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COMPUTATIONAL ELECTROMAGNETISM AND GAUGES

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Abstract

In three dimensional electromagnetic field calculations, the magnetic vector potential is the most orthodox and popular solution variable. However, there exists the problem of how to select the most suitable gauge for the problem considered, since the electromagnetic field is invariant under the gauge transformation. In this paper, formulations implicitly using the Lorentz gauge and the Coulomb gauge magnetic vector potential are presented and numerically tested for three dimensional eddy current problems. In addition, two applications of the formulations to practical problems are presented.

1. Introduction

The magnetic vector potential associated with the electric scalar potential is the most orthodox and popular solution variable for three dimensional electromagnetic field calculations.[1)-13)] The electric field intensity and the magnetic flux density, which are the fundamental quantities of electromagnetic field, are expressed in terms of the magnetic vector potential and the electric scalar potential which reduce the degree of freedom of solution variables by two. Moreover, the potentials enable us to discriminate the electric field due to magnetic induction from the electric field due to electric charge, and the numerical calculation becomes transparent.[14)] However, when the vector potential is applied to a three dimensional electromagnetic field problem, it is necessary to impose the gauge condition on the vector potential in order to obtain a unique solution. Furthermore, since the electric field is expressed as the sum of the vector potential and the scalar potential, the gauge transformation can be applied and consequently there can exist many formulations. Therefore, it is very important to select the most suitable gauge for the problem.

In this paper, formulations are given for typical gauges: the Lorentz gauge and the Coulomb gauge. They are numerically tested for three dimensional eddy current problems, and applications to practical problems are presented.

2. Helmholtz's Theorem for a Vector Field

When vectors are placed at all points of a space, the space is called a vector field. Assume that the vector field is simply connected, unbounded, and smooth everywhere, and vanishes at infinity. By Helmholtz's theorem, the vector field $v(r)$ is written as[15]

$$v(r) = \mathrm{rot}\ u(r) - \mathrm{grad}\ \mathrm{f}(r),$$

$$u(r) = \int_\Omega \frac{\mathrm{rot}\ v(r')}{4\pi|r-r'|}\mathrm{d}r', \quad \mathrm{f}(r) = \int_\Omega \frac{\mathrm{div}\ v(r')}{4\pi|r-r'|}\mathrm{d}r' \tag{2.1}$$

For self-containedness, the theorem is proved as follows. Consider the equations:

$$\mathrm{rot\ rot}\ u(r) = \mathrm{rot}\ v(r)\,, \quad \mathrm{div}\ u(r) = 0 \quad \text{in } \Omega\,. \tag{2.2}$$

Equation (2.2) has a unique solution: Since rot rot = grad div $-\ \nabla^2$, it follows from (2.2)

$$\nabla^2 u(r) = -\mathrm{rot}\ v(r)\,, \quad \text{therefore}\,, \quad u(r) = \int_\Omega \frac{\mathrm{rot}\ v(r')}{4\pi|r-r'|}\mathrm{d}r'\,.$$

From rot $[v(r) - \mathrm{rot}\ u(r)] = 0$ and the simply- connectedness of Ω follows

$$v(r) = \mathrm{rot}\ u(r) - \mathrm{grad}\ \mathrm{f}(r)\,, \quad \text{therefore}\,, \quad \mathrm{f}(r) = \int_\Omega \frac{\mathrm{div}\ v(r')}{4\pi|r-r'|}\mathrm{d}r'$$

Q.E.D. From the Helmholtz theorem, a vector field can be expressed by a vector potential and a scalar potential, and in order to determine the vector field uniquely it is necessary to specify its rotation and divergence.

3. Electromagnetic Field Equations in Free Space

Electromagnetic field equations in free space are given by the following Maxwell equations[3] and constitutive relations:

$$\mathrm{rot}\ E + \partial B/\partial \mathrm{t} = 0 \qquad (3.1), \qquad \mathrm{div}\ B = 0\,, \tag{3.2}$$

$$\mathrm{rot}\ H - \partial D/\partial \mathrm{t} = j_0 \qquad (3.3), \qquad \mathrm{div}\ D = \rho_0\,, \tag{3.4}$$

$$B = \mu_0 H\,, \quad D = \varepsilon_0 E \tag{3.5}$$

where $B, H, E, D, j_0, \rho_0, \mu_0$, and ε_0 are the magnetic flux density, the magnetic field intensity, the electric field intensity, the electric flux density, the current density, the electric charge density, the permeability of free space, and the permittivity of free space, respectively. From (3.2) and the Helmholtz theory, B is expressed as

$$B = \text{rot}\, A \tag{3.6}$$

where A is called the magnetic vector potential. From (3.1) and (3.6) follows

$$E = -\, \partial A/\partial t - \text{grad}\, \phi \tag{3.7}$$

where ϕ is called the electric scalar potential. Using rot rot = grad div $- \nabla^2$, the remaining equations (3.3) and (3.4) are written in terms of A and ϕ as

$$\nabla^2 A - \mu_0 \varepsilon_0 \partial^2 A/\partial t^2 - \text{grad}\,(\text{div}\, A + \mu_0 \varepsilon_0 \partial\phi/\partial t) = -\, \mu_0 j_0\,, \tag{3.8}$$

$$\nabla^2 \phi + \text{div}\, \partial A/\partial t = -\, \rho_0/\varepsilon_0\,. \tag{3.9}$$

The vector A is not determined completely by the magnetic field B. Since, for any scalar function ξ, rot grad $\xi = 0$, we can add to A the gradient of an arbitrary function ξ. According to (3.7), however, we have to replace ϕ by $\phi + \partial\xi/\partial t$ if we replace A by $A -$ grad ξ, in order that E should not be changed. This freedom in the choice of the potentials can be used to simplify the field equations (3.8) and (3.9). If A_0 and ϕ_0 represent certain possible values of A and ϕ, we determine ξ from the equation

$$\nabla^2 \xi - \mu_0 \varepsilon_0 \partial^2 \xi/\partial t^2 = \text{div}\, A_0 + \mu_0 \varepsilon_0 \partial\phi_0/\partial t \tag{3.10}$$

If we now put $\quad A = A_0 - \text{grad}\, \xi\,,$

$$\phi = \phi_0 + \partial\xi/\partial t\,,$$

we obtain

$$\text{div}\, A + \mu_0 \varepsilon_0 \partial\phi/\partial t = 0\,. \tag{3.11}$$

(3.11) represents a relation between the potentials and is called the Lorentz relation. The field equations (3.8) and (3.9) then become simply

$$\nabla^2 A - \mu_0 \varepsilon_0 \partial^2 A/\partial t^2 = -\, \mu_0 j_0\,, \tag{3.12}$$

$$\nabla^2 \phi - \mu_0 \varepsilon_0 \partial^2 \phi/\partial t^2 = -\, \rho_0/\varepsilon_0\,. \tag{3.13}$$

A and ϕ satisfy, therefore, the inhomogeneous wave equation. They are coupled by the Lorentz condition (3.11) only.

The different possible choice one can make for A and ϕ, leaving E and B unchanged, are called gauges, and the invariance of E and B under these transformations is called gauge invariance. In particular, the class of gauges satisfying (3.11) will be called the Lorentz gauge. Another important gauge which will be called the Coulomb gauge is determined by

$$\text{div}\, A = 0 \tag{3.14}$$

with the field equations (from (3.8) and (3.9)):

$$\nabla^2 A - \mu_0 \varepsilon_0 \partial^2 A/\partial t^2 - \mu_0 \varepsilon_0 \mathrm{grad}\ \partial\phi/\partial t = -\mu_0 j_0\ , \tag{3.15}$$

$$\nabla^2 \phi = -\rho_0/\varepsilon_0\ . \tag{3.16}$$

(3.16) is the Poisson equation. The scalar potential is then determined from the charges as if the latter were at rest. (Hence the name Coulomb gauge.)

4. Electromagnetic Field Equations in a Material

We assume the following constitutive relations in a material.

$$B = \mu H\ , \quad D = \varepsilon E\ , \quad j = \sigma E \tag{4.1}$$

where μ, ε, and σ are the permeability, the permittivity, and the conductivity of the material, respectively. μ, ε, and σ are all constant in the material. The field equations in the material are written in terms of the potentials as

$$\nabla^2 A - \mu\sigma\partial A/\partial t - \mu\varepsilon\partial^2 A/\partial t^2 - \mathrm{grad}(\mathrm{div}\ \mathrm{A} + \mu\sigma\phi + \mu\varepsilon\partial\phi/\partial t)$$

$$= -\mu j_0\ , \tag{4.2}$$

$$\nabla^2 \phi + \mathrm{div}\ \partial A/\partial t = -\rho_0/\varepsilon\ . \tag{4.3}$$

We define the Lorentz gauge in the material as

$$\mathrm{div}\ \mathrm{A} + \mu\sigma\phi + \mu\varepsilon\partial\phi/\partial t = 0 \tag{4.4}$$

with the field equations (from (4.2) and (4.3)):

$$\nabla^2 A - \mu\sigma\partial A/\partial t - \mu\varepsilon\partial^2 A/\partial t^2 = -\mu j_0\ , \tag{4.5}$$

$$\nabla^2 \phi - \mu\sigma\partial\phi/\partial t - \mu\varepsilon\partial^2\phi/\partial t^2 = -\rho_0/\varepsilon\ . \tag{4.6}$$

The Coulomb gauge in the material is the same as (3.14), with the field equations (from (4.2) and (4.3)):

$$\nabla^2 A - \mu\sigma\partial A/\partial t - \mu\varepsilon\partial^2 A/\partial t^2 - \mathrm{grad}(\mu\sigma\phi + \mu\varepsilon\partial\phi/\partial t)$$

$$= -\mu j_0\ , \tag{4.7}$$

$$\nabla^2 \phi = -\rho_0/\varepsilon\ . \tag{4.8}$$

5. The Lorentz Gauge Formulation for 3D Eddy Current Problems

In usual eddy current problems in which low frequencies are used, the following relation is satisfied:

$$|\partial D/\partial \mathrm{t}| \ll |\sigma E| \,. \tag{5.1}$$

We, therefore, set $\partial D/\partial \mathrm{t} = 0$ in the following discussion. The field equations then become, in the material (from (4.5) and (4.6)):

$$\nabla^2 A - \mu\sigma\partial A/\partial \mathrm{t} = -\,\mu j_0 \,, \tag{5.2}$$

$$\nabla^2 \phi - \mu\sigma\partial\phi/\partial \mathrm{t} = 0 \,,^{(*)} \tag{5.3}$$

and in free space (from (3.12) and (3.13)):

$$\nabla^2 A = -\,\mu_0 j_0 \,, \tag{5.4}$$

$$\nabla^2 \phi = 0 \,.^{(**)} \tag{5.5}$$

(*) Since rot $H = j = \sigma E$, we obtain $\mathrm{div}(\sigma E) = \sigma \mathrm{div}\, E = 0$, therefore, div $E = 0$. Then, $\rho =$ div $D = \varepsilon$div $E = 0$. There, therefore, exists no charge inside the material. However, in general, there exist charges on the surface of the material.
(**) In a usual eddy current problem, there exists no charge in free space.

The interface conditions between the material and free space are given as follows.

$$A_1 = A_2 \,, \quad (5.6); \qquad \phi_1 = \phi_2 \,, \tag{5.7}$$

$$1/\mu \text{ rot } A_1 \times n = 1/\mu_0 \text{ rot } A_2 \times n, \tag{5.8}$$

$$\partial(A_1 \,.\, n)/\partial \mathrm{n} + \mu\sigma\phi_1 = \partial(A_2 \,.\, n)/\partial \mathrm{n} \,, \tag{5.9}$$

$$n \,.\, (\partial A_1/\partial \mathrm{t} + \mathrm{grad}\ \phi_1) = 0 \,,^{(***)} \tag{5.10}$$

$$\int_\Gamma n \,.\, (\partial A_2/\partial \mathrm{t} + \mathrm{grad}\ \phi_2)\, \mathrm{d}\Gamma = 0^{\,(\#)} \tag{5.10$'$}$$

where suffix 1 and 2 denote the material and free space, respectively, and Γ and n are the interface and a unit normal vector at the interface pointing into free space, respectively. By Stokes's theorem, (5.6) involves $B_1 \,.\, n = B_2 \,.\, n$, and (3.7), (5.6) and (5.7) give $E_1 \times n = E_2 \times n$. (5.9) expresses the continuity of the gauge across the interface. (5.10) indicates that eddy currents do not run out from the material, and (5.10)′ indicates that the total of surface charges is zero.
(***) Since div $E = 0$ in the material (see (*)), $\int_\Gamma E_1 \,.\, n\, \mathrm{d}\Gamma = 0$. Therefore, (5.10) is independent except one point on the interface, leaving ϕ_1 within a constant.
(#) (5.10) and (5.10)′ make up the complete interface conditions for the flux of the scalar potential. (If the scalar potential is given at least at one point on the interface, e.g. from geometrical symmetry, (5.10)′ is not necessary.)

The boundary conditions at infinity are given as

$$A_2(r) = \mathrm{O}(1/|r|^2)\,, \quad (5.11)^{(\#\#)}\,; \quad \phi_2(r) = \mathrm{O}(1/|r|^2) \tag{5.12}$$

as $r \to \infty$. ("O" means "order of".)

(##) The vector potential A due to the current density j is

$$A(r) = \int_{\Omega_0} \mu j(r')/4\pi|r - r'|\,\mathrm{d}\Omega_0$$

where Ω_0 is the current region. Since Ω_0 is bounded in the eddy current problem and the current paths are closed loop, A becomes like a dipole field at infinity, and is expressed as (5.11). The scalar potential ϕ due to the charge density ρ is

$$\phi(r) = \int_{\Omega_0} \rho(r')/4\pi\varepsilon|r - r'|\,\mathrm{d}\Omega_0\,.$$

As far as (5.10)′ is satisfied, the situation is the same as A, and ϕ is expressed as (5.12).

The system of equations (5.2) through (5.12) yields a unique solution, and satisfies the Lorentz gauge:

$$\mathrm{div}\,A_1 + \mu\sigma\phi_1 = 0\,, \quad \text{and} \quad \mathrm{div}\,A_2 = 0 \tag{5.13}$$

(note that $\sigma_0 = 0$ in free space). First, we prove that the formulation given above satisfies the Lorentz gauge (5.13). From (5.2) follows

$$\begin{aligned}&\mathrm{rot}\,(1/\mu\,\mathrm{rot}\,A_1) - 1/\mu\,\mathrm{grad}\,(\mathrm{div}\,A_1 + \mu\sigma\phi_1)\\ &+ \sigma\,(\partial A_1/\partial \mathrm{t} + \mathrm{grad}\,\phi_1) = j_0\,.\end{aligned} \tag{5.14}$$

(5.3) can be written as

$$\mathrm{div}\,[\sigma\,(\partial A_1/\partial \mathrm{t} + \mathrm{grad}\,\phi_1)] - \sigma\,\partial(\mathrm{div}\,A_1 + \mu\sigma\phi_1)/\,\partial \mathrm{t} = 0\,. \tag{5.15}$$

Taking the divergence of (5.14) and subtracting (5.15) gives (note that $\mathrm{div}\,j_0 = 0$),

$$\nabla^2 \mathrm{p}_1 - \mu\sigma\partial \mathrm{p}_1/\partial \mathrm{t} = 0 \qquad \text{in } \Omega_1 \tag{5.16}$$

where $\mathrm{p}_1 = \mathrm{div}\,A_1 + \mu\sigma\phi_1$. From (5.4) follows

$$\mathrm{rot}\,(1/\mu_0\,\mathrm{rot}\,A_2) - 1/\mu_0\,\mathrm{grad}\,\mathrm{div}\,A_2 = j_0\,. \tag{5.17}$$

Taking the divergence of (5.17) gives

$$\nabla^2 \mathrm{p}_2 = 0 \qquad \text{in } \Omega_2 \tag{5.18}$$

where $p_2 = \text{div}\, A_2$. (5.6) and (5.9) give

$$p_1 = p_2 \qquad \text{on } \Gamma . \tag{5.19}$$

(Ω_1, Ω_2, and Γ denote the material region, free space region, and the interface between the material and free space region, respectively.) (5.8), (5.10), (5.14) and (5.17) give

$$1/\mu\ \partial p_1/\partial n = 1/\mu_0\ \partial p_2/\partial n \qquad \text{on } \Gamma \tag{5.20}$$

since, by Stokes's theorem, (5.8) means n . rot $(1/\mu$ rot $A_l) = n$. rot $(1/\mu_0$ rot $A_2)$ on Γ. From (5.16) follows

$$\begin{aligned} 0 &= \int_{\Omega_1} p_1[\text{div}\,(1/\mu\ \text{grad}\ p_1) - \sigma\ \partial p_1/\partial t]\ d\Omega_1 \\ &= -\int_{\Omega_1} [1/\mu\ (\text{grad}\ p_1)^2 + \sigma/2\ \partial p_1^2/\partial t]\ d\Omega_1 \\ &\quad + \int_{\Gamma} p_1 \,.\, 1/\mu \,.\, \partial p_1/\partial n\ d\Gamma . \end{aligned} \tag{5.21}$$

From (5.18) and (5.11) follows

$$\begin{aligned} 0 &= \int_{\Omega_2} p_2[\text{div}\,(1/\mu_0\ \text{grad}\ p_2)]\ d\Omega_2 \\ &= -\int_{\Omega_2} 1/\mu_0\ (\text{grad}\ p_2)^2\ d\Omega_2 - \int_{\Gamma} p_2 \,.\, 1/\mu_0\ \partial p_2/\partial n\ d\Gamma . \end{aligned} \tag{5.22}$$

We obtain from (5.19), (5.20), (5.21) and (5.22)

$$\begin{aligned} &\int_{\Omega_1} [1/\mu\ (\text{grad}\ p_1)^2 + \sigma/2\ \partial p_1^2/\partial t]\ d\Omega_1 \\ &+ \int_{\Omega_2} 1/\mu_0\ (\text{grad}\ p_2)^2\ d\Omega_2 = 0 . \end{aligned} \tag{5.23}$$

Since the potentials are zero at $t = 0$ (t denotes time),

$$\partial p_1^2/\partial t \geqq 0 \qquad \text{at } t = 0 . \tag{5.24}$$

From (5.11), (5.23) and (5.24) gives

$$p_1 = 0 \quad \text{in } \Omega_1 , \quad p_2 = 0 \quad \text{in } \Omega_2 , \quad \text{and} \quad \partial p_1^2/\partial t = 0 \quad \text{in } \Omega_1 ,$$

$$\text{therefore} , \quad \partial p_1/\partial t = 0 \quad \text{in } \Omega_1 , \quad \text{at } t = 0 . \tag{5.25}$$

From (5.25) we obtain (5.13). Q.E.D.

Second, we prove the uniqueness of the solution. Suppose we have two solutions and let (A^*, ϕ^*) be the difference between them. From (5.13), (5.14) and (5.17) follows

$$\mathrm{rot}\,(1/\mu\ \mathrm{rot}\ A_I^*) + \sigma\,(\partial A_I^*/\partial \mathrm{t} + \mathrm{grad}\ \phi_1^*) = 0\,, \tag{5.26}$$

$$\mathrm{rot}\,(1/\mu_0\ \mathrm{rot}\ A_2^*) = 0\,. \tag{5.27}$$

Now let us define A' as

$$A_I'(r, \mathrm{t}) = A_I(r, \mathrm{t}) + \mathrm{grad}\ \psi_1(r, \mathrm{t}) \tag{5.28}$$

where $\psi_1(r, \mathrm{t}) = \int_0^{\mathrm{t}} \phi_1(r, \mathrm{t}')\,\mathrm{dt}'$. From (5.26) and (5.28)

$$\mathrm{rot}\,(1/\mu\ \mathrm{rot}\ A_I'^*) + \sigma\,\partial A_I'^*/\partial \mathrm{t} = 0\,. \tag{5.26$'$}$$

We obtain from (5.26)′ and (5.27)

$$\begin{aligned} 0 &= \int_{\Omega_1} A_I'^* \,.\, [\mathrm{rot}\,(1/\mu\ \mathrm{rot}\ A_I'^*) + \sigma\,\partial A_I'^*/\partial \mathrm{t}]\ \mathrm{d}\Omega_1 \\ &= \int_{\Omega_1} [1/\mu\ (\mathrm{rot}\ A_I'^*)^2 + \sigma/2\ \partial (A_I'^*)^2/\partial \mathrm{t}]\ \mathrm{d}\Omega_1 \\ &\quad - \int_\Gamma A_I'^* \,.\, (1/\mu\ \mathrm{rot}\ A_I'^* \times n_I)\ \mathrm{d}\Gamma\,, \end{aligned} \tag{5.29}$$

$$\begin{aligned} 0 &= \int_{\Omega_2} A_2^* \,.\, \mathrm{rot}\,(1/\mu_0\ \mathrm{rot}\ A_2^*)\ \mathrm{d}\Omega_2 \\ &= \int_{\Omega_2} 1/\mu_0\ (\mathrm{rot}\ A_2^*)^2\ \mathrm{d}\Omega_2 - \int_\Gamma A_2^* \,.\, (1/\mu_0\ \mathrm{rot}\ A_2^* \times n_2)\ \mathrm{d}\Gamma \\ &\quad - \int_{\Gamma^\infty} A_2^* \,.\, (1/\mu_0\ \mathrm{rot}\ A_2^* \times n_2)\ \mathrm{d}\Gamma^\infty \end{aligned} \tag{5.30}$$

where Γ^∞ is the boundary at infinity. Since

$$\begin{aligned} &\int_\Gamma \mathrm{grad}\ \psi_1^* \,.\, (H_I^* \times n_I)\ \mathrm{d}\Gamma = \int_\Gamma \mathrm{div}\ [\psi_1^*\ (H_I^* \times n_I)]\ \mathrm{d}\Gamma \\ &- \int_\Gamma \psi_1^*\ \mathrm{div}\ (H_I^* \times n_I)\ \mathrm{d}\Gamma = 0^{(\#\#\#)} \end{aligned} \tag{5.31}$$

where $H_I^* = 1/\mu\ \mathrm{rot}\ A_I^* = 1/\mu\ \mathrm{rot}\ A_I'^*$, and the boundary integral at infinity in (5.30) vanishes from (5.11), we obtain from (5.29) and (5.30) (and (5.6), (5.8), and (5.28))

$$\int_{\Omega_1} [1/\mu\,(\mathrm{rot}\,A_1'^*)^2 + \sigma/2\;\partial(A_1'^*)^2/\partial t]\,\mathrm{d}\Omega_1$$
$$+\int_{\Omega_2} 1/\mu_0\,(\mathrm{rot}\,A_2^*)^2\,\mathrm{d}\Omega_2 = 0\,. \tag{5.32}$$

(###) The first integral in the right hand side in (5.31) is zero since Γ is the closed surface of the material. The second integral is also zero as is shown below: We assume that Γ is a smooth surface. Then, at every point on Γ there exists a tangential plane, and we can define local Descartes coordinates with the z axis coinciding with the normal vector n. In this coordinates

$$\mathrm{div}\,(H_1^* \times n_1) = \partial \mathrm{H_1{}^*y}/\partial \mathrm{x} - \partial \mathrm{H_1{}^*x}/\partial \mathrm{y} = n_1\,.\,\mathrm{rot}\,H_1^*.$$

And, from (5.10) and (5.26) gives $n_1\,.\,\mathrm{rot}\,H_1^* = 0$.

Applying to (5.32) the same discussion as in the preceding section gives

$$A_1'^* = 0 \quad \text{in } \Omega_1\,, \qquad (5.33); \qquad \mathrm{rot}\,A_2^* = 0 \quad \text{in } \Omega_2\,, \tag{5.34}$$

for all $t \geqq 0$. From (5.13), (5.28), (5.33) and (5.34) gives

$$\nabla^2\psi_1^* - \mu\sigma\partial\psi_1^*/\partial t = 0 \quad \text{in } \Omega_1\,, \tag{5.35}$$
$$A_2^* = \mathrm{grad}\;\xi_2^*, \quad \text{and} \quad \nabla^2\xi_2^* = 0 \quad \text{in } \Omega_2\,. \tag{5.36}$$

From (5.6), (5.28), (5.33) and (5.36)

$$-\,\mathrm{grad}\;\psi_1^* = \mathrm{grad}\;\xi_2^* \quad \text{on } \Gamma\,. \tag{5.37}$$

From (5.37) follows

$$-\,\psi_1^*(r, t) = \xi_2^*(r, t) + c(t), \quad \text{and}$$
$$-\,\partial\psi_1^*(r, t)/\partial n = \partial\xi_2^*(r, t)/\partial n \quad \text{on } \Gamma\,, \tag{5.38}$$

and from (5.11) and (5.36)

$$\int_{\Gamma} \partial\xi_2^*/\partial n_2\,\mathrm{d}\Gamma = \int_{\Omega_2} \nabla^2\xi_2^*\,\mathrm{d}\Omega_2 = 0\,. \tag{5.39}$$

From (5.35), (5.36), (5.38) and (5.39) gives

$$\int_{\Omega_1} [(\mathrm{grad}\;\psi_1^*)^2 + \mu\sigma/2\;\partial(\psi_1^*)^2/\partial t]\,\mathrm{d}\Omega_1$$
$$+\int_{\Omega_2} (\mathrm{grad}\;\xi_2^*)^2\,\mathrm{d}\Omega_2 = 0\,. \tag{5.40}$$

Applying to (5.40) the same discussion as in the preceding section gives

$$\psi_1{}^* = 0 \quad \text{in } \Omega_1 , \quad \text{and} \quad \text{grad } \xi_2{}^* = 0 \quad \text{in } \Omega_2 \tag{5.41}$$

for all t $\geqq$ 0. From (5.28), (5.34), (5.36) and (5.41), we obtain the uniqueness of A_1, A_2 and ϕ_1. The uniqueness of ϕ_2 is easily derived from (5.5), (5.7), (5.12), and the uniqueness of ϕ_1. Q.E.D.

6. The Coulomb Gauge Formulation for 3D Eddy Current Problems

In this chapter, we assume, as in the preceding chapter, $\partial D/\partial \mathrm{t} = 0$. The field equations in the material are written as (from (4.7) and (4.8)):

$$\nabla^2 A - \mu\sigma\, \partial A/\partial \mathrm{t} - \mu\sigma \,\mathrm{grad}\, \phi = -\, \mu\, j_0 , \tag{6.1}$$

$$\nabla^2 \phi = 0 , \tag{6.2}$$

and in free space (from (3.12) and (3.13)):

$$\nabla^2 A = -\, \mu_0\, j_0 , \tag{6.3}$$

$$\nabla^2 \phi = 0 . \tag{6.4}$$

The interface conditions between the material and free space are given as follows:

$$A_1 = A_2 , \qquad (6.5); \qquad \phi_1 = \phi_2 , \tag{6.6}$$

$$1/\mu \,\mathrm{rot}\, A_1 \times n = 1/\mu_0 \,\mathrm{rot}\, A_2 \times n , \tag{6.7}$$

$$\partial(A_1 \,.\, n)/\partial \mathrm{n} = \partial(A_2 \,.\, n)/\partial \mathrm{n} , \tag{6.8}$$

$$n \,.\, (\partial A_1/\partial \mathrm{t} + \mathrm{grad}\, \phi_1) = 0 , \tag{6.9}$$

$$\int_\Gamma n \,.\, (\partial A_2/\partial \mathrm{t} + \mathrm{grad}\, \phi_2)\, \mathrm{d}\Gamma = 0 . \tag{6.10}$$

The boundary conditions at infinity are given as:

$$A_2\, (r) = \mathrm{O}(1/|r|^2) , \qquad (6.11); \quad \phi_2(r) = \mathrm{O}(1/|r|^2) \tag{6.12}$$

as $r \to \infty$.

Remark. See the comments (*) through (##) in Chapter 4. These are also applied to the above formulation.

Remark. The Coulomb gauge condition is not explicitly included in the system of equations (6.1) through (6.12) as well as the Lorentz gauge condition is not explicitly included in (5.2) through (5.12).

The system of equations (6.1) through (6.12) yields a unique solution, and satisfies the Coulomb gauge. This is proved in the same way as in Chapter 4.

7. The Boundary Integral Equations for the Lorentz Gauge and Coulomb Gauge Formulations

In this chapter, we consider a time-harmonic 3D eddy current problem:

$$\partial/\partial t = j\omega\,, \quad j = \sqrt{-1}\,, \quad \omega = 2\pi f \tag{7.1}$$

where f is the frequency. In the Lorentz gauge formulation, the field equations in the material are written as (from (5.2), (5.3) and (7.1)):

$$\nabla^2 A + k^2 A = -\,\mu\, j_0\,, \quad \text{and} \quad \nabla^2\phi + k^2\phi = 0 \tag{7.2}$$

where $k^2 = -\,j\omega\mu\sigma$. The Green function, or the fundamental solution to the Helmholtz equation (7.2) is written as:

$$G(r, r') = \exp\,(jk|r - r'|)/4\pi|r - r'|\,. \tag{7.3}$$

The boundary integral equation corresponding to the Helmholtz equation is expressed in Descartes coordinates as:

$$c\,\xi(r) = \int_\Omega u(r')\,.\,G(r, r')\,d\Omega' - \int_\Gamma \xi(r')\,.\,\partial G(r, r')/\partial n'\,d\Gamma'$$
$$+ \int_\Gamma \partial\xi(r')/\partial n'\,.\,G(r, r')\,d\Gamma' \tag{7.4}$$

where $c = 1$, $r\,\varepsilon\,\Omega$; $= 1/2$, $r\,\varepsilon\,\Gamma$; $u = \mu j_{0x}$, μj_{0y}, μj_{0z}, or ρ_0/ε; and $\xi = Ax$, Ay, Az, or ϕ. The field equations in free space are written as (from (5.4) and (5.5)):

$$\nabla^2 A = -\,\mu_0 j_0\,, \quad \text{and} \quad \nabla^2\phi = 0\,. \tag{7.5}$$

The Green function, or the fundamental solution to the Poisson (and the Laplace) equation (7.5) is written as:

$$G(r, r') = 1/4\pi|r - r'|\,. \tag{7.6}$$

The boundary integral equation corresponding to the Poisson equation is expressed in Descartes coordinates by the same form as (7.4).

In the Coulomb gauge formulation, the field equations in the material are written as (from (6.1) and (6.2)):

$$\nabla^2 A - k^2 A - \mu\sigma \operatorname{grad} \phi = -\mu j_0 , \tag{7.7}$$

$$\nabla^2 \phi = 0 . \tag{7.8}$$

(7.7) is inconvenient for the boundary integral formulation since it requires the volume integration of the unknown scalar potential. We, therefore, introduce the variable defined as:

$$A' = A + 1/j\omega \operatorname{grad} \phi \tag{7.9}$$

((7.9) is not a gauge transformation since ϕ is not changed). From (7.7), (7.8) and (7.9) follows

$$\nabla^2 A' + k^2 A' = -\mu j_0 . \tag{7.10}$$

The interface conditions change into:

$$A_1' - 1/j\omega \operatorname{grad} \phi_1 = A_2 , \tag{6.5)'$$

$$1/\mu \partial[(A_1' - 1/j\omega \operatorname{grad} \phi_1) \,.\, t_i]/\partial n$$

$$= 1/\mu_0 \, \partial(A_2 \,.\, t_i)/\partial n + (1/\mu - 1/\mu_0)\partial(A_2 \,.\, n)/\partial t_i ,$$

$$i = 1, 2 \tag{6.7)'$$

where t_1 and t_2 are independent tangential unit vectors,

$$\partial[(A_1' - 1/j\omega \operatorname{grad} \phi_1) \,.\, n]/\partial n = \partial(A_2 \,.\, n)/\partial n , \tag{6.8)'$$

$$n \,.\, A_1' = 0 . \tag{6.9)'$$

The remaining interface conditions and boundary conditions are the same as in Chapter 6. The boundary integral equation for (7.10) is the same as for (7.2), and that for (7.8) is the same as for (7.5). The field equations in free space are the same as (7.5).

8. Discretization of the Boundary Integral Equations

For numerical calculations, the boundary integral equations and interface conditions should be discretized. In this paper, the discretization is carried out by using the zero-order boundary elements. In a zero-order boundary element, the potential and the flux (the normal derivative) of the potential take respectively only one value. The node point of a zero-order element is located at the center of geometry. The integration is carried out by using the Gauss-Legendre formula[16] for i $\neq$ j and the analytical formula given below for i = j, where i and j denote the i-th and the j-th boundary element, respectively. The analytical formula as $r - r'$ tends to 0 is given as (we assume that the boundary elements are rectangular):

$$\int_{\Gamma_i} 1/4\pi|r - r'|\, d\Gamma_i' = 1/\pi\, [L_1 \log ((L_2 + \sqrt{(L_1^2 + L_2^2)})/L_1)$$
$$+ L_2 \log((L_1 + \sqrt{(L_1^2 + L_2^2)})/L_2)]\ , \quad \text{and} \tag{8.1}$$

$$\int_{\Gamma_i} \partial G(r, r')/\partial n'\, d\Gamma_1' = 0\ , \tag{8.2}$$
$$r\, \varepsilon\, \Gamma_i\ , \quad r'\, \varepsilon\, \Gamma_i$$

where $\Gamma_i = [-L_1 \leqq x \leqq L_1, -L_2 \leqq y \leqq L_2]$. (8.1) is derived in Appendix 1 and (8.2) follows from that $\partial/\partial n' = n'$. grad and n' is perpendicular to $r - r'$. We define the matrices G and H as:

$$G = [g_{ij}]\ , \quad \text{and} \quad H = [h_{ij}]\ , \quad i, j = 1, \cdots, N;$$
$$g_{ij} = \int_{\Gamma_j} G(r_i, r_j')\, d\Gamma_j'\ , \quad \text{and}$$
$$h_{ij} = 1/2\ \delta_{ij} + \int_{\Gamma_j} \partial G(r_i, r_j')/\partial n_j'\, d\Gamma_j' \tag{8.3}$$

where $\delta_{ij} = 1$ for $i = j$, $= 0$ for $i \neq j$, and N is the total number of the boundary elements. The coefficient matrix of the linear simultaneous equations corresponding to the boundary integral equations and interface conditions takes the form shown in Fig. 1 in which $qx = \partial Ax/\partial n$, $qy = \partial Ay/\partial n$, $qz = \partial Az/\partial n$, and $qv = \partial\phi/\partial n$.

Ax	Ay	Az	φ	q_1x	q_1y	q_1z	q_1v	q_2x	q_2y	q_2z	q_2v	
H_1				$-G_1$								
	H_1				$-G_1$							R1
		H_1				$-G_1$						
			H_1'				$-G_1'$					
H_2								$-G_2$				
	H_2								$-G_2$			R2
		H_2								$-G_2$		
			H_2'								$-G_2'$	
dd	dd	dd		d	d	d		d	d	d		R3
dd	dd	dd		d	d	d		d	d	d		
			d	d	d	d		d	d	d		R4
d d	d d	d d		d	d	d		d	d	d		R5

Fig. 1 The coefficient matrix.

The coefficient matrix is a 12N × 12N square matrix in which R1, R2, R3, R4 and R5 (see Fig. 1) correspond to the boundary integral equations in the material, the boundary integral equations in free space, eq. (5.8) (eq. (6.7)), eq. (5.9) (eq. (6.8)), and eqs. (5.10) and (5.10)′ (eqs. (6.9) and (6.10)), respectively. The coefficient matrix is made up of 12 × 12 (= 144) block matrices. The block matrix is a N × N square matrix, and the d-block matrix, the dd-block matrix and the blank block matrix denote the diagonal matrix, the bidiagonal-like matrix and the zero matrix, respectively. (In R5 in Fig. 1, the first N −1 rows correspond to eq. (5.10) (eq. (6.9)) and the Nth row corresponds to eq. (5.10)′ (eq. (6.10)).

9. Numerical Experiments

Numerical experiments for the Lorentz gauge and Coulomb gauge formulations were carried out for a test problem shown in Fig. 2. The problem is the three dimensional eddy current calculation of an aluminum sphere placed in a one-turn square coil. The coil and the equator of the sphere are in the same plane. The physical properties used are as follows:

$$\mu = \mu_0 = 4\pi \times 10^{-7}\ \mathrm{H/m}\,, \quad \sigma = 2.5 \times 10^{7}\ \mathrm{S/m}\,,$$

$$\mathrm{f} = 50 \text{ and } 100\ \mathrm{Hz}\,, \quad \mathrm{I}_0 = 1/(4\pi \times 10^{-7}) + \mathrm{j}0\ \mathrm{A}\,. \tag{9.1}$$

The computation was carried out by using the boundary integral equation method described in Chapter 7 and 8. The discretization of the surface of the sphere is shown in Fig. 3. The total number of boundary elements is 800, and the shape of the element is spherical quadrangle. The interpolation used is of zero-order.

The computed results are shown in Fig. 4 through Fig. 7 and Table 1 through Table 8. The correspondence between the figures and the tables are as follows: Fig. 4 — Table 1 and 2, Fig. 5 — Table 3 and 4, Fig. 6 — Table 5 and 6, and Fig. 7 — Table 7 and 8. As is seen from the figures, there are almost no discrepancies in the computed results by the Lorentz gauge and Coulomb gauge formulations. (The dotted lines completely overlap the solid lines in the figures.) For example, in Table 3 and 4, the value of eddy current at the point $\theta = 19\pi/40$ and $\phi = \pi/40$ on the surface is (θ: zenith angle and ϕ: azimuth angle):

$$\mathrm{J}_\phi = [0.3149746213 + \mathrm{j}1.460785998] \times 10^{9}\ \mathrm{A/m^2}$$

for the Lorentz gauge, and

$$\mathrm{J}_\phi = [0.3160025548 + \mathrm{j}1.460745861] \times 10^{9}\ \mathrm{A/m^2}$$

for the Coulomb gauge.

On the other hand, the computation time using an IBM-RS/6000-320H (11.7 MFLOPS) is:

383 sec for the Lorentz gauge, and

823 sec for the Coulomb gauge.

The computation time with the Coulomb gauge is about twice as large as with the Lorentz gauge. This is due to the calculation of the normal derivative of grad ϕ_1 by using the boundary integral equations. (See (6.8)′.)

It may be concluded from the numerical experiments that the Lorentz gauge is preferable for the formulation using the boundary integral equations.

The computer program in FORTRAN for the Lorentz gauge formulation is given in Appendix 2. (The length of the FORTRAN program for this test problem is: 820 statements for the Lorentz gauge, and 1,320 statements for the Coulomb gauge.)

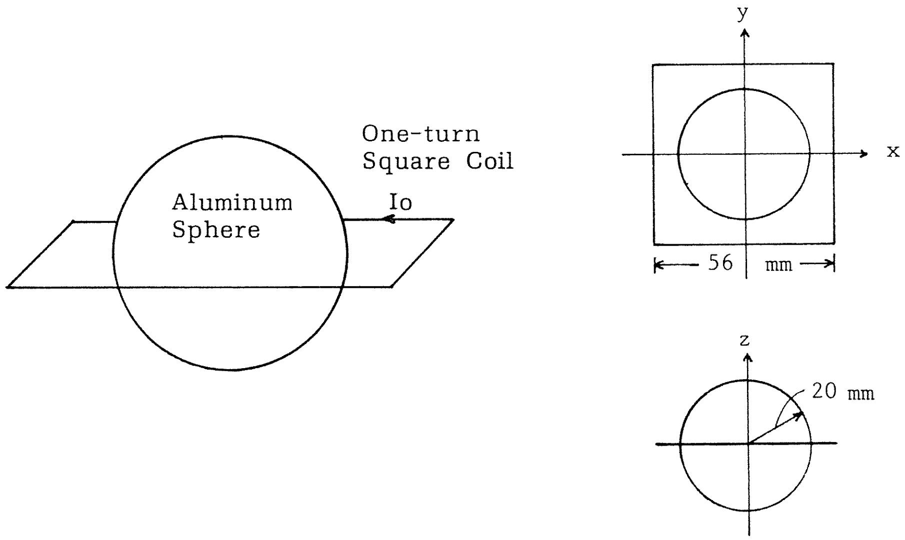


Fig. 2 The geometry of a sphere-square coil system.

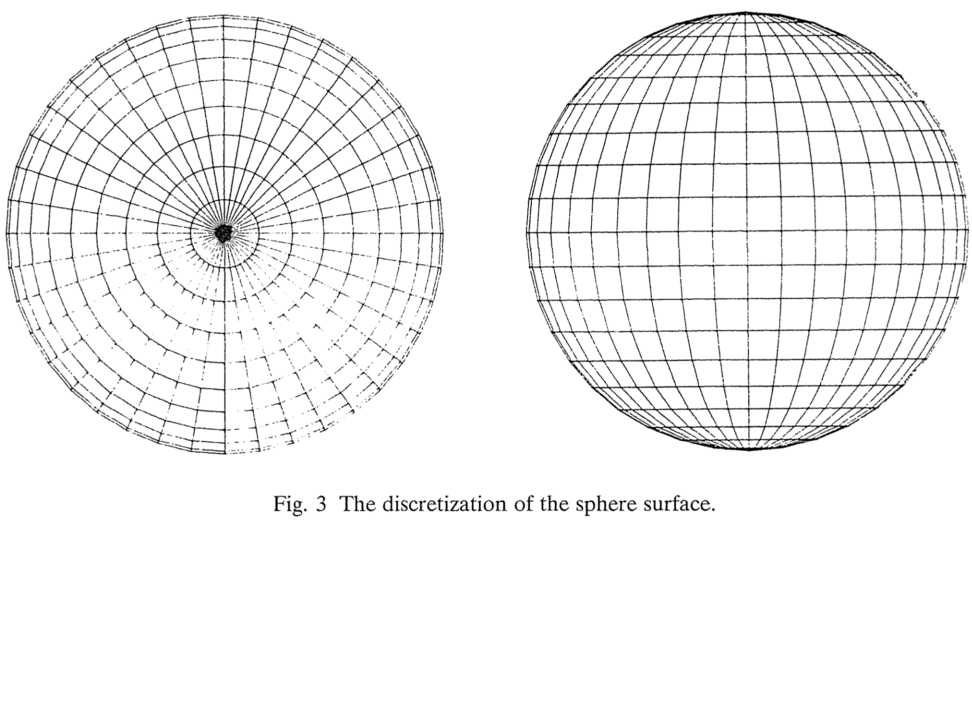


Fig. 3 The discretization of the sphere surface.

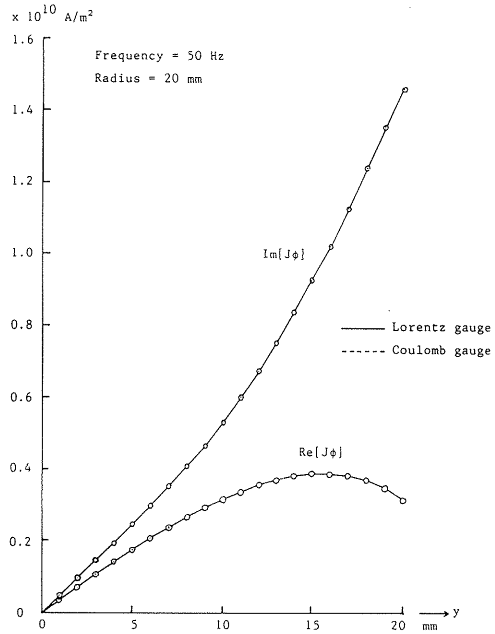


Fig. 4 The computed eddy current density $J\phi$ on the y axis in the plane $z=0$, $f=50$ Hz.

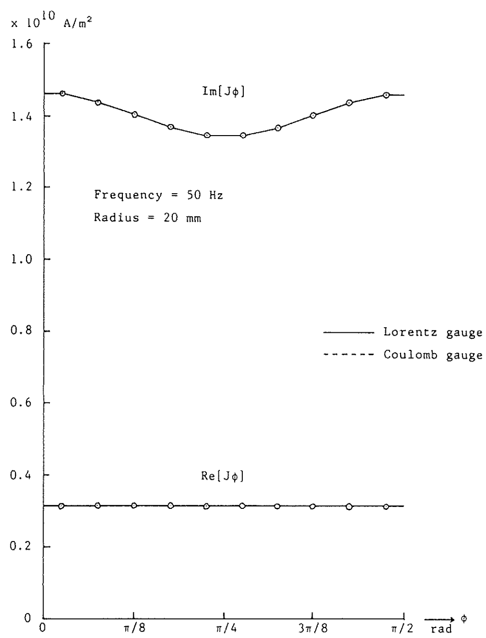


Fig. 5 The computed eddy current density $J\phi$ on the surface of the sphere at $\theta = 19\pi/40$, $f=50$ Hz.

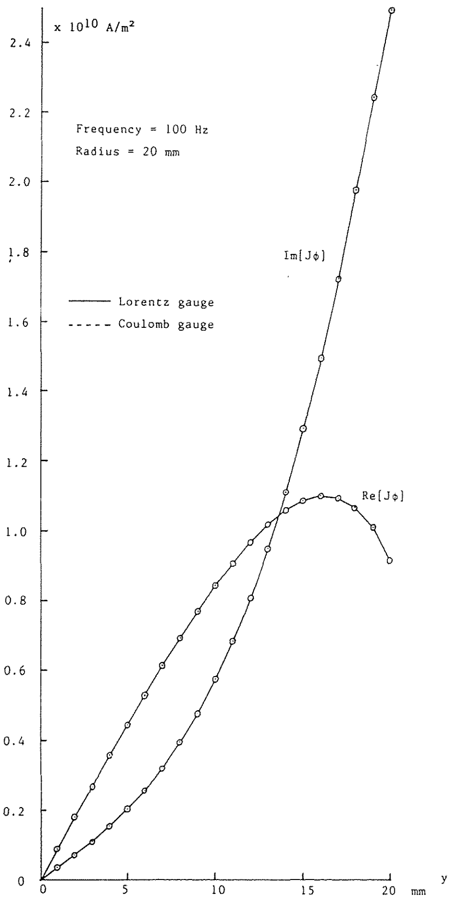


Fig. 6 The computed eddy current density $J\phi$ on the y axis in the plane $z = 0$, $f = 100$ Hz.

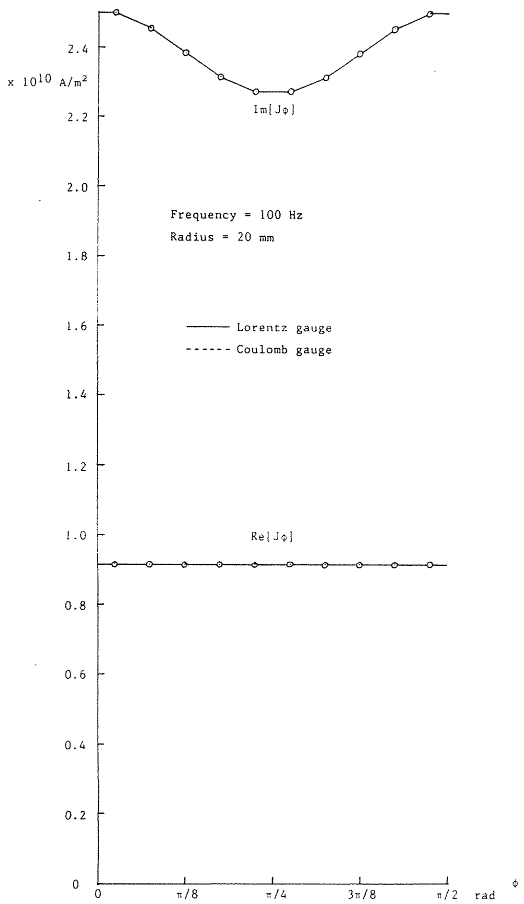


Fig. 7 The computed eddy current density $J\phi$ on the surface of the sphere at $\theta = 19\pi/40$, $f = 100$ Hz.

Table 1. The computed eddy current density J_ϕ on the y axis in the plane z = 0, f = 50 Hz. The Lorentz gauge

-.3556917303D+08	-.4694851213D+08
-.7090489624D+08	-.9426377534D+08
-.1057710487D+09	-.1423140489D+09
-.1399261455D+09	-.1914707088D+09
-.1731206002D+09	-.2421100654D+09
-.2050939161D+09	-.2946155174D+09
-.2355717745D+09	-.3493801980D+09
-.2642629817D+09	-.4068103159D+09
-.2908562248D+09	-.4673294584D+09
-.3150165697D+09	-.5313842236D+09
-.3363816172D+09	-.5994516863D+09
-.3545571975D+09	-.6720494148D+09
-.3691124413D+09	-.7497490414D+09
-.3795739865D+09	-.8331946760D+09
-.3854188218D+09	-.9231265472D+09
-.3860635762D+09	-.1020398229D+10
-.3808354986D+09	-.1125883561D+10
-.3688809472D+09	-.1239847831D+10
-.3524017671D+09	-.1375705788D+10

Table 2. The computed eddy current density J_ϕ on the y axis in the plane z = 0, f = 50 Hz. The Coulomb gauge

-.3556923199D+08	-.4694819377D+08
-.7090538446D+08	-.9426128947D+08
-.1057727887D+09	-.1423060029D+09
-.1399305714D+09	-.1914527806D+09
-.1731299855D+09	-.2420779318D+09
-.2051116558D+09	-.2945661053D+09
-.2356027045D+09	-.3493132714D+09
-.2643137109D+09	-.4067304156D+09
-.2909354530D+09	-.4672482170D+09
-.3151353578D+09	-.5313230849D+09
-.3365535710D+09	-.5994450692D+09
-.3547985206D+09	-.6721483725D+09
-.3694418060D+09	-.7500254893D+09
-.3800121665D+09	-.8337463125D+09
-.3859880426D+09	-.9240828953D+09
-.3867866897D+09	-.1021930556D+10
-.3817361225D+09	-.1128238020D+10
-.3699916797D+09	-.1243527133D+10
-.3538720911D+09	-.1383327668D+10

Table 3. The computed eddy current density J_ϕ and electric scalar potential ϕ on the surface of the sphere at $\theta = 19\pi/40$ and $\theta = \pi/40$, f = 50 Hz.
The Lorentz gauge

θ = π/40		θ = 19π/40	
Electric Scalar Potential at the Surface			
-.9416397789D-08	-.1339679577D-06	-.4927368809D-03	-.9589842306D-02
-.2465244988D-07	-.3507327362D-06	-.1290459036D-02	-.2512185532D-01
-.3047210469D-07	-.4335296432D-06	-.1596120281D-02	-.3109059977D-01
-.2465245122D-07	-.3507329616D-06	-.1292235240D-02	-.2519191993D-01
-.9416398618D-08	-.1339680970D-06	-.4938373529D-03	-.9633351612D-02
.9416398618D-08	.1339680970D-06	.4938373529D-03	.9633351612D-02
.2465245122D-07	.3507329616D-06	.1292235240D-02	.2519191993D-01
.3047210469D-07	.4335296432D-06	.1596120281D-02	.3109059977D-01
.2465244988D-07	.3507327362D-06	.1290459036D-02	.2512185532D-01
.9416397789D-08	.1339679577D-06	.4927368809D-03	.9589842306D-02
Eddy Current Jp at the Surface			
-.2014843904D+08	-.4964578022D+08	-.3149746213D+09	-.1460785998D+10
-.2014862432D+08	-.4964954609D+08	-.3148039903D+09	-.1438707743D+10
-.2014892411D+08	-.4965563941D+08	-.3145322567D+09	-.1403470607D+10
-.2014922390D+08	-.4966173272D+08	-.3142653078D+09	-.1368672324D+10
-.2014940919D+08	-.4966549860D+08	-.3141023696D+09	-.1347294191D+10
-.2014940919D+08	-.4966549860D+08	-.3141023696D+09	-.1347294191D+10
-.2014922390D+08	-.4966173272D+08	-.3142653078D+09	-.1368672324D+10
-.2014892411D+08	-.4965563941D+08	-.3145322567D+09	-.1403470607D+10
-.2014862432D+08	-.4964954609D+08	-.3148039903D+09	-.1438707743D+10
-.2014843904D+08	-.4964578022D+08	-.3149746213D+09	-.1460785998D+10

Table 4. The computed eddy current density J_ϕ and electric scalar potential ϕ on the surface of the sphere at $\theta = 19\pi/40$ and $\theta = \pi/40$, f = 50 Hz.
The Coulomb gauge

θ = π/40		θ = 19π/40	
Electric Scalar Potential at the Surface			
-.7133789976D-09	-.1348173421D-06	-.2158156273D-04	-.9621645331D-02
-.1867650740D-08	-.3529564537D-06	-.5652793044D-04	-.2520512403D-01
-.2308543800D-08	-.4362783092D-06	-.6992842395D-04	-.3119354236D-01
-.1867651597D-08	-.3529566792D-06	-.5662145184D-04	-.2527521818D-01
-.7133795307D-09	-.1348174815D-06	-.2163939559D-04	-.9665172965D-02
.7133795307D-09	.1348174815D-06	.2163939559D-04	.9665172965D-02
.1867651597D-08	.3529566792D-06	.5662145184D-04	.2527521818D-01
.2308543800D-08	.4362783092D-06	.6992842395D-04	.3119354236D-01
.1867650740D-08	.3529564537D-06	.5652793044D-04	.2520512403D-01
.7133789976D-09	.1348173421D-06	.2158156273D-04	.9621645331D-02
Eddy Current Jp at the Surface			
-.2014837856D+08	-.4964578058D+08	-.3160025548D+09	-.1460745861D+10
-.2014858694D+08	-.4964954631D+08	-.3154401698D+09	-.1438682947D+10
-.2014892411D+08	-.4965563941D+08	-.3145336119D+09	-.1403470612D+10
-.2014926128D+08	-.4966173250D+08	-.3136299711D+09	-.1368697123D+10
-.2014946967D+08	-.4966549825D+08	-.3130722385D+09	-.1347334320D+10
-.2014946967D+08	-.4966549825D+08	-.3130722385D+09	-.1347334320D+10
-.2014926128D+08	-.4966173250D+08	-.3136299711D+09	-.1368697123D+10
-.2014892411D+08	-.4965563941D+08	-.3145336119D+09	-.1403470612D+10
-.2014858694D+08	-.4964954631D+08	-.3154401698D+09	-.1438682947D+10
-.2014837856D+08	-.4964578058D+08	-.3160025548D+09	-.1460745861D+10

Table 5. The computed eddy current density J_ϕ on the y axis in the plane z = 0, f = 100 Hz. The Lorentz gauge

-.8989397302D+08	-.3504600342D+08
-.1795367795D+09	-.7142514718D+08
-.2686584085D+09	-.1104739427D+09
-.3569510629D+09	-.1535355998D+09
-.4440500235D+09	-.2019632817D+09
-.5295142126D+09	-.2571232928D+09
-.6128063474D+09	-.3203982593D+09
-.6932725554D+09	-.3931904450D+09
-.7701213018D+09	-.4769254737D+09
-.8424014431D+09	-.5730569075D+09
-.9089791625D+09	-.6830724069D+09
-.9685134539D+09	-.8085025894D+09
-.1019429680D+10	-.9509342669D+09
-.1059890472D+10	-.1112030289D+10
-.1087762388D+10	-.1293556545D+10
-.1100571439D+10	-.1497394351D+10
-.1095402138D+10	-.1725346750D+10
-.1068601487D+10	-.1977935950D+10
-.1025085362D+10	-.2277432036D+10

Table 6. The computed eddy current density J_ϕ on the y axis in the plane z = 0, f = 100 Hz. The Coulomb gauge

-.8989414945D+08	-.3504523735D+08
-.1795382594D+09	-.7141913595D+08
-.2686637844D+09	-.1104543175D+09
-.3569650534D+09	-.1534912933D+09
-.4440804197D+09	-.2018823464D+09
-.5295730601D+09	-.2569953837D+09
-.6129112910D+09	-.3202178311D+09
-.6934482567D+09	-.3929607279D+09
-.7704008180D+09	-.4766633028D+09
-.8428274322D+09	-.5727984990D+09
-.9096047562D+09	-.6828801871D+09
-.9694026124D+09	-.8084732097D+09
-.1020656813D+10	-.9512079671D+09
-.1061539080D+10	-.1112801867D+10
-.1089922536D+10	-.1295088768D+10
-.1103336429D+10	-.1500039229D+10
-.1098868840D+10	-.1729612797D+10
-.1072902132D+10	-.1984849218D+10
-.1030809184D+10	-.2292216592D+10

Table 7. The computed eddy current density J_ϕ and electric scalar potential ϕ on the surface of the sphere at $\theta = 19\pi/40$ and $\theta = \pi/40$, f = 100 Hz.
The Lorentz gauge

Electric Scalar Potential at the Surface

$\theta = \pi/40$		$\theta = 19\pi/40$	
-.3632824897D-07	-.2631658360D-06	-.1927655916D-02	-.1900004964D-01
-.9510859221D-07	-.6889772427D-06	-.5048493540D-02	-.4977337187D-01
-.1175606885D-06	-.8516229856D-06	-.6244370483D-02	-.6159972819D-01
-.9510859756D-07	-.6889776936D-06	-.5055575374D-02	-.4991333384D-01
-.3632825228D-07	-.2631661147D-06	-.1932043459D-02	-.1908696475D-01
.3632825228D-07	.2631661147D-06	.1932043459D-02	.1908696475D-01
.9510859756D-07	.6889776936D-06	.5055575374D-02	.4991333384D-01
.1175606885D-06	.8516229856D-06	.6244370483D-02	.6159972819D-01
.9510859221D-07	.6889772427D-06	.5048493540D-02	.4977337187D-01
.3632824897D-07	.2631658360D-06	.1927655916D-02	.1900004964D-01

Eddy Current Jp at the Surface

$\theta = \pi/40$		$\theta = 19\pi/40$	
-.5449102033D+08	-.6914902558D+08	-.9168815088D+09	-.2500291202D+10
-.5449174016D+08	-.6915649630D+08	-.9161801247D+09	-.2456224377D+10
-.5449290488D+08	-.6916858419D+08	-.9150625807D+09	-.2385895265D+10
-.5449406959D+08	-.6918067210D+08	-.9139640490D+09	-.2316443970D+10
-.5449478943D+08	-.6918814284D+08	-.9132932332D+09	-.2273777567D+10
-.5449478943D+08	-.6918814284D+08	-.9132932332D+09	-.2273777567D+10
-.5449406959D+08	-.6918067210D+08	-.9139640490D+09	-.2316443970D+10
-.5449290488D+08	-.6916858419D+08	-.9150625807D+09	-.2385895265D+10
-.5449174016D+08	-.6915649630D+08	-.9161801247D+09	-.2456224377D+10
-.5449102033D+08	-.6914902558D+08	-.9168815088D+09	-.2500291202D+10

Table 8. The computed eddy current density J_ϕ and electric scalar potential ϕ on the surface of the sphere at $\theta = 19\pi/40$ and $\theta = \pi/40$, f = 100 Hz.
The Coulomb gauge

Electric Scalar Potential at the Surface

$\theta = \pi/40$		$\theta = 19\pi/40$	
-.2758158499D-08	-.2692451710D-06	-.8385558026D-04	-.1923221543D-01
-.7220953738D-08	-.7048931483D-06	-.2196430179D-03	-.5038124700D-01
-.8925591739D-08	-.8712961267D-06	-.2717171815D-03	-.6235122582D-01
-.7220957077D-08	-.7048935989D-06	-.2200159362D-03	-.5052141630D-01
-.2758160568D-08	-.2692454495D-06	-.8408619199D-04	-.1931925896D-01
.2758160568D-08	.2692454495D-06	.8408619199D-04	.1931925896D-01
.7220957077D-08	.7048935989D-06	.2200159362D-03	.5052141630D-01
.8925591739D-08	.8712961267D-06	.2717171815D-03	.6235122582D-01
.7220953738D-08	.7048931483D-06	.2196430179D-03	.5038124700D-01
.2758158499D-08	.2692451710D-06	.8385558026D-04	.1923221543D-01

Eddy Current Jp at the Surface

$\theta = \pi/40$		$\theta = 19\pi/40$	
-.5449074979D+08	-.6914906221D+08	-.9208905599D+09	-.2500060887D+10
-.5449157296D+08	-.6915651894D+08	-.9186613716D+09	-.2456082156D+10
-.5449290488D+08	-.6916858420D+08	-.9150679717D+09	-.2385895392D+10
-.5449423680D+08	-.6918064946D+08	-.9114861547D+09	-.2316586269D+10
-.5449505997D+08	-.6918810622D+08	-.9092754386D+09	-.2274007677D+10
-.5449505997D+08	-.6918810622D+08	-.9092754386D+09	-.2274007677D+10
-.5449423680D+08	-.6918064946D+08	-.9114861547D+09	-.2316586269D+10
-.5449290488D+08	-.6916858420D+08	-.9150679717D+09	-.2385895392D+10
-.5449157296D+08	-.6915651894D+08	-.9186613716D+09	-.2456082156D+10
-.5449074979D+08	-.6914906221D+08	-.9208905599D+09	-.2500060887D+10

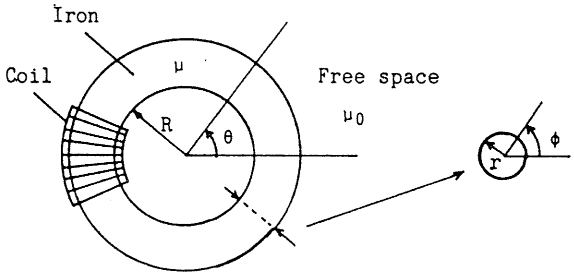


Fig. 8 A gapless magnetic circuit.

10. Application to a Magnetostatic Field Problem

Three dimensional magnetostatic field calculations for gapless magnetic circuits are very sensitive to discretization. A very small change of the element size causes us a great deal of computational errors. In this chapter, a gapless magnetic circuit like a power transformer is considered. The geometry of the circuit is shown in Fig. 8. The permeability of the iron is assumed to be constant. We will use the boundary integral equation based on surface magnetization currents which is given as:

$$(1 + 1/\chi)\ k(r) = \int_{\Omega} j_0(r') \times (r - r') \times n(r)/4\pi\ |r - r'|^3\ \mathrm{d}\Omega$$
$$+ \int_{\Gamma} k(r') \times (r - r') \times n(r)/4\pi\ |r - r'|^3\ \mathrm{d}\Gamma \tag{10.1}$$

where $k = M \times n$, M, and χ are the surface magnetization current density, the magnetization, and the susceptibility, respectively. (10.1) is derived from the vector potential A as follows: From (7.4) and (7.6), together with (5.6), follows

$$A_1(r) = \int_{\Omega_2} \mu_0 j_0(r')/4\pi\ |r - r'|\ \mathrm{d}\Omega_2$$
$$+ \int_{\Gamma} [\partial A_1(r')/\partial \mathrm{n}' - \partial A_2(r')/\partial \mathrm{n}']/4\pi\ |r - r'|\ \mathrm{d}\Gamma, \quad r \,\varepsilon\, \Omega_1 . \tag{10.2}$$

Let n be [1 0 0], then, from (5.8)

$$\partial \mathrm{A}_2\mathrm{y}/\partial \mathrm{x} = 1/\mu_\mathrm{r}\ \partial \mathrm{A}_1\mathrm{y}/\partial \mathrm{x} + (1 - 1/\mu_\mathrm{r})\ \partial \mathrm{A}_1\mathrm{x}/\partial \mathrm{y}\ , \quad \text{therefore,}$$
$$\partial \mathrm{A}_1\mathrm{y}/\partial \mathrm{x} - \partial \mathrm{A}_2\mathrm{y}/\partial \mathrm{x} = (1 - 1/\mu_\mathrm{r})(\partial \mathrm{A}_1\mathrm{y}/\partial \mathrm{x} - \partial \mathrm{A}_1\mathrm{x}/\partial \mathrm{y})$$
$$= (1 - 1/\mu_\mathrm{r})\ \mathrm{B}_1\mathrm{z} = \chi/\mu_\mathrm{r}\ .\ \mu_0\mu_\mathrm{r}\ \mathrm{H}_1\mathrm{z}\ = \mu_0\mathrm{Mz}\ . \tag{10.3}$$

We obtain in the same way,

$$\partial A_1 z/\partial x - \partial A_2 z/\partial x = -\mu_0 My\,. \tag{10.4}$$

From (5.9) gives (in a magnetostatic field problem, the Lorentz gauge reduces to the Coulomb gauge since the electric scalar potential does not exist)

$$\partial A_1 x/\partial x - \partial A_2 x/\partial x = 0\,. \tag{10.5}$$

(10.3), (10.4) and (10.5) give

$$\partial A_1/\partial n - \partial A_2/\partial n = \mu_0\, M \times n = \mu_0 k \tag{10.6}$$

Substituting (10.6) into (10.2) gives

$$A_1(r) = \int_{\Omega_2} \mu_0 j_0(r')/4\pi\,|r - r'|\,d\Omega_2 + \int_\Gamma \mu_0 k(r')/4\pi\,|r - r'|\,d\Gamma\,. \tag{10.7}$$

Taking the rotation of (10.7) gives

$$B_1(r) = \int_{\Omega_2} \mu_0 j_0(r') \times (r - r')/4\pi\,|r - r'|^3\,d\Omega_2 + \int_\Gamma \mu_0 k(r') \times (r - r')/4\pi\,|r - r'|^3\,d\Gamma\,. \tag{10.8}$$

Multiplying $n(r)$ into (10.8) and taking the limit $r\,(\varepsilon\,\Omega_1) \to \Gamma$ gives (10.1). Q.E.D.

The evaluation of the second term in the right hand side of (10.1) over the subregion Γ_s containing a singular point: $r - r' \to 0$, is carried out analytically as follows: Let $r = [-\varepsilon\ 0\ 0]$, $r' = [0\ y\ z]$, $n(r) = [1\ 0\ 0]$, and $\Gamma_s = (-\alpha \leqq y \leqq \alpha, -\beta \leqq z \leqq \beta)$. Then,

$$\begin{aligned}&\lim_{r \to 0} \int_\Gamma e \times (r - r') \times n(r)/4\pi\,|r - r'|^3\,d\Gamma_S \\ &= \lim_{\varepsilon \to 0} \int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta} \varepsilon/4\pi\sqrt{(\varepsilon^2 + y^2 + z^2)^3}\,dydz\;e \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 1/4\pi\sqrt{(1 + y'^2 + z'^2)^3}\,dy'dz'\;e = 1/2\;e\end{aligned} \tag{10.9}$$

where e denotes either [0 1 0] or [0 0 1]. The singular integral takes the value of 1/2, independent of the size of the subregion containing the singular point. The size of Γ_s, therefore, may give a serious influence on the numerical computation.

The computed results. (10.1) was applied to the problem shown in Fig. 8. The permeability of iron is assumed to be 1,000 μ_0. The inner radius R of the torus is 0.8 m, and the cross section radius r is 0.2 m. The solenoid has a radius of 0.21 m, and spans an angle of $\pi/16$ radian. The applied magnetomotive force is 16 ampere-turn. The torus is divided into 64 sections in the θ direction and 16 sections in the ϕ direction. The computed results are shown in Table 9 and

Fig. 9. δ in Table 9 denotes the width in the ϕ direction of the singular element in which the integration is carried out analytically (see (10.9)). "Sum" in Table 9 denotes the row-wise sum of the matrix elements derived from the second term in the right hand side of (10.1).
Remark. $k\theta$ (the θ component of k) is negligibly small, compared with $k\phi$ (the ϕ component of k): $k\theta/k\phi < 0.002$ for $\mu = 1{,}000\,\mu_0$, and < 0.0002 for $\mu = 10{,}000\,\mu_0$.

A method of reducing the computational error. As is seen from Table 9 and Fig. 9, a small change of the width of the singular element induces a great numerical error. The gapless problem, therefore, is very sensitive to the discretization. The proposed method is made up of two steps, and in the first step the optimal value of $2\pi r/\delta$ is searched by trial and error, checking the computed ampere-turn. For the case of Table 9, the optimal value is 48, since the corresponding computed ampere-turn is 13.26712 and is the closest to the applied ampere-turn: 16 AT. In the second step, the matrix elements are adjusted to satisfy the following equation for $k\,(=[k\phi\ k\theta]) = [k\phi\ 0]$

$$k(r) = \int_\Gamma k(r') \times (r - r') \times n(r)/4\pi\,|r - r'|^3\,\mathrm{d}\Gamma\,. \tag{10.10}$$

(10.10) is equivalent to that the respective row-wise sum of the matrix elements is equal to 1. The computed result corresponding to this adjustment is shown in Table 9 by symbol*. (A slightly smaller number: 0.9999990 was used in the computation.) The computed results by using this method are shown in Table 10 and Fig. 10 for the relative permeability of 2,000, 4,000, and 8,000.

Conclusions. 3D magnetostatic field calculations for gapless magnetic circuits are strongly affected by the discretization. In this chapter, this fact is analyzed for an iron torus like a transformer by using a boundary integral equation based on the surface magnetization current method. The main cause of the computational error is the imperfect cancelation of the permeability-free terms in the boundary integral equation (10.1) due to the improper size of the analytical integration region containing a singular point. A method of decreasing the computational error is presented.

Table 9. The width of the singular element vs the computed ampere-turn in the iron torus.

$2\pi r/\delta$	Computed AT	Sum
40	3.35184	0.9962248
42	4.18826	0.9971786
44	5.49706	0.9980887
46	7.84141	0.9989594
48	13.26712	0.9997943
50	39.58445	1.0005965
52	−43.53237	1.0013687
48*	15.97702	0.9999990

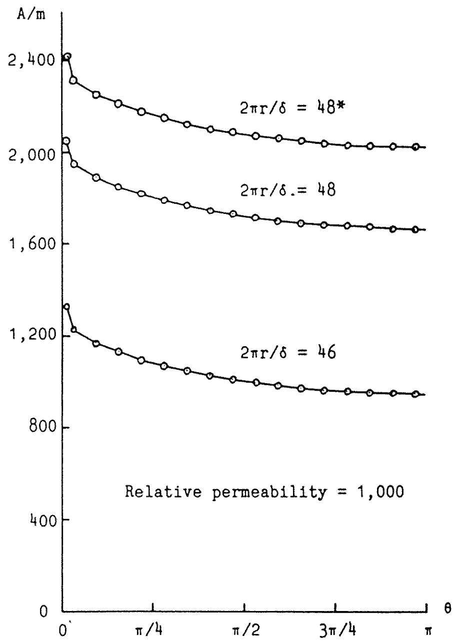


Fig. 9 The computed surface magnetization current density, kϕ.

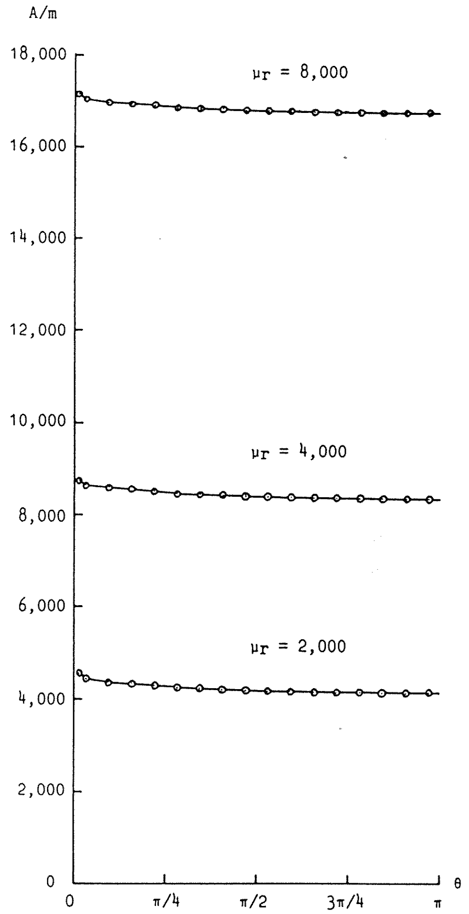


Fig. 10 The computed surface magnetization current density kϕ vs the relative permeability μr.

Table 10. The computed ampere-turn vs the relative permeability ($2\pi r/\delta = 48^*$).

μ_r	Computed AT	Sum
2,000	15.96108	0.9999990
4,000	15.92928	0.9999990
8,000	15.86607	0.9999990

11. Application to an Electromagnetic Field Problem

In this chapter, the TEAM workshop problem: "Coil Above A Crack, A Problem In Non Destructive Testing"[17] is analytically solved by using the principle of superposition for a linear system and the Fourier transform method.

Description of the problem. A block of austenitic stainless steel contains a rectangular slot, representing a flaw shown in Fig. 11. A differential probe moves across the surface of the block. The probe shown in Fig. 11 is a cylinder with an inducing solenoid and two smaller receptive solenoids. Each of these two solenoids is in a branch of a Wheatstone's bridge. The voltage at the bridge point is proportional to the difference of magnetic flux in the two receivers. An amplifier and a dephasor generate signals and send these to the two pairs of plates in an oscilloscope, representing the real and imaginary parts of the differential impedance between the block and the receivers. Variation of the signals is obtained by moving the prove above the face containing the flaw.

The solution method. Like a stress concentration problem in structural mechanics, the problem seems to require either a very fine FEM (or FDM) mesh near the flaw or an analytical approach. In this paper, the analytical approach is used. The principle of superposition for a linear system and the Fourier transform method are used as the solution method.

There are two driving forces in the problem:

j_0 : The exciting current density in the inducing solenoid,

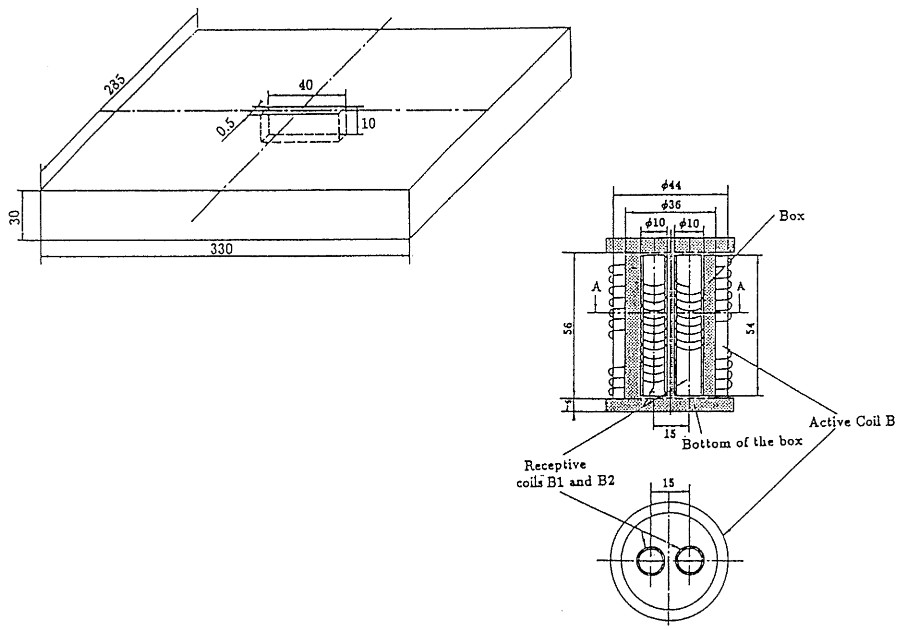


Fig. 11 The block with a flaw and the probe.

ϕ : The electric scalar potential along the surface of the flaw.

(The scalar potential should exist to cancel out the solenoidal vector potential, since the normal component of the eddy current at the surface of the flaw is zero.) Since the system is linear, the problem is decomposed into the following two problems each of which has only one driving force:

$$\text{Problem I} \quad : \quad j_1 = j_0 \quad \text{and} \quad \phi_1 = 0\,, \tag{11.1}$$

$$\text{Problem II} \quad : \quad j_2 = 0 \quad \text{and} \quad \phi_2 = \phi\,. \tag{11.2}$$

Since the width of the flaw is very small (0.5 mm), Problem I may be regarded as the problem without flaws. Furthermore, since the diameter of the inducing solenoid (44 mm) is about 1/7 of the lengths of the block (330 × 285 × 30 mm), the block may be regarded as an infinitely wide plate with a finite thickness. Under these assumptions, Problem I reduces to an axisymmetric problem. In Problem II, the electric scalar potential ϕ is not known in advance. ϕ is determined by the following equation:

$$(\mathrm{j}\omega A_2 + \operatorname{grad}\phi)\,.\,n = -\,\mathrm{j}\omega A_1\,.\,n \tag{11.3}$$

where the magnetic vector potential A_1 is already obtained in Problem I. In Problem II, the block may, as in Problem I, be regarded as an infinitely wide plate with a finite thickness, since the flaw is very small (40 × 10 × 0.5 mm) compared with the block. Under this assumption, the Fourier transform method is effective to solve the problem.

The solution to Problem I. The geometry of Problem I is axisymmetric (see Fig. 12), therefore, there exists no electric scalar potential, and the vector potential has only one component:

$$A = [\mathrm{Ar}\ \mathrm{A}\theta\ \mathrm{Az}] = [0\ \mathrm{A}\theta(\mathrm{r},\mathrm{z})\ 0] \tag{11.4}$$

The field equation in the conductor is written in cylindrical coordinates as:

$$\partial/\partial\mathrm{r}\ [1/\mathrm{r}\ \partial/\partial\mathrm{r}\ (\mathrm{rA}\theta)] + \partial^2\mathrm{A}\theta/\partial\mathrm{z}^2 - \mathrm{j}\omega\mu_0\sigma\mathrm{A}\theta = 0\,, \tag{11.5}$$

and the field equation in free space is written as:

$$\partial/\partial\mathrm{r}\ [1/\mathrm{r}\ \partial/\partial\mathrm{r}\ (\mathrm{rA}\theta)] + \partial^2\mathrm{A}\theta/\partial\mathrm{z}^2 = -\,\mu_0\mathrm{j}_0\theta\,. \tag{11.6}$$

The solution is obtained by using the Bessel-Fourier Transform method. The Bessel-Fourier transform is defined as:

$$\mathrm{g}(\xi) = \int_0^\infty \mathrm{f}(\mathrm{r})\mathrm{r}\mathrm{J}_1(\xi\mathrm{r})\,\mathrm{dr}\,, \tag{11.7}$$

$$\mathrm{f}(\mathrm{r}) = \int_0^\infty \mathrm{g}(\xi)\xi\mathrm{J}_1(\xi\mathrm{r})\,\mathrm{d}\xi \tag{11.8}$$

where $J_1(.)$ is the Bessel function of the first kind and the first degree, and has the following properties:

$$d/dr\,[1/r\; d/dr\,(rJ_1(r))] = -\,J_1(r)\,,$$

$$\int_0^\infty d/dr\,[1/r\; d/dr\,(ru(r))]rJ_1(r)\,dr = -\int_0^\infty u(r)rJ_1(r)\,dr\,. \qquad (11.9)$$

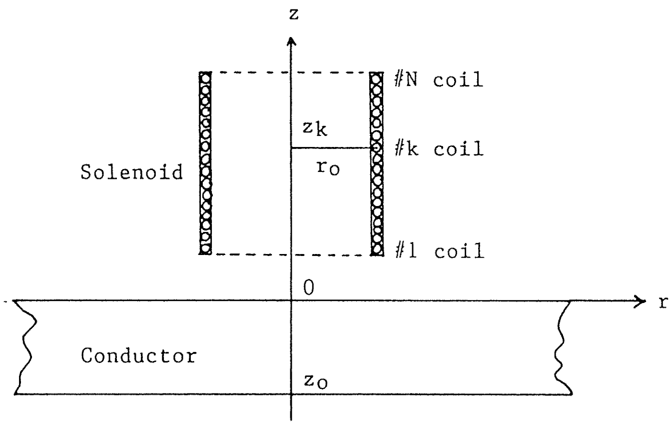


Fig. 12 The geometry of Problem I.

We obtain from (11.5) through (11.9)

$$-\,\xi^2 A\theta^* + d^2 A\theta^*/dz^2 - j\omega\mu_0\sigma A\theta^* = 0\,, \qquad (11.5)'$$

$$-\,\xi^2 A\theta^* + d^2 A\theta^*/dz^2 + \mu_0 j_0\theta^* = 0 \qquad (11.6)'$$

where * denote the Bessel-Fourier transform (see (11.7)). First, we consider the case of a one-turn inducing coil carrying $1 + j0$ A current:

$$j_0\theta(r, z) = \delta(r - r_1)\delta(z - z_1) \qquad (11.10)$$

where $\delta(.)$ is the Dirac delta function, and (r_1,z_1) is the coil location. From (11.10) and (11.7)

$$j_0\theta^*(\xi, z) = r_1 J_1(\xi r_1)\delta(z - z_1)\,. \qquad (11.11)$$

From (11.6)′ and (11.11) follows

$$dA\theta^*(r_1, z)/dz\Big|_{z = z_1 + 0} - dA\theta^*(r_1, z)/dz\Big|_{z = z_1 - 0}$$

$$= -\,\mu_0 r_1 J_1(\xi r_1) \qquad (11.12)$$

(11.5)′ and (11.6)′ give

$$
\begin{aligned}
&A\theta^*(\xi, z) = a(\xi)\exp(-\xi z) \quad \text{for} \quad z > z_1\,,\\
&A\theta^*(\xi, z) = b(\xi)\exp(\xi z) + c(\xi)\exp(-\xi z) \quad \text{for} \quad 0 < z < z_1\,,\\
&A\theta^*(\xi, z) = d(\xi)\exp(\eta z) + e(\xi)\exp(-\eta z) \quad \text{for} \quad z_0 < z < 0\,,\\
&A\theta^*(\xi, z) = f(\xi)\exp(\xi z) \quad \text{for} \quad z < z_0
\end{aligned}
\tag{11.13}
$$

where $\eta^2 = \xi^2 + j\omega\mu_0\sigma$. The coefficients $a(\xi)$ through $f(\xi)$ are determined by the interface conditions (5.6) and (5.8), and by (11.12). Applying the inverse Bessel-Fourier transform (11.8) to (11.13) gives $A\theta(r, z)$. Second, the solution to Problem I is obtained from the above result and the principle of superposition as:

$$
\begin{aligned}
A\theta(r, z) = &\sum_{k=1}^{N} \int_0^\infty [d(k, \xi)\exp(\eta z) + e(k, \xi)\exp(-\eta z)]\,\xi\\
&\times J_1(\xi r)\, d\xi
\end{aligned}
\tag{11.14}
$$

where N is the total number of the coil-turns, r_0 and $-z_0$ are the radius of the inducing solenoid and the thickness of the block, respectively. The solenoid ampere-turn is $N \times (1 + j0)$ A. The coefficients $a(k, \xi)$ through $f(k, \xi)$ are given as:

$$
\begin{bmatrix} a(k,\xi)\\ b(k,\xi)\\ c(k,\xi)\\ d(k,\xi)\\ e(k,\xi)\\ f(k,\xi) \end{bmatrix}
= -\mu_0 r_0 J_1(\xi r_0)
\begin{bmatrix}
\exp(-\xi z_k) & -\exp(\xi z_k) & -\exp(-\xi z_k) & 0 & 0 & 0\\
-\exp(-\xi z_k) & -\exp(\xi z_k) & \exp(-\xi z_k) & 0 & 0 & 0\\
0 & 1 & 1 & -1 & -1 & 0\\
0 & \xi & -\xi & -\eta & \eta & 0\\
0 & 0 & 0 & \exp(\eta z_0) & \exp(-\eta z_0) & -\exp(\xi z_0)\\
0 & 0 & 0 & \eta\exp(\eta z_0) & -\eta\exp(-\eta z_0) & -\xi\exp(\xi z_0)
\end{bmatrix}^{-1}
\begin{bmatrix} 0\\ 1\\ 0\\ 0\\ 0\\ 0 \end{bmatrix}
\tag{11.15}
$$

The computed eddy currents are shown in Fig. 13 and Fig. 14 for a frequency f = 5 kHz, and a conductivity $\sigma = 0.14 \times 10^7$ S/m.

The solution to Problem II. We assume that the vertical component of the eddy currents in the block is negligible since the inducing solenoid current has no vertical component. Consequently, the potentials are expressed as:

$$A = [Ax\ Ay\ Az] = [Ax(x, y, z)\ Ay(x, y, z)\ 0]\ , \quad \text{and} \tag{11.16}$$

$$\phi = \phi(x, y)\ . \tag{11.17}$$

We use the Coulomb gauge for Problem II. The equation for the scalar potential is written from (6.2) and (11.17) as:

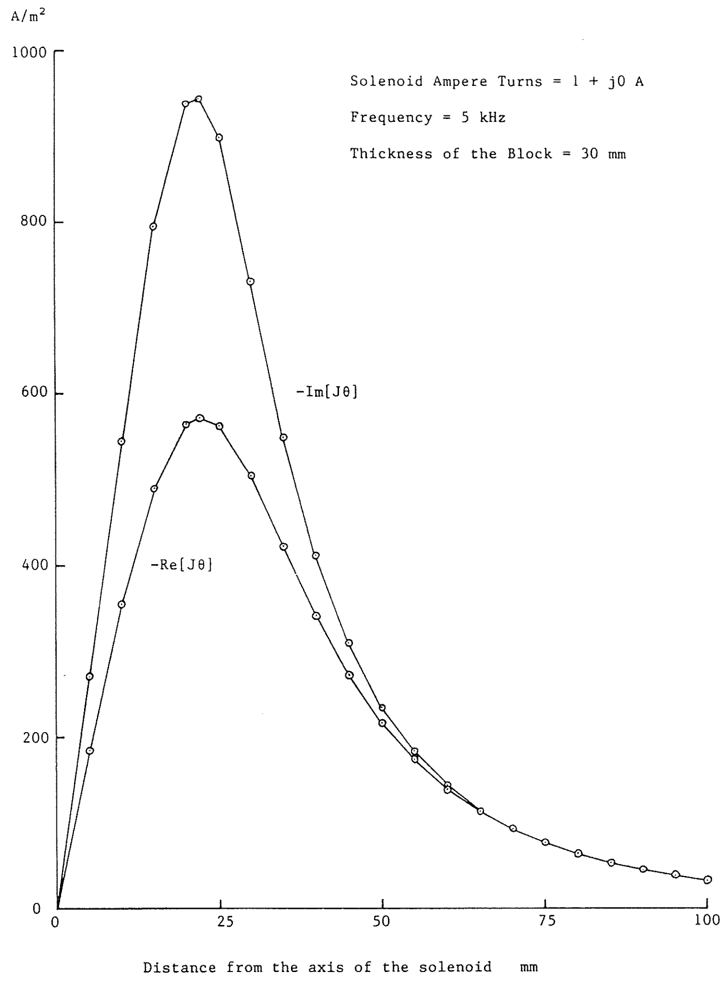


Fig. 13 The computed eddy current density on the surface of the block without flaws.

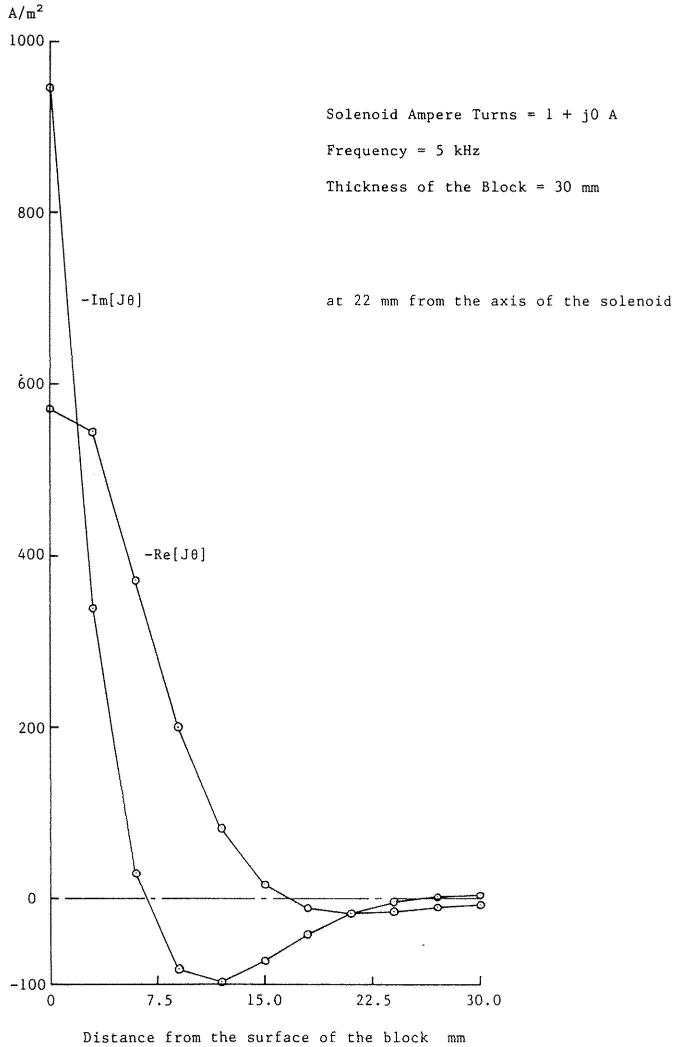


Fig. 14 The computed eddy current density inside the block without flaws.

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0\,. \tag{11.18}$$

The scalar potential, therefore, does not diffuse in the vertical direction under the assumption (11.17), and exists only in the region denoted by Conductor I in Fig. 15. The equation for the vector potential is written in the conductor, from (6.1) and (11.16), as:

$$\partial^2 Ak/\partial x^2 + \partial^2 Ak/\partial y^2 + \partial^2 Ak/\partial z^2 - j\omega\mu_0\sigma Ak - \mu_0\sigma\partial\phi/\partial k = 0\,, \tag{11.19}$$

and in free space, from (6.3), (11.2) and (11.16), as:

$$\partial^2 Ak/\partial x^2 + \partial^2 Ak/\partial y^2 + \partial^2 Ak/\partial z^2 = 0 \tag{11.20}$$

where k is x or y. First, we consider the solution of ϕ to the following boundary condition:

$$\phi(x, y)\Big|_{y=0} = \delta(x - x_0)\ . \tag{11.21}$$

(This is called an impulse response.) Applying to (11.18) the Fourier transform with respect to the x coordinate gives

$$-\xi^2\phi^* + d^2\phi^*/dy^2 = 0 \tag{11.22}$$

where

$$\phi^*(\xi, y) = \int_{-\infty}^{\infty} \phi(x, y)\exp(j\xi x)\,dx\ . \tag{11.23}$$

The inverse Fourier transform of $\phi^*(\xi, y)$ is written as:

$$\phi(x, y) = 1/2\pi \int_{-\infty}^{\infty} \phi^*(\xi, y)\exp(-j\xi x)\,d\xi. \tag{11.24}$$

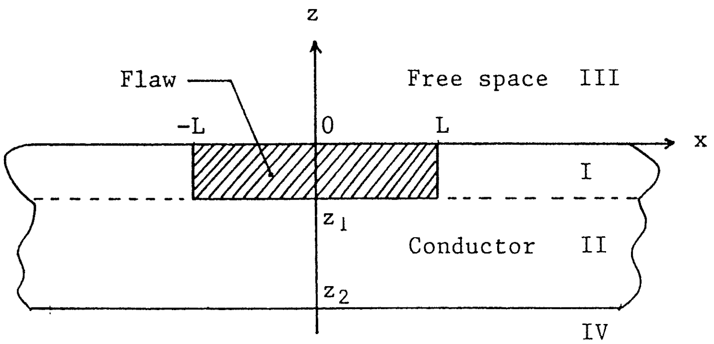


Fig. 15 The geometry of Problem II.

Since the Fourier transform of $\delta(x - x_0)$ is exp $(j\xi x_0)$, from (11.21) and (11.22) follows $\phi^*(\xi, y) = \exp(j\xi x_0)\exp(-\xi y)$ for $\xi \geqq 0$ and $y > 0$, therefore, $\phi(x, y)$ is written for $y > 0$ as:

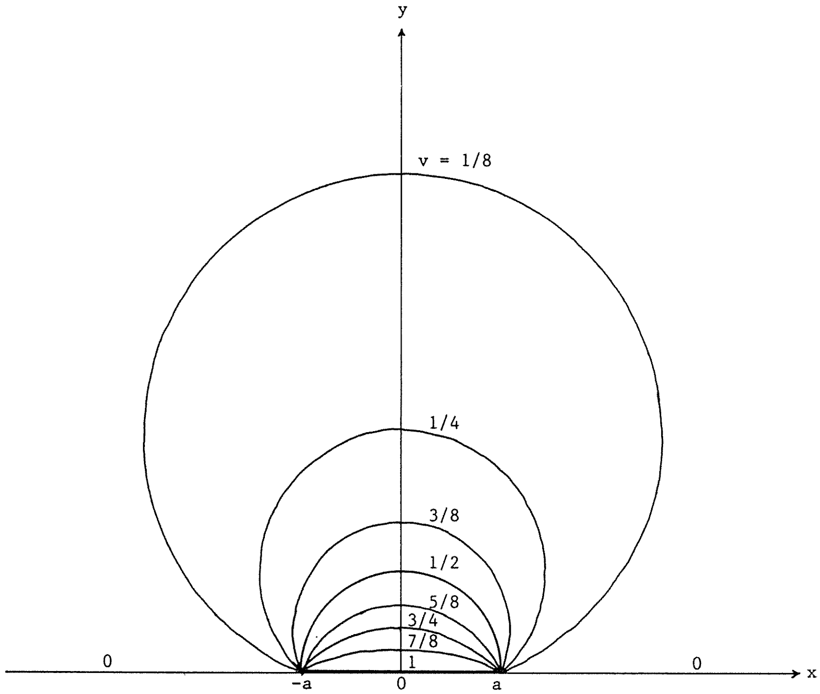


Fig. 16 The step response of the electric scalar potential (The equi-potential lines).

$$\phi(x, y) = 1/\pi \int_0^\infty \exp(-\xi y) \cos[\xi(x - x_0)]\, d\xi\,. \tag{11.25}$$

(From the geometrical symmetry, $\phi(x, -y) = -\,\phi(x, y)$, $y > 0$.) We obtain from (11.25) and the principle of superposition the solution of the scalar potential to Problem II as:

$$\phi(x, y) = \int_{-L}^{L} \phi(x_0, 0)\, [1/\pi \int_0^\infty \exp(-\xi y) \cos[\xi(x - x_0)]\, d\xi]\, dx_0\,,$$

$$-\infty < x < \infty\,, \quad y > 0 \tag{11.26}$$

where 2L is the length of the flaw (see Fig. 15). The solution corresponding to the boundary condition:

$$\phi(x, 0) = 1\,, \quad -a \leqq x \leqq a\,; \quad = 0\,, \quad |x| \geqq a \tag{11.27}$$

is given as (see Appendix 3):

$$\phi(x, y) = 1/\pi\, [\tan^{-1}((a - x)/y) + \tan^{-1}((a + x)/y)]\,. \tag{11.28}$$

$\phi(x, y)$ is symmetric with respect to the y axis. The equipotential lines are shown in Fig. 16. (The solution corresponding to the boundary condition (11.27) is called a step response.) Next, we consider the solution of A to a given scalar potential. Applying the double Fourier transform to (11.19) and (11.20) gives

$$- \xi^2 Ak^* - \eta^2 Ak^* + d^2 Ak^*/dz^2 - j\omega\mu_0\sigma Ak^*$$

$$- \mu_0\sigma[\partial\phi/\partial k]^* = 0 , \tag{11.19)'$$

$$- \xi^2 Ak^* - \eta^2 Ak^* + d^2 Ak^*/dz^2 = 0 \tag{11.20)'$$

where k = x or y, and

$$Ak^*(\xi, \eta, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Ak(x, y, z) \exp[j(\xi x + \eta y)]\, dxdy ,$$

$$[\partial\phi/\partial k]^*(\xi, \eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \partial\phi(x, y)/\partial k \exp[j(\xi x + \eta y)]\, dxdy . \tag{11.29}$$

From (11.19)′ and (11.20)′ follows:

$$Ak^*(\xi, \eta, z) = ak(\xi, \eta) \exp(-\alpha z) \quad \text{for} \quad z \geqq 0 , \tag{11.30}$$

$$Ak^*(\xi, \eta, z) = fk(\xi, \eta) \exp(\alpha z) \quad \text{for} \quad z \leqq z_2 , \tag{11.31}$$

$$Ak^*(\xi, \eta, z) = bk(\xi, \eta) \exp(\beta z) + ck(\xi, \eta) \exp(-\beta z)$$

$$+ Fk(\xi, \eta, z) \quad \text{for} \quad 0 \geqq z \geqq z_1 , \tag{11.32}$$

$$Ak^*(\xi, \eta, z) = dk(\xi, \eta) \exp(\beta z) + ek(\xi, \eta) \exp(-\beta z)$$

$$\text{for} \quad z_1 \geqq z \geqq z_2 \tag{11.33}$$

where $\alpha^2 = \xi^2 + \eta^2$, $\beta^2 = \xi^2 + \eta^2 + j\omega\mu_0\sigma$, and

$$Fk(\xi, \eta, z) = \mu_0\sigma[\partial\phi/\partial k]^* \int_0^z [1\ 0] \exp[(z - z')P] \begin{bmatrix} 0 \\ 1 \end{bmatrix} dz',$$

$$P = \begin{bmatrix} 0 & 1 \\ \beta^2 & 0 \end{bmatrix} . \tag{11.34}$$

Remark. For the scalar potential given by (11.28),

$$[\partial\phi/\partial x]^*(\xi, \eta) = 4 \sin(\xi a)\, \eta/(\xi^2 + \eta^2) ,$$

$$[\partial\phi/\partial y]^*(\xi, \eta) = -4 \sin(\xi a)\, \xi/(\xi^2 + \eta^2) . \tag{11.35}$$

The coefficients $ak(\xi, \eta)$ through $fk(\xi, \eta)$ are determined by the interface conditions: (6.5) and (6.7). For Problem II, these conditions reduce to that Ak^* and dAk^*/dz are continuous across the interfaces. Ak(x, y, z) is obtained by the inverse Fourier transform of Ak*(ξ, η, z):

$$\mathrm{Ak}(x, y, z) = (1/2\pi)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{Ak}^*(\xi, \eta, z) \exp[-j(\xi x + \eta y)]\, d\xi d\eta. \quad (11.36)$$

Remark. For the scalar potential given by (11.28), Ax and Ay are expressed as follows:

$$\mathrm{Ax}(x, y, z) = -1/\pi^2 \int_{0}^{\infty} \int_{0}^{\infty} \mathrm{Ax}^*(\xi, \eta, z) \sin(\xi x) \sin(\eta y)\, d\xi d\eta\,,$$

$$\mathrm{Ay}(x, y, z) = 1/\pi^2 \int_{0}^{\infty} \int_{0}^{\infty} \mathrm{Ay}^*(\xi, \eta, z) \cos(\xi x) \cos(\eta y)\, d\xi d\eta\,. \quad (11.37)$$

Since the scalar potential on the surface of the flaw is not given in advance, we assume it in the form of:

$$\phi(x, y)\Big|_{y=0} = \sum_{i=1}^{N} \phi_i u_i(x) \quad (11.38)$$

where $u_i(x) = 1$, for $x_{i-1} \le x \le x_i$; and $= 0$, for $x < x_{i-1}$, or $x > x_i$. ($-L \le x_{i-1} < x_i \le L$, $i = 1, \cdots, N$.) We, then, calculate the step responces to $u_i(x)$: $\mathrm{Ak}_i(x, y, z)$ and $\phi_i(x, y)$. The coefficients ϕ_i, $i = 1, \cdots, N$, in (11.38) are determined by (11.3).

The computed results. The computation was carried out for the simplified block shown in Fig. 18. (Conductor II in Fig. 15 may be negligible for high frequencies since the skin depth is about 5 mm at the frequency of 5 kHz.) The response of the eddy current to a step electric scalar potential is shown in Fig. 17 for a frequency f = 500 Hz. The admittance matrix of the eddy current density to the step electric scalar potential applied to the surface of the flaw is shown in Fig. 18. The matrix is derived from the data shown in Fig. 17: The flaw is divided into 8 sections, and each section has the same length. The matrix is a 8×8 square matrix and the entries of which are made up of the averages of the data (Fig. 17) over the successive intervals of 5 mm length. An example of the electric scalar potential across the flaw necessary to cancel out the eddy current due to the inducing solenoid (see Problem I) is shown in Fig. 19. In Fig. 20, the response of the electromotive force of the receptive solenoid to a step electric scalar potential applied to the surface of the flaw is given, and the differential electromotive forces in the receptive solenoids corresponding to the two different movements of the probe are shown in Fig. 21. The experimental results by Takagi[18] are shown in Fig. 22. It may be concluded from Fig. 21 and Fig. 22 that the computed results agree well with the experimental results, despite using the thin block for the computation. (In the experimental results there appear the edge effects due to finite dimension of the block, while in the computed results there is no edge effect since the block is regarded as an infinitely wide plate.)

Conclusions. The method using the principle of superposition and the Fourier transform is effective to the eddy current problem in non destructive testing. It provides us with higher accuracy and less computation time than the conventional FEM (FDM) or BIEM.

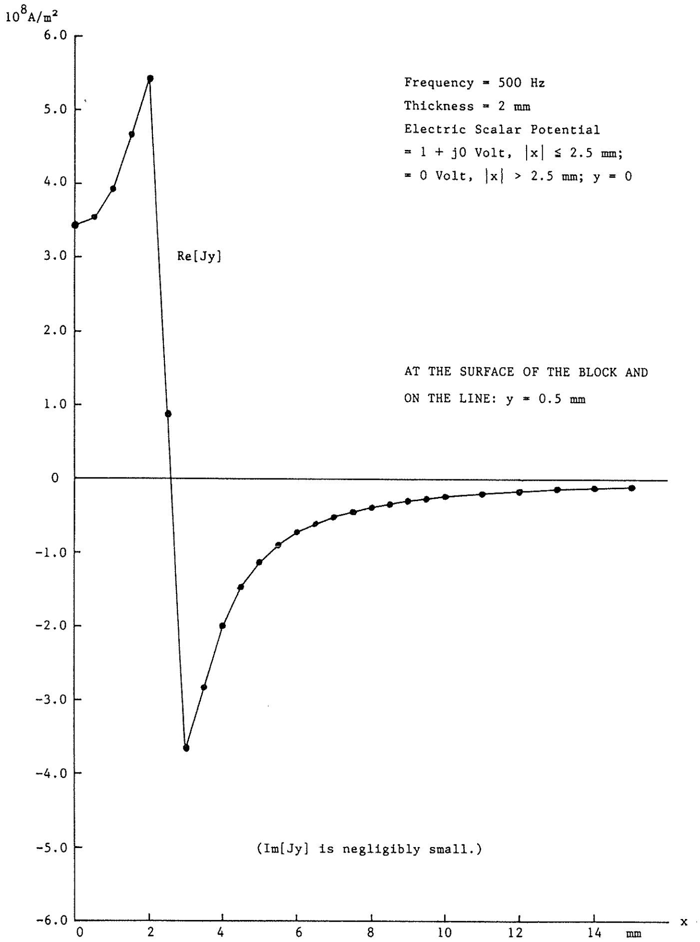
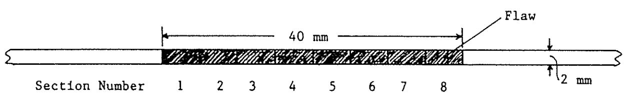


Fig. 17 The response of the eddy current to a step electric scalar potential applied to the flaw.



Admittance Matrix: [J]= Y [V], [J] = [J1,J2,---,J8]^T, [V] = [V1,V2,---,V8]^T

Jk: mean current density in the k-th section, Vk: electric scalar potential in the k-th section

Y (x 10^8A/m^2/V) :

$$
\begin{bmatrix}
3.854 & -1.402 & -0.2549 & -0.1053 & 0 & 0 & 0 & 0 \\
-1.402 & 3.854 & -1.402 & -0.2549 & -0.1053 & 0 & 0 & 0 \\
-0.2549 & -1.402 & 3.854 & -1.402 & -0.2549 & -0.1053 & 0 & 0 \\
-0.1053 & -0.2549 & -1.402 & 3.854 & -1.402 & -0.2549 & -0.1053 & 0 \\
0 & -0.1053 & -0.2549 & -1.402 & 3.854 & -1.402 & -0.2549 & -0.1053 \\
0 & 0 & -0.1053 & -0.2549 & -1.402 & 3.854 & -1.402 & -0.2549 \\
0 & 0 & 0 & -0.1053 & -0.2549 & -1.402 & 3.854 & -1.402 \\
0 & 0 & 0 & 0 & -0.1053 & -0.2549 & -1.402 & 3.854
\end{bmatrix}
$$

Fig. 18 The admittance matrix.

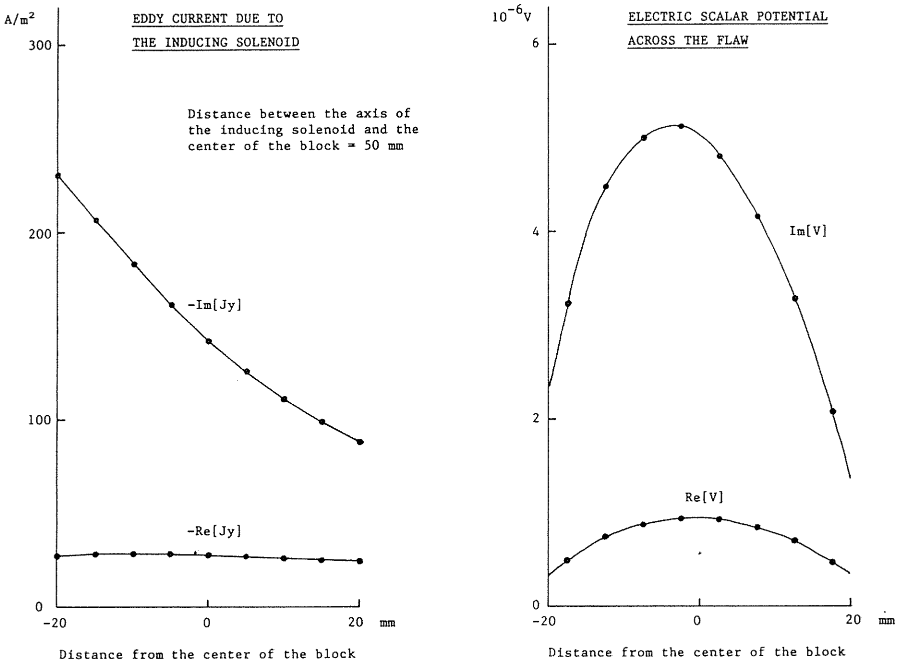


Fig. 19 An example of the electric scalar potential across the surface of the flaw necessary to cancel out the eddy current due to the inducing solenoid.

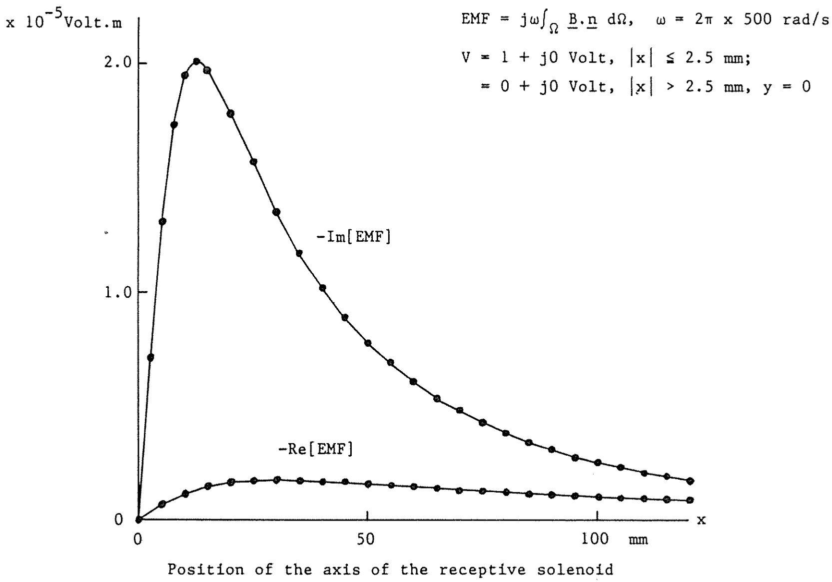


Fig. 20 The response of the electromotive force of the receptive solenoid to a step electric scalar potential applied to the flaw.

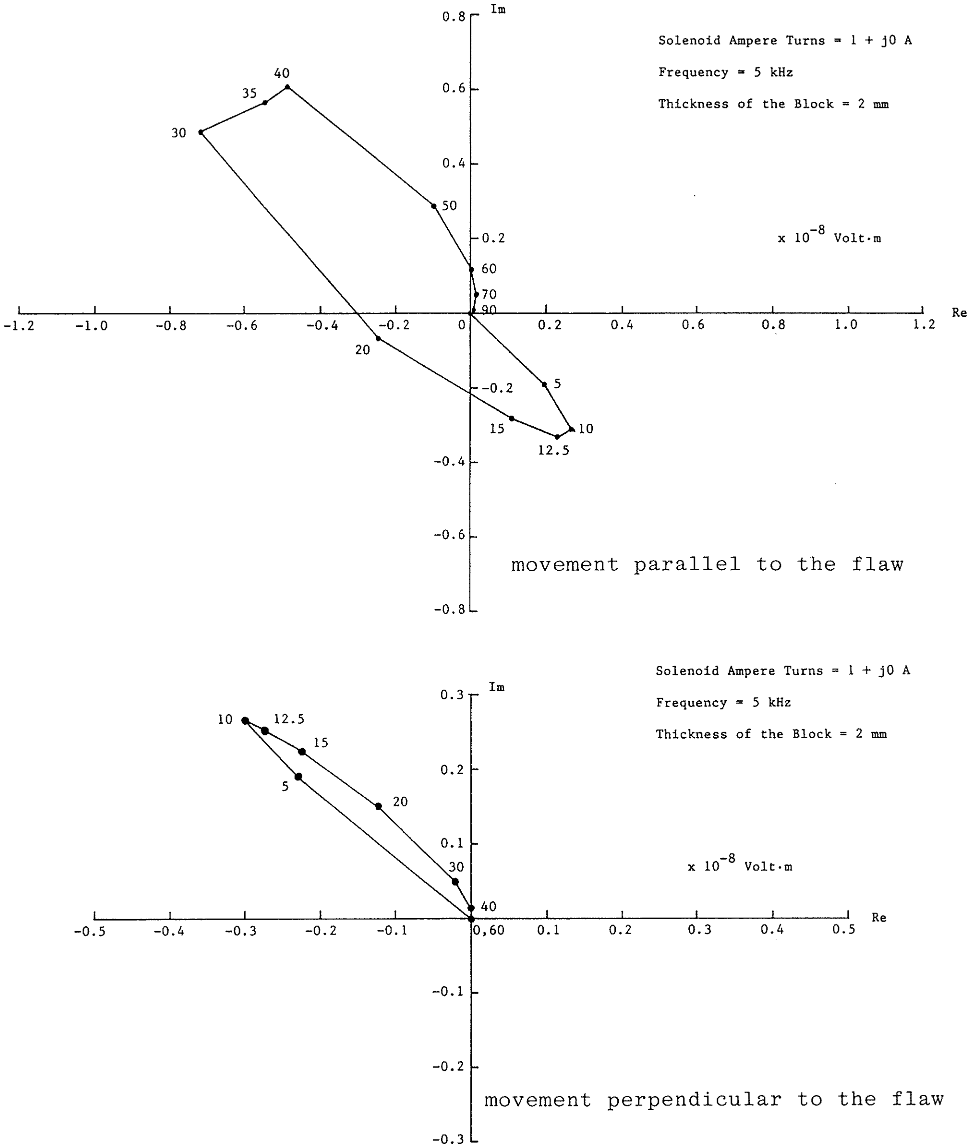


Fig. 21 The computed differential electromotive force in the receptive solenoids.

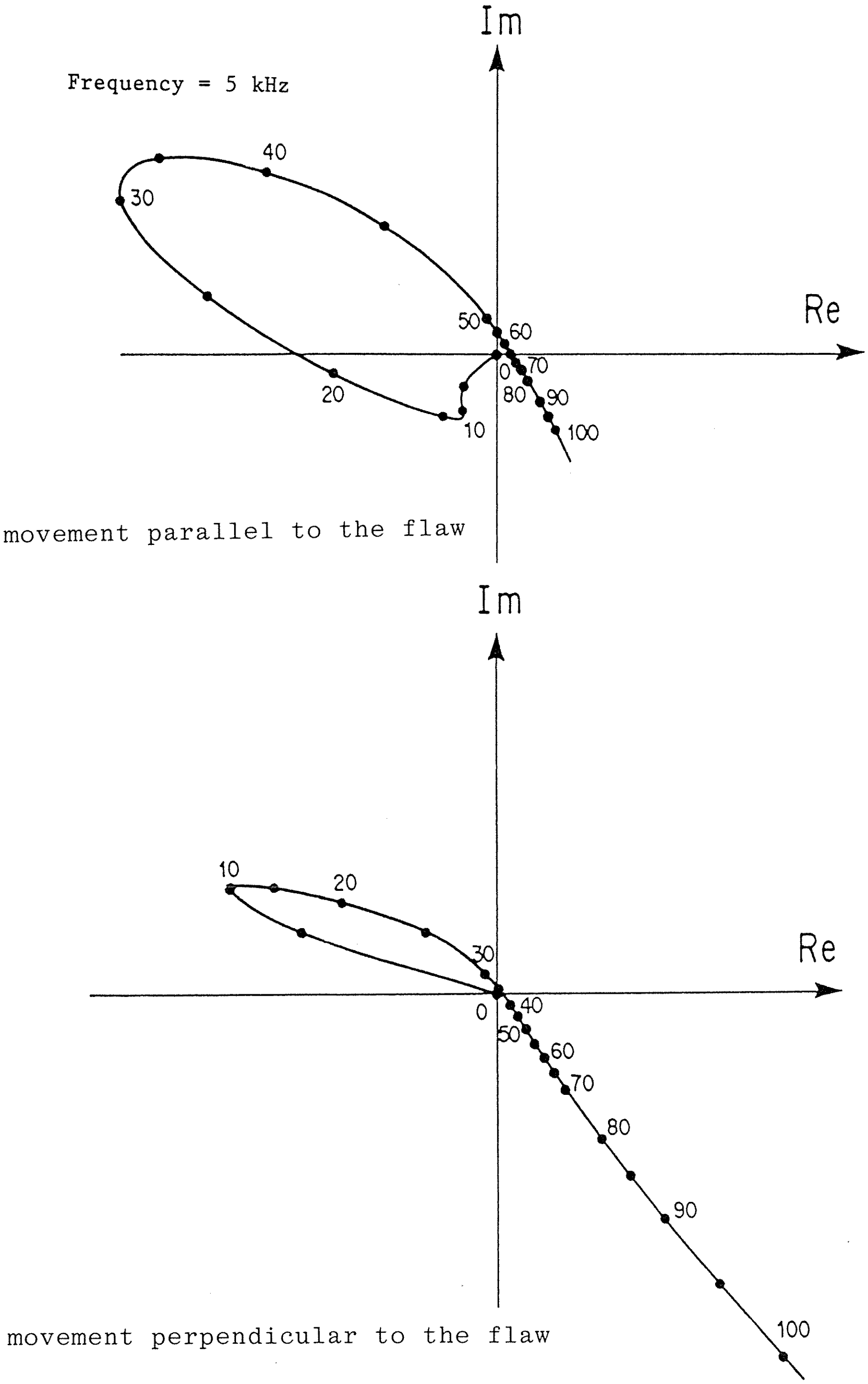


Fig. 22 The experimental results by Takagi of the differential electromotive force in the receptive solenoids.

12. Conclusions

The magnetic vector potential associated with the electric scalar potential is a most orthodox and popular solution variable for three dimensional electromagnetic field calculations. However, there exists the problem of how to select the most suitable gauge for the problem to be solved, since the electromagnetic field is invariant under the gauge transformation. In this paper, two solution methods are formulated for the Lorentz gauge and the Coulomb gauge. The methods include the gauge condition implicitly, and, therefore, are very conveniently applicable to the boundary integral equation method as well as to the finite element method or the finite difference method. The methods satisfy the gauge condition over the entire region and yield a unique solution to the problem. This fact is verified theoretically and by numerical experiments. Two examples of the application of the solution methods are presented.

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Appendix 1

(8.1) is derived by using the following integral formulas:

$$\int 1/\sqrt{(\xi^2 + a^2)}\, d\xi = \log[\xi + \sqrt{(\xi^2 + a^2)}]\,, \quad \xi > 0\,, \quad a > 0\,;$$

$$\int \sqrt{[(\xi - a)/(\xi + a)]}\, d\xi = \sqrt{(\xi^2 - a^2)}$$

$$+ a \log[a/(\xi + \sqrt{(\xi^2 - a^2)})]\,, \quad a > 0\,, \quad \xi \geqq a\,. \tag{A1.1}$$

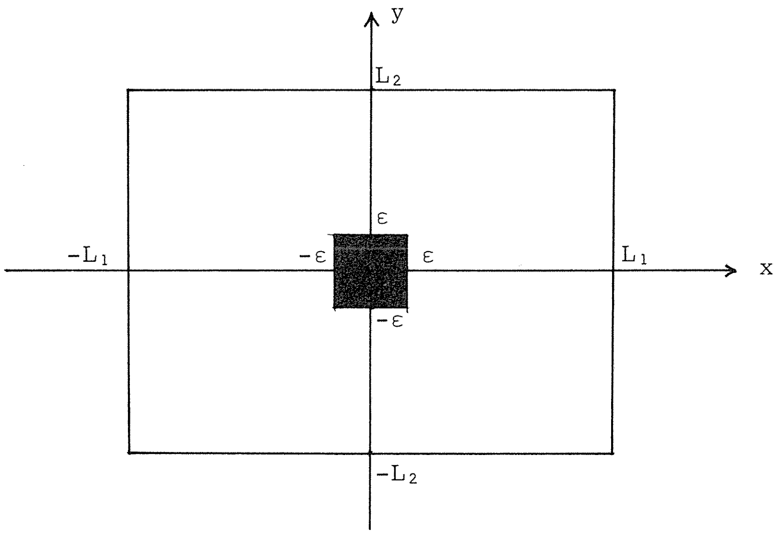

Fig. 23

$$g_{ii} = \lim_{\varepsilon \to 0} 1/\pi \, [\int_{\varepsilon}^{L_1} \int_{0}^{L_2} 1/\sqrt{(x^2 + y^2)}\, dxdy$$

$$+ \int_{0}^{\varepsilon} \int_{\varepsilon}^{L_2} 1/\sqrt{(x^2 + y^2)}\, dxdy] \tag{A1.2}$$

$$\int_{o}^{L_2} 1/\surd(x^2+y^2)\,dy = \log[L_2+\surd(x^2+L_2^2)] - \log x\,. \tag{A1.3}$$

$$\int_{\varepsilon}^{L_1} \log[L_2+\surd(x^2+L_2^2)) - \log x]\,dx$$

$$= \int_{\varepsilon}^{L_1} \log[L_2+\surd(x^2+L_2^2)]\,dx - (x\log x - x)\Big|_{\varepsilon}^{L_1}\,. \tag{A1.4}$$

$$\int_{o}^{L_1} \log[L_2+\surd(x^2+L_2^2)]\,dx = x\log[L_2+\surd(x^2+L_2^2)]\Big|_{o}^{L}$$

$$-\int_{o}^{L_1} x^2/[L_2\surd(x^2+L_2^2)+x^2+L_2^2]\,dx\,. \tag{A1.5}$$

$$\int_{o}^{L_1} x^2/[L_2\surd(x^2+L_2^2)+x^2+L_2^2]\,dx =$$

$$\int_{L_2}^{\surd(L_1^2+L_2^2)} [(\xi-L_2)/(\xi+L_2)]\,d\xi = \surd(\xi^2-L_2^2)$$

$$+ L_2\log[L_2/(\xi+\surd(\xi^2-L_2^2))]\Big|_{L_2}^{\surd(L_1^2+L_2^2)}$$

$$= L_1 + L_2\log[L_2/(\surd(L_1^2+L_2^2)+L_1]\,. \tag{A1.6}$$

$$\int_{o}^{\varepsilon} 1/\surd(x^2+y^2)\,dx = \log[\varepsilon+\surd(\varepsilon^2+y^2)] - \log y\,. \tag{A1.7}$$

$$\int_{\varepsilon}^{L_2} [\log(\varepsilon+\surd(\varepsilon^2+y^2)) - \log y]\,dy$$

$$= \int_{\varepsilon}^{L_2} \log[\varepsilon+\surd(\varepsilon^2+y^2)]\,dy - (y\log y - y)\Big|_{\varepsilon}^{L_2} \tag{A1.8}$$

$$\int_{\varepsilon}^{L_2} \log[\varepsilon+\surd(\varepsilon^2+y^2)]\,dy = y\log[\varepsilon+\surd(\varepsilon^2+y^2)]\Big|_{\varepsilon}^{L_2}$$

$$-\int_{\varepsilon}^{L_2} y^2/[\varepsilon\surd(\varepsilon^2+y^2)+\varepsilon^2+y^2]\,dy\,. \tag{A1.9}$$

$$\int_{\varepsilon}^{L_2} y^2/[\varepsilon\surd(\varepsilon^2+y^2)+\varepsilon^2+y^2]\,dy$$

$$= \int_{\surd 2\varepsilon}^{\surd(\varepsilon^2+L_2^2)} [(\xi-\varepsilon)/(\xi+\varepsilon)]\,d\xi$$

$$= [\surd(\xi^2-\varepsilon^2) + \varepsilon\log(\varepsilon/(\xi+\surd(\xi^2-\varepsilon^2)))]\Big|_{\surd 2\varepsilon}^{\surd(\varepsilon^2+L_2^2)}$$

$$= L_2 + \varepsilon\log[\varepsilon/(\surd(\varepsilon^2+L_2^2)+L_2)] - \varepsilon$$

$$- \varepsilon\log[\varepsilon/(\surd 2\ \varepsilon+\varepsilon)]\,. \tag{A1.10}$$

$$\lim_{\varepsilon \to 0} \varepsilon \log \varepsilon = 0 \,. \tag{A1.11}$$

(A1.2) through (A1.11) give (8.1).

Appendix 2

A FORTRAN program for the Lorentz gauge formulation using the boundary integral equations are shown in Table 11. Several remarks are listed below:
Remark 1. [Ar Aθ Aϕ] is related to [Ax Ay Az] as:

$$\begin{aligned} \mathrm{Ar} &= \mathrm{Ax} \sin\theta \cos\phi + \mathrm{Ay} \sin\theta \sin\phi + \mathrm{Az} \cos\theta \,, \\ \mathrm{A}\theta &= \mathrm{Ax} \cos\theta \cos\phi + \mathrm{Ay} \cos\theta \sin\phi - \mathrm{Az} \sin\theta \,, \\ \mathrm{A}\phi &= - \mathrm{Ax} \sin\phi + \mathrm{Ay} \cos\phi \,. \end{aligned} \tag{A2.1}$$

Remark 2. Symmetry relations:

$$\mathrm{Ay}(x, y, z) = - \mathrm{Ax}(y, x, z) \,. \tag{A2.2}$$

Therefore, the independent components of A are Ax and Az.

$$\begin{aligned} &\mathrm{Ax}(-x, y, z) = \mathrm{Ax}(x, y, z) \,, \quad \mathrm{Ax}(x, -y, z) = - \mathrm{Ax}(x, y, z) \,, \\ &\mathrm{Ax}(x, y, -z) = \mathrm{Ax}(x, y, z) \,; \end{aligned} \tag{A2.3}$$

$$\begin{aligned} &\mathrm{Az}(-x, y, z) = - \mathrm{Az}(x, y, z) \,, \quad \mathrm{Az}(x, -y, z) = - \mathrm{Az}(x, y, z) \,, \\ &\mathrm{Az}(x, y, -z) = - \mathrm{Az}(x, y, z) \,; \end{aligned} \tag{A2.4}$$

$$\begin{aligned} &\phi(-x, y, z) = - \phi(x, y, z) \,, \quad \phi(x, -y, z) = - \phi(x, y, z) \,, \\ &\phi(x, y, -z) = \phi(x, y, z) \,. \end{aligned} \tag{A2.5}$$

Remark 3. From (A2.5) follows that $\phi(0, y, z) = \phi(x, 0, z) = 0$. Therefore, the interface condition (5.10)′ ((6.10)) is not necessary, and, consequently, the calculation of the electric scalar potential in free space is not required.
Remark 4. The treatment of a singular point:

$$[\exp(\mathrm{jk}|r - r'|)]/|r - r'| \to 1/|r - r'| + \mathrm{jk} \tag{A2.6}$$

as $|r - r'| \to 0$.

Table 11. FORTRAN program of the Lorentz gauge formulation using the boundary integral equations.

```
      program eddy3600
c
c     3d eddy currents in a sphere
c     magnetic vector potential & electric scalar potential
c     Lorentz gauge
c     boundary integral equation method
c     eddy3600, 1992.03.23
c     ------------------------------------------------
      implicit real*8(a-h,o-z)
      complex*16 gx(100,100),hx(100,100),gz(100,100),hz(100,100)
      complex*16 vg(100,100),vh(100,100),area(10)
      complex*16 a(600,600),b(600),aw(600,50)
      complex*16 ax(10,10),ay(10,10),az(10,10),vv(10,10),ep(10,10)
      complex*16 ar(10,10),at(10,10),ap(10,10)
      complex*16 c1,c2,c3,c4,c5,c6,c7,c8,c9,c10
      complex*16 pc1,pc2,pc3,pc4,pc5,pc6,pc7,pc8
      complex*16 qc1,qc2,qc3,qc4,qc5,qc6,qc7,qc8
      dimension ggx(100,100),hhx(100,100)
      dimension ggz(100,100),hhz(100,100)
      dimension e(3),w(3)
      dimension xw(3,3),yw(3,3),zw(3,3)
      dimension xv(3,3),yv(3,3),zv(3,3)
      dimension anx(3,3),any(3,3),anz(3,3)
      dimension bnx(3,3),bny(3,3),bnz(3,3)
      dimension st(3),zeta(10),ss(10),cc(10)
c     ------------------------------------------------
      pi=atan(1.d0)*4.d0
      amu=4.d-7*pi
      sigma=2.5d+7
      freq=10.d1
      along=.028d0
      radius=.02d0
      nl=10
c     ------------------------------------------------
      write(6,*) '3D Eddy Currents in an Aluminum Sphere'
      write(6,*)
      write(6,*)
      write(6,*) 'radius = 20 mm, mu = 4.d-7*pi H/m, sigma = 2.5d+7 S/m'
      write(6,*)
      write(6,*) 'frequency = 100 Hz'
      write(6,*)
      write(6,*) 'one-turn square coil (56 mm x 56 mm), Io = 1.d+7/4pi A
     1'
      write(6,*)
      write(6,*) 'BIEM using Magnetic Vector Potential'
      write(6,*)
      write(6,*) 'Lorentz Gauge'
      write(6,*)
c     ------------------------------------------------
      e(3)=sqrt(3.d0/5.d0)
      e(2)=0.d0
      e(1)=-e(3)
      w(1)=5.d0/9.d0
      w(2)=8.d0/9.d0
      w(3)=5.d0/9.d0
c     --------------------
      omega=2.d0*pi*freq
      sigmu=sigma*amu
      oms=omega*amu*sigma
      pp=sqrt(oms/2.d0)
      c1=cmplx(-pp,pp)
      c2=cmplx(-pp,-pp)
```

```
      c3=cmplx(0.d0,0.d0)
      c10=cmplx(0.d0,omega)
c     --------------------
      dt=pi/(2.d0*n1)
      dth=dt/2.d0
      dp=dt
      dph=dth
      beta=radius*radius*dth*dph/(4.d0*pi)
      d1=radius*dth
      do i=1,n1
      tt=dt*(i-.5d0)
      d2=radius*sin(tt)*dph
      p1=sqrt(d1*d1+d2*d2)
      p2=d1*log((d2+p1)/d1)+d2*log((d1+p1)/d2)
      zeta(i)=p2/pi
      area(i)=c2*d1*d2/pi
      ss(i)=sin(tt)
      cc(i)=cos(tt)
      end do
c     --------------------
      n11=n1+1
      n12=n1-1
      nh=n1/2
      nn1=n1*n1
      nnh=n1*nh
      is1=nn1
      is2=is1+nnh
      is3=is2+nnh
      is4=is3+nn1
      is5=is4+nnh
      is6=is5+nn1
      is7=is6+nnh
      n=is7+nnh
c     ----------------------------------------
c     Green matrix
      do i=1,n1
      tt=dt*(i-.5d0)
      zk=radius*cos(tt)
      ri=radius*sin(tt)
      do j=1,n1
      pp=dp*(j-.5d0)
      xi=ri*cos(pp)
      yj=ri*sin(pp)
      l=i+n1*(j-1)
c     --------------------
      do ii=1,n1
      t2=dt*(ii-.5d0)
      t1=t2+e(1)*dth
      t3=t2+e(3)*dth
      ct1=cos(t1)
      ct2=cos(t2)
      ct3=cos(t3)
      st1=sin(t1)
      st2=sin(t2)
      st3=sin(t3)
      zz=radius*ct1
      zw(1,1)=zz
      zw(1,2)=zz
      zw(1,3)=zz
      anz(1,1)=ct1
      anz(1,2)=ct1
      anz(1,3)=ct1
      zz=radius*ct2
      zw(2,1)=zz
      zw(2,2)=zz
      zw(2,3)=zz
```

```
      anz(2,1)=ct2
      anz(2,2)=ct2
      anz(2,3)=ct2
      zz=radius*ct3
      zw(3,1)=zz
      zw(3,2)=zz
      zw(3,3)=zz
      anz(3,1)=ct3
      anz(3,2)=ct3
      anz(3,3)=ct3
      r1=radius*st1
      r2=radius*st2
      r3=radius*st3
c     --------------------
      do jj=1,n1
      p2=dp*(jj-.5d0)
      p1=p2+e(1)*dph
      p3=p2+e(3)*dph
      cp1=cos(p1)
      cp2=cos(p2)
      cp3=cos(p3)
      sp1=sin(p1)
      sp2=sin(p2)
      sp3=sin(p3)
      xw(1,1)=r1*cp1
      xw(1,2)=r1*cp2
      xw(1,3)=r1*cp3
      xw(2,1)=r2*cp1
      xw(2,2)=r2*cp2
      xw(2,3)=r2*cp3
      xw(3,1)=r3*cp1
      xw(3,2)=r3*cp2
      xw(3,3)=r3*cp3
      yw(1,1)=r1*sp1
      yw(1,2)=r1*sp2
      yw(1,3)=r1*sp3
      yw(2,1)=r2*sp1
      yw(2,2)=r2*sp2
      yw(2,3)=r2*sp3
      yw(3,1)=r3*sp1
      yw(3,2)=r3*sp2
      yw(3,3)=r3*sp3
      anx(1,1)=st1*cp1
      anx(1,2)=st1*cp2
      anx(1,3)=st1*cp3
      anx(2,1)=st2*cp1
      anx(2,2)=st2*cp2
      anx(2,3)=st2*cp3
      anx(3,1)=st3*cp1
      anx(3,2)=st3*cp2
      anx(3,3)=st3*cp3
      any(1,1)=st1*sp1
      any(1,2)=st1*sp2
      any(1,3)=st1*sp3
      any(2,1)=st2*sp1
      any(2,2)=st2*sp2
      any(2,3)=st2*sp3
      any(3,1)=st3*sp1
      any(3,2)=st3*sp2
      any(3,3)=st3*sp3
c     --------------------
      ll=ii+n1*(jj-1)
      st(1)=st1
      st(2)=st2
      st(3)=st3
      if (l.eq.ll) then
```

```
      p1=0.d0
      q1=0.d0
      pc1=c3
      qc1=c3
      goto 100
      else
      do k=1,3
      do m=1,3
      xv(k,m)=xw(k,m)
      yv(k,m)=yw(k,m)
      zv(k,m)=zw(k,m)
      bnx(k,m)=anx(k,m)
      bny(k,m)=any(k,m)
      bnz(k,m)=anz(k,m)
      end do
      end do
      p=0.d0
      q=0.d0
      c4=c3
      c5=c3
      do k=1,3
      do m=1,3
      xx=xi-xv(k,m)
      yy=yj-yv(k,m)
      zz=zk-zv(k,m)
      rr2=xx*xx+yy*yy+zz*zz
      rr=sqrt(rr2)
      rr3=rr*rr*rr
      c6=exp(c2*rr)
      p=p+w(k)*w(m)*st(k)/rr
      c4=c4+w(k)*w(m)*st(k)*c6/rr
      qq=bnx(k,m)*xx+bny(k,m)*yy+bnz(k,m)*zz
      q=q+w(k)*w(m)*st(k)*qq/rr3
      c5=c5-w(k)*w(m)*st(k)*c6*(c2-1.d0/rr)*qq/rr2
      end do
      end do
      p1=p*beta
      q1=-q*beta
      pc1=c4*beta
      qc1=c5*beta
      goto 100
      end if
c     --------------------
  100 continue
      do k=1.3
      do m=1,3
      xv(k,m)=-xw(k,m)
      yv(k,m)=yw(k,m)
      zv(k,m)=zw(k,m)
      bnx(k,m)=-anx(k,m)
      bny(k,m)=any(k,m)
      bnz(k,m)=anz(k,m)
      end do
      end do
      p=0.d0
      q=0.d0
      c4=c3
      c5=c3
      do k=1,3
      do m=1,3
      xx=xi-xv(k,m)
      yy=yj-yv(k,m)
      zz=zk-zv(k,m)
      rr2=xx*xx+yy*yy+zz*zz
      rr=sqrt(rr2)
      rr3=rr*rr*rr
```

```
      c6=exp(c2*rr)
      p=p+w(k)*w(m)*st(k)/rr
      c4=c4+w(k)*w(m)*st(k)*c6/rr
      qq=bnx(k,m)*xx+bny(k,m)*yy+bnz(k,m)*zz
      q=q+w(k)*w(m)*st(k)*qq/rr3
      c5=c5-w(k)*w(m)*st(k)*c6*(c2-1.d0/rr)*qq/rr2
      end do
      end do
      p2=p*beta
      q2=-q*beta
      pc2=c4*beta
      qc2=c5*beta
c     --------------------
      do k=1,3
      do m=1,3
      xv(k,m)=-xw(k,m)
      yv(k,m)=-yw(k,m)
      zv(k,m)=zw(k,m)
      bnx(k,m)=-anx(k,m)
      bny(k,m)=-any(k,m)
      bnz(k,m)=anz(k,m)
      end do
      end do
      p=0.d0
      q=0.d0
      c4=c3
      c5=c3
      do k=1,3
      do m=1,3
      xx=xi-xv(k,m)
      yy=yj-yv(k,m)
      zz=zk-zv(k,m)
      rr2=xx*xx+yy*yy+zz*zz
      rr=sqrt(rr2)
      rr3=rr*rr*rr
      c6=exp(c2*rr)
      p=p+w(k)*w(m)*st(k)/rr
      c4=c4+w(k)*w(m)*st(k)*c6/rr
      qq=bnx(k,m)*xx+bny(k,m)*yy+bnz(k,m)*zz
      q=q+w(k)*w(m)*st(k)*qq/rr3
      c5=c5-w(k)*w(m)*st(k)*c6*(c2-1.d0/rr)*qq/rr2
      end do
      end do
      p3=p*beta
      q3=-q*beta
      pc3=c4*beta
      qc3=c5*beta
c     --------------------
      do k=1,3
      do m=1,3
      xv(k,m)=xw(k,m)
      yv(k,m)=-yw(k,m)
      zv(k,m)=zw(k,m)
      bnx(k,m)=anx(k,m)
      bny(k,m)=-any(k,m)
      bnz(k,m)=anz(k,m)
      end do
      end do
      p=0.d0
      q=0.d0
      c4=c3
      c5=c3
      do k=1,3
      do m=1,3
      xx=xi-xv(k,m)
      yy=yj-yv(k,m)
```

```
      zz=zk-zv(k,m)
      rr2=xx*xx+yy*yy+zz*zz
      rr=sqrt(rr2)
      rr3=rr*rr*rr
      c6=exp(c2*rr)
      p=p+w(k)*w(m)*st(k)/rr
      c4=c4+w(k)*w(m)*st(k)*c6/rr
      qq=bnx(k,m)*xx+bny(k,m)*yy+bnz(k,m)*zz
      q=q+w(k)*w(m)*st(k)*qq/rr3
      c5=c5-w(k)*w(m)*st(k)*c6*(c2-1.d0/rr)*qq/rr2
      end do
      end do
      p4=p*beta
      q4=-q*beta
      pc4=c4*beta
      qc4=c5*beta
c     --------------------
      do k=1,3
      do m=1,3
      xv(k,m)=xw(k,m)
      yv(k,m)=yw(k,m)
      zv(k,m)=-zw(k,m)
      bnx(k,m)=anx(k,m)
      bny(k,m)=any(k,m)
      bnz(k,m)=-anz(k,m)
      end do
      end do
      p=0.d0
      q=0.d0
      c4=c3
      c5=c3
      do k=1,3
      do m=1,3
      xx=xi-xv(k,m)
      yy=yj-yv(k,m)
      zz=zk-zv(k,m)
      rr2=xx*xx+yy*yy+zz*zz
      rr=sqrt(rr2)
      rr3=rr*rr*rr
      c6=exp(c2*rr)
      p=p+w(k)*w(m)*st(k)/rr
      c4=c4+w(k)*w(m)*st(k)*c6/rr
      qq=bnx(k,m)*xx+bny(k,m)*yy+bnz(k,m)*zz
      q=q+w(k)*w(m)*st(k)*qq/rr3
      c5=c5-w(k)*w(m)*st(k)*c6*(c2-1.d0/rr)*qq/rr2
      end do
      end do
      p5=p*beta
      q5=-q*beta
      pc5=c4*beta
      qc5=c5*beta
c     --------------------
      do k=1,3
      do m=1,3
      xv(k,m)=-xw(k,m)
      yv(k,m)=yw(k,m)
      zv(k,m)=-zw(k,m)
      bnx(k,m)=-anx(k,m)
      bny(k,m)=any(k,m)
      bnz(k,m)=-anz(k,m)
      end do
      end do
      p=0.d0
      q=0.d0
      c4=c3
      c5=c3
```

```
      do k=1,3
      do m=1,3
      xx=xi-xv(k,m)
      yy=yj-yv(k,m)
      zz=zk-zv(k,m)
      rr2=xx*xx+yy*yy+zz*zz
      rr=sqrt(rr2)
      rr3=rr*rr*rr
      c6=exp(c2*rr)
      p=p+w(k)*w(m)*st(k)/rr
      c4=c4+w(k)*w(m)*st(k)*c6/rr
      qq=bnx(k,m)*xx+bny(k,m)*yy+bnz(k,m)*zz
      q=q+w(k)*w(m)*st(k)*qq/rr3
      c5=c5-w(k)*w(m)*st(k)*c6*(c2-1.d0/rr)*qq/rr2
      end do
      end do
      p6=p*beta
      q6=-q*beta
      pc6=c4*beta
      qc6=c5*beta
c     --------------------
      do k=1,3
      do m=1,3
      xv(k,m)=-xw(k,m)
      yv(k,m)=-yw(k,m)
      zv(k,m)=-zw(k,m)
      bnx(k,m)=-anx(k,m)
      bny(k,m)=-any(k,m)
      bnz(k,m)=-anz(k,m)
      end do
      end do
      p=0.d0
      q=0.d0
      c4=c3
      c5=c3
      do k=1,3
      do m=1,3
      xx=xi-xv(k,m)
      yy=yj-yv(k,m)
      zz=zk-zv(k,m)
      rr2=xx*xx+yy*yy+zz*zz
      rr=sqrt(rr2)
      rr3=rr*rr*rr
      c6=exp(c2*rr)
      p=p+w(k)*w(m)*st(k)/rr
      c4=c4+w(k)*w(m)*st(k)*c6/rr
      qq=bnx(k,m)*xx+bny(k,m)*yy+bnz(k,m)*zz
      q=q+w(k)*w(m)*st(k)*qq/rr3
      c5=c5-w(k)*w(m)*st(k)*c6*(c2-1.d0/rr)*qq/rr2
      end do
      end do
      p7=p*beta
      q7=-q*beta
      pc7=c4*beta
      qc7=c5*beta
c     --------------------
      do k=1,3
      do m=1,3
      xv(k,m)=xw(k,m)
      yv(k,m)=-yw(k,m)
      zv(k,m)=-zw(k,m)
      bnx(k,m)=anx(k,m)
      bny(k,m)=-any(k,m)
      bnz(k,m)=-anz(k,m)
      end do
      end do
```

```
      p=0.d0
      q=0.d0
      c4=c3
      c5=c3
      do k=1,3
      do m=1,3
      xx=xi-xv(k,m)
      yy=yj-yv(k,m)
      zz=zk-zv(k,m)
      rr2=xx*xx+yy*yy+zz*zz
      rr=sqrt(rr2)
      rr3=rr*rr*rr
      c6=exp(c2*rr)
      p=p+w(k)*w(m)*st(k)/rr
      c4=c4+w(k)*w(m)*st(k)*c6/rr
      qq=bnx(k,m)*xx+bny(k,m)*yy+bnz(k,m)*zz
      q=q+w(k)*w(m)*st(k)*qq/rr3
      c5=c5-w(k)*w(m)*st(k)*c6*(c2-1.d0/rr)*qq/rr2
      end do
      end do
      p8=p*beta
      q8=-q*beta
      pc8=c4*beta
      qc8=c5*beta
c     --------------------
      ggx(l,ll)=p1+p2-p3-p4+p5+p6-p7-p8
      hhx(l,ll)=q1+q2-q3-q4+q5+q6-q7-q8
      ggz(l,ll)=p1-p2+p3-p4-p5+p6-p7+p8
      hhz(l,ll)=q1-q2+q3-q4-q5+q6-q7+q8
      gx(l,ll)=pc1+pc2-pc3-pc4+pc5+pc6-pc7-pc8
      hx(l,ll)=qc1+qc2-qc3-qc4+qc5+qc6-qc7-qc8
      gz(l,ll)=pc1-pc2+pc3-pc4-pc5+pc6-pc7+pc8
      hz(l,ll)=qc1-qc2+qc3-qc4-qc5+qc6-qc7+qc8
      vg(l,ll)=pc1-pc2+pc3-pc4+pc5-pc6+pc7-pc8
      vh(l,ll)=qc1-qc2+qc3-qc4+qc5-qc6+qc7-qc8
      end do
      end do
c     --------------------
      end do
      end do
c     ----------------------------------------
      do i=1,n1
      do j=1,n1
      ij=i+n1*(j-1)
      ggx(ij,ij)=ggx(ij,ij)+zeta(i)
      hhx(ij,ij)=hhx(ij,ij)+.5d0
      ggz(ij,ij)=ggz(ij,ij)+zeta(i)
      hhz(ij,ij)=hhz(ij,ij)+.5d0
      gx(ij,ij)=gx(ij,ij)+zeta(i)+area(i)
      hx(ij,ij)=hx(ij,ij)+.5d0
      gz(ij,ij)=gz(ij,ij)+zeta(i)+area(i)
      hz(ij,ij)=hz(ij,ij)+.5d0
      vg(ij,ij)=vg(ij,ij)+zeta(i)+area(i)
      vh(ij,ij)=vh(ij,ij)+.5d0
      end do
      end do
c     ----------------------------------------
c     matrix forming
      do i=1,nn1
      do j=1,nn1
      jj=j+is5
      a(i,j)=hx(i,j)
      a(i,jj)=-gx(i,j)
      end do
      end do
c     --------------------
```

```
      do i=1,nn1
      ii=i+is3
      do j=1,nn1
      jj=j+is3
      a(ii,j)=hhx(i,j)
      a(ii,jj)=-ggx(i,j)
      end do
      end do
c     --------------------
      do i=1,n1
      do j=1,nh
      k=i+n1*(j-1)
      ki=k+is1
      do ii=1,n1
      do jj=1,nh
      kk=ii+n1*(jj-1)
      km=ii+n1*(n11-jj-1)
      j1=kk+is1
      j2=kk+is6
      a(ki,j1)=hz(k,kk)-hz(k,km)
      a(ki,j2)=gz(k,km)-gz(k,kk)
      end do
      end do
      end do
      end do
c     --------------------
      do i=1,n1
      do j=1,nh
      k=i+n1*(j-1)
      ki=k+is2
      do ii=1,n1
      do jj=1,nh
      kk=ii+n1*(jj-1)
      km=ii+n1*(n11-jj-1)
      j1=kk+is2
      j2=kk+is7
      a(ki,j1)=vh(k,kk)-vh(k,km)
      a(ki,j2)=vg(k,km)-vg(k,kk)
      end do
      end do
      end do
      end do
c     --------------------
      do i=1,n1
      do j=1,nh
      k=i+n1*(j-1)
      ki=k+is4
      do ii=1,n1
      do jj=1,nh
      kk=ii+n1*(jj-1)
      km=ii+n1*(n11-jj-1)
      j1=kk+is1
      j2=kk+is4
      a(ki,j1)=hhz(k,kk)-hhz(k,km)
      a(ki,j2)=ggz(k,km)-ggz(k,kk)
      end do
      end do
      end do
      end do
c     --------------------
      do i=1,n1
      do j=1,nh
      k=i+n1*(j-1)
      kit=k+is5
      km=i+n1*(n11-j-1)
      j1=k+is3
```

```
      j2=km+is3
      j3=k+is4
      j4=k+is5
      j5=km+is5
      j6=k+is6
      a(kit,j1)=cc(i)*cc(j)
      a(kit,j2)=-cc(i)*ss(j)
      a(kit,j3)=-ss(i)
      a(kit,j4)=cc(i)*cc(j)
      a(kit,j5)=-cc(i)*ss(j)
      a(kit,j6)=-ss(i)
      end do
      end do
c     --------------------
      do i=1,n1
      do j=1,nh
      k=i+n1*(j-1)
      kip=k+is5+nnh
      km=i+n1*(n11-j-1)
      j1=k+is3
      j2=km+is3
      j3=k+is5
      j4=km+is5
      a(kip,j1)=-ss(j)
      a(kip,j2)=-cc(j)
      a(kip,j3)=-ss(j)
      a(kip,j4)=-cc(j)
      end do
      end do
c     --------------------
      do i=1,n1
      do j=1,nh
      k=i+n1*(j-1)
      ki=k+is6
      km=i+n1*(n11-j-1)
      j1=k+is2
      j2=k+is3
      j3=km+is3
      j4=k+is4
      j5=k+is5
      j6=km+is5
      j7=k+is6
      a(ki,j1)=sigmu
      a(ki,j2)=ss(i)*cc(j)
      a(ki,j3)=-ss(i)*ss(j)
      a(ki,j4)=cc(i)
      a(ki,j5)=ss(i)*cc(j)
      a(ki,j6)=-ss(i)*ss(j)
      a(ki,j7)=cc(i)
      end do
      end do
c     --------------------
      do i=1,n1
      do j=1,nh
      k=i+n1*(j-1)
      ki=k+is7
      km=i+n1*(n11-j-1)
      j1=k
      j2=km
      j3=k+is1
      j4=k+is7
      a(ki,j1)=c10*ss(i)*cc(j)
      a(ki,j2)=-c10*ss(i)*ss(j)
      a(ki,j3)=c10*cc(i)
      a(ki,j4)=1.d0
      end do
```

```
      end do
c     --------------------
c     column swapping
      do i=1,n
      do j=1,nnh
      jj=j+is5+nnh
      aw(i,j)=a(i,jj)
      end do
      end do
      do i=1,n
      do ii=1,n1
      do jj=1,nh
      kk=ii+n1*(jj-1)
      km=ii+n1*(nh-jj)
      j=kk+is5+nnh
      a(i,j)=-aw(i,km)
      end do
      end do
      end do
c     ----------------------------------------
c     external field
      nc=100
      dl=2.d0*along/nc
      do i=1,n1
      zk=radius*cc(i)
      ri=radius*ss(i)
      do j=1,n1
      xi=ri*cc(j)
      yj=ri*ss(j)
      ij=i+n1*(j-1)+is3
      yy1=yj-along
      yy2=yj+along
      yz1=yy1*yy1+zk*zk
      yz2=yy2*yy2+zk*zk
      p=0.d0
      q=0.d0
      do k=1,nc
      xkk=dl*(k-.5d0)-along
      xx=xi-xkk
      xx2=xx*xx
      r1=sqrt(xx2+yz1)
      r2=sqrt(xx2+yz2)
      p=p+1.d0/r1
      q=q+1.d0/r2
      end do
      qp=(q-p)*dl/(4.d0*pi)
      b(ij)=cmplx(qp,0.d0)
      end do
      end do
c     ----------------------------------------
c     linear simultaneous equations
      nn=n-1
      do i=1,nn
      i1=i+1
      do j=i1,n
      c5=a(j,i)/a(i,i)
      do k=i1,n
      a(j,k)=a(j,k)-c5*a(i,k)
      end do
      b(j)=b(j)-c5*b(i)
      end do
      end do
      b(n)=b(n)/a(n,n)
      do i=1,nn
      ii=n-i
      i1=ii+1
```

```
      do j=i1,n
      b(ii)=b(ii)-a(ii,j)*b(j)
      end do
      b(ii)=b(ii)/a(ii,ii)
      end do
c     ----------------------------------------
c     vector potential and scalar potential
      do i=1,n1
      do j=1,n1
      jm=n11-j
      k=i+n1*(j-1)
      km=i+n1*(jm-1)
      ax(i,j)=b(k)
      ay(i,j)=-b(km)
      end do
      end do
c     --------------------
      do i=1,n1
      do j=1,nh
      k=i+n1*(j-1)
      az(i,j)=b(k+is1)
      vv(i,j)=b(k+is2)
      jm=n11-j
      az(i,jm)=-az(i,j)
      vv(i,jm)=-vv(i,j)
      end do
      end do
c     ----------------------------------------
c     magnetic vector potential Ar, At, and Ap
      do i=1,n1
      do j=1,n1
      ar(i,j)=ax(i,j)*ss(i)*cc(j)+ay(i,j)*ss(i)*ss(j)+az(i,j)*cc(i)
      at(i,j)=ax(i,j)*cc(i)*cc(j)+ay(i,j)*cc(i)*ss(j)-az(i,j)*ss(i)
      ap(i,j)=-ax(i,j)*ss(j)+ay(i,j)*cc(j)
      end do
      end do
  600 format(2d18.10,3x,2d18.10)
c     ----------------------------------------
c     electric scalar potential
      write(6,*)
      write(6,*) 'Electric Scalar Potential at the Surface '
      write(6,*)
      do j=1,n1
      write(6,600) vv(1,j),vv(n1,j)
      end do
c     ----------------------------------------
c     eddy current Jp
      write(6,*)
      write(6,*) 'Eddy Current Jp at the Surface'
      write(6,*)
      do i=1,n1
      ddp=radius*ss(i)*dp*2.d0
      do j=2,n12
      c5=(vv(i,j+1)-vv(i,j-1))/ddp
      c6=-ax(i,j)*ss(j)+ay(i,j)*cc(j)
      ep(i,j)=sigma*(-c10*c6-c5)
      end do
      c5=(vv(i,2)+vv(i,1))/ddp
      c6=-ax(i,1)*ss(1)+ay(i,1)*cc(1)
      ep(i,1)=sigma*(-c10*c6-c5)
      c5=-(vv(i,n1)+vv(i,n12))/ddp
      c6=-ax(i,n1)*ss(n1)+ay(i,n1)*cc(n1)
      ep(i,n1)=sigma*(-c10*c6-c5)
      end do
      do j=1,n1
      write(6,600) ep(1,j),ep(n1,j)
```

```
      end do
c     ---------------------------------------
      write(6,*)
      call time
      stop
      end
c     ---------------------------------------
      subroutine time
      itime=mclock()/100
      write(6,800) itime
  800 format('***** Time Accounting : ',i10,' sec *****')
      return
      end

 * * *  End of File  * * *
```

Appendix 3

From (11.25), after a simple calculation, we obtain $(y > 0)$

$$\int_0^\infty \exp(-\xi y) \cos[\xi(x - x_0)]\, d\xi = y/[y^2 + (x - x_0)^2]\,. \tag{A3.1}$$

From (11.26), (11.27) and (A3.1) follows

$$\phi(x, y) = 1/\pi \int_{-a}^{a} y/[y^2 + (x - x_0)^2]\, dx_0$$

$$= 1/\pi\, [\tan^{-1}((a - x)/y) + \tan^{-1}((a + x)/y)]\,. \tag{A3.2}$$

Note. $\int 1/(1 + x^2)\, dx = \tan^{-1} x$.