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A BASIC STUDY ON A HYBRID STRUCTURE CONSISTING OF CABLES AND RIGID STRUCTURES

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Abstract

In the paper, a new type of hybrid structure consisting of cables and rigid structures is proposed. In the hybrid structure, cables are used as tensile members, rigid structures are used as compressive members. Rigid structures in the hybrid structure may have arbitrary shapes, such as lines, circles, cubes, and so on. Initial stress is introduced in the system to provide enough geometric stiffness, so that the proposed hybrid structure is named "Hybrid Structure Stabilized by Tension".

In the first step of the paper, kinematic relations and equilibrium relations are formulated for cables and rigid structures and then for the hybrid structure. The kinematic equations and the equilibrium equations are called basic equations. In the basic equations, the degrees of freedom of rigid structures are not considered by using the displacements of the centers of gravity as usual, but by using the displacements of the nodes on their surfaces which are connected with cable members. To this end, generalized inverse is effectively used in the paper.

By using these basic relations, the following items about the hybrid structure are studied.

(1) Classification of stable and unstable hybrid structures
(2) Introduction of pre-stress for stability and initial stiffness
(3) Stress and displacement analysis under static loads
(4) Vibration analysis
(5) In order to examine the validity of the theoretical analyses, experiments on a hybrid structure model are done as follows: i) the introduction of pre-stress, ii) static loading experiment.

Keywords: hybrid structure, tension structure, cable structure, rigid structure, generalized inverse, equilibrated stress, pre-stress introduction, stability, static analysis, vibration analysis

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Contents

1. Introduction

Under the increasing demands for long span spatial structures and light weight spatial structures, cable structures, membrane structures and tension structures, have been developed rapidly in recent years. Tension structures consist of both tensile members and compressive members. Cables are often used as tensile members while rods with large cross sections or posts can be used as compressive members.

In the unstable state there exists a structural problem for cable structures, tensegric structures, hybrid structures, etc., and the primary structural characteristics of the problem are in the existence of rigid body displacement without stain and in the possibility of the introduction of pre-stress to change an unstable into a stable state. The reason why these structures in the unstable state, so called "unstable structures", can be adopted for real structures is that a positive geometric stiffness matrix can be constructed by introducing the pre-stresses. In the paper, the term "rigid body displacement" means "displacement without strain", and is used for the inextensional displacement in the internal mechanism.

Theoretical methods to investigate the structural behaviors of unstable structures have been proposed in the context of a discrete multi-degrees-of-freedom system[1-3], the investigation of self-equilibrated stress system and structural behaviors of truss structures stabilized by cable tension[4] and others. In the design field, a variety of tensegrity system has been proposed[5-7], and the most successful application is the cable dome proposed by Geiger[8].

In the paper, an analytical method for examining structural behaviors of a hybrid structure which consists of cables and rigid structures (see Fig. 1) is proposed by extending the above mentioned methods. In the analytical method, the Moor-Penrose generalized inverse for the rectangular coefficient matrices is effectively used for the theoretical treatment of the followings: (1) the classification of stable and unstable hybrid structures, (2) the existence condition and the analytical method of self-equilibrated stress systems, (3) stability condition of hybrid structures after the introduction of pre-stress, (4) the stress and displacement analysis under static loads and (5) the vibration analysis. In order to examine the validity of the proposed analytical method, an experiment on a hybrid structure model is reported, taking into consideration the introduction of pre-stress and the deformation under the static load.

2. Basic Equations for Hybrid Structure

In this chapter, the kinematic equations and equilibrium equations for hybrid structure are formulated. For a cable member, kinematic equations can be derived by using the coordinates of the nodes of the member, equilibrium equations can be derived by using the axial force of the member and the forces on its nodes. The kinematic equations for a rigid structure are derived by using the displacement components of the nodes on the surface rather than its center of gravity. In order to eliminate the freedom of the center of gravity, a theoretical method based on the generalized inverse is proposed. The equilibrium equations of a rigid structure are derived by considering the rigid structure to be in the equilibrium state with the axial forces of cable members. In the

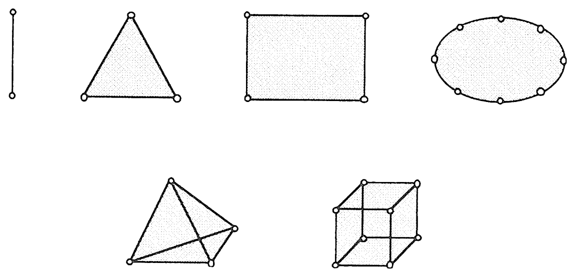


Fig. 1 Examples of rigid structures

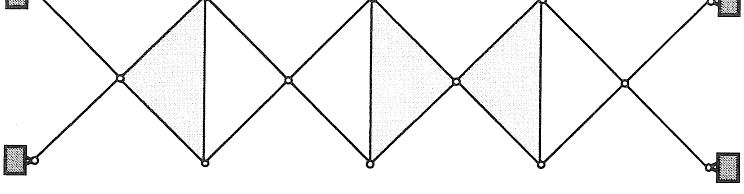


Fig. 2 An example of hybrid structure

equilibrium equations, forces on the nodes which are connected with cable members are used.

By collecting the kinematic equations for all the cable members and all the rigid structures, and collecting the equilibrium equations of all the cable members and all the rigid structures, the basic equations for hybrid structure are obtained.

2.1 Kinematic equations

Consider a hybrid structure which consists of cables and rigid structures as shown in Fig. 2. Cables and rigid structures are connected at nodes on the surface of rigid structures. Pre-stress is introduced and all cable members are in the tension state. Let a be a cable member ($a = 1,2,\ldots, m_a$, m_a is the total number of cable member), and b be a rigid structure ($b = 1,2,\ldots, m_b$, m_b is the total number of rigid structure).

Fig. 3 (a) shows a cable member a whose nodal points are i and j in a Cartesian coordinate system. Let the coordinates of i and j be represented in the vector form as

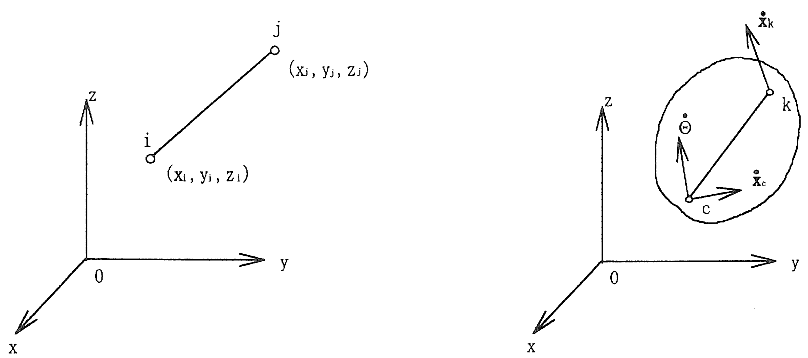
$$\mathbf{x}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}, \quad \mathbf{x}_j = \begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} \tag{1}$$

Then the length and the direction cosines of the element are

$$l_a = \left[\left(\mathbf{x}_j - \mathbf{x}_i\right)^T \left(\mathbf{x}_j - \mathbf{x}_i\right) \right]^{1/2} \tag{2}$$

$$\boldsymbol{\lambda}_a = \frac{1}{l_a}\left(\mathbf{x}_j - \mathbf{x}_i\right) \tag{3}$$

Let $\boldsymbol{\lambda}_a$ and l_a be functions of an arbitrary parameter t, such as functions of time. The first and



(a) A cable member

(b) A rigid structure

Fig. 3 Kinematic relation

the second derivatives of l_a with respect to t take the form

$$\left[-\boldsymbol{\lambda}_a^T \ \boldsymbol{\lambda}_a^T\right]\begin{bmatrix}\dot{\mathbf{x}}_i\\ \dot{\mathbf{x}}_j\end{bmatrix} = \dot{l}_a \tag{4}$$

$$\left[-\boldsymbol{\lambda}_a^T \ \boldsymbol{\lambda}_a^T\right]\begin{bmatrix}\ddot{\mathbf{x}}_i\\ \ddot{\mathbf{x}}_j\end{bmatrix} + \left[-\dot{\boldsymbol{\lambda}}_a^T \ \dot{\boldsymbol{\lambda}}_a^T\right]\begin{bmatrix}\dot{\mathbf{x}}_i\\ \dot{\mathbf{x}}_j\end{bmatrix} = \ddot{l}_a \tag{5}$$

Eqs. (4) and (5) are kinematic relations of cable member a.

In the case of a rigid member which means the elongation is zero, $\ddot{l}_a = \dot{l}_a = 0$ holds, and then Eqs. (4) and (5) become

$$\left[-\boldsymbol{\lambda}_a^T \ \boldsymbol{\lambda}_a^T\right]\begin{bmatrix}\dot{\mathbf{x}}_i\\ \dot{\mathbf{x}}_j\end{bmatrix} = \mathbf{0} \tag{6}$$

$$\left[-\boldsymbol{\lambda}_a^T \ \boldsymbol{\lambda}_a^T\right]\begin{bmatrix}\ddot{\mathbf{x}}_i\\ \ddot{\mathbf{x}}_j\end{bmatrix} + \ddot{\Phi}_a = \mathbf{0} \tag{7}$$

where

$$\ddot{\Phi}_a = \left[-\dot{\boldsymbol{\lambda}}_a^T \ \dot{\boldsymbol{\lambda}}_a^T\right]\begin{bmatrix}\dot{\mathbf{x}}_i\\ \dot{\mathbf{x}}_j\end{bmatrix} = \frac{1}{l_a}\left(\dot{\mathbf{x}}_j - \dot{\mathbf{x}}_i\right)^T\left(\dot{\mathbf{x}}_j - \dot{\mathbf{x}}_i\right) \tag{8}$$

Consider a rigid structure b which has nodes k (k = 1, 2,..., k_b, k_b is the total number of nodes on rigid structure b) shown in Fig. 3 (b). Let $\mathbf{x}_k$ be the coordinate vector of node k and $\boldsymbol{\Theta}$ the rotation vector of the rigid structure with respect to x,y,z axes. Let $\mathbf{x}_c$ be the coordinate vector of an arbitrary point c in the rigid structure, and $\dot{\mathbf{X}}$ the velocity vector such as

$$\mathbf{x}_k = \begin{bmatrix}x_k\\ y_k\\ z_k\end{bmatrix} \quad \mathbf{x}_c = \begin{bmatrix}x_c\\ y_c\\ z_c\end{bmatrix} \quad \Theta = \begin{bmatrix}\theta_x\\ \theta_y\\ \theta_z\end{bmatrix} \quad \dot{\mathbf{X}} = \begin{bmatrix}\dot{\mathbf{x}}_c\\ \dot{\Theta}\end{bmatrix} \tag{9}$$

Then the velocity of node k can be represented by

$$\dot{\mathbf{x}}_k = \dot{\mathbf{x}}_c + \dot{\Theta} \times \left(\mathbf{x}_k - \mathbf{x}_c\right) \tag{10}$$

which takes the form

$$\dot{\mathbf{x}}_k = \mathbf{H}_k\dot{\mathbf{X}} \ \ (k = 1,2,....,k_b) \tag{11}$$

where

$$\mathbf{H}_k = \begin{bmatrix} 1 & 0 & 0 & 0 & z_k - z_c & -(y_k - y_c) \\ 0 & 1 & 0 & -(z_k - z_c) & 0 & x_k - x_c \\ 0 & 0 & 1 & y_k - y_c & -(x_k - x_c) & 0 \end{bmatrix} (k = 1,2,\ldots,k_b) \tag{12}$$

Writing Eq. (11) for all nodes, and then collecting them into a single matrix equation, we obtain

$$\dot{\mathbf{x}}_b = \mathbf{H}\dot{\mathbf{X}} \tag{13}$$

in which

$$\dot{\mathbf{x}}_b = \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \cdot \\ \cdot \\ \dot{\mathbf{x}}_{k_b} \end{bmatrix} \quad \mathbf{H} = \begin{bmatrix} \mathbf{H}_1 \\ \cdot \\ \cdot \\ \mathbf{H}_{k_b} \end{bmatrix} \tag{14}$$

$\mathbf{H}$ is a $3k_b \times 6$ matrix. The necessary and sufficient condition to have solution can be used in order to eliminate $\dot{\mathbf{X}}$ in Eq. (13), that is[9)]

$$\left[\mathbf{I} - \mathbf{H}\mathbf{H}^+\right] \dot{\mathbf{x}}_b = 0 \tag{15}$$

where $\mathbf{H}^+$ is the Moore-Penrose generalized inverse matrix of $\mathbf{H}$. Eq. (15) is the first order kinematic equation of rigid structure b.
Differentiating Eq. (13) with respect to t leads to

$$\ddot{\mathbf{x}}_b = \mathbf{H}\ddot{\mathbf{X}} + \dot{\mathbf{H}}\dot{\mathbf{X}} \tag{16}$$

which can be written as

$$\mathbf{H}\ddot{\mathbf{X}} = \left(\ddot{\mathbf{x}}_b - \dot{\mathbf{H}}\dot{\mathbf{X}}\right) \tag{17}$$

the necessary and sufficient condition to have solution of Eq. (17) gives us

$$\left[\mathbf{I} - \mathbf{H}\mathbf{H}^+\right]\left(\ddot{\mathbf{x}}_b - \dot{\mathbf{H}}\dot{\mathbf{X}}\right) = 0 \tag{18}$$

From Eq. (13), we get

$$\dot{\mathbf{X}} = \mathbf{H}^+\dot{\mathbf{x}}_b \tag{19}$$

The introduction of Eq. (19) into Eq. (18) leads to

$$\left[\mathbf{I} - \mathbf{H}\mathbf{H}^+\right] \ddot{\mathbf{x}}_b + \ddot{\Phi}_b = 0 \tag{20}$$

where

$$\ddot{\boldsymbol{\Phi}}_b = -\left[\mathbf{I} - \mathbf{H}\mathbf{H}^+\right]\dot{\mathbf{H}}\mathbf{H}^+\dot{\mathbf{x}}_b \tag{21}$$

Eq. (20) is the second order kinematic equation.

Writing Eq. (6) for all cable members and Eq. (15) for all rigid structures, and then collecting them into a single matrix equation, after deleting boundary nodes, we get

$$\mathbf{A}\dot{\mathbf{x}} = 0 \tag{22}$$

where $\mathbf{x}$ is the coordinate vector of all nodes except boundary fixed nodes, $\mathbf{A}$ a $m \times n$ matrix, n the total number of degrees of freedom, and m is

$$m = m_a + 3\sum_{b=1}^{m_b} k_b \tag{23}$$

In a similar manner, from Eq. (7) and Eq. (20), we get

$$\mathbf{A}\ddot{\mathbf{x}} + \ddot{\boldsymbol{\Phi}} = 0, \quad \ddot{\boldsymbol{\Phi}} = \left[\ddot{\boldsymbol{\Phi}}_a \ \ddot{\boldsymbol{\Phi}}_b\right]^T \tag{24}$$

Eqs. (22) and (24) are the first and the second order kinematic equations respectively for hybrid structure.

2.2 *Equilibrium equations*

Let us denote the tensile force of a cable member a by n_a and nodal forces by $\mathbf{f}_{ia}$ and $\mathbf{f}_{ja}$ as shown in Fig. 4 (a):

$$\mathbf{f}_{ia} = \begin{bmatrix} f_{ix} \\ f_{iy} \\ f_{iz} \end{bmatrix}_a \quad \mathbf{f}_{ja} = \begin{bmatrix} f_{jx} \\ f_{jy} \\ f_{jz} \end{bmatrix}_a \tag{25}$$

The equilibrium equation for a member can be written as

$$\begin{bmatrix} -\boldsymbol{\lambda}_a \\ \boldsymbol{\lambda}_a \end{bmatrix} n_a = \begin{bmatrix} \mathbf{f}_{ia} \\ \mathbf{f}_{ja} \end{bmatrix} \tag{26}$$

Let us consider a rigid structure b as shown in Fig. 4 (b). Let $\mathbf{f}_k$ be the force vector of node k, and $\mathbf{F}$ be the vector which includes both forces and moments applied at an arbitrary point c in the rigid structure as

$$\mathbf{f}_k = \left[f_{kx} \ f_{ky} \ f_{kz}\right]^T \quad (k = 1,2,\ldots,k_b) \tag{27}$$

$$\mathbf{F} = \left[F_x \ F_y \ F_z \ m_x \ m_y \ m_z\right]^T \tag{28}$$

The equilibrium equation of rigid body b can be written as

$$\mathbf{F} + \mathbf{G}\mathbf{f}_b = 0 \tag{29}$$

where

$$\mathbf{f}_b = \begin{bmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_{k_b} \end{bmatrix} \tag{30}$$

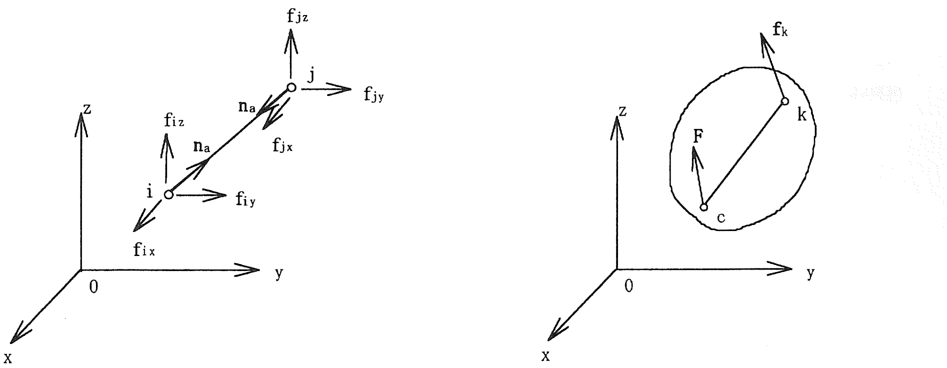
$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 1 \\ 0 & -(z_1 - z_c) & y_1 - y_c & \cdots & 0 & -(z_{k_b} - z_c) & y_{k_b} - y_c \\ z_1 - z_c & 0 & -(x_1 - x_c) & \cdots & z_{k_b} - z_c & 0 & -(x_{k_b} - x_c) \\ -(y_1 - y_c) & x_1 - x_c & 0 & \cdots & -(y_{k_b} - y_c) & x_{k_b} - x_c & 0 \end{bmatrix} \tag{31}$$

By comparing $\mathbf{G}$ with $\mathbf{H}$ we get $\mathbf{G} = \mathbf{H}^T$.

Consider a rigid structure to be in the equilibrium state with nodal forces, $\mathbf{F} = \mathbf{0}$ is satisfied. In this case, Eq. (29) becomes

$$\mathbf{G}\mathbf{f}_b = \mathbf{0} \tag{32}$$

By using generalized inverse, we get the solution of Eq. (32).



(a) A cable member

(b) A rigid structure

Fig. 4 Equilibrium relation

$$\mathbf{f}_b = \left[\mathbf{I} - \mathbf{G}^{+}\mathbf{G}\right]\eta \tag{33}$$

where η is an arbitrary vector. Because of $\mathbf{G} = \mathbf{H}^T$, Eq. (33) can be written as

$$\mathbf{f}_b = \left[\mathbf{I} - \mathbf{H}\mathbf{H}^{+}\right]\eta \tag{34}$$

where $(\mathbf{H}^T)^+ \mathbf{H}^T = (\mathbf{H}^+)^T \mathbf{H}^T = (\mathbf{H}\,\mathbf{H}^+)^T = \mathbf{H}\,\mathbf{H}^+$ is used[9]. Eq. (34) is the equilibrium equation of rigid structure b.

Collecting Eq. (26) for all cable members and Eq. (34) for all rigid structures into a single matrix equation gives us

$$\mathbf{B}\mathbf{p} = \mathbf{f} \qquad \mathbf{p} = \begin{bmatrix} \mathbf{p}_c \\ \mathbf{p}_r \end{bmatrix} \tag{35}$$

where $\mathbf{p}_c$ is a vector represents tensile forces for all cable members, and $\mathbf{p}_r$ is a vector which includes all vectors η in Eq. (34) for all rigid structures. By comparing Eqs. (4) and (15) with Eqs. (26) and (34), we get the relation $\mathbf{B} = \mathbf{A}^T$. Then Eq. (35) takes the form

$$\mathbf{A}^T\mathbf{p} = \mathbf{f} \tag{36}$$

which is the equilibrium equation of a hybrid structure.

If a hybrid structure is in an equilibrium state under tensile forces of cable members and no external force exists, we call this state as a self-equilibrated state. In a self-equilibrated state, self-equilibrated stress is introduced into the hybrid structure. In order to calculate the self-equilibrated stress modes of the system, we can get the equilibrium relation by setting $\mathbf{f} = \mathbf{0}$ in Eq. (36) as follow

$$\mathbf{A}^T\mathbf{p} = \mathbf{0} \tag{37}$$

2.3 *Example of basic equations*

Let us consider a simple hybrid structure consisting of three cable members and a triangular rigid structure as show in Fig. 5. Here we us two dimensional space instead of three dimensional space. The coordinates of the nodes are listed as follows:

$$\begin{cases} x_1 = 1 \\ y_1 = -1 \end{cases} \quad \begin{cases} x_2 = 0 \\ y_2 = 0 \end{cases} \quad \begin{cases} x_3 = -1 \\ y_3 = -1 \end{cases} \quad \begin{cases} x_4 = 3 \\ y_4 = -2 \end{cases} \quad \begin{cases} x_5 = 0 \\ y_5 = 1 \end{cases} \quad \begin{cases} x_6 = -3 \\ y_6 = -2 \end{cases}$$

Let the member of node 1 to node 4 be cable no.1, the member of node 2 to node 5 be cable no.2 and the member of node 3 to node 6 be cable no.3. Then the direction cosines of the cable members can be written as

$$\boldsymbol{\lambda}_1 = \frac{1}{\sqrt{5}}\begin{Bmatrix} 2 \\ -1 \end{Bmatrix} \quad \boldsymbol{\lambda}_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad \boldsymbol{\lambda}_1 = \frac{1}{\sqrt{5}}\begin{Bmatrix} -2 \\ -1 \end{Bmatrix} \tag{38}$$

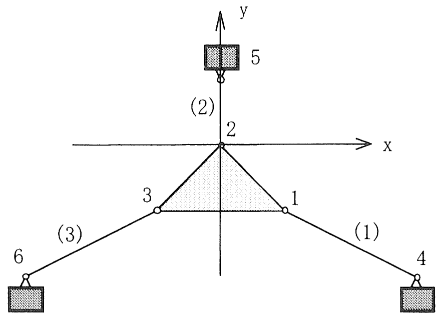


Fig. 5 Example of a hybrid structure

Using Eq. (6) we get kinematic equation of cable members.

$$\begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_2 \\ \dot{y}_2 \\ \dot{x}_3 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{39}$$

The items about node 4, node 5 and node 6 are eliminated in the equation because they are fixed as shown in figure. Also, we take cable members as rigid members: $\dot{l} = 0$.

Next, let us consider the rigid structure. Let $\mathbf{x}_c$ be coordinate vector of an arbitrary point c in the triangle, for example, $x_c = 0$, $y_c = -2/3$. From Eqs. (12) and (14),

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1 \\ 1 & 0 & -2/3 \\ 0 & 1 & 0 \\ 1 & 0 & 1/3 \\ 0 & 1 & -1 \end{bmatrix} \tag{40}$$

The Moor-Penrose generalized inverse matrix of $\mathbf{H}$ and $\mathbf{I} - \mathbf{HH}^{+}$ can be obtained as

$$\mathbf{H}^{+} = \begin{bmatrix} 1/3 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \\ 1/8 & 3/8 & -1/4 & 0 & 1/8 & -3/8 \end{bmatrix} \tag{41}$$

$$\mathbf{I}-\mathbf{H}\mathbf{H}^{+}=\begin{bmatrix} 5/8 & -1/8 & -1/4 & 0 & -3/8 & 1/8 \\ & 7/24 & 1/4 & -1/3 & -1/8 & 1/24 \\ & & 1/2 & 0 & -1/4 & -1/4 \\ & & & 2/3 & 0 & -1/3 \\ & sym. & & & 5/8 & 1/8 \\ & & & & & 7/24 \end{bmatrix} \tag{42}$$

From Eq. (15), we get the kinematic equation of the rigid structure.

$$\begin{bmatrix} 5/8 & -1/8 & -1/4 & 0 & -3/8 & 1/8 \\ & 7/24 & 1/4 & -1/3 & -1/8 & 1/24 \\ & & 1/2 & 0 & -1/4 & -1/4 \\ & & & 2/3 & 0 & -1/3 \\ & sym. & & & 5/8 & 1/8 \\ & & & & & 7/24 \end{bmatrix}\begin{bmatrix}\dot{x}_1\\ \dot{y}_1\\ \dot{x}_2\\ \dot{y}_2\\ \dot{x}_3\\ \dot{y}_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix} \tag{43}$$

Writing Eq. (39) and Eq. (43) into one equation,

$$\begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/\sqrt{5} & 1/\sqrt{5} \\ 5/8 & -1/8 & -1/4 & 0 & -3/8 & 1/8 \\ -1/8 & 7/24 & 1/4 & -1/3 & -1/8 & 1/24 \\ -1/4 & 1/4 & 1/2 & 0 & -1/4 & -1/4 \\ 0 & -1/3 & 0 & 2/3 & 0 & -1/3 \\ -3/8 & -1/8 & -1/4 & 0 & 5/8 & 1/8 \\ 1/8 & 1/24 & -1/4 & -1/3 & 1/8 & 7/24 \end{bmatrix}\begin{bmatrix}\dot{x}_1\\ \dot{y}_1\\ \dot{x}_2\\ \dot{y}_2\\ \dot{x}_3\\ \dot{y}_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix} \tag{44}$$

Eq. (44) is the kinematic equation of the hybrid structure. From Eq. (36) or Eq. (37), we can also get the equilibrium equation. According to Fig. 5, no external force exists, thus

$$\begin{bmatrix} -2/\sqrt{5} & 0 & 0 & 5/8 & -1/8 & -1/4 & 0 & -3/8 & 1/8 \\ 1/\sqrt{5} & 0 & 0 & -1/8 & 7/24 & 1/4 & -1/3 & -1/8 & 1/24 \\ 0 & 0 & 0 & -1/4 & 1/4 & 1/2 & 0 & -1/4 & -1/4 \\ 0 & -1 & 0 & 0 & -1/3 & 0 & 2/3 & 0 & -1/3 \\ 0 & 0 & 2/\sqrt{5} & -3/8 & -1/8 & -1/4 & 0 & 5/8 & 1/8 \\ 0 & 0 & 1/\sqrt{5} & 1/8 & 1/24 & -1/4 & -1/3 & 1/8 & 7/24 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{45}$$

where, $n_1 \sim n_3$ are axial forces of cable members, $\eta_1 \sim \eta_6$ are components of vector $\boldsymbol{\eta}$.

3. Classification of Stable and Unstable Structures and Introduction of Self-Equilibrated Stress

In this chapter, by using the generalized inverse theory, the rigid body displacement modes and the self-equilibrated stress modes of hybrid structure are calculated. Classification of a hybrid structure into stable or unstable and, classification of rigid body displacement into small or finite type are proposed. The self-equilibrated stress system of a hybrid structure, and the condition for stabilizing an unstable hybrid structure into stable state by introducing self-equilibrated stress, are discussed.

3.1 Classification of stable and unstable hybrid structures

In order to classify a hybrid structure, let us examine rigid body displacements first. In this case, cable members are considered as rigid members $\dot{l}_a = \ddot{l}_a = 0$. For rigid members, , Eq. (22) holds. By introducing an arbitrary n-dimensional vector $\dot{\boldsymbol{\alpha}}$, the solution of Eq. (22) can be written as

$$\dot{\mathbf{x}} = \left[\mathbf{I}_n - \mathbf{A}^{+}\mathbf{A}\right]\dot{\boldsymbol{\alpha}} \tag{46}$$

where $\mathbf{I}_n$ is a $n \times n$ identify matrix. If the rank of $\mathbf{A}$ is r, then the rank of coefficient matrix of Eq. (46) is $p = rank(\mathbf{I}_n - \mathbf{A}^{+}\mathbf{A}) = n - r$. p denotes the degrees of rigid body displacements and is called "degree of geometrically unstable". Let $q = m - r$, then q represents the number of compatibility conditions, and is called "degree of statically indeterminate".

Write Eq. (46) into the form as

$$\dot{\mathbf{x}} = \dot{\alpha}_1\mathbf{h}_1 + \dot{\alpha}_2\mathbf{h}_2 + \ldots + \dot{\alpha}_p\mathbf{h}_p \tag{47}$$

where h_1, h_2,..., h_p represent independent modes of rigid body displacement in the sense of the small displacement and $\dot{\alpha}_1$, $\dot{\alpha}_2$,..., $\dot{\alpha}_p$ are components of vector $\dot{\boldsymbol{\alpha}}$. Using method of reference[1)], we classify a hybrid structure as Table 1.

Next, let us assume that the rigid body displacements exist in the small displacements and examine the existence condition of finite rigid body displacements.

By introducing Eq. (47) into Eq. (24), $\ddot{\boldsymbol{\Phi}}$ can be expressed as a function of unknowns $\dot{\alpha}_1$, $\dot{\alpha}_2$,..., $\dot{\alpha}_p$. Then Eq. (24) takes the form

$$\mathbf{A}\ddot{\mathbf{x}} = -\ddot{\Phi}\left(\dot{\alpha}_1, \dot{\alpha}_2, \ldots, \dot{\alpha}_p\right) \tag{48}$$

Table 1 Classification of a hybrid structure

$q \backslash p$	$p=0$	$p>0$
$q=0$	Geometrically Stable	Geometrically Unstable
	Statically Determinate	Statically Determinate
$q>0$	Geometrically Stable	Geometrically Unstable
	Statically Indeterminate	Statically Indeterminate

The necessary and sufficient condition of Eq. (48) to have a solution is

$$\left[\mathbf{I}_m - \mathbf{A}\mathbf{A}^+\right]\ddot{\Phi}\left(\dot{\alpha}_1, \dot{\alpha}_2, \ldots, \dot{\alpha}_p\right) = 0 \tag{49}$$

If we have some unknowns $\dot{\alpha}_1$, $\dot{\alpha}_2$,..., $\dot{\alpha}_p$ which satisfy Eq. (49) trivially, then the finite rigid body displacement corresponding $\mathbf{h}_1$, $\mathbf{h}_2$,..., $\mathbf{h}_p$ exists.

3.2 Self-equilibrated stress

Let us examine self-equilibrated stress system on the condition of $\mathbf{f} = \mathbf{0}$. From Eq. (37) we get

$$\mathbf{p} = \left[\mathbf{I}_m - \left(\mathbf{A}^T\right)^+ \mathbf{A}^T\right] \boldsymbol{\beta} = \beta_1 \mathbf{g}_1 + \beta_2 \mathbf{g}_2 + \ldots + \beta_q \mathbf{g}_q \tag{50}$$

in which $q = m - rank(\mathbf{A})$, $\boldsymbol{\beta}$ is an arbitrary vector of q dimension and $\mathbf{g}_1$, $\mathbf{g}_2$,..., $\mathbf{g}_q$ are independent self-equilibrated stress modes. It is clear that the number of independent modes of self-equilibrated stress equals to the degrees of statically indeterminate.

Every self-equilibrated stress mode contains both tensile forces $\mathbf{p}_c$ of cable members and $\mathbf{p}_r$ of rigid structures as shown in Eq. (35). If $\mathbf{p}_c = \mathbf{0}$, it means it is impossible to introduce tensile forces into cable members by this self-equilibrated stress mode. On the other hand, if $\mathbf{p}_c \neq \mathbf{0}$, it is possible to introduce tensile forces into cable members.

As an example of calculating self-equilibrated stress, let us examine the example discussed in chapter 2 (see Fig. 5). In chapter 2, the equilibrium equation has been given by Eq. (45). By using Eq. (50) we get the self-equilibrated stress modes.

$$\left[\mathbf{g}_1\ \mathbf{g}_2\ \mathbf{g}_3\ \mathbf{g}_4\right] = \begin{bmatrix} \sqrt{5}/\sqrt{91} & 0 & 0 & 0 \\ 2/\sqrt{91} & 0 & 0 & 0 \\ \sqrt{5}/\sqrt{91} & 0 & 0 & 0 \\ 4/\sqrt{91} & 0 & 1/\sqrt{3} & 0 \\ -6/\sqrt{91} & 2/\sqrt{6} & 0 & 0 \\ 5/\sqrt{91} & 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{6} & 0 & 1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & 0 & 2/\sqrt{6} \end{bmatrix} \tag{51}$$

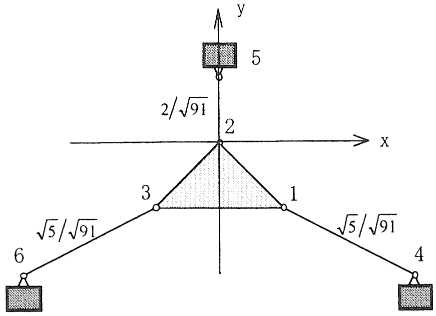


Fig. 6 One equilibrated stress mode exists

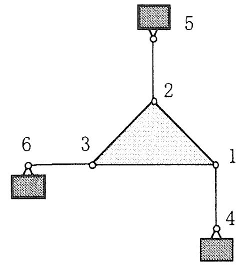


Fig. 7 No equilibrated stress mode exists

From Eq. (51), we can find that there are four self-equilibrated stress modes, but only $\mathbf{g}_1$ shows it is possible to introduce tensile forces into cable members (see Fig. 6).

$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \beta_1 \begin{bmatrix} \sqrt{5}/\sqrt{91} \\ 2/\sqrt{91} \\ \sqrt{5}/\sqrt{91} \end{bmatrix} \tag{52}$$

If we let the coordinates of node 4 and node 6 be (see Fig. 7)

$$\begin{cases} x_4 = 1 \\ y_4 = -2 \end{cases} \quad \begin{cases} x_6 = -2 \\ y_6 = -1 \end{cases} \tag{53}$$

then the self-equilibrated stress modes become

$$[\mathrm{g}_1 \ \mathrm{g}_2 \ \mathrm{g}_3] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 2/\sqrt{6} & 0 & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{6} & 0 & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix} \tag{54}$$

There is no self-equilibrated stress mode which tensile forces can be introduced into cable members.

3.3 Stability after introducing pre-stress

Let us examine the stability of a hybrid structure after introduction of pre-stress. Eq. (37) represents the equilibrium equation for a hybrid structure before deformation. Give the system a disturbance, after small deformation, nodal force $-\mathbf{A}^T\mathbf{p}$ occurs. Consider mass m_i attached to node i of a hybrid structure, the equation of motion of the mass is given by

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{A}^T\mathbf{p} \tag{55}$$

where $\mathbf{M} = diag(m_i)$. The kinetic energy can be written as

$$T = \frac{1}{2}\dot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}} \tag{56}$$

The derivative of the kinetic energy T respect to time t is expressed as

$$\dot{T} = \dot{\mathbf{x}}^T\mathbf{M}\ddot{\mathbf{x}} \tag{57}$$

By using Eq. (55), Eq. (57) becomes

$$\dot{T} = -\dot{\mathbf{x}}^T\mathbf{A}^T\mathbf{p} \tag{58}$$

Let the system be supposed to be staying at the initial equilibrium position for $t = 0$ and give the kinetic energy $T(0)$ by a small disturbance. Then the condition for stability can be written as

$$T(t) - T(0) < 0 \tag{59}$$

Maclaurin expansion of $T(t)$ of $|t| < 1$ leads

$$T(t) - T(0) = \dot{T}(0)\,t + \frac{1}{2}\ddot{T}(0)\,t^2 + \ldots \tag{60}$$

where $\dot{T}(0)$ equals zero because $\mathbf{A}^T\mathbf{p}$ equals zero according to Eq. (37) and Eq. (58). Thus Eq. (59) becomes

$$\ddot{T}(0) < 0 \tag{61}$$

Time derivative of Eq. (58) yields

$$\ddot{T}(0) = -\ddot{\mathbf{x}}^T\mathbf{A}^T\mathbf{p} - \dot{\mathbf{x}}^T\dot{\mathbf{A}}^T\mathbf{p} - \dot{\mathbf{x}}^T\mathbf{A}^T\dot{\mathbf{p}} \tag{62}$$

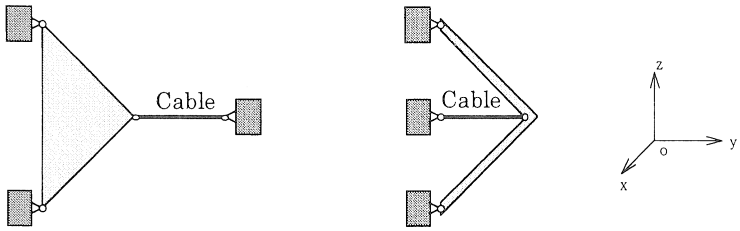
Because $\dot{\mathbf{x}}$ and $\mathbf{p}$ are rigid body displacement and self-equilibrated stress respectively, $\mathbf{A}\dot{\mathbf{x}} = \mathbf{0}$ and $\mathbf{A}^T\mathbf{p} = \mathbf{0}$ are satisfied. Then Eq. (62) becomes

$$\ddot{T}(0) = -\dot{\mathbf{x}}^T\dot{\mathbf{A}}^T\mathbf{p} \tag{63}$$

Hence the condition for stability is given by

$$\dot{\mathbf{x}}^T\dot{\mathbf{A}}^T\mathbf{p} > 0 \tag{64}$$

By using $\dot{\mathbf{x}}^T\dot{\mathbf{A}}^T = \dot{\mathbf{\Phi}}^T(\dot{\alpha})$ and $\mathbf{p} = [\mathbf{I} - \mathbf{A}\mathbf{A}^+]\,\boldsymbol{\beta}$, we get



a) stable (b) unstable

Fig. 8 Stability after introduction of pre-stress (in three dimensional space)

$$\ddot{\Phi}^T(\dot{\alpha})\left[\mathbf{I} - \mathbf{A}\mathbf{A}^+\right]\beta > 0 \tag{65}$$

Fig. 8 shows two simple hybrid structure modes under the introduction of pre-stress. If two models are in three dimensional space, then (a) will be stable under the pre-stress of the cable while (b) remains unstable under the pre-stress of the cable.

4. Stress and Displacement Analysis of Hybrid Structure

In this chapter, an analytical method to calculate stress and displacement under static load is proposed. In the chapter, a hybrid structure is divided into cable members and rigid structures. The stiffness matrix of cable members which consists of elastic stiffness and geometric stiffness is obtained, while rigid structures which have definite stiffness can be considered as displacement constraint conditions. The kinematic equation of rigid structures which formulated in chapter 2 is considered as displacement constraint conditions. The Bott-Duffin inverse method[10] is used to solve the problem. In the end, a numerical example is given and the validity of the proposed analytical method is examined.

4.1 Introduction of analytical method

Let us examine an example of hybrid structure as shown in Fig. 9(a). This hybrid structure contains of two rigid structures (with triangular shape) and four cable members. There are five movable nodes. Let the displacement vector of nodes be $\dot{\mathbf{x}}$, the load vector be $\dot{\mathbf{f}}$, and the stiffness matrix be $\mathbf{K}$. Consider $\dot{\mathbf{f}}$ as a nodal force vector, then

$$\dot{\mathbf{f}} = \mathbf{K}\dot{\mathbf{x}} \tag{66}$$

Stiffness matrix $\mathbf{K}$ can be written as

$$\mathbf{K} = \mathbf{K}_E + \mathbf{K}_C \tag{67}$$

where $\mathbf{K}_E$ is elastic stiffness matrix and $\mathbf{K}_G$ is geometric stiffness matrix. We divide the hybrid structure into cable members and rigid structures as shown in Fig. 9(b)(c). For cable members, the stiffness matrix $\mathbf{K}_C$ ($\mathbf{K}_C = \mathbf{K}_{EC} + \mathbf{K}_{GC}$) can be obtained. Substitute $\mathbf{K}$ in Eq. (66) by $\mathbf{K}_C$, Eq. (66) becomes

$$\dot{\mathbf{f}} = \mathbf{K}_C \dot{\mathbf{x}} \tag{68}$$

In the case of Fig. 9, Eq. (68) can be written as

$$\begin{bmatrix}\dot{\mathbf{f}}_1\\ \dot{\mathbf{f}}_2\\ \dot{\mathbf{f}}_3\\ \dot{\mathbf{f}}_4\\ -\\ \dot{\mathbf{f}}_5\end{bmatrix} = \left[\begin{array}{c|c} \mathbf{K}_{GC} + \mathbf{K}_{GC} & 0 \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix}\dot{\mathbf{x}}_1\\ \dot{\mathbf{x}}_2\\ \dot{\mathbf{x}}_3\\ \dot{\mathbf{x}}_4\\ -\\ \dot{\mathbf{x}}_5\end{bmatrix} \tag{69}$$

For rigid structures, it is impossible to obtain stiffness matrix because of their infinite stiffness. By using kinematic relation Eq. (15), rigid structures can be taken as constraint conditions of displacements. Write Eq. (15) for all rigid structures and find out independent conditions, we get

$$\bar{\mathbf{C}}\dot{\mathbf{x}} = \mathbf{0} \tag{70}$$

where $\bar{\mathbf{C}}$ is a (r,n) matrix, n is the number of freedom, r is the number of independent conditions, $r = rank(\bar{\mathbf{C}})$.

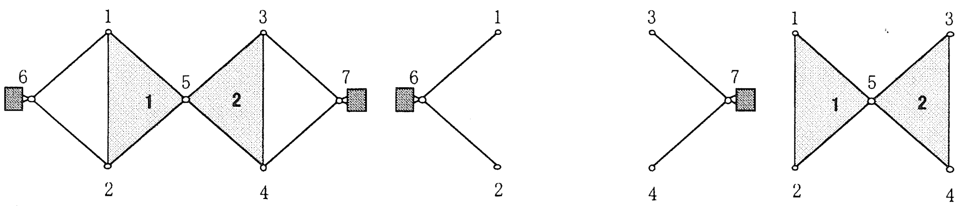
4.2 Stiffness matrix of cable member

In section 2.2, we have analyzed the equilibrium equation of cable members. Rewrite Eq. (26) here

$$\begin{bmatrix}-\boldsymbol{\lambda}_a\\ \boldsymbol{\lambda}_a\end{bmatrix} n_a = \begin{bmatrix}\mathbf{f}_{ia}\\ \mathbf{f}_{ja}\end{bmatrix} \tag{71}$$

Differentiating Eq. (71) with respect to t leads to

$$\begin{bmatrix}-\boldsymbol{\lambda}_a\\ \boldsymbol{\lambda}_a\end{bmatrix} \dot{n}_a + \begin{bmatrix}-\dot{\boldsymbol{\lambda}}_a\\ \dot{\boldsymbol{\lambda}}_a\end{bmatrix} n_a = \begin{bmatrix}\dot{\mathbf{f}}_{ia}\\ \dot{\mathbf{f}}_{ja}\end{bmatrix} \tag{72}$$



(a) a hybrid structure (b) cable members (c) rigid structures

Fig. 9 Analytical method for hybrid structure

Consider the first item of the left side of Eq. (72). According to the Hooke's law,

$$\dot{n}_a = \frac{EA}{l_a}\dot{l}_a \tag{73}$$

where E is elastic modulus and A is area of cross section. By using Eq. (4) for $\dot{l}_a$, we get

$$\begin{bmatrix} -\boldsymbol{\lambda}_a \\ \boldsymbol{\lambda}_a \end{bmatrix} \dot{n}_a = \begin{bmatrix} -\boldsymbol{\lambda}_a \\ \boldsymbol{\lambda}_a \end{bmatrix} \frac{EA}{l_a} \begin{bmatrix} -\boldsymbol{\lambda}_a^T & \boldsymbol{\lambda}_a^T \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_i \\ \dot{\mathbf{x}}_j \end{bmatrix} \tag{74}$$

Let the coefficient matrix of the right side of Eq. (74) be $(\mathbf{K}_{EC})_a$

$$\left(\mathbf{K}_{EC}\right)_a = \frac{EA}{l_a} \begin{bmatrix} \boldsymbol{\lambda}_a\boldsymbol{\lambda}_a^T & -\boldsymbol{\lambda}_a\boldsymbol{\lambda}_a^T \\ -\boldsymbol{\lambda}_a\boldsymbol{\lambda}_a^T & \boldsymbol{\lambda}_a\boldsymbol{\lambda}_a^T \end{bmatrix} \tag{75}$$

$(\mathbf{K}_{EC})_a$ is called elastic stiffness matrix of member a.
Next, consider the second item of the left side of Eq. (72). From Eq. (3), we can get

$$\dot{\boldsymbol{\lambda}}_a = \frac{1}{l_a}\begin{bmatrix} -\mathbf{I} + \boldsymbol{\lambda}_a\boldsymbol{\lambda}_a^T & \mathbf{I} - \boldsymbol{\lambda}_a\boldsymbol{\lambda}_a^T \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_i \\ \dot{\mathbf{x}}_j \end{bmatrix} \tag{76}$$

then

$$\begin{bmatrix} -\dot{\boldsymbol{\lambda}}_a \\ \dot{\boldsymbol{\lambda}}_a \end{bmatrix} n_a = \frac{n_a}{l_a} \begin{bmatrix} \mathbf{I} - \boldsymbol{\lambda}_a\boldsymbol{\lambda}_a^T & -\mathbf{I} + \boldsymbol{\lambda}_a\boldsymbol{\lambda}_a^T \\ -\mathbf{I} + \boldsymbol{\lambda}_a\boldsymbol{\lambda}_a^T & \mathbf{I} - \boldsymbol{\lambda}_a\boldsymbol{\lambda}_a^T \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_i \\ \dot{\mathbf{x}}_j \end{bmatrix} \tag{77}$$

Also let the coefficient matrix of the right side of Eq. (77) be $(\mathbf{K}_{GC})_a$

$$\left(\mathbf{K}_{GC}\right)_a = \frac{n_a}{l_a} \begin{bmatrix} \mathbf{I} - \boldsymbol{\lambda}_a\boldsymbol{\lambda}_a^T & -\mathbf{I} + \boldsymbol{\lambda}_a\boldsymbol{\lambda}_a^T \\ -\mathbf{I} + \boldsymbol{\lambda}_a\boldsymbol{\lambda}_a^T & \mathbf{I} - \boldsymbol{\lambda}_a\boldsymbol{\lambda}_a^T \end{bmatrix} \tag{78}$$

$(\mathbf{K}_{GC})_a$ is geometric stiffness matrix of member a.
By using $(\mathbf{K}_{EC})_a$ and $(\mathbf{K}_{GC})_a$, Eq. (72) becomes

$$\begin{bmatrix} \dot{\mathbf{f}}_{ia} \\ \dot{\mathbf{f}}_{ja} \end{bmatrix} = \left[\left(\mathbf{K}_{EC}\right)_a + \left(\mathbf{K}_{GC}\right)_a\right] \begin{bmatrix} \dot{\mathbf{x}}_i \\ \dot{\mathbf{x}}_j \end{bmatrix} \tag{79}$$

After writing Eq. (79) for all cable members, collecting them into a single matrix equation and considering boundary conditions, we get Eq. (66) for the hybrid structure.

4.3 Analytical method of Bottin-Duffin inverse

As shown in section 4.1, stress and displacement analysis can be made through the stiffness matrix of cable members and the constraint conditions of rigid structures. Here we use Bott-Duffin inverse method[10)] to solve equilibrium equation Eq. (66) under the constraint condition of Eq. (70).

Using Lagrange multiplier method, the potential energy function of the problem becomes

$$\Pi_k = \frac{1}{2}\dot{\mathbf{x}}^T\mathbf{K}\dot{\mathbf{x}} - \dot{\mathbf{f}}\dot{\mathbf{x}} + \boldsymbol{\lambda}^T\bar{\mathbf{C}}\dot{\mathbf{x}} \tag{80}$$

where $\boldsymbol{\lambda}$ a is Lagrange multiplier. Partially differentiate Eq. (80) with respect to $\dot{\mathbf{x}}$ and and $\boldsymbol{\lambda}$ then set to zero, we get

$$\mathbf{K}\dot{\mathbf{x}} - \dot{\mathbf{f}} + \bar{\mathbf{C}}^T\boldsymbol{\lambda} = 0 \tag{81}$$

$$\bar{\mathbf{C}}\dot{\mathbf{x}} = 0 \tag{82}$$

where $\mathbf{K}^T = \mathbf{K}$ is used. Let a vector $\mathbf{r}$ be

$$\mathbf{r} = \bar{C}^T\boldsymbol{\lambda} \tag{83}$$

then Eq. (81) becomes the next equation.

$$\mathbf{K}\dot{\mathbf{x}} + \mathbf{r} = \dot{\mathbf{f}} \tag{84}$$

By now, the problem is to solve Eq. (84) of unknowns $\mathbf{r}$ and $\dot{\mathbf{x}}$ with the constrain condition of Eq. (82). To do this, the Bott-Duffin inverse method is introduced.

Make a fundamental transformation for matrix $\bar{\mathbf{C}}$ as follow

$$\mathbf{P}_{(r\times r)}\bar{\mathbf{C}}_{(r\times n)}\mathbf{Q}_{(n\times n)} = \begin{bmatrix}\mathbf{I}_r & \mathbf{0}\end{bmatrix}_{(r\times n)} \tag{85}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are normal matrix. Also make transformations for $\dot{\mathbf{x}}$, $\mathbf{r}$ and $\dot{\mathbf{f}}$ as

$$\dot{\mathbf{x}} = \mathbf{Q}\dot{\mathbf{u}} \quad \mathbf{t} = \mathbf{Q}^T\mathbf{r} \quad \dot{\mathbf{q}} = \mathbf{Q}^T\dot{\mathbf{f}} \tag{86}$$

Vector $\dot{\mathbf{u}}$ and vector $\mathbf{t}$ satisfy orthogonality condition which can be proved as follow.

$$\dot{\mathbf{u}}^T\mathbf{t} = \dot{\mathbf{u}}^T\mathbf{Q}^T\mathbf{r} = \dot{\mathbf{x}}^T\mathbf{r} = \dot{\mathbf{x}}^T\bar{\mathbf{C}}^T\boldsymbol{\lambda} = \left(\bar{\mathbf{C}}\dot{\mathbf{x}}\right)^T\boldsymbol{\lambda} = 0 \tag{87}$$

Multiplying Eq. (84) from left side by $\mathbf{Q}^T$ and Using Eq. (86), Eq. (84) takes the form

$$\mathbf{L}\dot{\mathbf{u}} + \mathbf{t} = \dot{\mathbf{q}} \tag{88}$$

where

$$\mathbf{L} = \mathbf{Q}^T\mathbf{K}\mathbf{Q} \tag{89}$$

Multiplying Eq. (82) from left side by $\mathbf{P}$

$$\left[\mathbf{P}\overline{\mathbf{C}}\ \mathbf{Q}\right]\dot{\mathbf{u}} = \mathbf{0} \tag{90}$$

where $\dot{\mathbf{x}} = \mathbf{Q}\,\dot{\mathbf{u}}$ is used. From Eq. (85), the above equation can be written as

$$\mathbf{B}\dot{\mathbf{u}} = \mathbf{0} \quad \mathbf{B} = \left[\mathbf{I}_r\ \ \mathbf{0}\right] \tag{91}$$

Because $\dot{\mathbf{u}}$ and $\mathbf{t}$ satisfy orthogonality condition as shown in Eq. (87), we can assume that

$$\dot{\mathbf{u}} = \mathbf{P}_L\dot{\mathbf{a}} \qquad \mathbf{t} = \mathbf{P}_{L^\perp}\dot{\mathbf{a}} \tag{92}$$

where $\mathbf{P}_L$ and $\mathbf{P}_{L^\perp}$ are orthogonal projectors on L and on $L^\perp$ respectively. L is a linear subspace in R^n, $L^\perp$ is orthogonal complement to L and $\dot{\mathbf{a}} \in R^n$. Substitute Eq. (92) into Eq. (88)

$$\left(\mathbf{L}\mathbf{P}_L + \mathbf{P}_{L^\perp}\right)\dot{\mathbf{a}} = \dot{\mathbf{q}} \tag{93}$$

Solving the above equation for $\dot{\mathbf{a}}$ and substituting $\dot{\mathbf{a}}$ into Eq. (92) lead to

$$\dot{\mathbf{u}} = \mathbf{L}_{(L)}^{-1}\dot{\mathbf{q}} \tag{94}$$

In which, $\mathbf{L}_{(L)}^{-1}$ is called Bott-Duffin inverse.

$$\mathbf{L}_{(L)}^{-1} = \mathbf{P}_L\left(\mathbf{L}\mathbf{P}_L + \mathbf{P}_{L^\perp}\right)^{-1} \tag{95}$$

From Eq. (88)

$$\mathbf{t} = \dot{\mathbf{q}} - \mathbf{L}\dot{\mathbf{u}} \tag{96}$$

For $\mathbf{P}_L$ and $\mathbf{P}_{L^\perp}$, we can choose them as

$$\mathrm{P} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{I}_{n-r} \end{bmatrix} \quad \mathrm{P}_{L^\perp} = \begin{bmatrix} \mathbf{I}_{n-r} & 0 \\ 0 & 0 \end{bmatrix} \tag{97}$$

By using the transformation of Eq. (86), $\dot{\mathbf{x}}$ and $\mathbf{r}$ can be calculated in the end.

4.4 Numerical example

Fig. 10 shows a simple hybrid structure. The shape of the rigid structure is equilateral triangle, its three nodes are connected by three cable members with the same length of l and the same stiffness of EA. Initial self-equilibrated tension force n_0 is introduced in every member. At each node of the rigid structure, external force f is applied at the direction of z. The vertical displacement d can be derived theoretically by removing the rigid structure. Fig. 11 shows the results given 1) by the analytical method proposed in the paper and 2) by the theoretical analysis. The results which have a good agreement verify the validity of the proposed method.

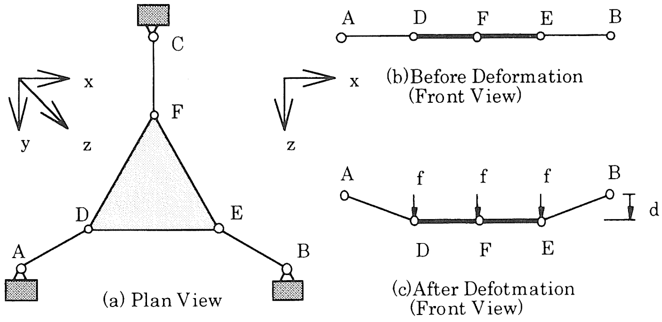


Fig. 10 A simple hybrid structure model

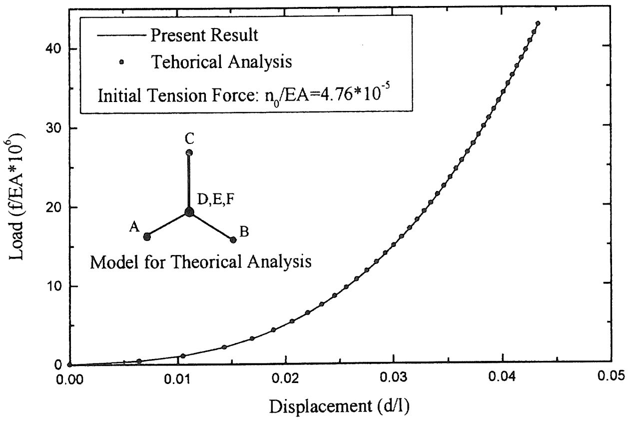


Fig. 11 Vertical displacement d

5. Vibration Analysis of Hybrid Structure

In this chapter, the equation of motion for hybrid structure is derived and the analytical method to solve the equation is given. The equation of motion derived in this chapter consists of two parts. One is the equation of motion for cable members. The mass matrix is the sum of the mass matrix of cable members and the mass matrix of rigid structures and, the stiffness matrix is the one which has been formulated in chapter 4. The other part of the equation of motion is the displacement constraint conditions. Rigid structures are considered as displacement constraint conditions as we have done in chapter 4.

The mass matrix of rigid structure used in the paper differs from the common mass matrix of a rigid body which contains of mass and moment of inertia. The new mass matrix is derived by using the freedom of the nodes of rigid structure, in this case, the motion of rigid structure is represented by the displacements of nodes.

In the end of the chapter, a method to solve this kind of equations is given and a numerical example is shown.

5.1 Equation of motion

In chapters 2~4, we take t as an arbitrary parameter to solve static problems. In dynamic analysis, we take parameter t as time. Let $\mathbf{d}$ be displacement vector of nodes, the equation of motion of cable members in a hybrid structure can be written as

$$\mathbf{M}_C\ddot{\mathbf{d}} + \mathbf{C}\dot{\mathbf{d}} + \left(\mathbf{K}_{EC} + \mathbf{K}_{GC}\right)\mathbf{d} = \mathbf{f}(t) \tag{98}$$

where $\mathbf{M}_C$ is the mass matrix of cable members, $\mathbf{C}$ is the damping matrix, $\mathbf{K}_{EC}$ and $\mathbf{K}_{GC}$ are elastic stiffness matrix and geometric stiffness matrix of cable members respectively, $\mathbf{f}(t)$ is a excitation vector.

Let us consider the kinematic relation and the motion equation of rigid structures. For a rigid structure b ($b = 1,..., m_b$), let $\mathbf{d}_k$ be the displacement vector of its node k ($k = 1,..., k_b$), $\mathbf{f}_k$ be the external force vector of node k, $\mathbf{D}_b$ be the coordinate vector of the center of mass and $\mathbf{F}_b$ be the force vector applied on the center of mass. In this case, the kinematic relation and equation of motion of rigid structure b become

$$\mathbf{H}\mathbf{D}_b = \mathbf{d}_k \tag{99}$$

$$\mathbf{M}_b\ddot{\mathbf{D}}_b + \mathbf{F}_b = \mathbf{G}\mathbf{f}_k \tag{100}$$

where matrix $\mathbf{H}$ and matrix $\mathbf{G}$ have been derived as Eqs. (12) and (31). Here, we try to describe the motion of rigid structure by using the displacements of nodes rather than its center of mass. In order to do this, $\mathbf{D}_b$ and $\ddot{\mathbf{D}}_b$ should be eliminated from Eqs. (99) and (100). From Eq. (99) we get

$$\left[\mathbf{I} - \mathbf{H}\mathbf{H}^+\right]\mathbf{d}_k = 0 \tag{101}$$

Writing Eq. (101) for all rigid structures gives

$$\bar{\mathbf{C}}\mathbf{d} = 0 \tag{102}$$

which has the same form as Eq. (70). Consider rigid structure b in an equilibrium state under node forces, we can let $\mathbf{F}_b = \mathbf{0}$. Thus, from Eqs. (99) and (100)

$$\mathbf{f}_k = \mathbf{G}^+\mathbf{M}_b\mathbf{H}^+\ddot{\mathbf{d}}_k = \overline{\mathbf{M}}_b\ddot{\mathbf{d}}_k \tag{103}$$

where $\overline{\mathbf{M}}_b$ is the generalized mass matrix. Eq. (103) is the equation of motion of rigid structure b by using coordinates of its nodes.

Writing $\overline{\mathbf{M}}_b$ for all rigid structures into $\mathbf{M}_r$ and collecting $\mathbf{M}_C$ and $\mathbf{M}_r$ into one matrix, Eq. (98) becomes

$$\mathbf{M}\ddot{\mathbf{d}} + \mathbf{C}\dot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{f}(t) \tag{104}$$

where

$$\mathbf{M} = \mathbf{M}_C + \mathbf{M}_r, \quad \mathbf{K} = \mathbf{K}_E + \mathbf{K}_G \tag{105}$$

Eq. (104) is the equation of motion for hybrid structure.

5.2 *Analytical method*

By solving the constraint condition of Eq. (102), we get

$$\mathbf{d} = \left[\mathbf{I}_n - \bar{\mathbf{C}}^{+}\bar{\mathbf{C}}\right]\boldsymbol{\alpha} \tag{106}$$

where $\boldsymbol{\alpha}$ is an arbitrary n-dimensional vector. Using normalized independent vectors $\mathbf{b}_1,\ldots, \mathbf{b}_p$ of the coefficient matrix of Eq. (106), Eq. (106) has the form

$$\mathbf{d} = \alpha_1\mathbf{b}_1 + \ldots + \alpha_p\mathbf{b}_p \tag{107}$$

where $\alpha_1,\ldots, \alpha_p$ are unknowns and p is

$$p = n - rank(\bar{\mathbf{C}}) = n - r \tag{108}$$

If we use

$$\mathbf{B} = \left[\mathbf{b}_1,\ldots,\mathbf{b}_p\right], \quad \mathbf{a} = \left[\alpha_1,\ldots,\alpha_p\right]^T \tag{109}$$

then, Eq. (107) becomes

$$\mathbf{d} = \mathbf{Ba} \tag{110}$$

Substituting Eq. (110) into Eq. (104) and left-multiplying Eq. (104) by $\mathbf{B}^{\mathrm{T}}$

$$\mathbf{M}_{\mathbf{B}}\ddot{\mathbf{a}} + \mathbf{C}_{\mathbf{B}}\dot{\mathbf{a}} + \mathbf{K}_{\mathbf{B}}\mathbf{a} = \mathbf{f}_{\mathbf{B}}(t) \tag{111}$$

where

$$\mathbf{M}_{\mathbf{B}} = \mathbf{B}^{\mathrm{T}}\mathbf{MB}, \quad \mathbf{C}_{\mathbf{B}} = \mathbf{B}^{\mathrm{T}}\mathbf{CB}, \quad \mathbf{K}_{\mathbf{B}} = \mathbf{B}^{\mathrm{T}}\mathbf{KB}, \quad \mathbf{f}_{\mathbf{B}} = \mathbf{B}^{\mathrm{T}}\mathbf{f} \tag{112}$$

Because **M**, **C** and **K** are symmetrical matrices, $\mathbf{M}_{\mathbf{B}}$, $\mathbf{C}_{\mathbf{B}}$ and $\mathbf{K}_{\mathbf{B}}$ are also symmetrical matrices. $\mathbf{M}_{\mathbf{B}}$, $\mathbf{C}_{\mathbf{B}}$ and $\mathbf{K}_{\mathbf{B}}$ are $p \times p$ matrices. **a** and $\mathbf{f}_{\mathbf{B}}$ are $p \times 1$ vectors.

Eq. (111) gives **a**, then **d** can be obtained by substituting **a** into Eq. (110). There are r constraint equations of rigid structures, so the number of unknowns decreases from n in Eq. (98) to $p = n - r$ in Eq. (111).

5.3 *Numerical example*

In order to examine the validity of the method discussed above, an example which given in Fig. 12 is solved by Newmark-β method with an iterative process to reduce unbalance force. The hybrid structure consists of eight cable members and a cube rigid structure. The cable members have the cross-section area $A = 5 \times 10^{-4}\ m^2$, the mass density $\bar{m} = 3.9\ kg/m$ and the elastic modulus $E = 2.1 \times 10^{11}\ Pa$. The mass of the rigid structure is 557 kg. The nodes of the rigid structure (nodes 1–8) are free while the other nodes (nodes 9–12) are fixed. In this case, $n = 24$, $rank(\bar{\mathbf{C}}) = 18$, $p = 6$.

Fig. 13 shows the natural frequencies of the hybrid structure and the corresponding modes when the initial tension force $N_0 = 5 \times 10^4\ N$ is introduced in every cable member. We use $\beta = 0.25$ and $\Delta t = 10^{-3}$ in Newmark-β method. Fig. 14 gives the response of free vibration with the damping of $\mathbf{C} = 10^{-3}\ (\mathbf{M} + \mathbf{K})$. The solid line represents the sum of the strain of the total cable members while the dot line represents the change of the sum of the distance between two nodes of the rigid structure. The dot line remains almost zero which means almost no error occurs during the numerical analysis process. Fig. 15 shows the history curve of stress.

In order to obtain response curve under harmonic excitation, harmonic load of $10^3 \sin 2\pi ft\,(N)$ (f is the frequency) is applied to every free nodes with z direction. We use Rayleigh damping as $\mathbf{C}_1 = 10^{-3}\ (\mathbf{M} + \mathbf{K})$, $\mathbf{C}_2 = 5 \times 10^{-3}\ (\mathbf{M} + \mathbf{K})$ and $\mathbf{C}_3 = 1 \times 10^{-2}\ (\mathbf{M} + \mathbf{K})$. Fig. 16 shows the results of vertical displacement response z.

6. Experiments on a Hybrid Structure Model

In this chapter, experiments of a hybrid structure model with 1/2 scale of size of a planning house are done as follows: i) pre-stress introducing experiment, ii) static loading experiment.

The experimental model consists of a rigid structure, which has six nodes on its surface. Cable members connected with the rigid structure by nodes. The roof of the model is covered with membrane. The hybrid structure is stabilized by introducing pre-stress into cable members. According to the analytical analysis based on chapter 3, there are two self-equilibrated stress modes in the experimental model, i.e., symmetrical mode and asymmetrical mode.

In pre-stress introducing experiment, the symmetrical self-equilibrated mode is introduced and the validity of the proposed analysis method in chapter 3 is verified by examining the self-equilibrated forces of cable members. Pre-stress is also introduced in membrane with about 200 kgf/m.

In static loading experiment, loads are applied to two nodes of the rigid structure. Displacements of nodes of the rigid structure and stresses of the cable members are measured. The data calculated by using the analytical method of chapter 4 have a good agreement with the results obtained from the experiment.

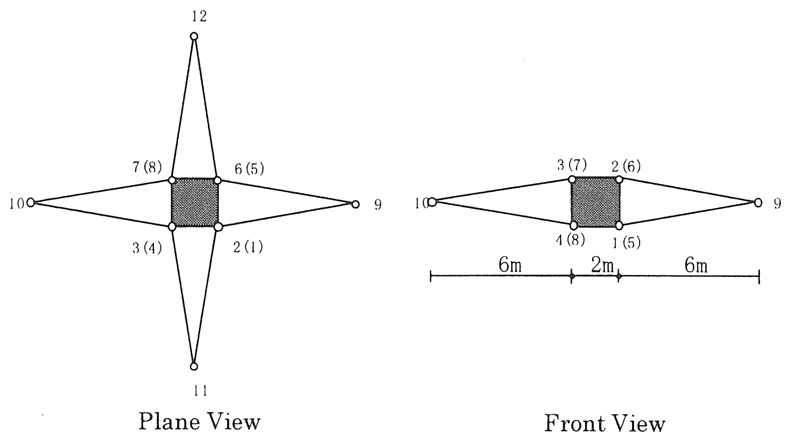


Fig. 12 Analytical model

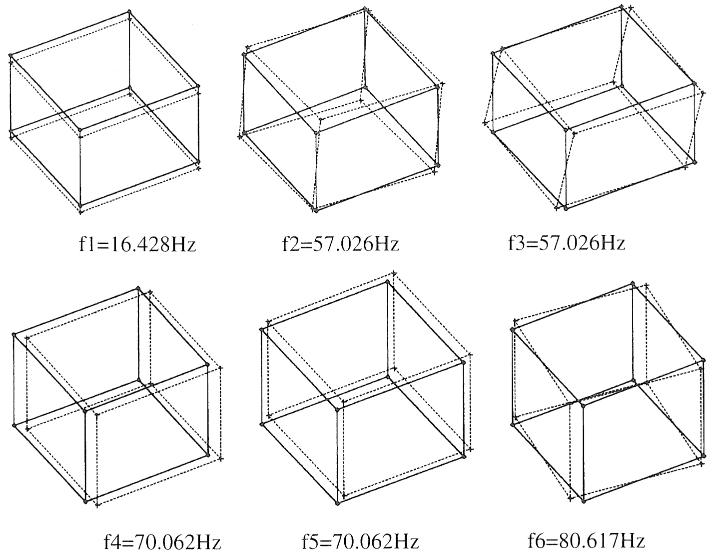


Fig. 13 Frequency (Hz) and mode ($N_0 = 5 \times 10^4$ N)

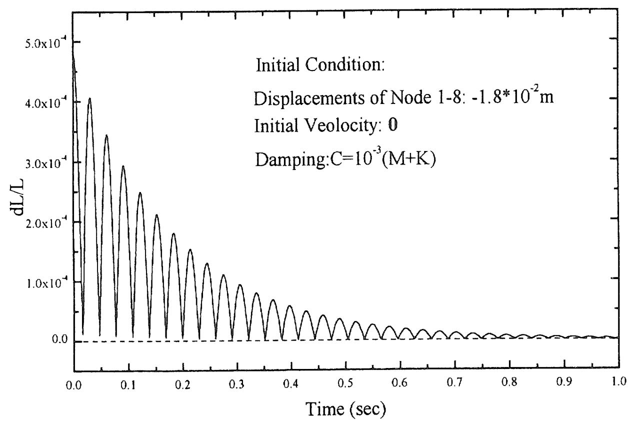


Fig. 14 Time history curve of strain

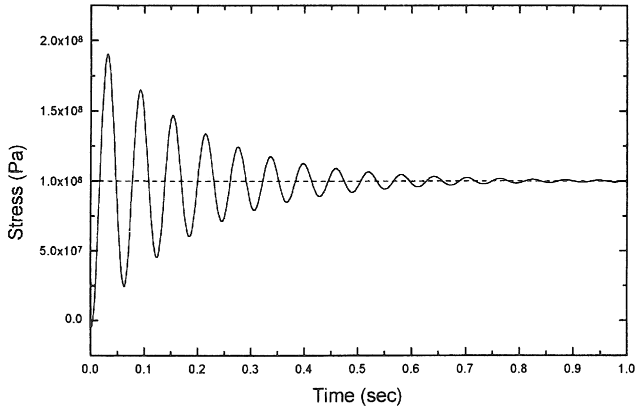


Fig. 15 Time history curve of stress

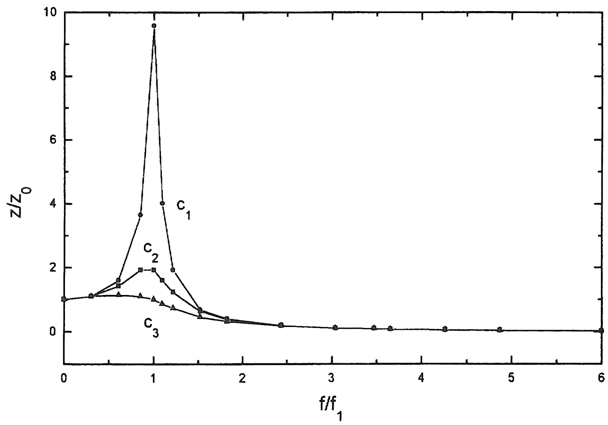


Fig. 16 Response under harmonic excitation

Table 2 Elements of experiment model

Item	Size
cable	tensile rod: diameter 25mm
rigid structure	pipe: diameter 48.6 mm with thickness 3.2 mm
out frame	H type: $H125 \times 125 \times 6.5 \times 9$

Membrane material	Direction	Tensile strength	Elongation after fracture
type A: PTFE thickness: 0.6 mm	warp	375 kgf/3 cm	3–10%
	fill	300 kgf/3 cm	6–15%

6.1 Experiment model

Fig. 17 gives the dimension of the experiment model. In the center part of the model, there is a rigid structure which is made by pipe. There are six nodes on the surface of the rigid structure. Through these nodes, the rigid structure is connected with cables and is stabilized by introducing pre-stresses into cables.

There are eight cable members with the same diameter of 25 mm and the numbers are indicated in Fig. 17 in parentheses. One end of every cable is linked with rigid structure and the other end is fixed at the out frame. Out frame is made by H type steel.

The model is covered with PTFE membrane. Membrane panel is fixed on the out frame and pre-stress is introduced. Photo 1 shows the experiment model.

From Eq. (50) the self-equilibrated stress modes of the hybrid structure model can be calculated. There are two independent modes, one is symmetrical mode and the other is asymmetrical mode. The axial forces of cable members can be given by means of the two modes as follows.

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \\ N_7 \\ N_8 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1.038 \\ 1.038 \\ 2.379 \\ 2.379 \\ 2.245 \\ 2.245 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1.296 \\ -1.29 \\ -1.422 \\ 1.422 \\ -1.819 \\ 1.819 \end{bmatrix} \tag{113}$$

where N_i is the axial force of cable member i. c_1 is coefficient of symmetrical mode and c_2 is coefficient of asymmetrical mode.

6.2 Pre-stress introducing experiment

In this experiment, first, we introduce symmetrical pre-stress to cable members by using turnbuckles step by step until the maximum tensile force is about 1000 *kgf*. The axial forces of cable members are obtained by measuring the strains. Next, membrane is set and pre-stress is introduced. After relaxation, the pre-stress of membrane is about 200 *kgf/m*.

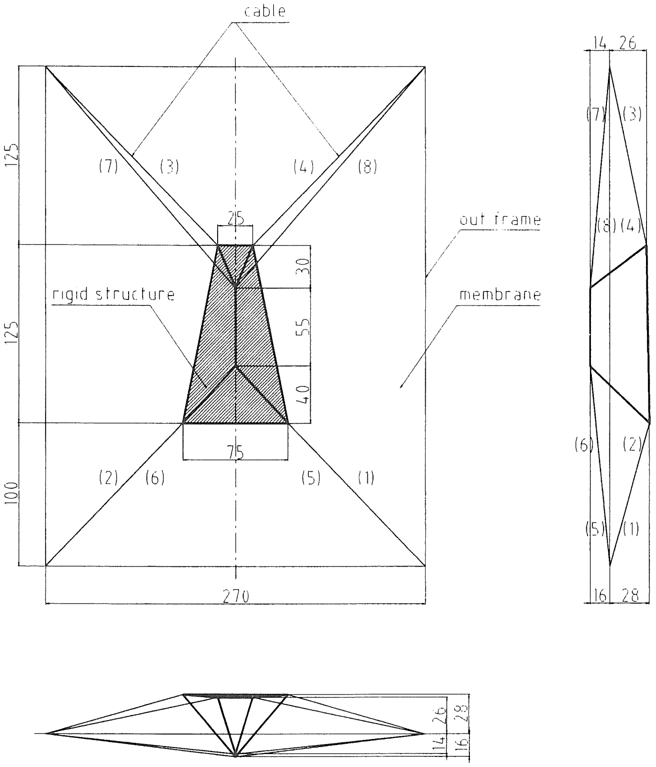


Fig. 17 Dimension of hybrid structure model (cm)

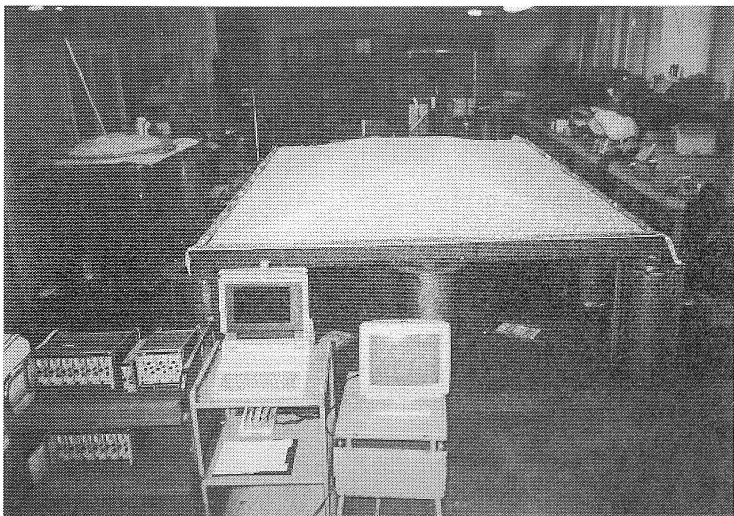


Photo 1 Hybrid structure model

Fig. 18 is the process of the introduction of initial tensile forces into cable members. It shows that proper stress is introduced for every member. After 14 steps, the maximum tensile force reaches 1100 *kgf* for member 5.

Fig. 19 shows the symmetrical self-equilibrated stress mode calculated from the experiment data. The experimental result has a good agreement with the analytical result. Fig. 20 shows the asymmetrical self-equilibrated stress mode.

Fig. 21 gives the changes of coefficient c_1 and c_2 in introducing process. We can see from Fig. 21 that c_1 increases in the process while c_2 remains small. This means from Eq. (113) that only symmetrical self-equilibrated stress mode is introduced.

When the introduction of pre-stress for cable members is over, membrane is set. In the process of pre-stress introducing for membrane, the axial forces of cable members vary with the increasing of the membrane stress as shown in Fig. 22. For membrane material, it has some visco-elastic properties such as stress relaxation and creep. Fig. 22 also shows the change of tensile forces caused by the stress relaxation of membrane.

6.3 Loding experiment

In loading experiment, we apply load to node 5 and node 6 of the rigid structure by using weights as shown in Photo 2. Load for node 5 is kept with the same value for node 6 in loading process. We increase load step by step till the total 12 steps. The total load of the two nodes is about 650 *kgf*. Displacements of the rigid structure and strains of the cable members are measured.

Fig. 24 gives the vertical displacements of nodes and Fig. 25 gives the axial forces of members 1~4. We can see from figures that the experimental results have a good agreement with the analytical results.

7. Summary

A new type of hybrid structures which consists of cables and rigid structures is proposed, and analytical methods for investigating structural behaviors such as 1) rigid body displacement modes and self-equilibrated stress system, 2) the introduction of pre-stress for a positive geometric stiffness matrix, 3) the stress and displacement analysis and 4) the vibration analysis are presented by using generalized inverse matrix.

Theoretical characteristic of the paper is that, freedoms of the rigid structures are considered by those of the nodes on the surface of the rigid structures, which are connected with cable members. In order to eliminate the freedom of any points in the rigid structures (such as the centers of gravity), a theoretical method was proposed.

In the end, the validity of the presented methods is examined by the experiment of a hybrid structure model.

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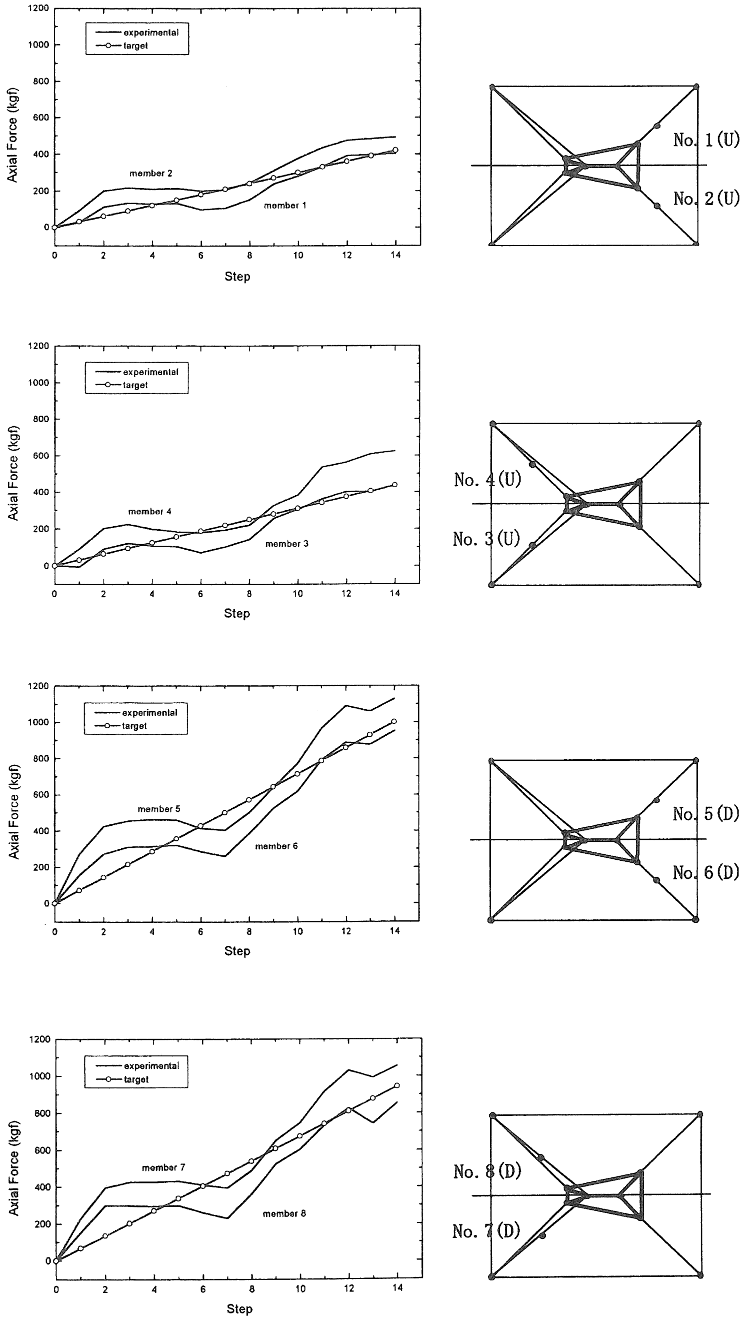

Fig. 18 Process of introduction of pre-stress (D means down member and U means upper member)

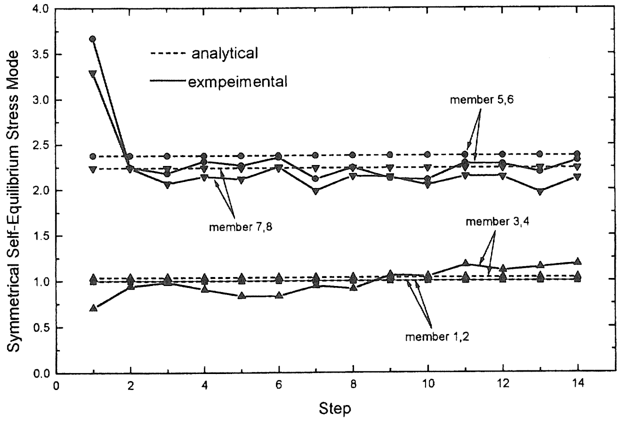

Fig. 19 Symmetrical self-equilibrated stress mode

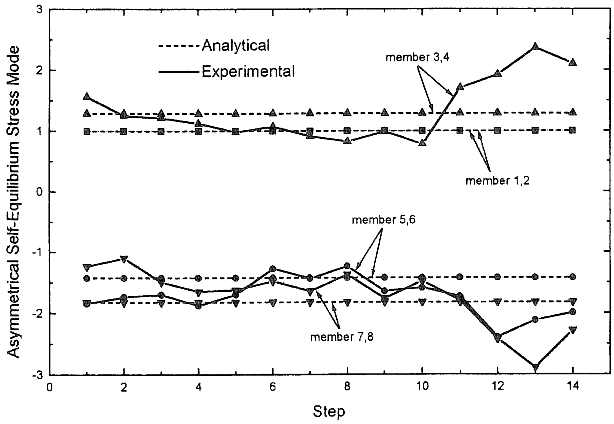

Fig. 20 Asymmetrical self-equilibrated stress mode

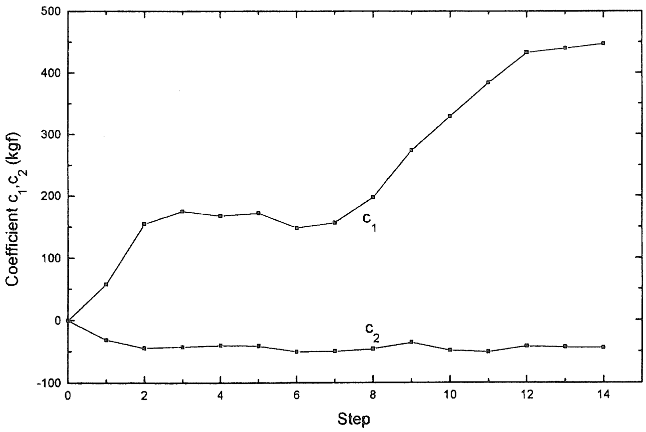

Fig. 21 Coefficients c_1 and c_2 during the introduction process of pre-stress

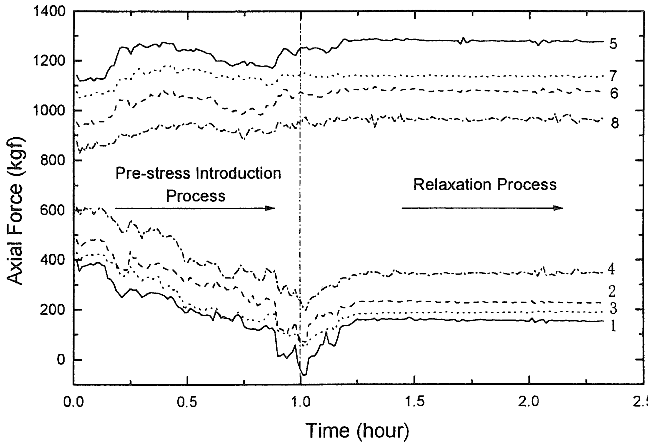

Fig. 22 The axial forces of cable members during the pre-stress introduction process and stress relaxation process of membrane

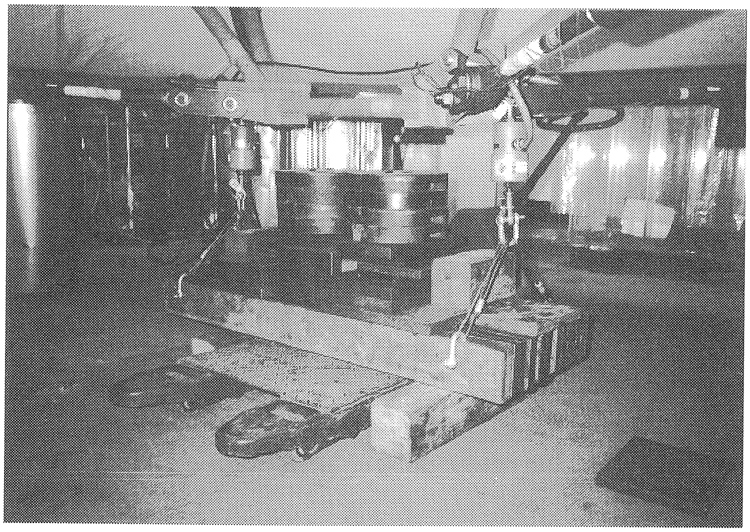

Photo2 Loading by weight

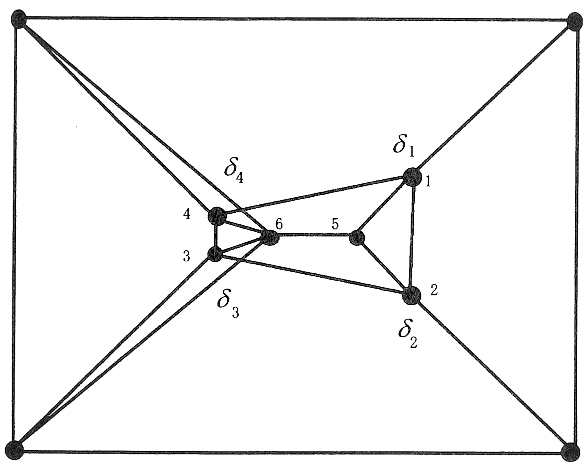

Fig. 23 Loading points 5 and 6 and displacement measuring points 1~4

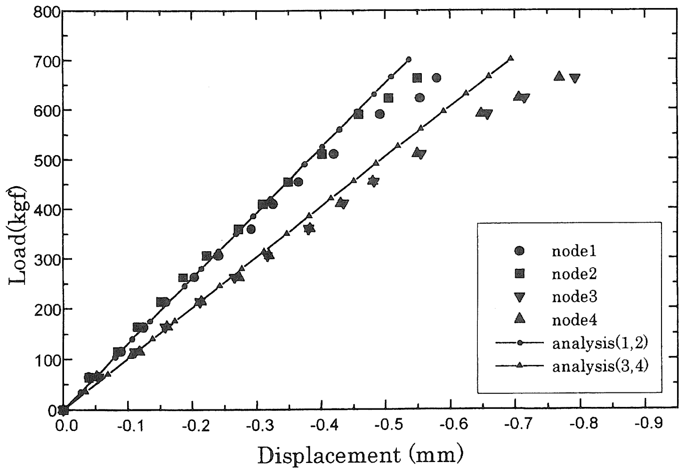

Fig. 24 Displacements of nodes

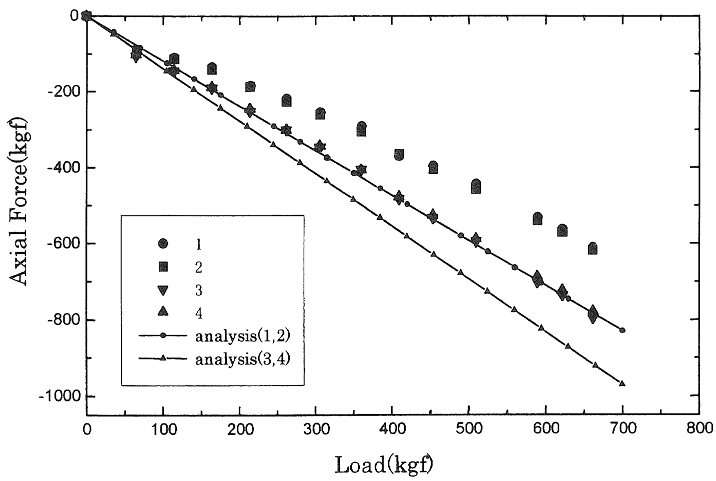

Fig. 25 Axial forces of members 1~4

References

1) Tanaka, H. and Hangai, Y. 'Rigid body displacement and stabilization condition of unstable truss structures', *Shell, Membrane and Space Frame, Proc. of IASS*, Osaka, 1986, 55-62.
2) Calladine, C.R. 'Buckminster Fuller's tensegrity structures and Clerk Maxwell's rules for the construction of stiff frame', *Int. J. Solids Structures*, 1978, **14**, 161-172.
3) Hangai, Y. and Kawaguchi, K. 'Shape-finding analysis of unstable link structures', *Journal of Structural and Construction Engineering, Trans. of AIJ*, 1987, **381**, 56-60.
4) Hangai, Y., Kawaguchi, K. and Oda, K. 'Self-equilibrated stress system and structural behavior of truss structures stabilized by cable tension', *Int. Jour. Space Structures*, 1992, **7**(2), 91-99.
5) Pugh, A. *An introduction to tensegrity*, University of California Press, 1976.
6) Motro, A. 'Tensegrity systems-latest developments and perspectives', *Proc. of IASS*, Madrid, 1989, **3**.
7) Vilnay, O. 'Structures made of infinite regular tensegric nets', *IASS Bulletin*, 1977, **XVIII-1**(63), 51-57
8) Geiger, D.H. 'Roof structure', U.S.Pat. 4,736,553, 1986.
9) Hangai, Y. and Kawaguchi, K. *Shape Analysis = Generalized Inverses and Applications*, Baifukan Press, 1991.
10) Hangai, Y., Guan, F.L. and Suzuki, T. 'Analytical method of membrane structures with constraint conditions of displacement by Bott.Duffin inverse', *Proc. of IASS, Madrid*, 1989, **5**, 157-173.
11) Domaszewski, M., Borkowski, A. 'Generalized inverse in elastic-plastic analysis of structures', *Journal of Structural Mechanics*, 1984, 219-244.
12) Calladine, C.R., and Pellegrino, S. 'First-order infinitesimal mechanisms', *Int. J. Solid Structures*, **27**(4), 1991, 505-515.
13) Kawaguchi, K., Hangai, Y. and Miyazaki, K. 'The dynamic analysis of kinematically indeterminate frameworks', Public Assembly Structures from Antiquity to the Present, *Proceedings IASS-Symposium 1993*, Istanbul, 569-576.
14) Hangai, Y. and Wu, M. 'Analytical method for hybrid structure of truss and rigid structures', *Proceedings of ASIA-PACIFIC Conference on Shell and Spatial Structures*, Beijing China, 1996, 209-216.
15) Wu, M. and Hangai, Y. 'Structural behaviour of a hybrid structure consisting of cables and rigid structures', *Journal of Structural and Construction Engineering*, Trans. of AIJ, 1997, **497**, 115-122.
16) Hangai, Y. and Wu, M. 'Analytical method of structural bahaviours of a hybrid structure consisting of cables and rigid structures', *Engineering Structures*, **21** 1999, 726-736.