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NUMERICAL METHOD FOR DISTRIBUTION OF EIGENVALUES IN LAMINAR PIPE FLOWS

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Abstract

In order to understand in more detail the dynamic structure of the linearized Navier-Stokes equation, a distribution of eigenvalues and classification of modes have been investigated especially for Poiseuille pipe flow. Generally, in numerical analysis of the eigenvalue problem, the accuracy of the eigenvalue gradually diminishes as its order increases. It is therefore difficult to obtain a correct eigenvalue distribution including higher order ones. This report aims to develop an improved numerical method for determining a distribution of eigenvalues with sufficient accuracy in general pipe flows. The method of expansion in orthogonal functions due to Salwen is reformulated in a Hilbert space. Then a finite dimensional approximate system is derived by the Galerkin method. In the Galerkin approximation, it is difficult do determine the dimension of approximate subspace a priori. A criterion for this truncation problem is presented based on a property of operator invariance of the subspace. The present numerical method is applied to Poiseuille pipe flow, and the result is compared with asymptotic method and other numerical methods.

1. Introduction

In order to understand the relationship between flow instability and transition to turbulence, the eigenvalue problem for the linearized Navier-Stokes equation has been investigated with respect to several fundamental laminar flows. In stability analysis, only the most unstable (or least stable) eigenvalue is usually of interest. For further comprehension of the linearized system, however, eigenvalues of higher order should also be considered. From this viewpoint, a distribution of eigenvalues and classification of modes have been investigated especially for Poiseuille pipe flow. Pekeris[1] decomposed an axisymmetric disturbance into *meridional mode* and *torsional mode* families. Corcos-Sellars[2] and Gill[3] further divided relatively lower-ordered eigenvalues of the meridional mode family into *slow mode* and *fast mode* families by their phase velocities. Davey-Drazin[4] distinguished the third mode of higher-ordered eigenvalues, which they called *mean mode*, from the above slow and fast modes. In the case of non-axisymmetric disturbances, Salwen-Grosch[5),6] obtained eigenvalues for the disturbances with the azimuthal wave number from 0 to 5, and Wadih[7] presented an analytical discussion on the classification of modes.

Generally speaking, in numerical analysis of the eigenvalue problem, the accuracy of the eigenvalue gradually diminishes as its order increases. It is therefore difficult to obtain a correct eigenvalue distribution including higher order ones. This report aims to develop an improved numerical mehtod for determining a distribution of eigenvalues with sufficient accuracy in general pipe flows. First, a formulation due to Salwen[5] is reformulated in a Hilbert space L^2 $(\tilde{\Omega})$, and a linearized system is obtained. Then a finite dimensional approximate system is derived by the Galerkin method. In the Galerkin approximation, it is difficult to determine the dimension of approximate subspace a priori. A criterion for this truncation problem is proposed based on a property of operator invariance of the subspace. Next, the present numerical method is applied to Poiseuille pipe flow and the result is compared with Davey-Drazin[4] and Corcos-Sellars.[2]

2. Formulation

2. 1. *Fundamental equation*

In this section, the fundamental equation for small perturbation of general pipe flows is introduced. Fig. 1 shows a domain and co-ordinate system. On the cylindrical co-ordinates $(r,\ \theta,\ z)$, the unbounded domain in a straight pipe is expressed as a product $(0,1)\times(0,2\pi)\times(-\infty,+\infty)$. Incompressible viscous fluid flow in the domain Ω is described by the following equations.

Equation of continuity: $\nabla\, \boldsymbol{u}=0$

Equation of motion: (Navier-Stokes equation)

$$\frac{\partial u}{\partial t}+\frac{1}{2}\nabla(\boldsymbol{u}\cdot\boldsymbol{u})-\boldsymbol{u}\times(\nabla\times\boldsymbol{u})=-\nabla p-\frac{1}{Re}\nabla\times(\nabla\times\boldsymbol{u}) \tag{1}$$

with

Boundary condition: $\boldsymbol{u}|_{\Gamma}=\boldsymbol{u}_B$

and

Initial condition: $\boldsymbol{u}|_{t=0}=\boldsymbol{u}_O(\mathrm{x})$

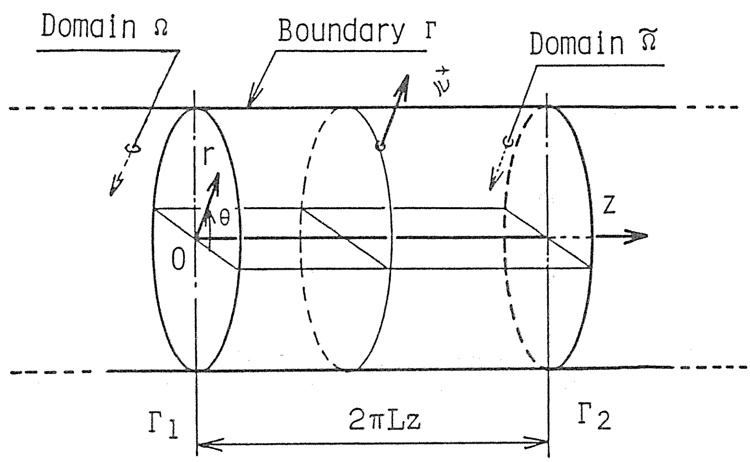


Fig. 1. Domain and coordinate system.

where, $u=(u_1,\ u_2,\ u_3)^T$ denotes the velocity vector, p is pressure, and $x=(r,\ \theta,\ z)^T$ represents an arbitrary point in the domain. All quantities in this report are non-dimensional; the length scale is r_0, the radius of the pipe and the velocity scale is W_0, the maximum axial velocity of the base flow. The Reynolds number is defined as,

$$Re=W_0r_0/\nu \tag{2}$$

The base flow is assumed to be a steady axisymmetric parallel flow,

$$U_S=(0,\ V_S(r),\ W_S(r))^T,\ P_S=P_S(r,\ z) \tag{3}$$

Evolution of a small perturbation of the base flow $(U_S,\ P_S)$ is described by the following linearized equations,

$$\frac{\partial \hat{u}}{\partial t}+L_c[\hat{u}]=-\nabla p-\frac{1}{Re}\nabla\times(\nabla\times\hat{u}) \tag{4.a}$$

$$\nabla\cdot\hat{u}=0 \tag{4.b}$$

with boundary condition:

$$\hat{\boldsymbol{u}}|_{\Gamma}=0 \tag{5.a}$$

and initial condition:

$$\hat{\boldsymbol{u}}|_{t=0}=\hat{\boldsymbol{u}}_0(\mathrm{x}) \tag{5.b}$$

In Eq. (4) L_c is a linear operator defined as

$$L_c[\hat{\boldsymbol{u}}]=\nabla(U_s\cdot\hat{\boldsymbol{u}})-U_s\times(\nabla\times\hat{\boldsymbol{u}})-\hat{\boldsymbol{u}}\times(\nabla\times U_s) \tag{6}$$

Under the assumption of an infinitely long pipe, upstream and downstream boundaries have no effect on the flow in consideration. Then the periodical boundary condition

$$\begin{aligned}\hat{\boldsymbol{u}}(t,\ r,\ \theta,\ z)&=\hat{\boldsymbol{u}}(t,\ r,\ \theta,\ z+2\pi L_z)\\ \hat{p}(t,\ r,\ \theta,\ z)&=\hat{p}(t,\ r,\ \theta,\ z+2\pi L_z)\end{aligned} \tag{7}$$

is applied, and the unbounded domain Ω is reduced to a bounded domain

$$\tilde{\Omega}=(0,1)\times(0,2\pi)\times(0,2\pi L_z),$$

where $2\pi L_z$ is a period in z-direction. Disturbance in this bounded domain $\tilde{\Omega}$ is discussed in the next section.

2. 2. Linearized evolution equation

Let $\boldsymbol{L}^2(\tilde{\Omega})$ be a 3-hold Cartesian product of the complex Hilbert space $L^2(\tilde{\Omega})$. Arbitary element of $\boldsymbol{L}^2(\tilde{\Omega})$ is written as

$$\boldsymbol{u}=(u_1,\ u_2,\ u_3)^{\mathrm{T}},$$

and each u_k is a square integrable complex valued function defined in the domain $\tilde{\Omega}$. The scalar product of elements u, v in $L^2(\tilde{\Omega})$ is defined as

$$<\boldsymbol{u},\ v>=\sum_{k=1}^{3}\int_{\Omega}\overline{u_k(x)}\ v_k(x)d_x \tag{8}$$

where $\bar{\bullet}$ denotes a complex conjugate. The induced norm is denoted by $\|\ \boldsymbol{u}\ \|=<\boldsymbol{u},\ \boldsymbol{u}>^{1/2}$.

By the Helmholtz theorem, the Hilbert space $\boldsymbol{L}^2(\tilde{\Omega})$ is described as a direct sum of orthogonal subspaces,

$$L^2(\tilde{\Omega}) = J(\tilde{\Omega}) \oplus G(\tilde{\Omega}).$$

J($\tilde{\Omega}$) and $G(\tilde{\Omega})$ are defined as

$$J(\tilde{\Omega}) = \{u \in L^2(\tilde{\Omega})\ ;\ \nabla \cdot u = 0,\ \vec{\nu} \cdot u|_\Gamma = 0\},$$
$$G(\tilde{\Omega}) = \{v \in L^2(\tilde{\Omega})\ ;\ v = \nabla p,\ p \in H^1(\tilde{\Omega})\} \tag{9}$$

where $\vec{\nu}$ is as outward unit normal on Γ, and $H^1(\tilde{\Omega})$ consists of function differentiable in $\tilde{\Omega}$ (Sobolev space of the first order). An arbitrary element w of L^2 ($\tilde{\Omega}$) is written uniquely as the sum of the solenoidal vector field u ($\nabla u=0$) and irrotational vector field ∇p. This relationship is shown in Fig. 2. Let ($\hat{u}$, p̂) be a solution of Eqs. (4), and P_J be an orthogonal projection of L^2 ($\tilde{\Omega}$) into $J(\tilde{\Omega})$. Solenoidal vector field $\hat{u}$ and irrotational vector field $\nabla \hat{p}$ are projected by P_J to itself and zero, respectively.

$$P_J\hat{u}(t,\cdot) = \hat{u}(t,\cdot),\ P_J \nabla p(t,\cdot) = 0 \tag{10}$$

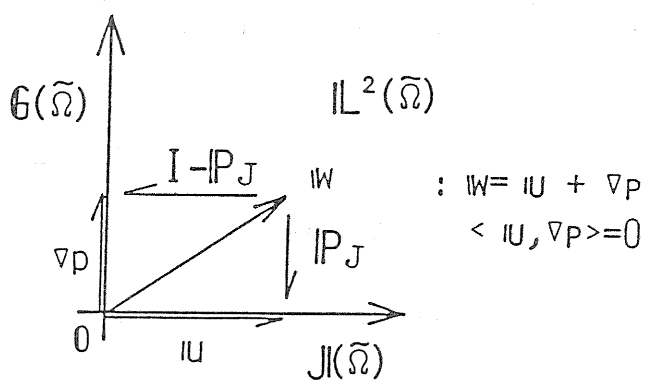


Fig. 2. Orthogonal direct sum decomposition of Hilbert space $L^2(\tilde{\Omega})$. "*I*" denotes identity operator.

Operating P_j to Eq. (4), we obtain a linear evolution equation,

$$\frac{d\hat{u}(t)}{dt} = A\,\hat{u}(t)$$
$$\hat{u}(0) = \hat{u}_0,\ \hat{u}(t) \in J_0(\tilde{\Omega}) \tag{11}$$

It should be noted that solution $\hat{u}(t)$ is an element of a function space $J_0(\tilde{\Omega})$, which is the subspace of $J(\tilde{\Omega})$ limited by the boundary condition (Eq. 5(a)). The linear operator A in Eq. (11) consists of A_c, the operator from the convective terms and A_D, the Stokes operator,

$$\begin{aligned} &A = A_c + (1/Re)A_D \\ \text{where} \quad &A_C = -P_J L_c \\ &A_D\hat{u} = -P_J[\nabla \times (\nabla \times \hat{u})] \end{aligned} \qquad (12)$$

2. 3. Wave form disturbance

In this section, Hilbert space $J_o(\tilde{\Omega})$ is subdivided for simplicity of the problem. An arbitary element $\hat{u}$ of $J_o(\tilde{\Omega})$ is expanded in double Fourier series.

$$\hat{u} = \sum_{n,m \in Z} \hat{u}(r;\ n,\ m)\exp[i(n\theta + mz/L_z)] \qquad (13)$$

where Z is a set of integers, and n and $\alpha = m/L_z$ are wavenumbers in the azimuthal and z-axial direction, respectively. Hilbert space $J_0(\tilde{\Omega})$ is then described as the orthogonal direct sum of J_{nm}.

$$J_o(\tilde{\Omega}) = \bigoplus_{n,m \in Z} J_{n,m} \qquad (14)$$

where J_{nm} is a closed subspace of $J_0(\tilde{\Omega})$ defined as

$$J_{n,m} = \{\hat{u} \in J_o(\tilde{\Omega})\ ;\ \hat{u} = \hat{u}(r)\exp[i(n\theta + mz/L_z)]\} \qquad (15)$$

The subspace J_{nm} has a property of operator A invariance, i.e., AJ_{nm} is contained in J_{nm}. Thus, the evolution of a wave form disturbance with fixed wave numbers $(n,\ \alpha)$

$$\hat{u}_0 = \tilde{u}(r)\exp[i(n\theta + \alpha z)]$$

is described by the linear dynamic system in the subspace J_{nm} of $J_0(\tilde{\Omega})$

$$\begin{aligned} &\frac{d\hat{u}(t)}{dt} = A\hat{u}(t) \\ &\hat{u}(0) = \hat{u}(r)\exp[i(n\theta + mz/L_z)],\ \hat{u}(t) \in J_{n,m} \end{aligned} \qquad (16)$$

The general system of Eq. (11) is now reduced to more simplified systsm of Eq. (16).

3. Eigenvalue Problem in Approximate Subspace

In the following, the subspace J_{nm} of $J_o(\tilde{\Omega})$ is simply written as "H". The linear dynamic system in H (Eq. (16)) is reduced to a finite dimensional approximate system by the Galerkin method. In this process, both reasonable choice of basis and truncation in an appropriate dimension are very important. A criterion for this truncation problem is presented based on the concept of operator invariance of the subspace.

3. 1. *Orthonormal bases*

First, we construct a basis of the Hilbert space H. Let us consider an eigenvalue problem of the Stokes operator A_D (Eq. 12)):

$$A_D\phi_k=\lambda_k\phi_k, \quad \phi_k \in D(A_D) \tag{17}$$

where the domain of A_D is $D(A_D)=H\cap H^2(\tilde{\Omega})$, and $H^2(\tilde{\Omega})$ is the Sobolev space of the second order. It is known that the Stokes operator has denumerable negative real eigenvalues

$$0>\lambda\geqq\lambda_2\geqq\cdots\geqq\lambda_k\rightarrow-\infty \quad (k\rightarrow\infty) \tag{18}$$

and that eigenvectors $\{\phi_k\}_{k=1}^{\infty}$ form a complete orthonormal basis of H (ϕ_k is normalized as $\|\phi_k\|=1$).[8] So an arbitrary element of H is represented uniquely as

$$\hat{u}=\sum_{j=1}^{\infty}<\phi_j, \ \hat{u}>\phi_j \tag{19}$$

3. 2. *Finite dimensional approximate system*

Let $\hat{u}_N(t)$ be an approximate solution of Eq. (16) obtained by truncation in the first N terms of the series in Eq. (19).

$$\hat{u}_N(t)=\sum_{j=1}^{N}a_j(t)\phi_j \tag{20}$$

where $a_j(t)s$ ($j=1, 2,\cdots$, N) are unknown coefficients. Introducing Eq. (20) into Eq. (16), we obtain a residual $R_E(\hat{u}_N)\in H$ as

$$R_E(\hat{u}_N)=d\hat{u}_N/\mathrm{d}t-A\hat{u}_N \tag{21}$$

The Galerkin method implies that the residual R_E must be orthogonal to all vectors ϕ_is ($i=1,\cdots$, N).

$$<\phi_i, \ R_E(\hat{u}_N)>=0 \quad (i=1, \ 2,\cdots, \ N) \tag{22}$$

Evolution of N-dimensional vector of unknown coefficients $\boldsymbol{a}(t)=[a_1(t,\cdots, a_N(t)]^T$ is determined by the linear dynamic system obtained from Eqs. (20)~(22).

$$\frac{d\boldsymbol{a}(t)}{dt}=A_N\boldsymbol{a}(t)$$
$$\boldsymbol{a}(0)=a_0,\ \ \boldsymbol{a}(t)\in C^N \tag{23}$$

where A_N is a N×N matrix whose (i, j) element is

$$(A_N)_{ij}=<\phi_i,\ A\phi_j>$$

and C^N is N-dimensional Enclidean space in the complex field. Structure of the linear dynamic system (23) is clarified from the eigenvalue problem.

$$A_N e_k=\sigma_k e_k(k=1,\ 2,\dots,\ N) \tag{24}$$

where σ_k is k'th eigenvalue and $e_k=[e_{k1}\cdots e_{kn}]^T$ is the corresponding eigenvector. The above result gives approximate lower N ordered eigenvalues $\Lambda(A)$ and eigenvectors $\{\Psi_k\}_{k=1}^N$ of the linear dynamic system of Eq. (16) in $\boldsymbol{H}$.

$$\Lambda_N(A)=\{\sigma_k;\ k=1,\ 2,\dots,\ N\}$$
$$\psi_k(r,\ \theta,\ z)=\sum_{j=1}^{N} e_{kj}\phi_j(r,\ \theta,\ z)(k=1,\ 2,\dots,\ N) \tag{25}$$

3. 3. Crietertion for determination of dimension

Le $\boldsymbol{H}_N$ be a subspace of $\boldsymbol{H}$ spanned by $\{\phi_k;\ k=1,\cdots,\ N\}$, and $\boldsymbol{P}_J$ an orthogonal projection of $\boldsymbol{H}$ onto $\boldsymbol{H}_N$. Then Eqs. (21) and (22) are equivalent to the following expression.

$$\frac{d\hat{\boldsymbol{u}}_N(t)}{dt}\boldsymbol{P}_N\cdot A\hat{\boldsymbol{u}}_N(t)$$
$$\hat{\boldsymbol{u}}_N(0)=\hat{\boldsymbol{u}}_0,\ \ \hat{\boldsymbol{u}}_N(t)\in\boldsymbol{H}_N \tag{26}$$

In general, the image of $\hat{\boldsymbol{u}}_N$ by A does not belong to $\boldsymbol{H}_N$ but to $\boldsymbol{H}$. In an approximate dynamic system of Eq. (26), the element $A\hat{\boldsymbol{u}}_N\in\boldsymbol{H}$ is projected into $\boldsymbol{H}_N$ by $\boldsymbol{P}_J$. In order that approximate system of Eq. (26) may conserve the dynamic structure of the original system of Eq. (16), the subspace $\boldsymbol{H}_N$ should have a property of operator A invariance, i.e., $A\boldsymbol{H}_N\subset\boldsymbol{H}$. In subsequent discussion, a measure of operator invariance is introduced to give a criterion for the determination of the dimension of the approximate subspace $\boldsymbol{H}_N$.

3. 3. 1. Representation of operator A

Let $V_N=[\phi_1 \cdots \phi_N]$ be a basis matrix of $\boldsymbol{H}_N$ and $V_N=[\phi_{N+1} \cdots]$ be the matrix of orthogonal complement of $\boldsymbol{H}_N$ with respect to H. Representation matrix $\boldsymbol{A}$ of the operator A is defined by

$$A[\boldsymbol{V}_N \vdots \ \boldsymbol{V}_N^{\perp}]=[\overset{\overset{N}{\leftrightarrow}}{\boldsymbol{V}_N} \ \vdots \ \overset{\overset{\infty}{\leftrightarrow}}{\boldsymbol{V}_N^{\perp}}]\begin{bmatrix} \overset{\overset{N}{\leftrightarrow}}{\boldsymbol{A}_{11}} & \overset{\overset{\infty}{\leftrightarrow}}{\boldsymbol{A}_{12}} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \tag{27}$$

where $\boldsymbol{A}$ consists of the submatrices $\boldsymbol{A}_{k\ell}$ ($1 \leqq k, \ \ell \leqq 2$). The (i, j) element of $\boldsymbol{A}$ is obtained from Eqs. (12) and (17) as:

$$(A)_{ij}=<A_C+(1/Re)A_D]\phi_j>=<\phi_i, \ A_C\phi_j>+(1/Re)\lambda_j\delta_{ij} \tag{28}$$

where δ_{ij} is the Kronecker's delta. Numerical calculation shows that the first term of the right hand side of Eq. (28) forms a band matrix in general, with a band width M of about 6 for Poiseuille pipe flow. The representation of the operator A therefore is also a band matrix of the band width M.

3. 3. 2. Measure ot operator A invariance

In order that the approximate subspace $\boldsymbol{H}_N$ of $\boldsymbol{H}$ be operator A invariant, a necessary and sufficient condition is that the submatrix $A_{21}=0$ (Eq. (27)). Since the respresentation $\boldsymbol{A}$ is the band matrix, this condition is mentioned more simply as follows. We introduce a $2M \times 2M$ submatrix $\boldsymbol{B}_N$ of $\boldsymbol{A}$ (Fig. 3).

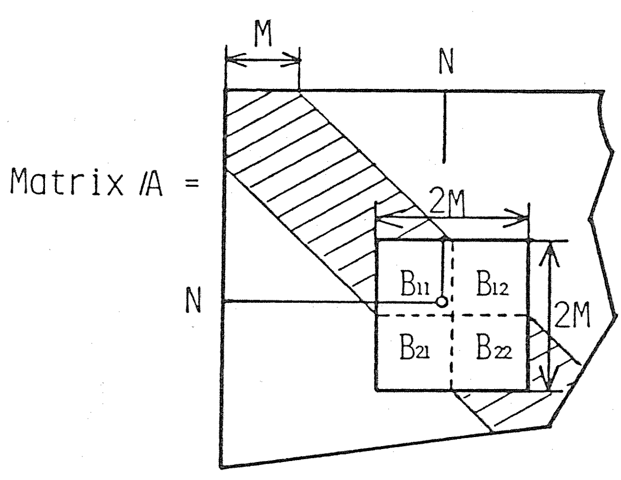


Fig. 3. Band matrix $\boldsymbol{A}$ and submatrix $\boldsymbol{B}$.

$$(\boldsymbol{B}_N)_{ij} = (\boldsymbol{A})_{N-M+i,N-M+j}, \quad (1 \leqq i, j \leqq 2M) \tag{29}$$

The condition $A_{21}=0$ is equivalent to $\boldsymbol{B}_{21}=0$ in block notation of $\boldsymbol{B}_N$ as,

$$B_N = \left[\begin{array}{c:c} \overset{\overset{M}{\leftrightarrow}}{B_{11}} & \overset{\overset{M}{\leftrightarrow}}{B_{12}} \\ \hdashline 0 & B_{22} \end{array}\right] \tag{30}$$

Operator A invariance of the subspace $\boldsymbol{H}_N$ for a given N value can therefore be measured by the proximity of matrix $\boldsymbol{B}_N$ to some $2M \times 2M$ matrix of the form in Eq. (30). The linear vector space of $2M \times 2M$ matrices Mat(2M) becomes finite dimensional Hilbert space,[9] if the inner product is defined for $\boldsymbol{B}_1$, $\boldsymbol{B}_2 \in$ Mat(2M) as:

$$\ll \boldsymbol{B}_1,\ \boldsymbol{B}_2 \gg = \mathrm{T_r}(\boldsymbol{B}_1^* \boldsymbol{B}_2) \tag{31}$$

where $(\bullet)^*$ denotes a transpose of complex conjugate and Tr indicates a trace of a matrix. The induced norm is defined as

$$\| \boldsymbol{B} \|_M = \ll \boldsymbol{B},\ \boldsymbol{B} \gg^{1/2}$$

Let $\boldsymbol{E}_1$ be subspace of Mat (2M) whose element has a block form of Eq. (30). Then the proximity of $\boldsymbol{B}_N$ Mat (2M) to $\boldsymbol{E}_1$ is measured by the value of ϱ_N defined as,

$$\begin{aligned} \varrho_N &= 1 - \cos\ \theta_N \\ &= 1 - \| \tilde{B}_N \|_M / \| B_N \|_M \end{aligned} \tag{32}$$

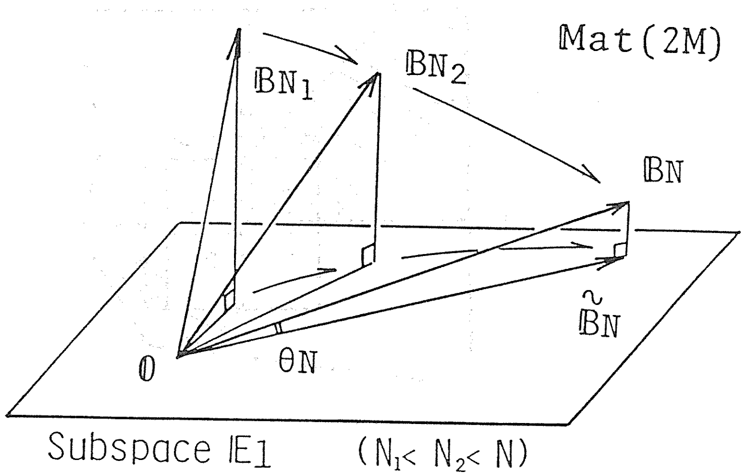


Fig. 4. Variation of operator invariance of subspace as a function of its dimension.

where $\tilde{B}_N$ is a projection of B_N to the subspace E_I. Fig. 4 shows a numerical result illustratively that B_N approaches the subspace E_I monotonically as the dimension N of the subspace H_N increases. Operator A invariance of the subspace H_N can therefore be measured by the smallness of the nonnegative value ϱ_N.

4. Numerical Calculation

4. 1. Eigenvalues for Poiseuille pipe flow

Eigenvalues of axisymmetric disturbance on Poiseuille pipe flow are calculated by the present method, and compared with those of former studies. The following results are all plotted in a plane of complex axial phase velocity, denoted by c, in relation to eigenvalue σ as

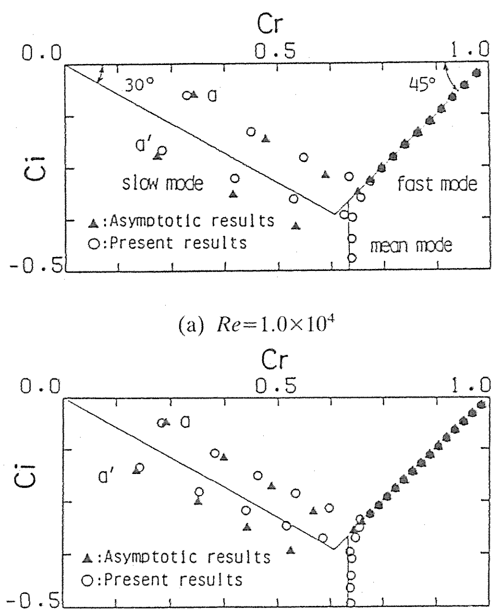
$$c = c_r + i c_i = i\sigma/\alpha \tag{33}$$

where the real part c_r is a physical phase velocity and the imaginary part c_i is an amplification factor divided by the axial wavenumber α.

As mentioned before, the bases $\phi_k (k=1, 2, \dots, \mathrm{N})$ of the function space H_N are the eigenvectors of the Stokes operator A_D. It is therefore expected that the present method gives the eigenvalues with sufficient accuracy for lower Reynolds number.

In the following, on the other hand, the validity of the present method for higher Reynolds number is examined. In Figs. 5(a) and (b), comparison is made between the present result denoted by ○ and the result of the asymptotic method[2)] denoted by ▲, with the axial wavenumber $\alpha=1.0$ and the Reynolds number $Re=1\times10^4$ and 2×10^4, respectively. The asymptotic method gives the eigenvalues which are accurate at the limit of $\alpha Re \to \infty$ and $c_r \to 1$, or $\alpha Re \to \infty$ and $c_r \to 0$; the former eigenvalues are customarily called *fast modes* and the latter ones *slow modes*. Agreement between both methods is very good for the fast mode eigenvalues, but not so good for the slow mode eigenvalues except for those with small phase velocity, denoted by a or a' in Fig. 5(a) and (b).

Davey and Drazin obtained eigenvalues of Poiseuille pipe flow by numerical analysis of the Orr-Sommerfeld equation, pointing out the existence of the third mode family named *mean mode*. Comparison is made in Figs. 6(a) and (b) for axisymmetric disturbance with $n=0$, $\alpha=1.0$, and the Reynolds number $Re=5.0\times10^3$ and 1.0×10^4, respectively. For relatively small Reynolds number in Fig. 6(a), the agreement is good for eigenvalues in fast mode, slow mode and lower-ordered mean mode with the imaginary part



(a) $Re=1.0\times10^4$

(b) $Re=2.0\times10^4$

Fig. 5. Comparison with asymptotic results[4)]. ($n=0$, $\alpha=1.0$)

c_i up to -1.0. For higher Reynolds number in Fig. 6(b), however, agreement is limited within lower-ordered eigenvalues of fast and slow-mode families with c_i up to -0.2, and the two methods yield completely different distributions of eigenvalues.

4. 2. Discussion of truncation problem

Eigenvalues are calculated by the present method with $n=0$, $\alpha=1.0$ and $Re=5.0\times10^3$. The results are shown in Figs. 7(a) to (c) for approximate subspace $\boldsymbol{H}_N$ with the dimension N of 20, 25 and 40, respectively. Variation of the measure ϱ_N of the operator invariance indicates that the increment of the dimension N makes the accuracy of eigenvalues better. Distribution of lower ordered 30 eigenvalues shown in Fig. 7(c) is no longer changed by an increment of N value over 40. It is therefore concluded that Davey's results in Figs. 6, whose distribution are essentially the same as the ones in Fig. 7(a) or (b), do not have sufficient accuracy for higher-ordered eigenvalues, and that distribution of eigenvalues for axisymmetric disturbance of Poiseuille pipe flow is given in Fig. 7(c).

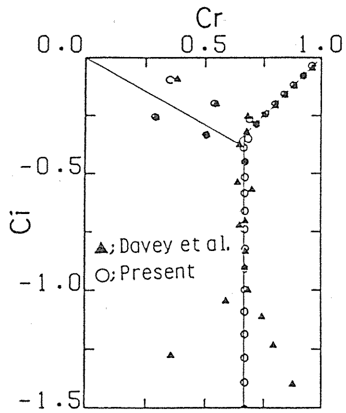
Calculations are conducted systematically with the azimuthal wavenumber n from 0 to 2, the axial wave numbers α from 0.01 to 10.0, and the Reynolds number Re from 0.1 to 5.0×10^4. For each 3-tuple (n, α, Re), accurate distribution of eigenvalues is obtained at some small value of ϱ_N corresponding to the subspace of dimension N. In the range of paramenters given above, the critical value of ϱ_N is constant:

$$(\varrho_N)_{crit} \fallingdotseq 0.15\times10^{-3} \qquad (34)$$

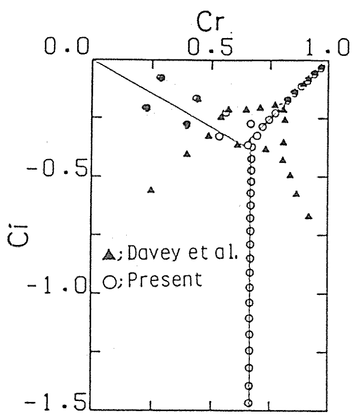
This property is conveniently used as a criterion for determining a dimension of the subspace.

"*Dimension of the subspace must be so determined that the value of ϱ_N is equal or less than the critical value $(\varrho_N)_{crit}$.*"

Table 1 shows that the appropriate dimension N of the subspace determined by the above criterion. Value of N increases from 42 up to 120 corresponding to the



(a) $Re=5.0\times10^3$

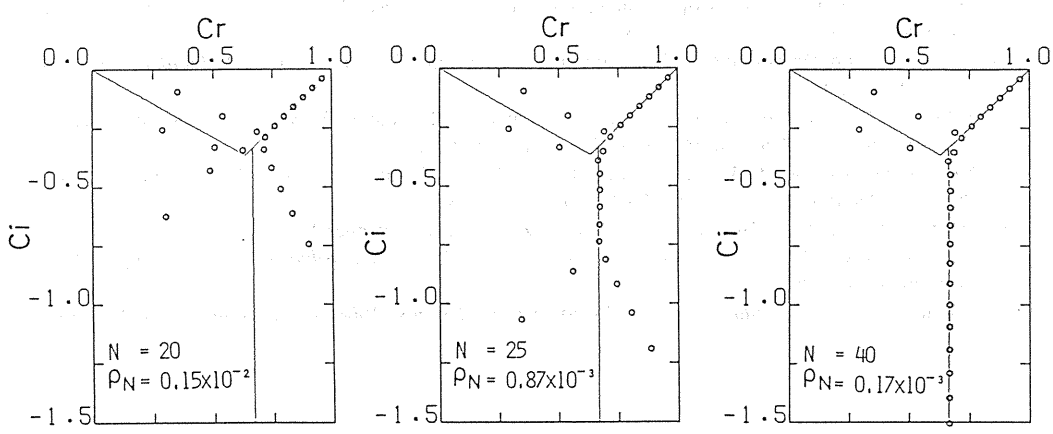


(b) $Re=1.0\times10^4$

Fig. 6. Comparison with numerical analysis of Orr-Sommerfeld equation[4]. ($n=0$, $\alpha=1.0$)

Table 1. Appropriate dimension of subspace as a function of the Reynolds number. ($n=0$, $\alpha=1.0$)

Re	N	CPU time(s)
5000	42	1.154
10000	59	1.232
20000	85	3.546
30000	104	6.787
40000	120	10.381



(a) N=20 (b) N=25 (c) N=40

Fig. 7. Eigenvalue distributions calculated in approximate subspaces of different dimensions.

Reynolds number from 500 to 40000. Dimensions of the subspace in the calculations of Figs. 5 and 6 are determined by the above criterion.

5. Conclusions

This report developed an improved numerical method for determining a distribution of eigenvalues of general pipe flows. The results are summarized as follows.

(1) Measure ϱ_N of the operator invariance of the subspace is introduced. The present numerical method gives accurate eigenvalue distributions if the value of ϱ_N is equal or smaller than a certain critical value.

(2) Calculations limited to the subspace of too small dimension yield an incorrect result, particularly in higher-ordered eigenvalues.

(3) The present numerical method can be applied for relatively large Reynolds numbers up to 5.0×10^4 by suitable choice of approximate subspace.

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