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DETERMINATION OF FUNDAMENTAL MODE DECAY CONSTANT OUT OF A DECAY CURVE WITH HIGHER MODE CONTAMINATION

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Abstract

A practical new method is presented for the determination of a fundamental mode decay constant out of a decay curve with higher mode contamination. Suppose one attempts to determine the decay constant of a certain physical quantity whose fundamental component decreases exponentially with time. If higher modes are included in the measured values the logarithmic curve of the quantity deviates from a straight line but approaches asymptotically to the latter, which represents the decay of the fundamental mode. The authors introduced an appropriate function which approximated the deviation, and devised a practical method to get the fundamental decay constant. The new method was applied successfully to the neutron population decay experiments, which was carried out with a neutron generator with pulsing function and a graphite stack. The calculational method may apply to any decaying phenomena.

Introduction

There are many physical phenomena in which observed quantities decay exponentially with time. Examples of such quantities are: activity of radio-isotopes, temperature of a cooling body, and neutron density in a moderator after a neutron pulse injection. The constant which characterises this decay is called a decay constant. Because one can explore the physical mechanism of the phenomena via these decay constants in general, it is worth while determining the constant with good accuracy. It is rather rare, however, to find a case where some quantity decays with a single exponential form right from the time origin as

$$A(t)=A_1 \exp(-\lambda_1 t). \tag{1}$$

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Generally, in the early time range of the decay, many transient components are observed, and therefore, $A(t)$ has to be expressed rather as

$$A(t)=A_1 \exp(-\lambda_1 t)+A_2 \exp(-\lambda_2 t)+A_3 \exp(-\lambda_3 t) + \cdots\cdots\cdots, \tag{2}$$

where $0<\lambda_1<\lambda_2<\lambda_3<\cdots$. The first term of Eq. (2) is the fundamental mode whose constant λ_1 we would like to determine, and the rest are the higher modes which we would like to eliminate.

The ordinary scheme of determing λ_1 is as follows. Taking advantage of the fact that the higher modes decay faster than the fundamental mode, and that the $\ln A(t)$ vs. t plot approaches asymptotically a straight line, one draws a tangent to the asymptotic line and obtain the value of λ_1. The disadvantage of this method is that one discards the data points of early time range, where the statistical errors are generally small. To improve this situation some calculating codes have been made by several authors,[1] but they appear rather difficult to use. The present authors devised a new, handy calculating methd of determining the fundamental mode which is effective even for the case of accompanying higher modes. In the following, the principle of the method will be described. The new method can be applied to any decay phenomena in physics, but here, for convenience, will be explained for the pulsed neutron experiment of a moderator system.

Pulsed Neutron Experiment in a Graphite Stack[2]

A neutron source with pulsing function is placed within or adjacent to a graphite system. Neutrons are slowed down to thermal energy in a few tens of micro-second. The slowed-down neutrons which are called thermal neutrons diffuse in the system. Some neutrons leak from the surface, and some are absorbed in the system. As a result we observe a decay of thermal neutron population. If we measure the thermal neutron flux at a fixed position in the system, it decay as

$$\phi(t)=\sum_{l,m,n} A_{lmn} \exp(-\lambda_{lmn} t), \tag{3}$$

where

$$\lambda_{lmn}=v(\Sigma_a+DB^2_{lmn}), \tag{4}$$

and

$$B^2_{lmn}=\pi^2[(l/a)^2+(m/b)^2+(n/c)^2]. \tag{5}$$

Here, the shape of the moderator system is assumed to be a parallelepiped of a dimension $a\times b\times c$ cm^3 including extrapolation distance.[2] The indices l, m, and n are positive integers; Σ_a is macroscopic absorption cross section of the medium and D is the diffusion constant. The A_{lmn}'s are constants depending on the position. The λ-value is the smallest when l, m, and n are all equal to unity, and increases with l, m, n values. Arranging the series of Eq. (3) in the order of increasing λ-values, we can rewrite the series as

$$\phi(t)=\sum_{k=1}^{\infty} \phi_k \exp(-\lambda_k t). \tag{6}$$

The term of $k=1$ is the fundamental mode, and the rest are the higher modes.

Neutron flux is measured by an appropriate detector placed at a certain position in the system, and neutron detections are ordinarily converted to a series of electric pulse train. The temporal decay of the counts of neutron is registered by a multi-channel-scalor (MCS) system.

The count in the i-th channel of the MCS is given as

$$C(i)=N\int_{(i-1)T}^{iT}\varepsilon\phi dt, \tag{7}$$

where T is the dwell time of each scalor, ε is overall detecting efficiency, and N is the repetition number of neutron source bursts. Each burst is assumed to be of the same intensity, and each yields the neutron flux given by Eq. (6). Using Eq. (6), we get

$$C(i)=N\int_{(i-1)T}^{iT}\sum_{k=1}^{\infty}\varepsilon\phi_k \exp(-\lambda_k t)dt$$

$$=N\sum_{k=1}^{\infty}\varepsilon(\phi_k/\lambda_k[\exp(\lambda_k T)-1]\exp(-\lambda_k T\cdot i), \tag{8}$$

which can be rewritten as

$$C(i)=\sum_{k=1}^{\infty}G_k \exp(-\lambda_k T\cdot i), \quad i=1,\ 2,\ \cdots,\ I, \tag{9}$$

where I is the largest channel number. In an actual experiment, we have to consider other disturbing factors such as background counts, and the dead time of the detector. We here assume that the corrections for them have already been done. Our problem is then how to get the most probable value of λ_1 using a data which is in the form of Eq. (9).

Principle of Calculation

(i) *The Case of a Single Mode*

In the case of a single exponential mode, the counts of the MCS are, according to Eq. (9), given as

$$C(i)=G_1 \exp(-\lambda_1 T\cdot i). \tag{10}$$

Taking the logarithm of Eq. (10), we have

$$\ln C(i)=\ln G_1-\lambda_1 T\cdot i. \tag{11}$$

Therefore the $\ln C(i)$ vs. i curve shows a linear relation as in Fig. 1. Defining the notations

$$C_l(i)\equiv\ln C(i), \tag{12}$$

$$R_a\equiv\ln G_1, \tag{13}$$

$$R_x\equiv-\lambda_1 T, \tag{14}$$

we write Eq. (11) as

$$C_l(i)=R_a+R_x\cdot i, \quad i=1,\ 2,\ \cdots,\ I, \tag{15}$$

and regard these equations as a set of observation equations* to which least square method is to be applied.[3] Using a conventional method of least squares, the most probable values of R_a, and R_x can be obtained. Since T is known, then, the value of λ_1 is obtained from R_x. This is an idealized case, and rarely found in real experiments.

(ii) *The Case of Accompanying Higher Modes*

In this case, the ln $C(i)$ vs. i curve appears as in Fig. 2. The region of large i-value may asymptotically become linear as already mentioned. Our objective is

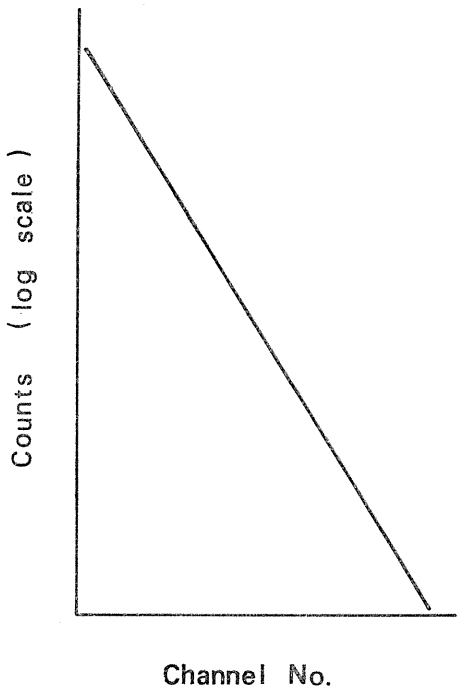

Fig. 1. The Characteristic of Single Mode Exponential Decay.

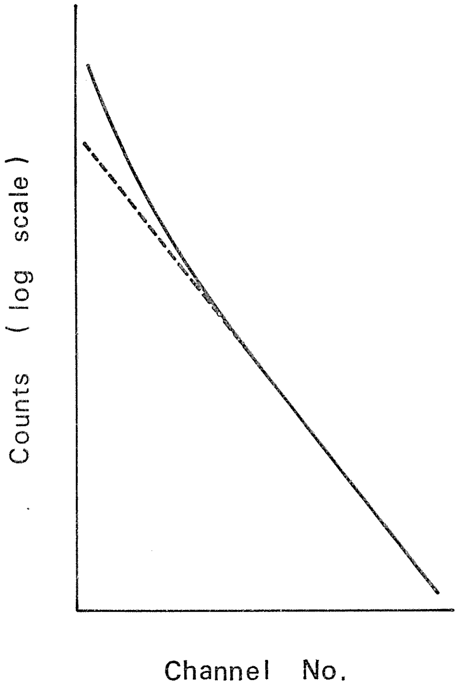

Fig. 2. The Characteristic of Multi-Mode Decay.

to obtain the tangent of this asymptotic line. For this purpose we define a function F which gives the deviation from the line, and adding this function to Eq. (15), we try to fit the experimental curve.

Suppose we want to fit the curve up to the j-th channel. We put the derivative of F at j equal to zero, and adjust the decay of the F-function with a parameter X. Then an observation equation takes a form

$$C_l(i) = R_a + R_x \cdot i + R_b \cdot F(X, i, j), \tag{16}$$

where we have introduced a parameter R_b in combination with the $F(X, i, j)$ function. Applying the method of least squares, we obtain the sum of the squared

*) The both sides of this equation are not necessarily equal for a chosen set of R_a and R_x, and measured values of $C_l(i)$.

residuals as

$$S(x)=[\sum_{i=1}^{j} W(i)\{C_l(i)-R_a-R_x\cdot i-R_b\cdot F(X,\ i,\ j)\}^2]\cdot[\sum_{i=1}^{j} W(i)]^{-1}, \qquad (17)$$

where, $W(i)$ is the weight function on the i-th channel. Using Eq. (17), we seek the minimum value of $S(X)$ by varying X. Once such an X is found, the values of R_a and R_x for that X would give the fundamental mode which we are just seeking.

Determination of the Function $F(X,\ i,\ j)$

The determination of the function "F" may be accomplished by several ways. The authors determined it by the following consideration. We note that when the higher modes coexist the counts of i-th channel are expressed as in Eq. (9), where $\lambda_1<\lambda_2<\lambda_3<\cdots$. Eq. (9) can be rearranged as

$$C(i)=G_1 \exp(-\lambda_1 T\cdot i)[1+(G_2/G_1)\exp\{-(\lambda_2-\lambda_1)T\cdot i\}+$$
$$+(G_3/G_1)\exp\{-(\lambda_3-\lambda_1)T\cdot i\}+\cdots]. \qquad (18)$$

As i increases, $\exp\{-(\lambda_k-\lambda_1)T\cdot i\}$ becomes small very quickly, because $\lambda_k-\lambda_1>0$. Then it would be legitimate to approximate the right side taking only the first two terms in the square bracket, and taking the logarithm as

$$\ln C(i)=\ln G_1-\lambda_1 T\cdot i+\ln[1+(G_2/G_1)\exp\{-(\lambda_2-\lambda_1)T\cdot i\}]. \qquad (19)$$

Furthermore, expanding the third term into Taylor series and taking its first term, we have

$$\ln C(i)=\ln G_1-\lambda_1 T\cdot i+(G_2/G_1)[\exp\{-(\lambda_2-\lambda_1)T\}]^i. \qquad (20)$$

Rewriting Eq. (20), we have

$$C_l(i)=R_a+R_x\cdot i+R_b\cdot y^i, \quad i=1,\ \cdots,\ I, \qquad (21)$$

where the parameters have been taken as

$$R_b\equiv(G_2/G_1), \qquad (22)$$

and

$$y\equiv\exp\{-(\lambda_2-\lambda_1)T\}. \qquad (23)$$

Comparing the form of Eq. (21) with Eq. (16), one immediately thinks of a possible choice of R_b and $F(X,\ i,\ j)$ as

with

$$\left.\begin{aligned} F(X,\ i,\ j)&=X^i \\ X&=y. \end{aligned}\right\} \qquad (24)$$

With this choice, Eq. (21) is a set of observation equations analogous to Eq. (15), and we could get the solutions of the normal equations derived form these equations, in principle. Provided that X is neither equal to one nor zero, one would then use $F(X,\ i,\ j)$ of Eq. (24) with X in the open interval $0<X<1$. Because we are dealing with experimental data with statistical fluctuations, however, we cannot exclude a possibility that X happens to be unity or zero. If $X=1$ and 0, the determinant constructed by the coefficients of the normal equations become 0, and we cannot obtain solutions.* Furthermore, when X is close to one or zero it happens that we lose many significant digits during the numerical calculation of the determinant.

Consider, for example, the case of X near unity. Putting $z=1-X$, we have

$$X^i=(1-z)^i=1-iz+\frac{1}{2}z^2 i(i-1)+\cdots$$

$$=1-\left(z+\frac{1}{2}z^2+\cdots\right)i+\frac{1}{2}(z^2+\cdots)i^2+\cdots. \tag{25}$$

Then, if $X\to 1$, Eq. (25) asymptotically approaches to a linear equation, since the third term of Eq. (21) becomes almost linear with i. This spoils the advantage of including the additional term $R_b\cdot F(X, i, j)$ in Eq. (16): the determinant of the coefficient of the normal equations approaches zero, and thus one cannot find healthy solutions.

For this reason, we have to select a more adequate function F, which can be used even at the points $X=1$ or 0. With such an F one can find without fear a value of X which makes $S(X)$ minimum. From this consideration, it would be reasonable to use a function which is constructed by substracting from the Eq. (21) the tangent equation at $i=j$. At the same time we want to utilize the form of Eq. (24) for X not equal either to unity or zero. As the result, we define the function F as follows:

$$\text{(a)}\quad X=0;\quad F(0, i, j)=\begin{cases}1 & \text{for } i=1\\ 0 & \text{for } i>1,\end{cases}$$

$$\text{(b)}\quad 0<X<1;\quad F(X, i, j)=\frac{(X^i-iX^j\ln X)-(X^j-jX^j\ln X)}{(X-X^j\ln X)-(X^j-jX^j\ln X)}, \tag{26}$$

$$\text{(c)}\quad X=1;\quad F(1, i, j)=\frac{(i-j)^2}{(1-j)^2}.$$

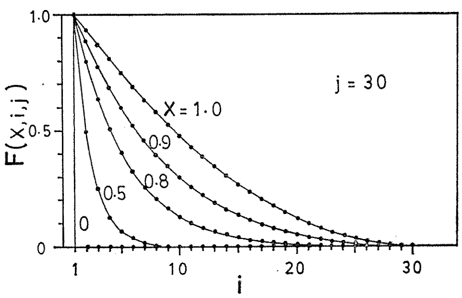

Fig. 3. Curves of $F(X, i, j)$ v.s. i, with $j=30$.

*) In fact, for the iteration scheme described later, the values $X=0$ and $X=1$ are intentionally used as the initial values.

These functions are shown in Fig. 3 for the closed interval $0 \leq X \leq 1$. Note that for the cases (b) and (c) the function is unity at $i=1$, zero at $i=j$, and its derivative equal to zero at $i=j$. For the case (c) where $X=1$, the curve is parabolic. As X decreases from one, the decay of the curve with i becomes more significant; when $X=0$ [case (a)], it makes a δ-function-like curve. We further point out that $F(0, i, j)$ of case (a), and $F(1, i, j)$ of case (c) are the limiting values of $F(X, i, j)$ [of case (b)] as X approaches unity and zero,* respectively.

We put Eq. (26) into the third term of the observation equation, Eq. (16), and determine R_a, R_x and R_b, which are respectively the initial value (at $i=0$) of the fundamental mode, decay constant λ_1 times T and the fraction of the higher mode with the assumed X-value.

Calculating Method

There are many possible methods to calculate λ_1 by the above principle. Out of them the following method chosen by the authors appears reliable. As Eq. (16) is linear with respect to the unknowns R_a, R_x and R_b, we can readily derive a normal equation, and get these values by conventional method of the least squares. Then, the problem is just the determination of X.

According to our experience, the following method for getting X has been practical and reliable. First, we select three points of X in the interval of $0 \leq X \leq 1$, as shown in the Fig. 4. The method of the selection may be arbitrary. The authors took these points in equal intervals in the scale of $(1-X)^{1/2}$. With these points and the additional two points at $X=0$ and 1, we calculate values of R_a, R_x and R_b from the observation equation (16), substitute these values into Eq. (17) and get the values of $S(X)$. Then out of the five points of X we determine the value $X_{1,min}$ for which $S(X)$ is the least. As shown in the figure, the five points for the next iteration are $X_{1,min}$, its two neighbours in the previous iteraton, and the midpoints of two intervals formed by the mentioned three points. We compare the $S(X)$ values again among the five points and determine the point $X_{2,min}$ for which $S(X)$ takes a minimum. These procedures are illustrated in the figure for five iterations. After the distance between the neighbouring points of X become sufficiently small, we can determine the most probable value of X assuming that the $S(X)$ vs. X curve is parabolic. From this X-value we can get the most probable values of R_a, R_x and R_b, and determine finally the decay constant of the fundamental mode.

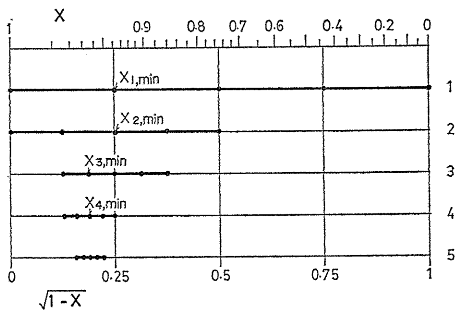

Fig. 4. The Procedure of Getting an Optimum X-value.

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*) One can show this by carefully taking the limits of the functions that appear in the expression of $F(X, i, j)$ in case (b). In the procedure, note that $\lim_{\varepsilon \to 0} \varepsilon^0 = 1$.

Results and Conclusion

To test the validity of this method, the authors carried out the following calculations.

(i) *A test with pseudo data*

For the test calculations we need raw data with reasonable statistical fluctuations. At the same time it is desirable to have a prior knowledge on the value of λ_1 for that raw data. For this reason the authors generated a set of pseudo data which simulate data of pulsed neutron experiments, and which fluctuate about the decay

$$A(t)=A_1 \exp(-\lambda_1 t)+A_2 \exp(-\lambda_2 t)+A_3 \exp(-\lambda_3 t), \qquad (27)$$

which has three decay components. If we put

$$t=T\cdot i, \qquad (i: \text{integer})$$

the count in the i-th channel of the MCS is given as

$$Q(i)=g_1 \exp(-\lambda_1 T\cdot i)+g_2 \exp(-\lambda_2 T\cdot i)+g_3 \exp(-\lambda_3 T\cdot i), \qquad (28)$$

which corresponds to Eq. (9). To give the statistical deviations to the function Q(i) in Eq. (28) we took for every i the Gaussian random distribution* about the center value $Q(i)$ with dispersion $Q(i)^{1/2}$. We call this curve $C(i)$ which is shown in Fig. 5. The parameters which characterise the pseudo data are tabulated in Table 1.

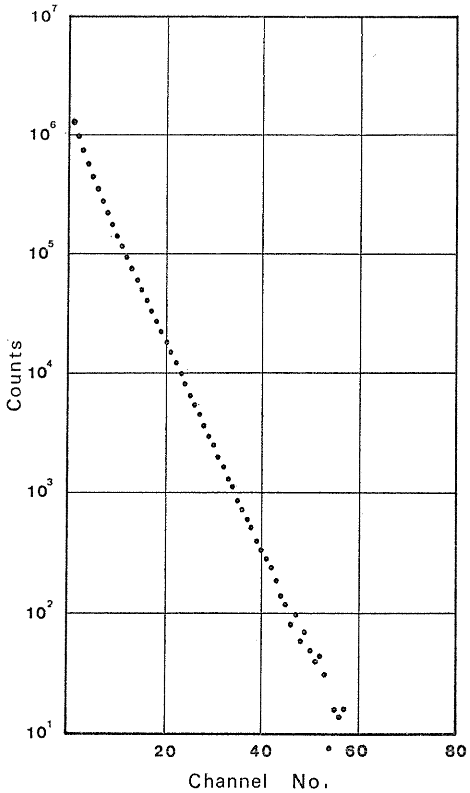

Fig. 5. Pseudo Data Simulating a Pulsed-Neutron Decay.

In applying the present method the validity of the result was tested by shifting the initial channel of fitting as other people have done with conventional methods. The fitting procedure was carried out between the i_0-th channel and the j-th channel, and i_0 was varied as j was fixed constant. The result is shown in Fig. 6–a. In the figure, the curve A is obtained by the conventional single exponential fitting method, while the curve B is obtained by the present method. In Fig. 6–b, the fraction of the higher mode component which is defined as R_b/R_a is shown for the present method. It is

*) For the generation of this curve, random numbers with Gaussian distributions were used.

Table 1.

A_1	1×10^6	λ_1	$1000/s$
A_2	$1/2\times10^6$	λ_2	$2000/s$
A_3	$1/3\times10^6$	λ_3	$3000/s$
T	$200\mu s$		

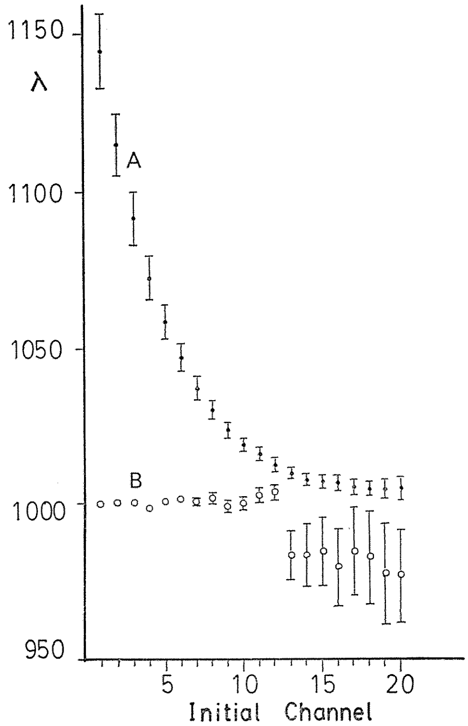

Fig. 6-a. Graph of the Calculated λ_1's,
Curve A: treated with the conventional single exponential fitting.
Curve B: treated with the present new method.

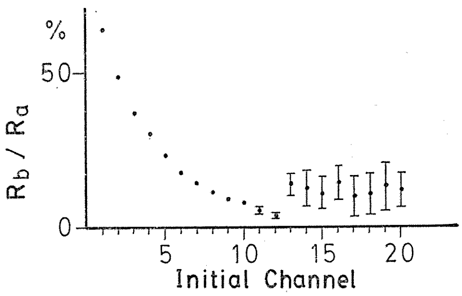

Fig. 6-b. The Fraction of Accompanying Higher Modes. The fraction is defined by R_b/R_a.

obvious that the result is improved significantly. It should be noted, however, that as the initial channel number of fitting is increased the error suddenly becomes large and calculated values fluctuate widely, as shown is the figure. It is considered that this is due to the increase of statistical error in the data.

(ii) *Test with the actual pulsed neutron experiment data*

As the next example we applied the present method to the data of a pulsed neutron experiment in our laboratory.[2] As shown in Fig. 7, a D–T neutron source is placed adjacent to the center of a surface of a 150 cm × 150 cm × 150 cm cubic graphite system. Neutron detectors were placed at three positions A, B and C as

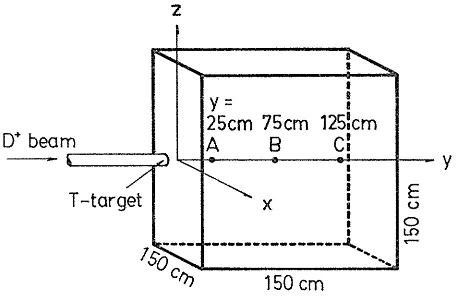

Fig. 7. The Position of the Neutron Detectors in the Pulsed Neutron Decay Experiment.

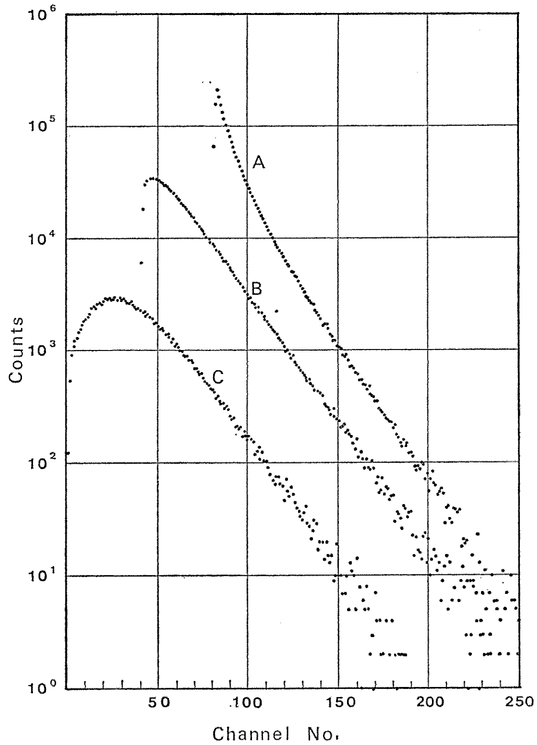

Fig. 8. The Graph of the MCS Counts in the Pulsed Neutron Experiment. A, B and C designate the positions of the neutron detectors indicated in Fig. 7.

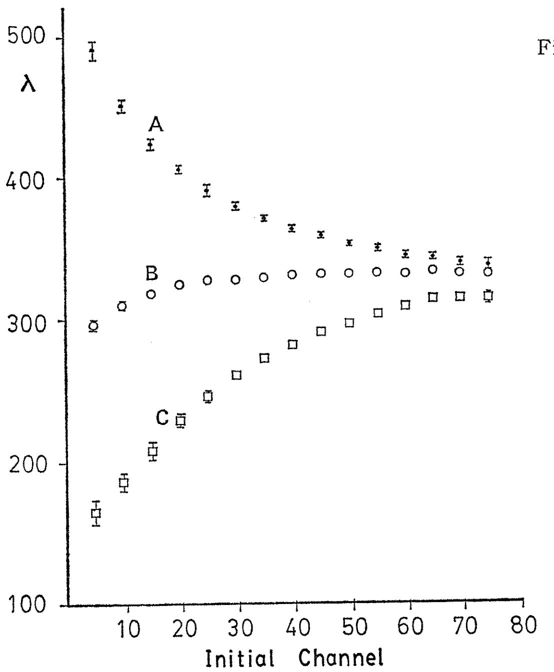

Fig. 9. The Graph of Calculated λ_1's. The values were obtained with the conventional single exponential fitting.

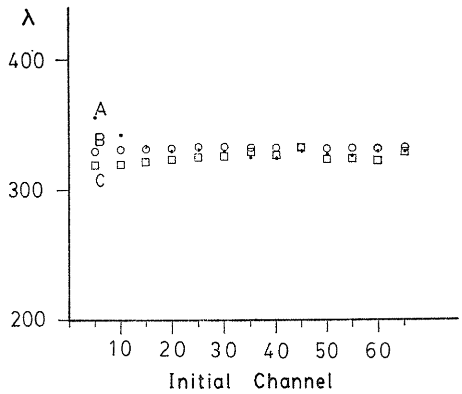

Fig. 10. The Graph of Calculated λ_1's. The values were obtained with the present new method.

shown in the figure.

Experimental data are shown in Fig. 8. In the figure, the time origin of the curves A and B were shifted to the 80th and 40 th channels for the ease of illustration. Because the position B is at the center of the cube, one can expect no second harmonic mode in the data, i. e., the values of A_{211}, A_{121} and A_{112} in Eq. (3) disappear. Thus for this case the single exponential approximation should be comparatively good, which indeed is the case in Fig. 9, where the results of conventional single exponential fitting is illustated. Fig. 10 is the results of the fitting by the present method. Comparing both figures it is obvious that the result is excellent. Thus, the usefulness of the present method has been proved for calculating the fundamental mode decay constant. The method was successfully applied to the above experiment of pulsed neutron with graphite, and the results have been reported in the Reference (2).

Acknowledgement

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The calculations were carried out with FACOM 230/60 computer at Nagoya University Computation Center.

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