Memoirs of the Faculty of Engineering, Nagoya University
Vol. 34, No. 1 (1982) — Research Report —

TREE LANGUAGE ACCEPTED BY PROBABILISTIC TREE AUTOMATON WITH ISOLATED CUT-POINT

KAZUE SUGINO

Department of Information Science

YASUYOSHI INAGAKI

Department of Electrical Engineering

and TERUO FUKUMURA

Department of Information Science

(Received May 20, 1982)

Abstract

In this paper we introduce a probabilistic tree automaton by extending the notion of a tree automaton. Let $\mathcal{A}$ be a probabilistic tree automaton and $\mathcal{R}_{\mathcal{T}(\mathcal{A};\lambda)}$ be a congruence relation defined by a tree language $\mathcal{T}(\mathcal{A};\lambda)$ which a probabilistic tree automaton $\mathcal{A}$ accepts with a cut-point λ, $0 \leqq \lambda < 1$. We prove that if λ is an isolated cut-point then the number of equivalence classes of $\mathcal{R}_{\mathcal{T}(\mathcal{A};\lambda)}$ is finite, that is, $\mathcal{T}(\mathcal{A};\lambda)$ is regular.

1. Introduction

A probabilistic automaton introduced by Rabin (1963) is a generalization of a finite automaton. Intuitively it is a finite automaton whose state transitions are stochastic. On the other hand, a tree automaton, which was introduced and investigated by Thatcher and Wright (1968), Brainerd (1968, 1969) and others (e. g., see Thatcher, 1973), can be considered as another generalization of a finite automaton. Combining these two directions of generalization, we can obtain the concept of a probabilistic tree automaton.

In this paper we give the formal definition of probabilistic tree automaton, and we prove that a tree language which a probabilistic tree automaton accepts with an isolated cut-point is regular. This result is a natural generalization of Rabin's theorem (Theorem 3 of Rabin, 1963) concerning a probabilistic automaton.

2. Preliminaries

In this section we first give some definitions and properties related to trees and tree automaton, which will be used in the discussions of this paper.

Definition 2. 1. A ranked alphabet (abbreviated as r. a.) is a pair (Σ, r), where Σ is a finite set of symbols and r is a mapping from Σ into N, the set of nonnegative integers. For a symbol A in Σ, $r(A)=k$ means that A has rank k.

Let Σ_k be the set of symbols with rank k, i. e., $\Sigma_k=\{A \mid A \in \Sigma, r(A)=k\}$. We presume that if $i \neq j$ then $\Sigma_i \cap \Sigma_j = \phi$. In the sequel we simply write Σ instead of r. a. (Σ, r).

Definition 2. 2. A tree over Σ is recursively defined as follows:

1) For any $a \in \Sigma_0$, a is a tree,

2) If $A \in \Sigma_k$ and $t_1, \cdots, t_k$ are trees over Σ then $At_1 \cdots t_k$ is a tree.

Symbols in Σ_0 are sometimes called leaf symbols, whereas the other symbols node symbols. Let T_Σ denote the set of all trees over Σ. The subset $\mathcal{T}$ of T_Σ is called a tree language.

Definition 2. 3. The depth $d(t)$ of a tree t is defined as follows:

1) $d(a)=0$ for $a \in \Sigma_0$,

2) $d(At_1 \cdots t_k)=1+\max_i\{d(t_i) \mid 1 \leqq i \leqq k\}$ for $t=At_1 \cdots t_k$, where $A \in \Sigma_k$ and $t_i \in T_\Sigma$; $i=1, \cdots, k$.

Definition 2. 4. Let $\Sigma=\{A_1, \cdots, A_m\}$ be a ranked alphabet. A tree automaton M over Σ is a system specified by a 3-tuple,

$$M=\langle S, \{\delta_{A_i} \mid 1 \leqq i \leqq m\}, F\rangle,$$

where

1) S is a finite set of states,

2) $\delta_{A_i}: S^{r(A_i)} \longrightarrow S (1 \leqq i \leqq m)$ is transition function, and particularly $\delta_a (a \in \Sigma_0)$ is an element of S, called an initial state,

3) $F \subseteqq S$ is a set of final states.

Definition 2. 5. The response function ρ: $T_\Sigma \longrightarrow S$ of a tree automaton M is defined as follows:

1) If $a \in \Sigma_0$ then $\rho(a)=\delta_a \in S$,

2) If $t=At_1 \cdots t_k$ with $A \in \Sigma_k$ and $t_i \in T_\Sigma$; $i=1, \cdots, k$ then $\rho(t)=\delta_A(\rho(t_1), \cdots, \rho(t_k))$.

Definition 2. 6. The subset $\mathcal{T}(M)=\{t \mid t \in T_\Sigma, \rho(t) \in F\}$ of T_Σ is called a tree language accepted by tree automaton M.

Definition 2. 7. A tree language $\mathcal{T} \subseteq T_\Sigma$ is said to be regular if it is accepted by a tree automaton.

Definition 2. 8. An equivalence relation E over T_Σ is called a congruence relation when for any $A \in \Sigma$ (here we suppose that $r(A)=k \geqq 1$) and for any trees $t_1, \cdots, t_k$; $t'_1, \cdots, t'_k$ over Σ if $t_1 E t'_1, \cdots, t_{k-1} E t'_{k-1}$ and $t_k E t'_k$ hold then $At_1 \cdots t_k E At'_1 \cdots t'_k$ holds.

The following lemma is well known:

Lemma 2. 1. (Brainerd, 1968) A tree language $\mathcal{T} \subseteq T_\Sigma$ is regular if and only if $\mathcal{T}$ is the union of some equivalence classes generated by a congruence relation E of finite index over T_Σ.

For any trees t and s in T_Σ, let $s \underset{a}{\circ} t$ denote a tree obtained by replacing one of leaves of symbol a in tree s by tree t.

Definition 2. 9. For a tree language $\mathcal{T} \subseteq T_\Sigma$, we define the relation $\mathcal{R}_\mathcal{T}$ on T_Σ as follows:

$$\forall t,\ \forall t' \in T_\Sigma,$$

$$t \mathcal{R}_\mathcal{T} t' \text{ iff } \forall s \in T_\Sigma,\ \bigwedge_{a_i}[s \underset{a_i}{\circ} t \in \mathcal{T} \text{ iff } s \underset{a_i}{\circ} t' \in \mathcal{T}],$$

where a_i is the symbol of i-th leaf node from left of tree s and we should understand that $s \underset{a_i}{\circ} t$ and $s \underset{a_i}{\circ} t'$ are two trees obtained from tree s by replacing the i-th leaf node of symbol a_i with trees t and t', respectively. Further, $\bigwedge_{a_i}$ means that the relation in the bracket [ ] holds for all leaf nodes of tree s.

Lemma 2. 2. $\mathcal{R}_\mathcal{T}$ is a congruence relation on T_Σ.

(proof) it is obvious that $\mathcal{R}_\mathcal{T}$ is an equivalence relation on T_Σ and it suffices to show that $\mathcal{R}_\mathcal{T}$ satisfies the condition of Definition 2. 8. Suppose that $t_l \mathcal{R}_\mathcal{T} t'_l$ $(1 \leqq l \leqq k)$ holds, namely, for any tree s,

$$\bigwedge_{a_i}[s \underset{a_i}{\circ} t_l \in \mathcal{T} \text{ iff } s \underset{a_i}{\circ} t'_l \in \mathcal{T}] \quad (1 \leqq l \leqq k)$$

holds.

Let u be an arbitrary tree in T_Σ and let A be an arbitrary symbol in Σ_k. Consider two trees $u \underset{a_i}{\circ} (At_1t_2 \cdots t_k)$ and $u \underset{a_i}{\circ} (At'_1t_2 \cdots t_k)$. These trees can be obtained from $u \underset{a_i}{\circ} (Abt_2 \cdots t_k)$ by replacing the leaf symbol b with t_1 and t'_1, respectively. Since $t_1 \mathcal{R}_\mathcal{T} t'_1$, we have

$$u \underset{a_i}{\circ} (At_1t_2 \cdots t_k) \in \mathcal{T} \text{ iff } u \underset{a_i}{\circ} (At'_1t_2 \cdots t_k) \in \mathcal{T}$$

Next, consider two trees $u \underset{a_i}{\circ} (At'_1t_2t_3 \cdots t_k)$ and $u \underset{a_i}{\circ} (At'_1t'_2t_3 \cdots t_k)$ and apply the same reasoning as above. Then we can obtain

$$u \underset{a_i}{\circ} (At'_1t_2t_3 \cdots t_k) \in \mathcal{T} \text{ iff } u \underset{a_i}{\circ} (At'_1t'_2t_3 \cdots t_k) \in \mathcal{T}$$

Proceeding in such a way as above we can conclude

$$\begin{aligned} u \underset{a_i}{\circ} (At_1t_2 \cdots t_k) \in \mathcal{T} &\text{ iff } u \underset{a_i}{\circ} (At'_1t_2 \cdots t_k) \in \mathcal{T} \\ &\text{ iff } u \underset{a_i}{\circ} (At'_1t'_2t_3 \cdots t_k) \in \mathcal{T} \\ &\qquad \cdots\cdots\cdots\cdots\cdots\cdots \\ &\text{ iff } u \underset{a_i}{\circ} (At'_1t'_2t'_3 \cdots t'_k) \in \mathcal{T} \end{aligned}$$

This means $At_1t_2 \cdots t_k \ \mathcal{R}_\mathcal{T}\ At'_1t'_2 \cdots t'_k$.

3. Probabilistic tree automaton

Here we introduce a probabilistic tree automaton (abbreviated as p. t. a.) and some notions related to it.

Intuitively, a probabilistic tree automaton may be considered as a tree automaton whose state transitions are stochastic. Formally, p. t. a. is defined as follows:

Definition 3. 1. Let Σ be a ranked alphabet. A probabilistic tree automaton $\mathcal{A}$ over Σ is a system specified by a 3-tuple,

$$\mathcal{A}=<S,\ \{P_A|A\in\Sigma\},\ F>.$$

Here $S=\{s_1, s_2, \cdots, s_n\}$ is a finite set of states. If $r(A)=k\geqq 1$ then P_A is a stochastic matrix with n^k rows and n columns. If $r(A)=0$ then P_A is a probability distribution over S, i.e., an n-dimensional probability distribution. F is a distinguished subset of S, called a set of final states.

Definition 3. 2. Let $C=(c_{ij})$ and $D=(d_{kl})$ be two matrices of order $p\times q$ and $u\times v$, respectively. Then the Kronecker product of C and D, denoted $C\otimes D$, is defined by

$$C\otimes D=\begin{pmatrix} c_{11}D & \cdots & c_{1q}D \\ \vdots & & \vdots \\ c_{p1}D & \cdots & c_{pq}D \end{pmatrix}$$

Lemma 3. 1. Let C's and D's be matrices. Then

1) $C\otimes(D_1+D_2)=C\otimes D_1+C\otimes D_2$
2) $(C_1+C_2)\otimes D=C_1\otimes D+C_2\otimes D$
3) $\alpha(C\otimes D)=(\alpha C)\otimes D=C\otimes(\alpha D)$; α is a constant.
4) $(C_1\otimes D_1)(C_2\otimes D_2)=(C_1C_2)\otimes(D_1D_2)$
5) $C_1\otimes(C_2\otimes C_3)=(C_1\otimes C_2)\otimes C_3$

Let Π be the set of all n-dimensional probability distributions, i.e., distributions over S.

Definition 3. 3. The behavior mapping β : $T_\Sigma \longrightarrow \Pi$ of a p. t. a. $\mathcal{A}$ is recursively defined as follows:

1) If $t=a\in\Sigma_0$ then $\beta(t)=\beta(a)=P_a$.
2) If $t=At_1\cdots t_k$ for some $A\in\Sigma_k$ and $t_1, \cdots, t_k\in T_\Sigma$ then $\beta(t)=[\beta(t_1)\otimes\cdots\otimes\beta(t_k)]\cdot P_A$

Definition 3. 4. Let $\eta=(\eta_1, \cdots, \eta_n)^T$ be a column vector such that the i-th entry η_i of η is 1 if $s_i\in F$ and 0 if $s_i\notin F$. Then let f: $T_\Sigma\longrightarrow[0, 1]$ be a function defined by

$$f(t)=\beta(t)\eta$$

for $t\in T_\Sigma$.

The function $f(t)$, called the output function of p. t. a. $\mathcal{A}$, gives the probability for $\mathcal{A}$ to accept an input tree t.

Definition 3. 5. Let λ be a real number such that $0\leqq\lambda<1$. Define a tree language $\mathcal{T}(\mathcal{A};\lambda)$ by

$$\mathcal{T}(\mathcal{A};\ \lambda)=\{t|f(t)>\lambda,\ t\in T_\Sigma\}$$

, which is called the tree language accepted by $\mathcal{A}$ with a cut-point λ.

Now we prove the following lemma concerning the behavior mapping β of Definition 3. 3.

Lemma 3. 2. Let β be the behavior mapping of a p. t. a. $\mathcal{A}=<S,\{P_A|A\in\Sigma\},$

$F>$ with n states. Then for any tree $t \in T_\Sigma$, $\beta(t)$ can be represented as

$$\beta(t)=[P_{a_1}\otimes\cdots\otimes P_{a_l}]\cdot P_t,$$

where P_t is an $n^l\times n$ stochastic matrix determined for the tree t and $P_{a_1}, \cdots, P_{a_l}$ are probability distributions over S corresponding to the leaf symbols $a_1, \cdots, a_l$ appeared in the tree t, respectively.

(Proof) We use induction on the depth $d(t)$ of a tree t.

Basis. Trivial, since if $d(t)=0$, then $t=a$ for some a in $\sum_0$.

Induction Step. Assume that the lemma holds for all trees of depth less than or equal to $m-1$. Let $t_1, t_2, \cdots, t_k$ be trees of depth less than m and let $t=At_1\cdots t_k$ for some $A\in\sum_k$. By the definition of β, we have

$$\beta(t)=[\beta(t_1)\otimes\cdots\otimes\beta(t_k)]\cdot P_A$$

On the other hand, by the induction hypothesis, we have

$$\beta(t_i)=(P_{a_{i1}}\otimes\cdots\otimes P_{a_{ih_i}})P_{t_i} \quad (1\leqq i\leqq k),$$

where $P_{a_{i1}}, \cdots, P_{a_{ih_i}}$ are probability distributions over S corresponding to leaf symbols $a_{i1}, \cdots, a_{ih_i}$ appeared in the tree t_i and P_{t_i} is an $n^{h_i}\times n$ stochastic matrix.

Thus, we obtain

$$\beta(t)=[(P_{a_{11}}\otimes\cdots\otimes P_{a_{1h_1}})P_{t_1}\otimes\cdots\otimes(P_{a_{k1}}\otimes\cdots\otimes P_{a_{kh_k}})P_{t_k}]\cdot P_A$$

Applying 4) of Lemma 3.1 repeatedly, we have

$$\begin{aligned}\beta(t)&=[(P_{a_{11}}\otimes\cdots\otimes P_{a_{1h_1}})\otimes\cdots\otimes(P_{a_{k1}}\otimes\cdots\otimes P_{a_{kh_k}})]\\&\quad\cdot[P_{t_1}\otimes\cdots\otimes P_{t_k}]\cdot P_A\\&=(P_{a_{11}}\otimes\cdots\otimes P_{a_{kh_k}})[P_{t_1}\otimes\cdots\otimes P_{t_k}]\cdot P_A\end{aligned}$$

Here $P_{t_1}\otimes\cdots\otimes P_{t_k}$ is an $n^{h_1+\cdots+h_k}\times n^k$ stochastic matrix, since P_{t_i} is an $n^{h_i}\times n$ stochastic matrix for each $i(1\leqq i\leqq k)$, and P_A is an $n^k\times n$ stochastic matrix. Hence $[P_{t_1}\otimes\cdots\otimes P_{t_k}]\cdot P_A$ is an $n^{h_1+\cdots+h_k}\times n$ stochastic matrix. Thus, putting $l=h_1+\cdots+h_k$ and $P_t=[P_{t_1}\otimes\cdots\otimes P_{t_k}]\cdot P_A$, we have proved that our lemma holds for the tree $t=At_1\cdots t_k$.

4. Regularity of tree language accepted by p.t.a. $\mathcal{A}$ with isolated cut-point

In this section we consider a congruence relation $\mathcal{R}_{\mathcal{T}(\mathcal{A};\lambda)}$ defined by a tree language $\mathcal{T}(\mathcal{A};\lambda)$ which a p. t. a. $\mathcal{A}$ accepts with a cut-point λ. we prove that the number of equivalence classes of the congruence relation $\mathcal{R}_{\mathcal{T}(\mathcal{A};\lambda)}$ is finite if a cut-point λ is isolated. Thus, we show that the tree language accepted by p. t. a. $\mathcal{A}$ with isolated cut-point is regular, i.e., it can be accepted by some tree automaton.

Definition 4. 1. A cut-point λ is called ε-isolated with respect to a p. t. a. $\mathcal{A}$ if there exists an $\varepsilon>0$ such that

$$|f(t)-\lambda|\geqq\varepsilon \quad \text{for all } t\in T_{\Sigma}$$

The following Lemmas 4.1 and 4.2 are given in Paz (1971), which will be used to obtain our Lemma 4.3.

Lemma 4. 1. (Paz, 1971) Let $\mathcal{P}_n$ be the set of all n-dimensional stochastic vectors [i.e., $\mathcal{P}_n=\{\xi=(\xi_i),\ \xi_i\geqq 0,\ \sum_{i=1}^{n}\xi_i=1\}$] and let U_ε be a subset of $\mathcal{P}_n$ such that for any pair of vectors ξ and ζ in U_ε the inequality $\sum_{i=1}^{n}|\xi_i-\zeta_i|\geqq\varepsilon$ [ε is a given positive real number] holds true. Then U_ε is a finite set containing at most $k(\varepsilon)$ elements where $k(\varepsilon)=(1+2/\varepsilon)^{n-1}$.

Notation : For a real number α, $\alpha^+=\max(\alpha,\ 0)$ and $\alpha^-=\min(\alpha,\ 0)$.

Lemma 4. 2. (Paz, 1971) If $\xi=(\xi_i)$ and $\zeta=(\zeta_i)$ are n-dimensional stochastic vectors, the following equalities hold:

1) $\sum(\xi_i-\zeta_i)^+=-\sum(\xi_i-\zeta_i)^-$

2) $\sum|\xi_i-\zeta_i|=2\sum(\xi_i-\zeta_i)^+$

Lemma 4. 3. Let $\mathcal{A}$ be a p. t. a. with n states. If λ is an ε-isolated cut-point for $\mathcal{A}$, then the number of equivalence classes of a congruence relation $\mathcal{R}_{\mathcal{T}(\mathcal{A};\lambda)}$ induced by a tree language $\mathcal{T}(\mathcal{A};\ \lambda)$ is at most $\left(1+\dfrac{1}{2\varepsilon}\right)^{n-1}$.

(Proof) Let T be a set of trees such that any pair of trees in T does not satisfy the relation $\mathcal{R}_{\mathcal{T}(\mathcal{A};\lambda)}$, and put $B=\{\beta(t_i)\,|\,t_i\in T\}$, where β is the behavior mapping of $\mathcal{A}$.

Assume that $t\in T$ and $t'\in T$. Then for some tree s either of the following two conditions holds:

i) $s\underset{a_i}{\circ}t\in\mathcal{T}(\mathcal{A};\ \lambda)$ and $s\underset{a_i}{\circ}t'\notin\mathcal{T}(\mathcal{A};\ \lambda)$ for some leaf symbol a_i,

ii) $s\underset{a_i}{\circ}t\notin\mathcal{T}(\mathcal{A};\ \lambda)$ and $s\underset{a_i}{\circ}t'\in\mathcal{T}(\mathcal{A};\ \lambda)$ for some leaf symbol a_i

equivalently,

i') $f(s\underset{a_i}{\circ}t)>\lambda$ and $f(s\underset{a_i}{\circ}t')\leqq\lambda$ for some leaf symbol a_i,

ii') $f(s\underset{a_i}{\circ}t)\leqq\lambda$ and $f(s\underset{a_i}{\circ}t')>\lambda$ for some leaf symbol a_i

Now we assume that i') holds. [In a similar way to the following arguments, we can also succeed to have the result for the case ii')]. Since λ is isolated, it follows that

$$|f(s\underset{a_i}{\circ}t)-\lambda|\geqq\varepsilon \text{ and } |f(s\underset{a_i}{\circ}t')-\lambda|\geqq\varepsilon$$

Hence we have

$$f(s\underset{a_i}{\circ}t)-f(s\underset{a_i}{\circ}t')\geqq 2\varepsilon \tag{a}$$

On the other hand, by Lemma 3.2 and the definition of behavior mapping β, we have

$$\beta(s\underset{a_i}{\circ}t)=[P_{a_1}\otimes\cdots\otimes P_{a_{i-1}}\otimes\beta(t)\otimes P_{a_{i+1}}\otimes\cdots\otimes P_{a_k}]\cdot P_A$$

and

$$\beta(s\underset{a_i}{\circ}t')=[P_{a_1}\otimes\cdots\otimes P_{a_{i-1}}\otimes\beta(t')\otimes P_{a_{i+1}}\otimes\cdots\otimes P_{a_k}]\cdot P_A,$$

where $a_1,\cdots,a_k$ are the leaf symbols of the tree s. Thus we obtain

$$
\begin{aligned}
& f(s \underset{a_i}{\circ} t)-f(s \underset{a_i}{\circ} t') \\
&= \beta(s \underset{a_i}{\circ} t)\eta-\beta(s \underset{a_i}{\circ} t')\eta \\
&= [(P_{a_1}\otimes\cdots\otimes P_{a_{i-1}})\otimes(\beta(t)-\beta(t'))\otimes(P_{a_{i+1}}\otimes\cdots\otimes P_{a_k})]\cdot P_A\eta \qquad \text{(b)}
\end{aligned}
$$

Here let $\beta(t)=(\xi_1, \cdots, \xi_n)$ and $\beta(t')=(\xi_1', \cdots, \xi_n')$, where $0\leqq\xi_i$, $\xi_i'\leqq 1$ and $\sum_{i=1}^{n}\xi_i=\sum_{i=1}^{n}\xi_i'=1$. For simplicity, put

$$
\begin{aligned}
& P_{a_1}\otimes\cdots\otimes P_{a_{i-1}}=\boldsymbol{\zeta}=(\zeta_1, \cdots, \zeta_{n^{i-1}}), \ \sum_{j=1}^{n^{i-1}}\zeta_j=1 \ \ (0\leqq\zeta_j\leqq 1), \\
& P_{a_{i+1}}\otimes\cdots\otimes P_{a_k}=\boldsymbol{\theta}=(\theta_1, \cdots, \theta_{n^{k-i}}), \ \sum_{j=1}^{n^{k-i}}\theta_j=1 \ \ (0\leqq\theta_j\leqq 1) \\
& \beta(t)-\beta(t')=\boldsymbol{\xi}=(\xi_1-\xi_1', \ \cdots, \ \xi_n-\xi_n')
\end{aligned}
$$

and

$$
P_A=\begin{pmatrix} Q_{11} \\ \vdots \\ Q_{1n} \\ \vdots \\ Q_{n^{i-1},\,1} \\ \vdots \\ Q_{n^{i-1},\,n} \end{pmatrix}, \text{ where } Q_{lm}\text{'s are } n\times n \text{ stochastic matrices.}
$$

Then we can estimate (b) as follows:

$$
\begin{aligned}
& f(s \underset{a_i}{\circ} t)-f(s \underset{a_i}{\circ} t') \\
&= [\boldsymbol{\zeta}\otimes\boldsymbol{\xi}\otimes\boldsymbol{\theta}]P_A\boldsymbol{\eta} \\
&= [\zeta_1\boldsymbol{\xi}\otimes\boldsymbol{\theta}, \ \cdots, \ \zeta_{n^{i-1}}\boldsymbol{\xi}\otimes\boldsymbol{\theta}]P_A\boldsymbol{\eta} \\
&= [\zeta_1(\xi_1-\xi_1')\boldsymbol{\theta}, \ \cdots, \ \zeta_1(\xi_n-\xi_n')\boldsymbol{\theta}, \ \cdots, \ \zeta_{n^{i-1}}(\xi_1-\xi_1')\boldsymbol{\theta}, \\
& \qquad \cdots, \ \zeta_{n^{i-1}}(\xi_n-\xi_n')\boldsymbol{\theta}]P_A\boldsymbol{\eta} \\
&= \sum_{m=1}^{n}(\xi_m-\xi_m')\sum_{l=1}^{n^{i-1}}\zeta_l\boldsymbol{\theta}Q_{lm}\boldsymbol{\eta} \\
&\leqq \sum_m(\xi_m-\xi_m')^+\max_m(\sum_l\zeta_l\boldsymbol{\theta}Q_{lm}\boldsymbol{\eta})+\sum_m(\xi_m-\xi_m')^-\min_m(\sum_l\zeta_l\boldsymbol{\theta}Q_{lm}\boldsymbol{\eta}) \\
&= \sum_m(\xi_m-\xi_m')^+\{\max_m(\sum_l\zeta_l\boldsymbol{\theta}Q_{lm}\boldsymbol{\eta})-\min_m(\sum\zeta_l\boldsymbol{\theta}Q_{lm}\boldsymbol{\eta})\}
\end{aligned}
$$

Here note that $0\leqq\sum_l\zeta_l\boldsymbol{\theta}Q_{lm}\boldsymbol{\eta}\leqq\sum_l\zeta_l=1$, since $\boldsymbol{\zeta}$ and $\boldsymbol{\theta}$ are stochastic vectors, Q_{lm}'s are stochastic matrices, and $\boldsymbol{\eta}$ is a vector whose elements are 0 or 1.

Thus, we have

$$f(s \underset{a_i}{\circ} t) - f(s \underset{a_i}{\circ} t')$$

$$\leqq \sum_m (\xi_m - \xi'_m)^+ = \frac{1}{2} \sum_m |\xi_m - \xi'_m| \tag{c}$$

From(a) and (c) we get

$$2\varepsilon \leqq f(s \underset{a_i}{\circ} t) - f(s \underset{a_i}{\circ} t') \leqq \frac{1}{2} \sum_m |\xi_m - \xi'_m|$$

Therefore we obtain

$$\sum_{m=1}^{n} |\xi_m - \xi'_m| \geqq 4\varepsilon$$

Thus it follows from Lemma 4.1 that the set B is a finite set containing at most $k=\left(1+\frac{1}{2\varepsilon}\right)^{n-1}$ elements. Hence we have proved Lemma 4.3.

By Lemma 4.3 and Lemma 2.1 we obtain our main theorem:

Theorem 4. 1. Let $\mathcal{A}$ be a probabilistic tree automaton with n states, and let f be its output function. Let λ be an isolated cut-point such that $|f(t)-\lambda| \geqq \varepsilon$ for all $t \in T_\Sigma$. Then the tree language $\mathcal{T}(\mathcal{A}; \lambda)$ accepted by $\mathcal{A}$ with λ is regular, that is, there exists a finite tree automaton with less than or equal to $\left(1+\frac{1}{2\varepsilon}\right)^{n-1}$ states that accepts $\mathcal{T}(\mathcal{A}; \lambda)$.

5. An illustrative example

we give a simple example to illustrate our previous discussions.

Let $\Sigma=\{A, a\}$ be a ranked alphabet, where $A \in \Sigma_2$ and $a \in \Sigma_0$. Let $\mathcal{A}=<\{s_0, s_1\}, \{P_A, P_a\}, \{s_1\}>$ be a probabilistic tree automaton over Σ, where P_a and P_A are the following:

$$\begin{array}{cc} & \begin{array}{cc} s_0 & s_1 \end{array} \\ P_a = & (0, \ \ 1) \end{array}$$

$$\begin{array}{rl} & \begin{array}{cc} s_0 & s_1 \end{array} \\ P_A = \begin{array}{r} s_0s_0 \\ s_0s_1 \\ s_1s_0 \\ s_1s_1 \end{array} & \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \\ 1 & 0 \\ 1/2 & 1/2 \end{pmatrix} \end{array}$$

The all trees over Σ can be represented as $t=At_1At_2\cdots At_ma$ for some $t_1, t_2, \cdots, t_m$ in T_Σ and for some $m \geqq 0$. They are topologically represented as in Fig. 1.

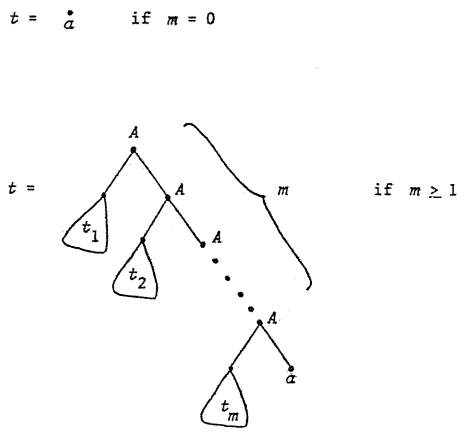

Fig. 1. Trees in T_Σ.

The output function f is computed as follows:

$$\beta(t)=\left(1-\left(\frac{1}{2}\right)^m,\ \left(\frac{1}{2}\right)^m\right)\ ;\quad m=0,\ 1,\ 2,\ \cdots\cdots,$$

$$\eta=(0,\ 1)^T,$$

$$f(t)=\beta(t)\eta=\left(\frac{1}{2}\right)^m\ ;\quad m=0,\ 1,\ 2,\ \cdots.$$

Here for any tree $t\in T_{\Sigma=\{A,a\}}$, $|f(t)-\lambda|\geqq\varepsilon$ holds for $\lambda=3/8$ and $\varepsilon=1/8$. Therefore, for an isolated cut-point $\lambda=3/8$ the tree language $\mathcal{T}(\mathcal{A};\ 3/8)=\{t\,|\,f(t)>3/8\}$ defined by $\mathcal{A}$ is $\{t\,|\,t=At_1a$ and t_1 is any tree over $\Sigma\}$. This tree language is accepted by the following tree automaton

$$M=<S,\ \{\delta_a,\ \delta_A\},\ F>,$$

where $S=\{q_0,\ q_1,\ q_2\}$, $F=\{q_1\}$, and δ is as follows.

$$\delta_a=q_0,$$

$$\delta_A(q,\ q_0)=q_1 \text{ for any } q\in S,$$

and

$$\delta_A(q,\ q')=q_2 \text{ for any } q\in S \text{ and any } q'\in S-\{q_0\}.$$

This tree automaton M has three states and $3<\left(1+\frac{1}{2\times(1/8)}\right)^{2-1}=5$.

Acknowledgment

The authors would like to thank Dr. N. Honda, Professor of Toyohashi University of Technology, for his encouragement and invaluable guidance to pursue this work. They also thank their colleagues for their helpful discussions.

References

Brainerd, W. S. (1968), The Minimalization of Tree Automata, Inform. Control, 13, 484–491.

Brainerd, W. S. (1969), Tree Generating Regular Systems, Inform. Control, 14, 217–231.

Paz, A. (1971), Introduction to Probabilistic Automata, Academic Press, New York.

Rabin, M. O. (1963), Probabilistic Automata, Inform. Control, 6, 230–245.

Thatcher, J. W. (1973), Tree Automata: An Informal Survey *in* "Currents in the Theory of Computing", (A. V. Aho, Ed.), pp. 143–172, Prentice-Hall, Inc., Englewood Cliffs, N. J.

Thatcher, J. W. and Wright, J. B. (1968), Generalized Finite Automata Theory with an Application to a Decision Problem of Second-Order Logic, Math. Systems Theory, 2, 1, 57–81.