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GENERAL TOOTH SURFACE WITH ZERO SPECIFIC SLIDING

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Abstract

The formula of a general tooth surface with zero specific sliding is known in the case of skew gear. Bur this formula has not been given expression of the instantaneous axis of rotation, and so we can not describe the tooth surface formula exactly. In this paper, we obtain a formula including the instantaneous axis of rotation in the most generalized case.

1. Introduction

In his report[2)] concerning the tooth surface with zero specific sliding, Ogino obtains a formula for skew.gear (i.e. (1, A(t), B(t), C(t))**C**(s; τ s/ω^2)) in the coordinate system (i.e. **P**(**U**) (t)) on the instantaneous axis of rotation.
In this paper, we obtain a formula of the instantaneous axis of rotation for arbitrary one parameter motion **U**(t), and we obtain a formula of the tooth surface with zero specific sliding.
U(t) is the relative motion between the coordinate system fixed on a gear and the other.
The formula satisfies a differential equation.
In the case of skew gear, the formula and the differential equation are shown exactly by an example.
The motion **U**(t) is equal to the following matrix.

$$\mathbf{U}(t)=\begin{pmatrix} 1 & T_1(t) & T_2(t) & T_3(t) \\ 0 & u_{11}(t) & u_{12}(t) & u_{13}(t) \\ 0 & u_{21}(t) & u_{22}(t) & u_{23}(t) \\ 0 & u_{31}(t) & u_{32}(t) & u_{33}(t) \end{pmatrix}$$

where, $\mathbf{T}=(T_1(t), T_2(t), T_3(t))$ is the translational part of this motion and

$$U=\begin{pmatrix} u_{11}(t) & u_{12}(t) & u_{13}(t) \\ u_{21}(t) & u_{22}(t) & u_{23}(t) \\ u_{31}(t) & u_{32}(t) & u_{33}(t) \end{pmatrix}$$

is the rotational part of this motion.

2. Notations

$$\mathbf{C}(t)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos t & \sin t & 0 \\ 0 & -\sin t & \cos t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ the rotation round Z-axis

$$\mathbf{C}_x(t)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{pmatrix}$$ the rotation round X-axis

$$\mathbf{C}_y(t)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos t & 0 & -\sin t \\ 0 & 0 & 1 & 0 \\ 0 & \sin t & 0 & \cos t \end{pmatrix}$$ the rotation round Y-axis

$$\mathbf{E}_x(t)=\begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ the linear motion in the direction of X-axis

$$\mathbf{E}_y(t)=\begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ the linear motion in the direction of Y-axis

$$\mathbf{E}_z(t)=\begin{pmatrix}1 & 0 & 0 & t\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$ the linear motion in the direction of Z-axis

$$\mathbf{C}(t;\ \lambda\ (t))=\begin{pmatrix}1 & 0 & 0 & \lambda\ (t)\\ 0 & \cos t & \sin t & 0\\ 0 & -\sin t & \cos t & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$ the spiral motion round Z-axis with pitch λ (t)

$$C(\theta)=\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\Omega\ (\mathbf{U})=d\mathbf{U}\cdot\mathbf{U}^{-1}=\begin{pmatrix}0 & d\mathbf{T}\cdot\mathrm{U}^{-1}\\ 0 & d\mathrm{U}\cdot\mathrm{U}^{-1}\end{pmatrix}=\begin{pmatrix}0 & \omega_1 & \omega_2 & \omega_3\\ 0 & 0 & \omega_{12} & \omega_{13}\\ 0 & -\omega_{12} & 0 & \omega_{23}\\ 0 & -\omega_{13} & -\omega_{23} & 0\end{pmatrix}dt$$

An infinitesimal motion of the orthogonal coordinate system ($\mathbf{A}$, $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$) is shown by the following.[4)]

$$\begin{pmatrix}d\mathbf{A}\\ d\mathbf{X}\\ d\mathbf{Y}\\ d\mathbf{Z}\end{pmatrix}=\Omega\ (\mathbf{U})\begin{pmatrix}\mathbf{A}\\ \mathbf{X}\\ \mathbf{Y}\\ \mathbf{Z}\end{pmatrix}$$

where, $\mathbf{A}$ is the origin of this system and $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ are unit vectors. Ω_θ ($\mathbf{U}$) is the differential operation with respect to θ.

3. Instantaneous Axis of Rotation

The motion $\mathbf{U}(t)$ is regarded as a spiral motion round an axis. This axis is called the instantaneous axis of rotation. We set a coordinate system $\mathbf{P}$ ($\mathbf{U}$) (t) whose Z-axis is the instantaneous axis, and whose origin lies on the X-Y plane of $\mathbf{U}(t)$ coordinate system.

Then, we put

$$\mathbf{P}(\mathbf{U})\ (t)=\begin{pmatrix}1 & f(t) & g(t) & 0\\ 0 & & P(t) & \end{pmatrix} \tag{1}$$

where, $\mathbf{P}(\mathbf{U})$ (t) is expressed with respect to the coordinate system of the motion $\mathbf{U}(t)$.

The infinitesimal motion of $\mathbf{U}(t)$ is decomposed into three motions, $\mathbf{P}(\mathbf{U})$ (t) (the motion to the instantaneous axis from the coordinate system of $\mathbf{U}(t)$), $\mathbf{C}(\theta;\ p(t)\theta)$ (the spiral motion on the instantaneous axis) and $\mathbf{P}(\mathbf{U})^{-1}(t)$ (the motion to the coordinate system of $\mathbf{U}(t)$ from the instantaneous axis).

$$\Omega\ (\mathbf{U}(t))=\Omega_\theta\ (\mathbf{P}(\mathbf{U})^{-1}(t)\mathbf{C}(\theta;\ p(t)\theta)\mathbf{P}(\mathbf{U}\ (t))$$

Furthermore, we put

$$\mathbf{P}(\mathbf{U})^{-1}(t)\mathbf{C}(\theta;\ p(t)\theta)\mathbf{P}(\mathbf{U})\ (t)=\begin{pmatrix}1 & \mathbf{T}_1(t,\ \theta)\\ 0 & U_1(t,\ \theta)\end{pmatrix}$$

then,

$$U_1(t,\ \theta)=P^{-1}(t)C(\theta)P(t)$$

$$\mathbf{T}_1(t,\ \theta)=(f(t),\ g(t),\ 0)+(0,\ 0,\ p(t)\theta)P(t)$$
$$-(f(t),\ g(t),\ 0)P^{-1}(t)C(\theta)P(t).$$

Because of

$$\Omega_\theta\ \left(\begin{pmatrix}1 & \mathbf{T}_1(t,\ \theta)\\ 0 & U_1(t,\ \theta)\end{pmatrix}\right)=\begin{pmatrix}0 & \partial\ \mathbf{T}_1/\partial\theta\ U_1{}^{-1}\\ 0 & \partial\ U_1/\partial\theta\ U_1{}^{-1}\end{pmatrix}\ d\theta$$

we have

$$\partial\ \mathbf{T}_1/\partial\ \theta\ U_1{}^{-1}=(0,\ 0,\ p(t))P(t)$$
$$-(f(t),\ g(t),\ 0)P^{-1}(t)dC(\theta)/d\theta\cdot C(-\theta)P(t) \tag{2}$$

$$\partial\ U_1/\partial\ \theta\ U_1{}^{-1}=P^{-1}(t)dC(\theta)/d\theta\cdot C(-\theta)P(t). \tag{3}$$

When we put

$$P(t)=\begin{pmatrix}a(t) & b(t) & c(t)\\ u(t) & v(t) & w(t)\\ l(t) & m(t) & n(t)\end{pmatrix} \tag{4}$$

P(t) is the normal orthogonal matrix.
This implies

$$l(t) = b(t)w(t) - c(t)v(t)$$

$$m(t) = c(t)u(t) - a(t)w(t)$$

$$n(t) = a(t)v(t) - b(t)u(t).$$

Because of

$$dC(\theta)/d\theta \cdot C(-\theta) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

the right side of (2) is equal to

$$(g(t)n(t) + l(t)p(t),\ -f(t)n(t) + m(t)p(t),$$
$$f(t)m(t) - g(t)l(t) + n(t)p(t)).$$

Similarly, the right side of (3) is equal to

$$\begin{pmatrix} 0 & n(t) & -m(t) \\ -n(t) & 0 & l(t) \\ m(t) & -l(t) & 0 \end{pmatrix}$$

Therefore, we have

$$\Omega_\theta\ (\mathbf{P}(\mathbf{U})^{-1}(t)\mathbf{C}(\theta;\ p(t)\theta)\mathbf{P}(\mathbf{U})\ (t))$$
$$= \begin{pmatrix} 0 & gn+lp & -fn+mp & fm-gl+np \\ 0 & 0 & n & -m \\ 0 & -n & 0 & l \\ 0 & m & -1 & 0 \end{pmatrix} d\theta$$

On the other hand,

$$\Omega_\theta\ (\mathbf{P}(\mathbf{U})^{-1}(t)\mathbf{C}(\theta;\ p(t)\theta)\mathbf{P}(\mathbf{U})\ (t)) = \Omega\ (\mathbf{U}(t)).$$

Hence, we have

$$\begin{cases} n(t) = \omega_{12} dt/d\theta \\ -m(t) = \omega_{13} dt/d\theta \\ l(t) = \omega_{23} dt/d\theta \end{cases} \tag{5}$$

$$\begin{cases} g(t)n(t)+l(t)p(t)=\omega_1 dt/d\theta \\ -f(t)n(t)+m(t)p(t)=\omega_2 dt/d\theta \\ f(t)m(t)-g(t)l(t)+n(t)p(t)=\omega_3 dt/d\theta \end{cases} \tag{6}$$

From (5), we have

$$l(t):\ m(t):\ n(t)=\omega_{23}:\ -\omega_{13}:\ \omega_{12}.$$

Thus, we can put

$$l(t)=\omega_{23}/\omega,\ m(t)=-\omega_{13}/\omega,\ n(t)=\omega_{12}/\omega$$

$$(\omega^2=\omega_{12}{}^2+\omega_{13}{}^2+\omega_{23}{}^2) \tag{7}$$

then $dt/d\theta=1/\omega$, and by substituting these into (6), we obtain the followings.

$$\begin{cases} f(t) = -(\omega^2\omega_2+\omega_{13}\tau)/\omega^2\omega_{12} \\ g(t) = (\omega^2\omega_1-\omega_{23}\tau)/\omega^2\omega_{12} \\ p(t) = \tau/\omega^2 \end{cases} \tag{8}$$

$$(\tau=\omega_{23}\omega_1-\omega_{13}\omega_2+\omega_{12}\omega_3)$$

If we suppose that X-axis of **P(U)** (t) coordinate system lies on X-Y plane of **U**(t) coordinate system, we obtain the followings.

$$\begin{cases} a(t) = \omega_{13}/(\omega_{13}{}^2+\omega_{23}{}^2)^{1/2} \\ b(t) = \omega_{23}/(\omega_{13}{}^2+\omega_{23}{}^2)^{1/2} \\ c(t) = 0 \\ u(t) = -\omega_{12}\omega_{23}/\omega\ (\omega_{13}{}^2+\omega_{23}{}^2)^{1/2} \\ v(t) = \omega_{12}\omega_{13}/\omega\ (\omega_{13}{}^2+\omega_{23}{}^2)^{1/2} \\ w(t) = (\omega_{13}{}^2+\omega_{23}{}^2)^{1/2}/\omega \end{cases} \tag{9}$$

Thus, **P(U)** (t) is given by the formulas (1), (4), (7), (8), (9).

4. Differential Equation

The tooth surface with specific sliding $(1-\beta\ (t,\ s))$ satisfies the following differential equation.[5)]

$$(1,\ (1-\beta\ (t,\ s))\partial \mathbf{X}/\partial t - a(t,\ s)\partial \mathbf{X}/\partial s) = -(1,\ \mathbf{X})\ \Omega\ (\mathbf{U}^{-1}) \tag{10}$$

If $1-\beta\ (t,\ s) \fallingdotseq 0$, using a function $\phi\ (t,\ s)$ such that

$$(1-\beta\ (t,\ s))\partial\phi/\partial t - a(t,\ s)\partial\phi/\partial s = 0,$$

the differential equation (10) yields

$$(1,\ 1-\beta\ (t,\ \phi^{-1}))\partial \mathbf{X}/\partial t) = -(1,\ \mathbf{X})\ \Omega\ (\mathbf{U}^{-1}). \tag{11}$$

If $1-\beta\ (t,\ s) \equiv 0$, the differential equation (10) reduces to

$$(1,\ a(t,\ s)\partial \mathbf{X}/\partial s) = (1,\ \mathbf{X})\ \Omega\ (\mathbf{U}^{-1}) = (1,\ -\mathbf{V}(t,\ s)) \tag{12}$$

where, $\mathbf{V}(t,\ s)$ is the velocity vector of $\overline{\mathbf{X}}(t,\ s)$ relative to $\mathbf{X}(t,\ s)$ and $\overline{\mathbf{X}}(t,\ s)$ is the pair of the tooth surface $\mathbf{X}(t,\ s)$.
This implies that the vector $\partial \mathbf{X}/\partial s$ and $\mathbf{V}(t,\ s)$ are parallel to each other.
We have the following relation between $\mathbf{X}(t,\ s)$ and $\overline{\mathbf{X}}(t,\ s)$[5)].

$$(1,\ \overline{\mathbf{X}}(t,\ s)) = (1,\ \mathbf{X}(t,\ s))\mathbf{U}(t)^{-1}$$

We put

$$(1,\ \mathbf{X}(t,\ s)) = (1,\ A(t),\ B(t),\ C(t))\mathbf{C}(s;\ \tau s/\omega^2)\mathbf{P}(\mathbf{U})\ (t)\mathbf{U}(t) \tag{13}$$

where, $A(t)$, $B(t)$, $C(t)$ are arbitrary functions.
When we regard the motion $\mathbf{U}(t)$ as a instantaneous spiral motion, the pitch of the instantaneous spiral motion is equal to the pitch of the spiral motion $\mathbf{C}(s;\ \tau s/\omega^2)$ by the formula (8).
When we fix the parameter t, the s-line in the formula (13) is a spiral line with pitch τ/ω^2 round the instantaneous axis of rotation. Therefore, the tangent vector at a point on s-line and the velocity vector of the point by the motion $\mathbf{U}(t)$ are parallel to each other.
Thus, we have $\partial \mathbf{X}/\partial s // \mathbf{V}(t,\ s)$ and $\mathbf{X}(t,\ s)$ satisfies the differential equation (12).

5. Example (The Case of Skew Gear)

$$\mathbf{U}(t)=\mathbf{C}(\lambda\ t)\mathbf{M}\mathbf{C}(t)$$

$$\mathbf{M}=\mathbf{E}x(d)\mathbf{C}x(-\pi/2)$$

d: the distance of the axis of rotation from the other axis
λ: the rotation ratio of two gears

$$\Omega\ (\mathbf{U}^{-1}(t))=\begin{pmatrix} 0 & 0 & 0 & -\lambda\ d \\ 0 & 0 & -1 & \lambda\ \cos\ tt \\ 0 & 1 & 0 & \lambda\ \sin\ tt \\ 0 & -\lambda\ \cos\ t & -\lambda\ \sin\ t & 0 \end{pmatrix}$$

$$\mathbf{P}(\mathbf{U})\ (t)=$$

$$\begin{pmatrix} 1 & -d\ \cos\ \lambda\ t/(\lambda^2+1) & d\ \sin\ \lambda\ t/(\lambda^2+1) & 0 \\ 0 & \cos\ \lambda\ t & -\sin\ \lambda t & 0 \\ 0 & \lambda\ \sin\ \lambda\ t/(\lambda^2+1)^{1/2} & \lambda\ \cos\ \lambda\ t/(\lambda^2+1)^{1/2} & 1/(\lambda^2+1)^{1/2} \\ 0 & -\sin\ \lambda\ t/(\lambda^2+1)^{1/2} & -\cos\ \lambda\ t/(\lambda^2+1)^{1/2} & \lambda/(\lambda^2+1)^{1/2} \end{pmatrix}$$

$$\mathbf{P}(\mathbf{U})\ (t)\mathbf{U}(t)=$$

$$\begin{pmatrix} 1 & \lambda^2 d\ \cos\ t/(\lambda^2+1) & \lambda^2 d\ \sin\ t/(\lambda^2+1) & 0 \\ 0 & \cos\ t & \sin\ t & 0 \\ 0 & -\sin\ \alpha\ \sin\ t & \sin\ \alpha\ \cos\ t & -\cos\ \alpha \\ 0 & -\cos\ \alpha\ \sin\ t & \cos\ \alpha\ \cos\ t & \sin\ \alpha \end{pmatrix}$$

where, $\tan\ \alpha=1/\lambda$
We put,

$$\mathbf{C}(s;\ \tau s/\omega^2)\mathbf{P}(\mathbf{U})\ (t)\mathbf{U}(t)=\begin{pmatrix} 1 & T_1 & T_2 & T_3 \\ 0 & U_{11} & U_{12} & U_{13} \\ 0 & U_{21} & U_{22} & U_{23} \\ 0 & U_{31} & U_{32} & U_{33} \end{pmatrix}$$

Then,

$$T_1 = \lambda^2 d \cos t/(\lambda^2+1) - \lambda\, ds \cos\alpha \sin t/(\lambda^2+1)$$

$$T_2 = \lambda^2 d \sin t/(\lambda^2+1) + \lambda\, ds \cos\alpha \cos t/(\lambda^2+1)$$

$$T_3 = \lambda\, ds \sin\alpha/(\lambda^2+1)$$

$$U_{11} = \cos t \cos s - \sin\alpha \sin t \sin s$$

$$U_{12} = \sin t \cos s + \sin\alpha \cos t \sin s$$

$$U_{13} = -\cos\alpha \sin s$$

$$U_{21} = -\cos t \sin s - \sin\alpha \sin t \cos s$$

$$U_{22} = -\sin t \sin s + \sin\alpha \cos t \cos s$$

$$U_{23} = -\cos\alpha \cos s$$

$$U_{31} = -\cos\alpha \sin t$$

$$U_{32} = \cos\alpha \cos t$$

$$U_{33} = \sin\alpha$$

Furthermore, we put

$$(1,\ \mathbf{X}(t,\ s)) = (1,\ A,\ 0,\ 0)\mathbf{C}(s;\ \tau s/\omega^2)\mathbf{P}(\mathbf{U})\ (t)\mathbf{U}(t)$$
$$= (1,\ X_1,\ X_2,\ X_3)\ \text{(where A is a constant)}$$

then,

$$X_1 = \lambda^2 d \cos t/(\lambda^2+1) - \lambda\, ds \cos\alpha \sin t/(\lambda^2+1) + A(\cos t \cos s - \sin\alpha \sin t \sin s)$$

$$X_2 = \lambda^2 d \sin t/(\lambda^2+1) + \lambda\, ds \cos\alpha \cos t/(\lambda^2+1) + A(\sin t \cos s + \sin\alpha \cos t \sin s)$$

$$X_3 = \lambda\, ds \sin\alpha/(\lambda^2+1) - A \cos\alpha \sin s$$

$\mathbf{X}(t,\ s)$ satisfies the following equation.

$$(1,\ -(\lambda^2+1)^{1/2}\partial\mathbf{X}/\partial s) = (1,\ \mathbf{X})\ \Omega\ (\mathbf{U}^{-1})$$

References

1) S. Ogino, "Study on the Meshing of Skew Gear" Trans. of Japan Soc. Mech. Engrs. (in Japanese), Vol. 30, No. 211, pp. 379–384.
2) S. Ogino, "Skew Gear Whose Specific Sliding is Equal to Zero" Trans. of Japan Soc. Mech. Engrs. (in Japanese), Vol. 30, No. 211, pp. 384–388.
3) T. Sakai, "On the Gears Whose Tooth Surfaces are Particular" Trans. of Japan Soc. Mech. Engrs. (in Japanese), Vol. 79, No. 696, pp. 1080–1085.
4) M. Kurita, "Differential Form and It's Applications" (in Japanese).
5) Y. Ohara, "On a Functional Equation Which is Satisfied with Tooth Surfaces" Memoirs of the Faculty of Engineering, Nagoya University, Vol. 38, No. 2, November, 1986, pp. 279–286.