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ON A FUNCTIONAL EQUATION WHICH IS SATISFIED WITH TOOTH SURFACES

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Abstract

It is very difficult problem how to clssify the tooth surfaces. In this paper, we define a functional (or differential) equation which is satisfied with the tooth surfaces. And by classifying the solutions of this equation, we can approach that classification problem.

1. Preliminaries

In this paper, we assume that one of the pair of gears is fixed, so the motion of gears is expressed relatively. A motion is expressed by a (4, 4) matrix

$$\boldsymbol{U}(t)=\begin{pmatrix}1 & T(t)\\ 0 & U(t)\end{pmatrix}$$

where, $T(t)$ is the row vector which expresses the position of the origin and $U(t)$ is the matrix of rotation.

A point $\overline{X}$ in the moving coordinate system is expressed as a point X in the standard coordinate system as follows;

$$(1,\ X)=(1,\ \overline{X})\boldsymbol{U}=(1,\ \overline{X})\begin{pmatrix}1 & T\\ 0 & U\end{pmatrix}=(1,\ T+\overline{X}U)$$

When the moving coordinate system is in motion shown by $\boldsymbol{U}_1$ against the standard coordinate system, another moving coordinate system which is in motion shown by $\boldsymbol{U}_2$ against the moving coordinate system, is in motion shown by $\boldsymbol{U}_2\ \boldsymbol{U}_1$ against the standard coordinate system.
i. e.

$$\boldsymbol{U}_2\boldsymbol{U}_1=\begin{pmatrix}1 & T_2\\ 0 & U_2\end{pmatrix}\begin{pmatrix}1 & T_1\\ 0 & U_1\end{pmatrix}=\begin{pmatrix}1 & T_1+T_2U_1\\ 0 & U_2U_1\end{pmatrix}$$

And the standard coordinate system is in motion shown by $\boldsymbol{U}^{-1}$ against the moving coordinate system.
i. e.

$$\boldsymbol{U}^{-1}=\begin{pmatrix} 1 & -TU^{-1} \\ 0 & U^{-1} \end{pmatrix}$$

2. Notations

$$\boldsymbol{C}(t)=\begin{pmatrix} 1 & 0 \\ 0 & C(t) \end{pmatrix} \qquad C(t)=\begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{C}_x(t)=\begin{pmatrix} 1 & 0 \\ 0 & C_x(t) \end{pmatrix} \qquad C_x(t)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix}$$

$$\boldsymbol{C}_y(t)=\begin{pmatrix} 1 & 0 \\ 0 & C_y(t) \end{pmatrix} \qquad C_y(t)=\begin{pmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{pmatrix}$$

$$\boldsymbol{E}_x(t)=\begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{E}_y(t)=\begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{E}_z(t)=\begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3. Functional Equation

We assume that a tooth surface $X(t, s)(=(x(t, s), y(t, s), z(t, s)))$ is fixed in the standard coordinate system and the pair surface $\overline{X}(t, s)$ is fixed in the moving coordinate system which is in motion shown by $\boldsymbol{U}(t)$.

Hence, the tooth surface $X(t, s)$ contacts with the surface $\overline{X}(t, s)$ along s-line $X(t,s)$ at time t.

A point $\overline{X}(t, s)$ and a vector $\overline{A}$ in the moving coordinate system is related to the point $X(t, s)$ and the vector A in the standard coordinate system as follows;

$$(1,\ \bar{X})=(1,\ X)U^{-1}(t)$$

$$(1,\ \bar{A})=(1,\ A)\begin{pmatrix}1 & 0\\ 0 & U^{-1}(t)\end{pmatrix}$$

$\partial\bar{X}/\partial t$ is the moving velocity vector of the contacting line $\bar{X}$ on the tooth surface $\bar{X}(t,\ s)$ in the moving coordinate system.

In the standard coordinate system, the expression of this vector $\partial\bar{X}/\partial t$ is following.

$$(1,\ \bar{X}_t)\begin{pmatrix}1 & 0\\ 0 & U\end{pmatrix}=(1,\ \bar{X}_t\ U)$$

When the surface X contacts with $\bar{X}$ along s-line $X(t,\ s)$, this vector $\bar{X}_tU$ lies on the tangent plane at the point $X(t,\ s)$. Hence, there are two scalars $\beta(t,\ s)$ and $a(t,\ s)$ as follows;

$$\bar{X}_tU=\beta(t,\ s)X_t+a(t,\ s)X_s \tag{1}$$

Moreover, when we assume that the contacting line $\bar{X}(t,\ s)$ is fixed in a moment in the moving coordinate system, the velocity vector V of $\bar{X}(t,\ s)$ is

$$(1,\ \bar{X})\begin{pmatrix}1 & dT\\ 0 & dU\end{pmatrix}.$$

Therefore,

$$\begin{aligned}(1,\ V)&=(1,\ \bar{X})\begin{pmatrix}1 & dT\\ 0 & dU\end{pmatrix}\\ &=(1,\ X)\begin{pmatrix}1 & -TU^{-1}\\ 0 & U^{-1}\end{pmatrix}\begin{pmatrix}1 & dT\\ 0 & dU\end{pmatrix}\\ &=(1,\ dT-TU^{-1}dU+XU^{-1}dU)\end{aligned}$$

On the other hand,

$$\begin{aligned}(1,\ \bar{X})&=(1,\ X)\begin{pmatrix}1 & -TU^{-1}\\ 0 & U^{-1}\end{pmatrix}\\ &=(1,\ -TU^{-1}+XU^{-1})\end{aligned}$$

Hence,

$$(1,\ \bar{X}_t)=(1,\ XdU^{-1}+dX\cdot U^{-1}-dT\cdot U^{-1}-TdU^{-1})$$

$$(1,\ \bar{X}_t)\begin{pmatrix}1 & 0\\ 0 & U\end{pmatrix}=(1,\ XdU^{-1}\cdot U+dX-dT-TdU^{-1}\cdot U)$$

$$(1,\ V)+(1,\ \bar{X}_t)\begin{pmatrix}1 & 0\\ 0 & U\end{pmatrix}$$

$$=(1,\ dT-TU^{-1}dU+XU^{-1}dU+XdU^{-1}\cdot U+dX-dT-TdU^{-1}\cdot U)$$

By using relation

$$U^{-1}dU=-dU^{-1}\cdot U,$$
$$=(1,\ X_t)$$

i. e.

$$X_t=V+\overline{X}_tU$$

By substituting this relation into (1)

$$X_t-V=\beta(t,\ s)X_t+a(t,\ s)X_s \tag{2}$$

On the other hand, we define $\Omega(U)$ as follows;

$$\Omega(U)=dU\cdot U^{-1}=\begin{pmatrix}0 & dT\cdot U^{-1}\\ 0 & dU\cdot U^{-1}\end{pmatrix}$$

Then,

$$(1,\ X)\Omega(U^{-1})=(1,\ X)\begin{pmatrix}0 & d(-TU^{-1})\cdot U\\ 0 & dU^{-1}\cdot U\end{pmatrix}$$
$$=(1,\ X)\begin{pmatrix}0 & -dT-TdU^{-1}\cdot U\\ 0 & dU^{-1}\cdot U\end{pmatrix}$$
$$=-dT+TU^{-1}dU-XU^{-1}dU$$
$$=-V \qquad \text{(we regard as a vector)}$$

By substituting this into (2), we obtain following relation.

$$(1-\beta(t,\ s))X_t-a(t,\ s)X_s=-(1,\ X)\Omega(U^{-1}) \tag{3}$$

This relation is a functional equation of tooth surfaces for the motion $U(t)$.
Where, $(1-\beta\ (t,\ s))$ and $a(t,\ s)$ are known or unknown, and $(1-\beta)$ is called "specific sliding".

If we rewrite (3) by using,

$$\Omega(U^{-1})=\begin{pmatrix}0 & \overline{\omega}_1 & \overline{\omega}_2 & \overline{\omega}_3\\ 0 & 0 & \overline{\omega}_{12} & \overline{\omega}_{13}\\ 0 & -\overline{\omega}_{12} & 0 & \overline{\omega}_{23}\\ 0 & -\overline{\omega}_{13} & -\overline{\omega}_{23} & 0\end{pmatrix}$$

we obtain following equation.

$$\begin{cases}(\beta-1)\dfrac{\partial x}{\partial t}+a\dfrac{\partial x}{\partial s} \qquad\qquad +\overline{\omega}_{12}y+\overline{\omega}_{13}z=\overline{\omega}_1\\ (\beta-1)\dfrac{\partial y}{\partial t}+a\dfrac{\partial y}{\partial s}-\overline{\omega}_{12}x \qquad\qquad +\overline{\omega}_{23}z=\overline{\omega}_2\\ (\beta-1)\dfrac{\partial z}{\partial t}+a\dfrac{\partial z}{\partial s}-\overline{\omega}_{13}x \quad -\overline{\omega}_{23}y \qquad\quad =\overline{\omega}_3\end{cases}$$

4. Functional Equation of the Pair Surface

$$(1,\ X)=(1,\ \overline{X})U=(1,\ \overline{X})\begin{pmatrix}1 & T\\ 0 & U\end{pmatrix}=(1,\ T+\overline{X}U)$$

then,

$$X_t=dT+d\overline{X}\cdot U+\overline{X}dU$$

$$X_s=\overline{X}_sU$$

By substituting this relations into (1),

$$\overline{X}_tU=\beta(dT+d\overline{X}\cdot U+\overline{X}dU)+a\overline{X}_sU$$

By multiplying U^{-1} from right side, and by multiplying $1/\beta$, we obtain,

$$\begin{aligned}(1-1/\beta)\overline{X}_t+a/\beta\cdot\overline{X}_s&=-dT\cdot U^{-1}-\overline{X}dU\cdot U^{-1}\\ &=-(1,\ \overline{X})\begin{pmatrix}0 & dT\cdot U^{-1}\\ 0 & dU\cdot U^{-1}\end{pmatrix}\\ &=-(1,\ \overline{X})\Omega(U)\end{aligned}\qquad(4)$$

This equation is the pair of the equation (3).

5. Functional Equation on a Invariant Surface

Similarly, we can define the functional equation of tooth curves on a invariant surface.

Invariant surface $Z(u,\ v)$ (in the standard coordinate system) is defined as follows.

When there are two functions $f(u,\ v,\ t)$ and $g(u,\ v,\ t)$, such that

$$(1,\ Z(f(u,\ v,\ t),\ g(u,\ v,\ t)))=(1,\ Z(u,\ v))U(t)$$

we call $Z(u,\ v)$ a invariant surface for the motion $U(t)$.

In this time, if a curve $Z(u(t),\ v(t))$ on the invariant surface $Z(u,\ v)$ satisfies following differential equation;

$$(1-\beta(t))\frac{d}{dt}Z(u(t),\ v(t))=-(1,\ Z(u(t),\ v(t))\Omega(U^{-1}(t))$$

we call $Z(u(t),\ v(t))$ a tooth curve for the motion $U(t)$ and it's invariant surface $Z(u,\ v)$.

6. Examples

a) Let $U(t)$ be the motion of a circle of radius r_2 which rotates round a circle

of radius r_1 without sliding.
That is,

$$U(t)=C(t/r_2)E_x(r_1+r_2)C(t/r_1)$$
$$=\begin{pmatrix} 1 & (r_1+r_2)\cos t/r_1, \ (r_1+r_2)\sin t/r_1, \ 0 \\ 0 & C(t/r_1+t/r_2) \end{pmatrix}$$

(1) Involute Curve

$$U_0(t)=E_y(-t)E_x(r_1)C(t/r_1)$$
$$=\begin{pmatrix} 1 & r_1\cos t/r_1+t\sin t/r_1, \ r_1\sin t/r_1-t\cos t/r_1, \ 0 \\ 0 & C(t/r_1) \end{pmatrix}$$
$$X_0(t)=(1,\ A,\ B,\ 0)U_0(t)$$
$$=(1,\ (r_1+A)\cos t/r_1+(t-B)\sin t/r_1,$$
$$(r_1+A)\sin t/r_1-(t-B)\cos t/r_1,\ 0)$$

where, A and B are constats.

$X_0(t)$ is called "involute curve" and satisfies following differential equation.

$$(1+r_1/r_2)\frac{dX_0}{dt}=-(1,\ X_0)\Omega(U^{-1}(t))$$

where,

$$\Omega(U^{-1}(t))=\begin{pmatrix} 0 & -(1+r_1/r_2)\sin t/r_1, & (1+r_1/r_2)\cos t/r_1, & 0 \\ 0 & 0 & -1/r_1-1/r_2 & 0 \\ 0 & 1/r_1+1/r_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(2) Cycloid Curve

$$U_0(t)=\begin{pmatrix} 1 & (r_1+R)\cos t/r_1, (r_1+R)\sin t/r_1, \ 0 \\ 0 & C(t/r_1+t/R) \end{pmatrix}$$

where, R is the radius of a rolling circle.

$$X_0(t)=(1,\ A,\ B,\ 0)U_0(t)$$
$$=((r_1+R)\cos t/r_1+A\cos(1/r_1+1/R)t-B\sin(1/r_1+1/R)t,$$
$$(r_1+R)\sin t/r_1+A\sin(1/r_1+1/R)t+B\cos(1/r_1+1/R)t,\ 0)$$

where, A and B are constants. $X_0(t)$ is called "cycloid curve".

$X_0(t)$ satisfies following differrential equation.

$$\left(1-\frac{r_1(r_2-R)}{r_1r_2+Rr_2}\right)\frac{dX_0}{dt}=-(1,\ X_0)\Omega(U^{-1}(t))$$

b) Let $U(t)$ be the motion of a circle of spherical radius r_2 which rotates round a circle of spherical radius r_1 without sliding on a spherical surface of radius R.

That is,

$$U(t)=C\left(\frac{t}{R\sin(r_2/R)}\right)C_y\left(\frac{r_1+r_2}{R}\right)C\left(\frac{t}{R\sin(r_1/R)}\right)$$

Moreover, let $U_0(t)$ and $X_0(t)$ be followings.

$$U_0(t)=C_x(t/R)C_y(r_1/R)C\left(\frac{t}{R\sin(r_1/R)}\right)$$

$$\begin{aligned}X_0(t)&=(1,\ 0,\ 0,\ R)U_0(t)\\&=R\Big(\sin(r_1/R)\cos(t/R)\cos\left(\frac{t}{R\sin(r_1/R)}\right)\\&\qquad+\sin(t/R)\sin\left(\frac{t}{R\sin(r_1/R)}\right),\\&\qquad\sin(r_1/R)\cos(t/R)\sin\left(\frac{t}{R\sin(r_1/R)}\right)\\&\qquad-\sin(t/R)\cos\left(\frac{t}{R\sin(r_1/R)}\right),\\&\qquad\cos(r_1/R)\cos(t/R)\Big)\end{aligned}$$

$X_0(t)$ is colled "spherical involute curve" and satisfies following differential equation.

$$\frac{\sin(r_1/R+r_2/R)}{\cos(r_1/R)\sin(r_2/R)}\frac{dX_0}{dt}=-(1,\ X_0)\Omega(U^{-1}(t))$$

where,

$$\Omega(U^{-1}(t))=\begin{pmatrix}0&\bar{\omega}_1&\bar{\omega}_2&\bar{\omega}_3\\0&0&\bar{\omega}_{12}&\bar{\omega}_{13}\\0&-\bar{\omega}_{12}&0&\bar{\omega}_{23}\\0&-\bar{\omega}_{13}&-\bar{\omega}_{23}&0\end{pmatrix}$$

$$\bar{\omega}_1=\bar{\omega}_2=\bar{\omega}_3=0$$

$$\bar{\omega}_{12}=-\frac{\cos(r_1/R)\sin(r_1/R+r_2/R)}{R\sin(r_1/R)\sin(r_2/R)}$$

$$\bar{\omega}_{13}=\frac{\sin(r_1/R+r_2/R)}{R\sin(r_2/R)}\sin\left(\frac{t}{R\sin(r_1/R)}\right)$$

$$\bar{\omega}_{23}=-\frac{\sin(r_1/R+r_2/R)}{R\sin(r_2/R)}\cos\left(\frac{t}{R\sin(r_1/R)}\right)$$

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