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PARAMETER DOMAINS FOR THE EXISTENCE OF NEUTRON-WAVE DISCRETE EIGENVALUES

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Abstract

The report begins with a brief introductory review on the past works of three eignvalue problems in neutron transport, namely the evaluation of pulsed-neutron decay constant, diffusion length and neutron-wave attenuation constant. Then by the scheme of Cercignani and Sernagiotto, the domain of eigenvalue existence is derived for neutron-wave attenuation constant, with various models of reactor physics parameters assumed for moderators.

While general functional analyses in the past have clarified the spectral structure of these eigenvalue problems, the present treatment is intended to give rather heuristic and practical illustration of the subject, giving explicit analytical expressions of the domain boundary.

The transport theory treatment of Duderstadt's predicts more stringent condition than the present diffusion theory versions. The variation of the diffusion constant D, and a constant term in the scattering cross section produce considerable shift of the domain boundary. Effect of fission neutrons on this problem is speculated.

I. Introduction

The eigenvalue problem of neutron transport equation for moderators was studied extensively during 1950's and 1960's.[1, 2, 3, 4] It was initiated in 1950's with the analyses of pulsed-neutron decay experiments where the discrete decay modes of the exp $(-\lambda t)$ form was the subject of investigation. Then in 1960's the disappearance of such discrete time-eigenvalue λ, and the upper bound in the values of λ were studied for moderators.

Such study on the existence of λ was made through the scrutiny on the locus of λ in Laplace-transform space,[5, 6, 7] or through intuitive analogy to Schroedinger's equation,[8] or through functional analysis.[9, 10, 11, 12] The energy dependence of cross sections and of neutron flux that appear in the transport equation was always the key factor in these investigations. The dimensions of the system studied were either infinite[8, 9, 10, 11] or finite,[5, 6, 7, 12] and the transport equation in some cases was treated by diffusion approximation[6] to extract essential, though simplified, nature of the problem. The medium was either gas, liquid or solid.[11] In the case of solid, the crystalline structure brings additional complication[6, 7] in the analyses.

Then analogous analyses were soon made on the eigenvalue spectrum of neutron diffusion experiments,[13, 14] in which steady state neutron flux is expected to attenuate spatially in exp $(-x/L)$ form, and thus equations take a form very similar to that of the pulsed-neutron experiments. A brief historical review was given in Ref. (15) on the typical works on pulsed-neutron experiments and on diffusion length experiments.

Quite naturally in retrospect, another extension of the analyses to neutron wave experiments was made,[16~21] where a modulation of the form

$$I(t, v)=I_0(v)+\Delta I\cdot F(v)\exp(i\omega t) \tag{1}$$

is assumed for the intensity of incident neutron beam, and a discrete decay mode of the form $\exp(-\kappa x)=\exp(-(\alpha+i\beta)x)$ is investigated. The first term in Eq. (1) denotes a steady component, while the second term a component with periodic temporal variation. The variable x denotes the distance along the direction of propagation of the disturbance. Because the eigenvalue is a complex number in this case, the analysis is more complicated than in the preceding two problems. Exhausitive reviews and references are given in Refs. (4) and (22) on the three fields mentioned.

While the possibility of eigenvalue disappearance has been examined and verified by these elaborate studies, the explicit expressions for critical values of the experimental parameters for which the disappearance just occurs have not been explored in detail so far. Confining our review on neutron wave studies, for example, Duderstadt[16] gave a parameter domain of eigenvalue existence, employing one-dimensional transport theory and a separable scattering kernel. The mathematical tool used was similar to that by Cercignani and Sernagiotto.[23] The parameters investigated were frequencies of the modulation and absorption cross section. Williams[18] obtained a minimum transverse size of a prism required for the existance of a discrete eigenvalue κ. His analysis was based on the integral form of transport equation.

Even with these conclusions available, it appears that there is a room for further studies of such parameter domain. Deeper physical insight would be gained if we could draw conclusions on the effect of $B_\perp^2$ (transverse buckling) variation, in combination with energy dependence of the diffusion coefficient D. It appears that the more rigorous the treatment is, the more difficult to draw such practical conclusions with it.

Thus in the present report we illustrate such effects on neutron wave experiments with a use of simple models: the analysis will be made with an energy-dependent diffusion equation combined with separable scattering kernels. Various forms of energy dependence will be employed for the separated factors in the kernel, and they result in large (mild condition), or small (strict) parameter domain. It turns out that the condition obtained by Duderstadt mentioned is the most stringent one.

II. Statement of the Problem

Suppose a neutron beam is incident on one end of an infinitely extended slab with a thickness a cm. If its intensity is modulated according to Eq. (1), the neutron flux well inside the slab is given by

$$\phi(x, v, t)=\phi_0(x, v)+\Delta I\cdot\exp(i\omega t)[f_0(v)\exp(-\kappa_0 x) + \int_C f(v, \kappa)\exp(-\kappa x)d\kappa]. \tag{2}$$

The first term $\phi_0(x, v)$ is a solution of a problem having only the $I_0(v)$ term in Eq. (1), and the second term corresponds to the periodic component in Eq. (1). Neutron-wave studies deal only with this second term, and in particular, with the discrete mode term $f_0(v)\exp(-\kappa_0 x)$. Its motivation is that it is this term that one can extract physical informations from.[24, 25, 26] The integration path C is chosen properly in the complex κ-plane.

The equation describing this problem is

$$\frac{1}{v}\frac{\partial\phi}{\partial t}(x, v, t)-D(v)\frac{\partial^2\phi}{\partial x^2}+\Sigma_t(v)\phi=\int_0^\infty dv'\Sigma_s(v'\to v)\phi(x, v', t) \tag{3}$$

according to diffusion theory, where

$D(v)$: diffusion coefficient
$\Sigma_t(v)$: macroscopic total cross section
$\Sigma_s(v'\to v)$: scattering kernel.

In the present study we assume the form of the scattering kernel as[6]

$$\Sigma_s(v'\to v)\equiv\gamma(v)\Sigma_i(v')+\Sigma_e(v')\delta(v-v'), \tag{4}$$

which can be interpreted as consisting of a separable component in the first term, and a component without any energy alteration in the second term. The cross section $\Sigma_i(v')$ gives the cross section for the energy alteration in a collision, and we often call it "inelastic" scattering cross section for convenience, although somewhat misleading. The quantity $\Sigma_e(v')$ is the cross section for no energy alteration, and is likewise called quite often "elastic" cross section. The symbol $\delta(v-v')$ denotes the Dirac delta function. The function $\gamma(v)$ is yet to be specified. Because it is required that

$$\int_0^\infty \Sigma_s(v'\to v)dv=\Sigma_s(v')\equiv\Sigma_i(v')+\Sigma_e(v') \tag{5}$$

the factor $\gamma(v)$ has to satisfy a condition

$$\int_0^\infty \gamma(v)dv=1. \tag{6}$$

By substituting into Eq. (3) this scattering kernel and the discrete mode term of Eq. (2) we obtain,

$$\{i\omega - D(v)\kappa_0^2 v + v\Sigma_{non}(v)\} f_0(v) = v\gamma(v)\int_0^\infty dv' \Sigma_i(v') f_0(v') \tag{7}$$

where we have introduced a notation

$$\Sigma_{non}(v) \equiv \Sigma_t(v) - \Sigma_e(v) = \Sigma_a(v) + \Sigma_i(v).$$

Assume $(i\omega - Dv\kappa_0^2 + v\Sigma_{non}) \neq 0$ for any v, and divide Eq. (7) by this quantity, then multiply both sides by $\Sigma_i(v)$, and integrate over v to obtain

$$\Lambda(i\omega,\ \kappa_0^2) \equiv 1 - \int_0^\infty \gamma(v) v \Sigma_i(v) dv / \{i\omega - D(v) v\kappa_0^2 + v\Sigma_{non}(v)\} = 0. \tag{8}$$

For $D(v)$, $\Sigma_{non}(v)$, and $\Sigma_i(v)$ given as functions of energy, the discrete mode propagation exists if there is a κ_0 satisfying this relation. Thus assuming some simple models on $\Sigma_i(v)$, $D(v)$, $\gamma(v)$ and $\Sigma_t(v)$ we investigate in the following sections the conditions for which this dispersion relation has a discrete root κ_0.

III. Domains of Discrete Mode Existence for Various Models

III-1. *Model-1 (1/v Model)*

In this model, we assume the energy dependence of the parameters as follows:

$$\left.\begin{aligned} &\Sigma_i(v) = \lambda_i/v, \quad \Sigma_a(v) = \lambda_a/v, \quad \Sigma_e(v) = 0, \\ &D(v) = (3\Sigma_t)^{-1} = v/[3(\lambda_i + \lambda_a)], \\ &\gamma(v) \equiv M(v) = (4/\sqrt{\pi})(v^2/v_m^3)\exp(-v^2/v_m^2), \end{aligned}\right\} \tag{9}$$

where λ_i and λ_a are constants, and v_m is the most probable speed of the Maxwellian distribution $M(v)$. Note that with this model, the scattering kernel (4) satisfies the detailed balance relation

$$v'M(v')\Sigma_s(v' \to v) = vM(v)\Sigma_s(v \to v').$$

The dispersion relation (8) takes a form

$$\Lambda(i\omega,\ \kappa_0^2) = 1 - \int_0^\infty \frac{\lambda_i M(v) dv}{i\omega - vD\kappa_0^2 + \lambda_{non}} = 0, \tag{10}$$

where $\lambda_{non} = \lambda_i + \lambda_a$.

In investigating the existence of κ_0 which satisfies the relation (10), we employ Nyquist's theorem or argument principle.[27] The number (N_0) of zeroes and the number (N_p) of poles of a function $f(z)$ inside a closed contour C satisfy the relation

$$\frac{1}{2\pi}\Delta_C[\mathrm{Arg} f(z)] = N_o - N_p \tag{11}$$

including multiplicities. Here $f(z)$ is assumed to be analytic inside and on the contour C, except for a finite number of poles interior to C. Further, it is assumed that $f(z)$ has no zeroes on C. Here $\Delta_C[\mathrm{Arg}\ f(z)]$ is the change in the argument of $f(z)$ as the point z describes C once in the positive sense. The value $\Delta_C[\mathrm{Arg}\ f(z)]/2\pi$ represents the number of times the point $f(z)=\omega$ winds around the origin in the ω-plane, as z describes C in the z-plane.

We rearrange Eq. (10) so that this theorem can readily be applied to the function $\Lambda(i\omega,\kappa_0^2)$. Put

$$\left.\begin{aligned} &u\equiv v/v_m, \quad z\equiv -\kappa_0^2/[3\lambda_{non}(\lambda_{non}+i\omega)], \\ &p\equiv\lambda_a/\lambda_i, \quad q\equiv\omega/\lambda_i. \end{aligned}\right\} \tag{12}$$

Then the dispersion relation becomes

$$\left.\begin{aligned} \Lambda(i\omega,\ z)=1-(4/\sqrt{\pi})[(1+p)+iq]^{-1}\cdot\int_0^\infty u^2\exp(-u^2)\cdot \\ \cdot(1+zu^2v_m^2)^{-1}du=0. \end{aligned}\right\} \tag{13}$$

The problem is then to investigate the existence of z satisfying the relation (13). In identifying $\Lambda(i\omega,\ z)$ as the function $f(z)$ in the theorem, and choosing an appropriate closed contour on the z-plane, we note that the integrand in Eq. (13) diverges for z given by

$$z=-(1/v_m^2u^2), \quad 0<u<\infty. \tag{14}$$

Furthermore, $\Lambda(z)$ is unity*) for z at infinity. Thus we choose the path AOBEA in Fig. 1 as the contour C of the theorem. Our attention has to be focused on the path described by $\Lambda(z)$ as z moves along AO and OB. To express z along these paths we put

$$z=-(v_m^2u_0^2)^{-1}+i\varepsilon \text{ on AO}, \quad 0<u_0<\infty \tag{15}$$

$$z=-(v_m^2u_0^2)^{-1}-i\varepsilon \text{ on OB}, \quad 0<u_0<\infty \tag{16}$$

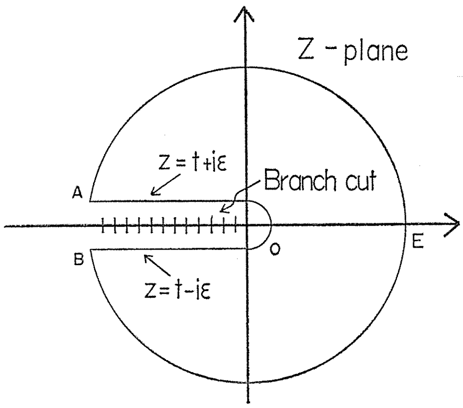

Fig. 1. The Contour Chosen for the Application of Argument Principle to the Dispersion Relation (13) in Model 1. The branch cut $z=(u^2v_m{}^2)^{-1}$ is indicated by the hatched line.

*) For brevity we will write $\Lambda(z)$ for $\Lambda(i\omega,\ z)$ in the following, whenever convenient.

where ε is made as small as possible, and where i is the imaginary unit. By substituting these expressions to Eq. (13) we obtain

$$\left.\begin{aligned} \Lambda(u_0)=1-(4/\sqrt{\pi})[(1+p)+iq]^{-1}\int_0^{\infty}[\exp(-u^2)/v_m^2] \\ \times[(uv_m)^{-2}(1-u^2/u_0^2)\pm i\varepsilon]^{-1}\cdot du=0, \end{aligned}\right\} \tag{17}$$

for which we employ a formula[28, 29, 30]

$$\lim_{\varepsilon\to 0}\frac{1}{y(x)\pm i\varepsilon}=P\frac{1}{y(x)}\mp i\pi\frac{\delta(x-x_0)}{|y'(x_0)|}. \tag{18}$$

Here P denotes a principal value integral, and $y(x)$ is a monotonic function of z. Thus we obtain

$$\left.\begin{aligned} \Lambda(u_0)=1-(4/\sqrt{\pi})u_0^2[(1+p)+iq]^{-1}[P\int_0^{\infty}u^2\exp(-u^2)du \\ \times(u_0^2-u^2)^{-1}\mp i\pi u_0\exp(-u_0^2)/2]=0. \end{aligned}\right\} \tag{19}$$

The principal value integral in the bracket can be rearranged to give

$$P\int_0^{\infty}u^2\exp(-u^2)du\cdot[u_0^2-u^2]^{-1}=(-\sqrt{\pi}/2)[1+u_0Z(u_0)], \tag{20}$$

where the function $Z(u_0)$ defined by

$$Z(u_0)\equiv(\sqrt{\pi})^{-1}P\int_{-\infty}^{\infty}\exp(-u^2)/(u-u_0)du \tag{21}$$

u_0 : real and positive

has been used, which can be related to plasma dispersion function, and can be written[31] as

$$Z(u_0)=-2\exp(-u_0^2)\int_0^{u_0}\exp(t^2)dt. \tag{22}$$

Thus the function $\Lambda(u_0)$ can now be written as

$$\left.\begin{aligned} \Lambda(u_0)=1-2u_0^2[(1+p)+iq]^{-1}\cdot\{-1+2u_0\exp(-u_0^2)\cdot \\ \cdot\int_0^{u_0}\exp(t^2)dt\mp i\sqrt{\pi}u_0\exp(-u_0^2)\}. \end{aligned}\right\} \tag{23}$$

The minus sign in the last term is for z along AO, and the plus sign for z along OB. By separating the real and imaginary parts of $\Lambda(i\omega, u_0)$ we have

$$R_e\Lambda(i\omega,\ u_0)=1-u_0[(1+p)^2+q^2]^{-1}\cdot[-(1+p)B(u_0)\mp(q/2)C(u_0)], \tag{24}$$

$$I_m\Lambda(i\omega,\ u_0)=u_0[(1+p)^2+q^2]^{-1}\cdot\{-qB(u_0)\pm[(1+p)/2]C(u_0)\}, \tag{25}$$

where we have employed the notations

$$B(u_0) \equiv 2u_0[1-2u_0\exp(-u_0^2)\int_0^{u_0}\exp(t^2)dt], \tag{26}$$

$$C(u_0) \equiv 4\sqrt{\pi}u_0^2\exp(-u_0^2). \tag{27}$$

We are seeking a critical set of p and q values, for which $A(i\omega, u_0)$ curve just passes the origin of A plane, and for which the root κ_0 just begins to disappear. To seek such a set we put

$$R_eA(i\omega, u_0)=0, \tag{28}$$

and

$$I_mA(i\omega, u_0)=0. \tag{29}$$

Substituting Eq. (29) into Eq. (28) we obtain

$$\left.\begin{aligned} p&=-1-u_0B(u_0), \\ q&=\mp u_0C(u_0)/2. \end{aligned}\right\} \tag{30}$$

Because q (frequency) is positive, we choose the lower sign for q in Eq. (30). This means that for a certain value of z along OB the point $A(i\omega, u_0)$ coincides with the origin.

Eq. (30) gives the boundary of the domain in terms of a variable u_0. It is the curve 1 of Fig. 2, and κ_0 exists for those sets of (p, q) inside this curve. In Fig. 3 is given a Nyquist plot of $A(i\omega, u_0)$ for a set on the curve: $p=0$ and $q=1.42$.*)

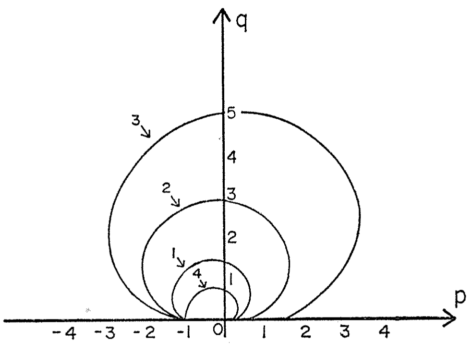

Fig. 2. Parameter Domain of Discrete Root Existence for Various Models.
Curve 1 : Model 1
Curve 2 : Model 2
Curve 3 : Model 3
Curve 4 : Model 4

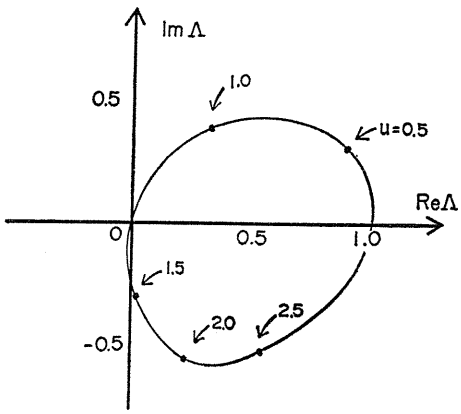

Fig. 3. Nyquist Diagram of $A(z)$ Given by Eqs. (24) and (25). The Diagram is given for Model 1 as z describes AOBEA in Fig. 1. with $p=0$ and $q=0.42$, which are at the intersection of the Curve 1 with the q-axis in Fig. 2.

*) For a set of (p, q) inside the domain, the root $z_0=-\kappa_0^2\cdot[3\lambda_{non}(\lambda_{non}+i\omega)]^{-1}$ must be found below the negative imaginary axis of the z-plane.

Actual wave experiments in a moderator are performed in the quadrant where $p>0$ and $q>0$. It is interesting, however, to observe the domain in the quadrant where $p<0$ and $q>0$. This domain corresponds to negative absorption, and therefore to fictitious fission process, in which the fission neutrons are emitted with the same energy as that of absorbed neutrons. The figure shows that too many of such fissions could also disturb the discrete mode, causing it to disappear.

The defective aspect of the curve (1) in Fig. 2 is its behavior near the p-axis. The present pattern implies that the root ceases to exist as q decreases with a certain positive p, and get close to zero. This is in contradiction with our experience in experiments, and we attempt to alter this behavior by changing our models in the following.

III-2. *Model-2 (1/v model with a constant D)*

The only difference in this model from Model 1 is the constancy of D, namely we put

$$\left.\begin{aligned} &D(v)=D^* \text{ (const)}, \quad \Sigma_i(v)=\lambda_i/v, \\ &\Sigma_a(v)=\lambda_a/v, \quad \Sigma_e(v)=0 \\ &\gamma(v)=M(v). \end{aligned}\right\} \tag{31}$$

Neither the scattering kernel, nor the total cross section $\Sigma_t(v)=\Sigma_i(v)+\Sigma_a(v)$ has been altered. The discussion goes just in parallel with the previous model, and the dispersion relation (8) for this case becomes

$$\Lambda(\kappa_0^2)=1-\lambda_i\int_0^\infty M(v)dv/(i\omega-vD^*\kappa_0^2+\lambda_{non})=0. \tag{32}$$

By a transformation

$$z=-D^*\kappa_0^2/(\lambda_{non}+i\omega), \tag{33}$$

it becomes

$$\Lambda(z)=1-(4/\sqrt{\pi})[(1+p)+iq]^{-1}\cdot\int_0^\infty u^2\exp(-u^2)du\times[1+zuv_m]^{-1}=0, \tag{34}$$

where we have employed the similar notations as in Model 1 except the definition of z in Eq. (33). This time we put

$$z=-1/(v_m u_0)\pm i\varepsilon \tag{35}$$

for z along the negative real axis, and the dispersion relation becomes

$$\Lambda(u_0)=1+u_0[(1+p)+iq]^{-1}\cdot\{A(u_0)+B(u_0)\mp iC(u_0)\}=0, \tag{36}$$

where we have used the notation

$$A(u)\equiv(2/\sqrt{\pi})\{1-u^2\exp(-u^2)E_i(u^2)\}, \tag{37}$$

and employed the Eqs. (21), (22), and the relation[32)]

$$\int_0^\infty \frac{\exp(-x^2)dx}{x+u} = \exp(-u^2)\{\sqrt{\pi}\int_0^u \exp(t^2)dt - E_i(u^2)/2\}, \tag{38}$$

where

$$E_i(x) \equiv P\int_{-\infty}^x (e^t/t)\,dt. \tag{39}$$

Then the real and imaginary parts of Λ can be given as

$$R_e\Lambda(u_0) = 1 + u_0[(1+p)\{A(u_0)+B(u_0)\} \pm qC(u_0)]\cdot[(1+p)^2+q^2]^{-1}, \tag{40}$$

$$I_m\Lambda(u_0) = -u_0[q\{A(u_0)+B(u_0)\} \pm (1+p)C(u_0)]\cdot[(1+p)^2+q^2]^{-1}. \tag{41}$$

The critical value of p and q can be obtained similarly as in Model 1 as

$$\left.\begin{aligned} q &= \mp u_0 C(u_0), \\ p &= -1 - u_0\{A(u_0)+B(u_0)\}. \end{aligned}\right\} \tag{42}$$

Again we take the lower sign for q, and Eqs. (42) gives the curve 2 of Fig. 2. We see that the condition for the existence of the root has been relaxed as compared with Model 1. This can be attributed to the smaller diffusion coefficient of Model 1 in large v range, which causes less leakage, less diffusion cooling, and thus a preservation of position-independent thermal energy spectrum.

It is interesting to note that the value of D^* does not manifests itself in the figure. It does influence the value of κ_0 through Eq. (33), however. Another point to be noted is the intersection of the curve with the p-axis. The intersection is still at the origin, leaving the paradox mentioned in the last example. It is to be alleviated in the next model.

III-3. *Model-3 (a constant term in $\Sigma_i(v)$)*

In this model we add a constant term to the $\Sigma_i(v)$ of the Model 2 and put

$$\Sigma_i(v) = (\lambda_i/v) + \Sigma_{io}. \tag{43}$$

Other quantities are assumed to have the same dependence as in Model 2:

$$\Sigma_a(v) = \lambda_a/v, \quad \Sigma_e(v) = 0,$$

$$D = D^* \text{ (const)}, \quad \gamma(v) = M(v).$$

Then with the transformation

$$z = -(D^*\kappa_0^2 + \Sigma_{io})/(\lambda_{non} + i\omega) \tag{44}$$

where

$$\lambda_{non} \equiv \lambda_a + \lambda_i\,,$$

the relation (8) becomes

$$\Lambda(z) = 1 - [(1+p)+iq]^{-1}\int_0^\infty (1+s_i u)u^2\exp(-u^2)\cdot[1+v_m uz]^{-1}\cdot du = 0, \tag{45}$$

where s_i defined by

$$s_i \equiv \Sigma_{io}/(\lambda_i/v_m)$$

expresses the proportion of the Σ_{io} term as compared to the $1/v$ term in $\Sigma_i(v)$.

The critical values of p and q are found to be

$$q=(1+s_i u_0)^2 u_0 C(u_0)/\{1+(2/\sqrt{\pi})s_i\}, \tag{46}$$

$$\left.\begin{aligned} p=-1-u_0[1+(2/\sqrt{\pi})s_i]^{-1}\cdot\{(1+s_i u_0)^2[A(u_0)+B(u_0)] \\ +2s_i(1+s_i/\sqrt{\pi}+s_i u_0/2)\}. \end{aligned}\right\} \tag{47}$$

The curve for $s_i=0.50$ is illustrated as the curve 3 in Fig. 2. The intersection of the curve with the p-axis is no more at the origin, but at the point

$$p_\infty \equiv \lim_{u_0\to\infty} p(u_0)=(2/\sqrt{\pi})s_i(1+3\sqrt{\pi}s_i/4)\times(1+2s_i/\sqrt{\pi})^{-1}, \tag{48}$$

which is equal to 0.601 in the case of $s_i=0.50$. It can be shown that the slope at p_∞ is zero. Furthermore, for this choice of s_i value the domain for the existence of the discrete root has been enlarged considerably.

III-4. *Model-4 (Duderstadt's model)*

For comparison we present Duderstadt's model.[16] It employs one-dimensional transport equation with the following energy-dependent cross sections;

$$\Sigma_a(v)=\lambda_a/v, \quad \Sigma_i(v)=\lambda_i/v, \quad \Sigma_e(v)=0.$$

The parameter region boundary in this case is given by the equations[16, 23]

$$\left.\begin{aligned} p&=-B(u_0)/2u_0, \\ q&=C(u_0)/4u_0. \end{aligned}\right\} \tag{49}$$

It is given by the curve 4 of Fig. 2, and is the most stringent one among the four models.

III-5. *Model-5 (Diffusion theory treament for Model 1 with transverse leakage)*

So far our treatment has been confined to the wave propagation in an infinite slab. We here investigate by diffusion theory the effect of the finiteness of transverse dimensions. If the wave is propagating in longitudinal (x-) direction of a rectangular prism, which is finite in the transverse (y- and z-) directions, we assume $\cos(B_y y)\cdot\cos(B_z z)$ distributions, and put the periodic component of Eq. (2) as

$$\phi(x, y, z, v, t)=\Delta I\cdot\exp(i\omega t)f_0(v)\exp(-\kappa_0 x)\cdot\cos B_y y\cos B_z z. \tag{50}$$

For this distribution the Laplacian operator is equal to

$$\nabla^2=\kappa_0^2-(B_y^2+B_z^2)\equiv\kappa_0^2-B_\perp^2. \tag{51}$$

If we employ the same energy-dependent cross sections as in Model 1, the dispersion relation (8) becomes

$$\Lambda(i\omega,\ \kappa_0^2)=1-\int_0^\infty \lambda_i M(v)dv\cdot[i\omega-vD(\kappa_0^2-B_\perp^2)+\lambda_{non}]^{-1}=0. \tag{52}$$

With a transformation

$$z=-(\kappa_0^2-B_\perp^2)[3\lambda_{non}(\lambda_{non}+i\omega)]^{-1}, \tag{53}$$

this dispersion relation reduces to Eq. (13), namely the relation for Model 1. Thus in this case the finiteness of the transverse dimension does not alter the domain of existence. This must be because our treatment is within the frame of diffusion approximation. In the next model we attempt to treat the finiteness by transport theory. Although the final form of p and q will not be derived, we will see what kind of procedures are required in the analyses.

III-6. *Model-6 (Transport theory treatment for Model 1 with transverse leakage)*

In this case we assume the discrete mode of neutron flux (corresponding to the $f_0(v)\cdot\exp(-\kappa_0 x)$ term of Eq. (2)) of the form[33]

$$\phi(x,\ y,\ z,\ \theta,\ \varphi,\ v,\ t)=\Delta I\cdot g(\theta,\ \varphi,\ v)\cdot\exp\{-\kappa_0 x+i(B_y y+B_z z)+i\omega t\}. \tag{54}$$

Then the transport equation becomes

$$\left.\begin{aligned}&[(i\omega/v)-\mu\kappa_0+iB_\perp\sqrt{1-\mu^2}\cos(\varphi-\alpha)+\Sigma_t(v)]g(\theta,\ \varphi,\ v)\\&=\int_{-1}^1 d\mu'\int_0^{2\pi}d\varphi'\int_0^\infty dv'\Sigma_s(\theta'\to\theta,\ \varphi'\to\varphi,\ v'\to v)g(\theta',\ \varphi',\ v'),\end{aligned}\right\} \tag{55}$$

where $\mu=\cos\theta$ is the direction cosine of neutron velocity with respect to x-axis, and φ is the azimuthal angle around x-axis, and $\alpha=\tan^{-1}(B_z/B_y)$. Assuming isotropy of scattering, and the energy-dependent cross section of Eq. (4) and Model 1 except D, the dispersion relation becomes

$$\left.\begin{aligned}\Lambda(\kappa_0)=1-(\lambda_i/4\pi v_m)(4/\sqrt{\pi})\int_{-1}^1 d\mu\int_0^\infty u\exp(-u^2)du\\ \cdot\int_0^{2\pi}d\varphi[(\lambda_{non}+i\omega)/uv_m-\mu\kappa_0+iB_\perp\sqrt{1-\mu^2}\cos(\varphi-\alpha)]^{-1}=0.\end{aligned}\right\} \tag{56}$$

The integration with respect to φ is written as

$$I(\mu,\ u)\equiv\int_0^{2\pi}d\varphi/[A_1+iA_2\cos(\varphi-\alpha)], \tag{57}$$

where

$$\left.\begin{aligned}A_1&=(\lambda_{non}+i\omega)/(uv_m)-\mu\kappa_0,\\A_2&=B_\perp\sqrt{1-\mu^2}.\end{aligned}\right\} \tag{58}$$

By using the variable $s=\exp(i\varphi)$ this integral can be transformed to the following closed contour integral.

$$I(\mu,u) = -(2/A_2)\exp(i\alpha)\oint_{\substack{\text{unit}\\\text{circle}}} ds[s^2 - 2A_1 i\exp(i\alpha)/A_2 + \exp(2i\alpha)]^{-1} \quad (59)$$

The residue is to be evaluated at one of the following two poles s_1 and s_2.[15)]

$$\left.\begin{aligned} \binom{s_1}{s_2} &= [(A_1/A_2) \pm \sqrt{1+(A_1/A_2)^2}]i\exp(i\alpha) \\ s_1 \cdot s_2 &= \exp(2i\alpha) \end{aligned}\right\} \quad (60)$$

In the present case, however, A_1 is a complex variable, and the relative positions of these poles with respect to the unit circle depends on the variables μ and u. This point has to be investigated further. At the present review we just proceed for the case where s_1 is within the circle. It may be invalid, but we can speculate what kind of functions are associated with the result of the present problem (If s_2 is inside, it is only necessary to reverse the sign of the right side of Eq. (61)). If s_1 is inside the circle, we have

$$I(\mu,\ u) = 2\pi/\sqrt{A_1^2 + A_2^2}. \quad (61)$$

In performing the integration with respect to μ, we borrow the general conclusion from diffusion theory that

$$\kappa_0^2 > B_\perp^2. \quad (62)$$

With this assumption we put

$$\nu \equiv \sqrt{\kappa_0^2 - B_\perp^2},$$

and obtain

$$\int_{-1}^{1} I(\mu,\ u)d\mu = \frac{2\pi}{\nu}\ln\left\{\frac{\Sigma_t + i\omega/v_m u + \nu}{\Sigma_t + i\omega/v_m u - \nu}\right\}. \quad (63)$$

Next we employ the speed-dependence of Σ_t given by Eq. (4), and carry out the following integration in Eq. (56) by parts.

$$\Lambda(i\omega,\ \nu) = 1 - \frac{2}{\sqrt{\pi}}\frac{\lambda_i}{v_m \nu}\int_0^\infty u\exp(-u^2)\ln\left\{\frac{(\lambda_{non}+i\omega)+\nu v_m u}{(\lambda_{non}+i\omega)-\nu v_m u}\right\}du.$$

The resulting dispersion relation is

$$\Lambda(i\omega,\ z) = 1 - \lambda_i \cdot [\sqrt{\pi}(\lambda_{non}+i\omega)]^{-1} \cdot \int_{-\infty}^{\infty} \exp(-u^2)du/(zu+1) = 0, \quad (64)$$

where we have defined z as

$$z = v_m \nu/(\lambda_{non} + i\omega).$$

If we proceed similarly as in the previous cases, putting

$$z = -u_0^{-1} \pm i\varepsilon,$$

we obtain the same equations of the domain boundary as Eq. (49).

This means that the domain has not been affected even by the introduction of finite transverse dimension. It contradicts the conclusion by Williams[18]. This contradiction was brought about by the assumption in evaluating I (μ, u) of Eq. (59) that s_1 is always inside the unit circle. Therefore, more careful treatment of this step is necessary: depending on the ranges of μ and u, the sign of (61) must be treated differently, and it could lead to more complicated expressions than (49).

Ⅳ. Discussions

In the Models 1 through 5, an unacceptable feature of the domain near the p-axis could not be removed. It is speculated[34] that this is caused by the separable kernel Eq. (4) we have used. This one-term kernel (with $\sum_e(v)=0$) implies that neutrons acquire Maxwellian distribution in a single collision, and this is obviously too effective tendency toward thermalization. It would be interesting to see the effect of improvement in kernels. Although the computational labor would be much increased, more terms in the kernel expansion could be attempted.[35] Higher modes of energy spectra in ordinary thermalization problem may give hints in selecting a proper higher term, if one wants to simulate slower thermalization than that by one-term kernel.

The evaluation of the domain by Model 4 have not been completed. Because of the complicated integraion over φ, μ and u, explicit expressions such as in the previous five cases may not be obtainable.

In conclusion, the present method is useful in seeing the effect of reactor physics parameters on the neutronwave eigenvalue problem of moderators. Although the model may not be rigorous in the present form, there is room for the inclusion of more realistic energy-dependence in cross sections treated.

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