]_{\tau=t'}^{\tau=t} \right| + \lambda^+_{2j^*-1} \int_{t'}^{t} \||\partial_x|^{m-j^*+1} P^+ u(\tau)\|^2 d\tau \\
&\le C \int_{t'}^{t} \||\partial_x|^{1/2} u(\tau)\|^2 d\tau + C \int_{t'}^{t} \||\partial_x|^{m-j^*+1/2} P^+ u(\tau)\|^2 d\tau \\
&\quad + \lambda^+_{2j^*-1} \int_{t'}^{t} \||\partial_x|^{m-j^*+1} P^+ u(\tau)\|^2 d\tau + \left| \Big[G^+(|\partial_x|^{1/2} u) \Big]_{\tau=t'}^{\tau=t} \right|.
\end{aligned}$$

(4.21), (4.23) and the dominated convergence theorem imply that the right-hand side goes to 0 as $|t - t'| \to 0$, which shows that $\||\partial_x|^{1/2} P^+ u(t)\|$ is continuous on $[t_*, t_1]$. The fact $P^+ u \in C([t_0, t_1]; L^2(\mathcal{M}))$ with $P^+ u \in L^\infty([t_*, t_1]; H^{1/2}(\mathcal{M}))$ yields $P^+ u \in C_w([t_*, t_1]; H^{1/2}(\mathcal{M}))$. Combining the continuity of $\||\partial_x|^{1/2} P^+ u(t)\|$ and the weak continuity of $P^+ u(t)$ in $H^{1/2}(\mathcal{M})$, we obtain $P^+ u \in C([t_*, t_1]; H^{1/2}(\mathcal{M}))$. Since we can take $\varepsilon > 0$ arbitrary small, we get $P^+ u \in C((t_0, t_1]; H^{1/2}(\mathcal{M}))$. We also obtain $P^- u \in C([t_0, t_1); H^{1/2}(\mathcal{M}))$ in the same manner. Therefore, $u = P^- u + P_0 u + P^+ u \in C((t_0, t_1); H^{1/2}(\mathcal{M}))$. By repeating this process, we also obtain $u \in C((t_0, t_1); H^{k/2}(\mathcal{M}))$ for any $k \in \mathbb{N}$, which yields $u \in C^\infty((t_0, t_1) \times \mathcal{M}))$ since u satisfies (4.1). □

Proof of "Twisted parabolic type" in Theorem 4.1. We use the argument from the proof of Theorem 1.2 in [32]. We consider only the case $\lambda_{2j^*-1} > 0$ since the case $\lambda_{2j^*-1} < 0$ follows from the same argument. Let $\varphi \in L^2(\mathcal{M})$ satisfy $P^+\varphi \notin H^{1/2}(\mathcal{M})$. We prove Theorem 4.1 by contradiction. We assume that there exists $u \in C([-\delta, 0]; L^2(\mathcal{M}))$ satisfying (4.1)–(4.2) on $[-\delta, 0]$. By Proposition 4.8, we have $P^+ u \in C((-\delta, 0]; H^{1/2}(\mathcal{M}))$, which contradicts to $P^+\varphi \notin H^{1/2}(\mathcal{M})$. This proof works even if we replace P^+ and $[-\delta, 0]$ with P^- and $[0, \delta]$, respectively. □

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