

STATISTICAL HYDRODYNAMICS FOR THE BURGERS TURBULENCE

SINZI KUWABARA

Department of Applied Physics

(Received November 1, 1978)

Abstract

Statistical hydrodynamics, which is the generalization of the classical statistical mechanics for the Hamiltonian dynamical system, is outlined (in I). Statistical hydrodynamics is much more complicated, because the hydrodynamical system has continuous, infinite degrees of freedom, although the Hamiltonian system has discrete, finite degrees of freedom. Three solution methods of the initial value problem for the Hopf equation, which is the basic equation of statistical hydrodynamics, are proposed (in II, III and IV). They are applied to the Burgers turbulence, which is the simplified model turbulence of the Navier-Stokes real turbulence. Numerical calculations are carried out for some examples of the homogeneous turbulence, in which the initial energy spectra are chosen as the typical ones. The three methods are compared and criticized (in V).

CONTENTS

I. Basic Concepts in Statistical Hydrodynamics	246
1. Introduction	246
2. Basic Equation of Statistical Hydrodynamics — The Hopf Equation	246
3. Homogeneous Turbulence	251
4. The Burgers Turbulence	254
II. Method for Solving the Hopf Equation by Expanding the Argument Function into Orthonormal Functions	255
5. Formulation of the Initial Value Problem of the Hopf Equation	255
6. Method of Solution by Expanding the Argument Function into Orthonormal Functions	256
7. The Application of the Method by Expanding the Argument Function into Orthonormal Functions to the Burgers Turbulence	257
8. Numerical Calculation	259
III. Semigroup-Theoretic Expression for the Solution and its Application to the Perturbation Method	264

9. Semigroup-Theoretic Expression for the Solution of the Hopf Equation	264
10. The Perturbation Method	265
11. Some Examples	268
IV. The Interaction Representation of the Hopf Equation and the Perturbation Method Based on This Representation	276
12. The Interaction Representation for the Hopf Equation	276
13. The Perturbation Method Based on the Interaction Representation	278
14. Some Examples	280
V. Conclusions	287
15. Comparison of the Three Perturbation Methods	287

I. Basic Concepts in Statistical Hydrodynamics

1. Introduction

Since O. Reynolds^{1,2)} investigated turbulent flow in a circular tube experimentally in 1883, many works on the turbulent flow have been done. In these, the Orr-Sommerfeld theory of the onset of turbulence based on the linear stability of laminar flow, the mixing length theories based on the analogy between momentum and energy transfers among molecules in kinetic theory of gases and those among fluid particles in turbulent motion and so on are the famous ones.

Many studies concerning statistical theory of homogeneous, isotropic turbulence have been made since Taylor's pioneering paper¹⁵⁾ in 1938. The Kármán-Howarth equation⁶⁾, Kolmogoroff's⁹⁾ and Heisenberg's²⁾ spectral theory and the theory of Proudman and Reid¹¹⁾ and Tatsumi¹³⁾ based on the quasi-normal distribution may be the most important contributions to this subject.

In 1953, E. Hopf³⁾ proposed an equation, which is called as the Hopf equation. It includes completely the statistical features of turbulence, but is mathematically very complicated. He combined hydrodynamics and the theory of probability, and we call his theory of turbulence as statistical hydrodynamics. In this paper, we propose three methods of solution for the Hopf equation.

J. E. Burgers¹⁾ proposed a simplified model equation which is one-dimensionalized one of the Navier-Stokes equation, in order to study the turbulence. In his equation, the nonlinear and dissipative properties, which play the essential role in the turbulent motion, are retained. We will apply our methods for solving the initial value problem of the Hopf equation to the model turbulence governed by the Burgers equation. Thus, we will call such a turbulence as the Burgers turbulence.

Other aspects of the Burgers turbulence, i. e. equilibrium spectra in the inertial subrange etc. are reported^{5,14)} and not touched here.

2. Basic Equation of Statistical Hydrodynamics — The Hopf Equation

We consider only the turbulent flows in incompressible viscous fluid, because the turbulent features are realized in such flows. The basic equations and the boundary condition of the incompressible fluid are written as

$$\operatorname{div} \mathbf{v} = 0, \quad \mathbf{x} \in V \quad (2.1)$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \operatorname{grad}) \mathbf{v} = -\frac{1}{\rho} \operatorname{grad} p + \nu \nabla^2 \mathbf{v}, \quad \mathbf{x} \in V \quad (2.2)$$

$$\mathbf{v} = \mathbf{v}_b, \quad \mathbf{x} \in \partial V \tag{2.3}$$

where $\mathbf{x} = (x_1, x_2, x_3) = (x, y, z)$ are the cartesian coordinats, t is the time, $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ the velocity vector, p the pressure, ρ the mass density, ν the kinematic viscosity, V the volume in which the fluid is filled, ∂V the boundary of V and \mathbf{v}_b the velocity vector on the boundary.

We can always construct the Green function:

$$\left. \begin{aligned} \nabla^2 G(\mathbf{x}, \mathbf{x}') &= \delta^3(\mathbf{x} - \mathbf{x}'), \quad \mathbf{x}, \mathbf{x}' \in V \\ G(\mathbf{x}, \mathbf{x}') &= 0, \quad \mathbf{x} \in V + \partial V, \mathbf{x}' \in V, \mathbf{x}' \neq \mathbf{x} \end{aligned} \right\} \tag{2.4}$$

where $\delta^3(\mathbf{x} - \mathbf{x}') = \delta(x_1 - x_1')\delta(x_2 - x_2')\delta(x_3 - x_3')$. If $f(\mathbf{x})$ is an arbitrary function differentiable up to the second order, we have

$$f(\mathbf{x}) = \int_V G(\mathbf{x}, \mathbf{x}') \nabla'^2 f' d^3 \mathbf{x}' + \int_{\partial V} \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial \mathbf{n}'} f' ds', \tag{2.5}$$

from the Green theorem. Here, the prime denotes that the argument of the function is \mathbf{x}' and the operator is that with respect to \mathbf{x}' , and \mathbf{n}' the outward normal on ∂V .

Taking grad div of (2.2), we have

$$\nabla^2(\text{grad } p/\rho) = -\text{grad}[\text{div}\{\mathbf{v} \cdot \text{grad}\mathbf{v}\}]. \tag{2.6}$$

Application of (2.5) for $f = \partial p/\partial x$, $\partial p/\partial y$, and $\partial p/\partial z$ gives

$$\begin{aligned} \text{grad } p/\rho &= - \int_V G(\mathbf{x}, \mathbf{x}') \text{grad}'[\text{div}'\{(\mathbf{v}' \cdot \text{grad}')\mathbf{v}'\}] d^3 \mathbf{x}' \\ &\quad - \int_{\partial V} \left[\frac{\partial \mathbf{v}'}{\partial t} + (\mathbf{v}' \cdot \text{grad}')\mathbf{v}' - \nu \nabla'^2 \mathbf{v}' \right] \frac{\partial G}{\partial \mathbf{n}'} ds'. \end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.2), we have

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \text{grad})\mathbf{v} - \nu \Delta^2 \mathbf{v} &= \int_V G \text{grad}'[\text{div}'\{(\mathbf{v}' \cdot \text{grad}')\mathbf{v}'\}] d^3 \mathbf{x}' \\ &\quad + \int_{\partial V} \left[\frac{\partial \mathbf{v}'}{\partial t} + (\mathbf{v}' \cdot \text{grad}')\mathbf{v}' - \nu \nabla'^2 \mathbf{v}' \right] \frac{\partial G}{\partial \mathbf{n}'} ds', \end{aligned} \tag{2.2'}$$

which is an integro-differential equation only for \mathbf{v} , but does not include p . From (2.2'), the initial data of \mathbf{v} in V and on ∂V and the boundary data of \mathbf{v} on ∂V in the time after the initial instant determines the complete development of \mathbf{v} . From $\frac{\partial}{\partial t} \text{div } \mathbf{v} = 0$, the incompressibility condition is always satisfied if it does at the initial instance.

If we choose a function space Ω of $\mathbf{v}(\mathbf{x})$, which satisfies

$$\text{div } \mathbf{v} = 0, \tag{2.8}$$

the turbulent state of one dynamical system is expressed by a point in Ω . Thus, the turbulent phenomena, i. e. the turbulent states of the dynamical system at all times correspond to a trajectory in Ω .

From the statistical point of view, it is convenient to introduce a phase space, and the probability distribution function in the phase space. In the turbulent case, the phase space should be the function space Ω from the above consideration and the distribution should be a functional in Ω :

$$P[\Omega]=1, \quad P[A] \leq 1 \text{ for } A \subset \Omega. \quad (2.9)$$

It is more convenient to use the probability distribution density D , which is a functional of $\mathbf{v}(\mathbf{x})$ and t . Thus, corresponding to (2.9), we have

$$\int_{\Omega} D[\mathbf{v}] \delta \mathbf{v} = 1, \quad \int_A D[\mathbf{v}] \delta \mathbf{v} \leq 1 \text{ for } A \subset \Omega, \quad (2.10)$$

where $\delta \mathbf{v}(\mathbf{x})$ is the measure in Ω . The ensemble mean $\langle F \rangle$ of a variable $F(\mathbf{v})$ concerning $\mathbf{v}(\mathbf{x})$ is expressed as

$$\langle F \rangle(t) = \int_{\Omega} F(\mathbf{v}) D[\mathbf{v}, t] \delta \mathbf{v}. \quad (2.11)$$

We should derive the basic equation for D . It comes from the law of conservation of probability. We consider a box turbulence. The box is divided into cubic cells and the flow field is represented by the L lattice points \mathbf{x}_l ($l=1, 2, \dots, L$). The box turbulence is assumed to be approximated by the $3L$ variables $\mathbf{v}_l = \mathbf{v}(\mathbf{x}_l)$. Approximating the differential and integral operators by the differences and sums, we have

$$\frac{\partial \mathbf{v}_l}{\partial t} = \mathbf{Q}_l(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_L) \quad l=1, 2, \dots, L \quad (2.12)$$

from (2.2'). Thus, we have a $3L$ -dimensional Euclid space as the phase space in this case.

In this space, the probability distribution is expressed by $D(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_L, t)$. The law of conservation of probability is the continuity equation in this space:

$$\frac{\partial D}{\partial t} + \sum_{l=1}^L \frac{\partial}{\partial \mathbf{v}_l} (\mathbf{Q}_l D) = \frac{\partial D}{\partial t} + \sum_{l=1}^L \sum_{\alpha=1}^3 \frac{\partial}{\partial v_{l,\alpha}} (\mathbf{Q}_{l,\alpha} D) = 0, \quad (2.13)$$

where $\alpha=1, 2, 3$ denote the vector components and $l=1, \dots, L$ the lattice points.

This consideration is assumed to be valid for arbitrary volume (even an infinite volume) and the infinitesimal cells. In this limit, the summation term in (2.13) is changed to

$$\lim_{L \rightarrow \infty} \lim_{|\Delta^3 \mathbf{x}_l| \rightarrow 0} \sum_{l=1}^L \frac{\partial}{\Delta^3 \mathbf{x}_l \partial \mathbf{v}_l} (\mathbf{Q}_l D) \Delta^3 \mathbf{x}_l = \int \frac{\partial}{\partial \mathbf{v}_l(\mathbf{x})} (\mathbf{Q}_l(\mathbf{x}) D) d^3 \mathbf{x} \quad (2.14)$$

where $\Delta^3 \mathbf{x}_l$ is the cell volume at \mathbf{x}_l , and $\partial/\partial \mathbf{v}(\mathbf{x}) = (\partial/\partial v_1(\mathbf{x}), \partial/\partial v_2(\mathbf{x}), \partial/\partial v_3(\mathbf{x}))$ is the functional derivative. In general, the functional derivative is defined as follows: $F[\mathbf{y}(\mathbf{x})]$ is a functional of a vector function $\mathbf{y}(\mathbf{x})$. If we change $\mathbf{y}(\mathbf{x})$ for $\mathbf{y}(\mathbf{x}) + \delta \mathbf{y}(\mathbf{x})$ by an infinitesimal difference $\delta \mathbf{y}(\mathbf{x})$, and the corresponding change of F : $\delta F = F[\mathbf{y}(\mathbf{x}) + \delta \mathbf{y}(\mathbf{x})] - F[\mathbf{y}(\mathbf{x})]$ is expressed by

$$\delta F = \int A_{\alpha}(\mathbf{x}) y_{\alpha}(\mathbf{x}) d^3 \mathbf{x} + O((\delta \mathbf{y})^2), \quad (2.15)$$

we call

$$A_\alpha = \frac{\delta F}{\delta y_\alpha(\mathbf{x})} \tag{2.16}$$

as the functional derivatives of F , with respect to $y_\alpha(\mathbf{x})$. In (2.15), the summation convention for α has been used. Analyzing the process of limit, (2.14) is recognized to be valid. Thus, the law of conservation of probability in $\mathcal{Q}(v(\mathbf{x}))$ can be written as

$$\frac{\partial D}{\partial t} + \int \frac{\delta}{\delta \mathbf{x}(v)} (\mathbf{Q}D) d^3\mathbf{x} = 0. \tag{2.17}$$

This is an analogue of the Liouville equation in classical statistical mechanics for the dynamical systems with discrete degrees of freedom, to that with infinite, continuous degrees of freedom. Thus, we call (2.17) as the generalized Liouville equation.

In the probability theory, the characteristic function, which is the Fourier transform of the probability distribution, is used widely. In statistical hydrodynamics, we can introduce a characteristic functional which is defined as

$$\left. \begin{aligned} \Phi[\mathbf{y}(\mathbf{x}), t] &= \int_{\mathbf{v}} D[\mathbf{v}(\mathbf{x}), t] \exp\{i(\mathbf{y}, \mathbf{v})\} \delta \mathbf{v}(\mathbf{x}), \\ (\mathbf{y}, \mathbf{v}) &= \int_{\mathbf{v}} \mathbf{y}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) d^3\mathbf{x}. \end{aligned} \right\} \tag{2.18}$$

The merit to use the characteristic functional is for the mean values to be derived not from the integration but from the differentiation. From (2.18), we have

$$i^l \langle v_{\alpha_1}(\mathbf{x}_1) \dots v_{\alpha_l}(\mathbf{x}_l) \rangle = \frac{\delta^l \Phi}{\delta y_{\alpha_1}(\mathbf{x}_1) \dots \delta y_{\alpha_l}(\mathbf{x}_l)} \Big|_{\mathbf{y}=0}. \tag{2.19}$$

From the definition of Φ , the following formulas can be obtained,

$$\left. \begin{aligned} \Phi[0, t] &= 1, \quad \Phi[\mathbf{y}(\mathbf{x}), t] \leq 1, \\ \Phi[-\mathbf{y}, t] &= \Phi[\mathbf{y}, t]^* \end{aligned} \right\} \tag{2.20}$$

where * denotes "the complex conjugate of". Since

$$\begin{aligned} (\text{grad } \phi, \mathbf{v}) &= \int_{\mathbf{v}} \left(\frac{\partial \phi}{\partial x} u + \frac{\partial \phi}{\partial y} v + \frac{\partial \phi}{\partial z} w \right) d^3\mathbf{x} \\ &= \int_{\partial \mathbf{v}} \phi \mathbf{v} \cdot \mathbf{n} ds - \int_{\mathbf{v}} \phi \text{div } \mathbf{v} d^3\mathbf{x} = 0, \end{aligned}$$

for arbitrary scalar function ϕ with the boundary condition $\phi=0$ on ∂V , we have

$$(\mathbf{y} + \text{grad } \phi, \mathbf{v}) = (\mathbf{y}, \mathbf{v}),$$

thus

$$\Phi[\mathbf{y} + \text{grad } \phi] = \Phi[\mathbf{y}], \tag{2.21}$$

for any ϕ which is subject to the boundary condition $\phi=0$ on ∂V . From the continuity equation and

$$\frac{\delta\Phi}{\delta y_\alpha} = i \langle v_\alpha(\mathbf{x}) \exp\{i(\mathbf{y}, \mathbf{v})\} \rangle,$$

we have

$$\frac{\partial}{\partial x_\alpha} \frac{\delta\Phi}{\delta y_\alpha} = 0. \quad (2.22)$$

(2.21) and (2.22) are the conditions imposed on Φ .

From (2.18) and (2.17),

$$\begin{aligned} \frac{\partial\Phi}{\partial t} &= \int \frac{\partial D}{\partial t} \exp\{i(\mathbf{y}, \mathbf{v})\} \delta\mathbf{v} = - \int \left\{ \int \frac{\delta}{\delta\mathbf{v}} (\mathbf{Q}D) d^3\mathbf{x} \right\} \exp\{i(\mathbf{y}, \mathbf{v})\} \delta\mathbf{v} \\ &= i \int \mathbf{y}(\mathbf{x}) \cdot \mathbf{Q} \left(\frac{\delta}{i\delta\mathbf{y}} \right) \Phi d^3\mathbf{x} \end{aligned} \quad (2.23)$$

is obtained. If we use

$$\mathbf{Q} = -(\mathbf{v} \cdot \text{grad})\mathbf{v} + \nu \nabla^2 \mathbf{v} - \frac{1}{\rho} \text{grad } p, \quad (2.24)$$

we have

$$\frac{\partial\Phi}{\partial t} = \int_{\mathbf{v}} y_\alpha \left[i \frac{\partial}{\partial x_\alpha} \frac{\delta^2\Phi}{\delta y_\alpha \delta y_\beta} + \nu \nabla^2 \frac{\delta\Phi}{\delta y_\alpha} - i \frac{\partial \Pi}{\partial x_\alpha} \right] d^3\mathbf{x} \quad (2.25)$$

where

$$\begin{aligned} \Pi &= \int_{\mathbf{v}} G \text{grad}' \left[\text{div}' \left\{ \frac{\partial}{\partial \mathbf{y}'} \text{grad}' \right\} \frac{\delta\Phi}{\delta \mathbf{y}'} \right] d^3\mathbf{x}' + \int_{\partial\mathbf{v}} \left[i \frac{\partial}{\partial t} \frac{\delta\Phi}{\delta \mathbf{y}'} \right. \\ &\quad \left. + \frac{\delta}{\delta \mathbf{v}'} \text{grad}' \frac{\delta\Phi}{\delta \mathbf{y}'} - i \nu \nabla'^2 \frac{\delta\Phi}{\delta \mathbf{y}'} \right] ds' \end{aligned} \quad (2.26)$$

If we choose $\mathbf{y}(\mathbf{x})$ as

$$\left. \begin{aligned} \text{div } \mathbf{y} &= 0 & \mathbf{x} \in V, \\ \mathbf{y} &= 0 & \mathbf{x} \in \partial V, \end{aligned} \right\} \quad (2.27)$$

the pressure term Π is dropped, and we have

$$\frac{\partial\Phi}{\partial t} = \int_{\mathbf{v}} y_\alpha \left[i \frac{\partial}{\partial x_\beta} \frac{\delta^2\Phi}{\delta y_\alpha \delta y_\beta} + \nu \nabla^2 \frac{\delta\Phi}{\delta y_\alpha} \right] d^3\mathbf{x}. \quad (2.28)$$

(2.25) and (2.28) are the equation derived first by E. Hopf in 1952. But our derivation is different from his. We emphasize for the Hopf equation to be the correspondence of the generalized Liouville equation.

3. Homogeneous Turbulence

If statistical mean value $F(\mathbf{x}, t)$ concerning a turbulence has a property,

$$F(\mathbf{x} + \mathbf{a}, t) = F(\mathbf{x}, t) \quad \text{for any } \mathbf{a}, \tag{3.1}$$

we call the turbulence homogeneous. We now consider a turbulent flow in an infinite volume. In this case, we can express the flow as the Fourier transform $\mathbf{u}(\mathbf{k}, t)$ of the velocity field $\mathbf{v}(\mathbf{x}, t)$:

$$\left. \begin{aligned} \mathbf{u}(\mathbf{k}, t) &= \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \mathbf{v}(\mathbf{x}, t) \exp\{-i\mathbf{k} \cdot \mathbf{x}\} d^3\mathbf{x}, \\ \mathbf{v}(\mathbf{x}, t) &= \iiint_{-\infty}^{\infty} \mathbf{u}(\mathbf{k}, t) \exp\{i\mathbf{k} \cdot \mathbf{x}\} d^3\mathbf{k}. \end{aligned} \right\} \tag{3.2}$$

From (3.2), we have

$$\mathbf{u}(-\mathbf{k}) = \mathbf{u}(\mathbf{k})^*. \tag{3.3}$$

$\mathbf{v}(\mathbf{x}, t)$ would be an ordinary vector function of \mathbf{x} . In conventional analysis, $\mathbf{v}(\mathbf{x})$ should belong to L_1 and L_2 , i. e.

$$\left. \begin{aligned} \iiint_{-\infty}^{\infty} |\mathbf{v}(\mathbf{x})| d^3\mathbf{x} < \infty, \\ \iiint_{-\infty}^{\infty} \mathbf{v}(\mathbf{x})^2 d^3\mathbf{x} < \infty, \end{aligned} \right\} \tag{3.4}$$

for the Fourier transform of $\mathbf{v}(\mathbf{x})$ to exist, according to the Plancherel theorem. But, (3.4) are not satisfied by the velocity field in homogeneous turbulence, because $\mathbf{v}(\mathbf{x})$ does not damp for $|\mathbf{x}| \rightarrow \infty$. Therefore, $\mathbf{u}(\mathbf{k})$ could be assumed not to be an ordinary vector function, but a distribution.

Carrying out the Fourier transform of (2.1), (2.2) and (2.6), we have

$$\mathbf{k} \cdot \mathbf{u}(\mathbf{k}) = 0, \tag{3.5}$$

$$\frac{\partial \mathbf{u}}{\partial t} + i \int \mathbf{u}(\mathbf{k}') \{ \mathbf{k} \cdot \mathbf{u}(\mathbf{k} - \mathbf{k}') \} d^3\mathbf{k}' + \nu \mathbf{k}^2 \mathbf{u} = -\mathbf{k} q(\mathbf{k}) / \rho, \tag{3.6}$$

$$\mathbf{k}^2 q(\mathbf{k}) / \rho = -\mathbf{k} \int \{ \mathbf{k} \cdot \mathbf{u}(\mathbf{k} - \mathbf{k}') \} \{ \mathbf{k} \cdot \mathbf{u}(\mathbf{k}') \} d^3\mathbf{k}', \tag{3.7}$$

where $q(\mathbf{k}, t)$ is the Fourier transform of $p(\mathbf{x}, t)$:

$$\left. \begin{aligned} q(\mathbf{k}, t) &= \frac{1}{(2\pi)^3} \int p(\mathbf{x}, t) \exp(-i\mathbf{k} \cdot \mathbf{x}) d^3\mathbf{x}, \\ p(\mathbf{x}, t) &= \int q(\mathbf{k}, t) \exp(i\mathbf{k} \cdot \mathbf{x}) d^3\mathbf{k}. \end{aligned} \right\} \tag{3.8}$$

Substituting q in (3.7) into (3.6), we have

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_\alpha(\mathbf{k}) + i \int \mathbf{k}' u_\gamma(\mathbf{k} - \mathbf{k}') \Delta_{\alpha\beta}(\mathbf{k}) u_\beta(\mathbf{k}') d^3\mathbf{k}', \tag{3.9}$$

$$\Delta_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta / k^2, \tag{3.10}$$

(3.9) is the basic dynamical equation for $\mathbf{u}(\mathbf{k})$.

For $\mathbf{u}(\mathbf{k})$, we can derive the generalized Liouville equation and the Hopf equation. In this case, the phase space is the space of distribution $\Omega(\mathbf{u}(\mathbf{k}))$. A turbulent state of a dynamical system corresponds to a point in $\Omega(\mathbf{u}(\mathbf{k}))$, and the turbulent phenomena is expressed by a trajectory in $\Omega(\mathbf{u}(\mathbf{k}))$.

If we introduce a functional of the probability distribution density $D[\mathbf{u}(\mathbf{k}), t]$ in $\Omega(\mathbf{u}(\mathbf{k}))$, we have similarly

$$\int_{\Omega} D[\mathbf{u}(\mathbf{k}), t] \delta \mathbf{u}(\mathbf{k}) = 1, \tag{3.11}$$

$$\int_A D[\mathbf{u}(\mathbf{k}), t] \delta \mathbf{u}(\mathbf{k}) \leq 1 \quad A \subset \Omega,$$

and the ensemble mean of $F = F(\mathbf{u}(\mathbf{k}), t)$

$$\langle F \rangle(t) = \int F(\mathbf{u}) D[\mathbf{u}, t] \delta \mathbf{u}. \tag{3.12}$$

In the similar manner to the former case, we have the generalized Liouville equation,

$$\left. \begin{aligned} \frac{\partial D}{\partial t} + \int \frac{\partial}{\partial \mathbf{u}(\mathbf{k})} (QD) d^3\mathbf{k} &= 0, \\ Q_\alpha &= -i \int \mathbf{k}' u_\gamma(\mathbf{k} - \mathbf{k}') \Delta_{\alpha\beta}(\mathbf{k}) u_\beta(\mathbf{k}') d^3\mathbf{k}' \\ &\quad - \nu k^2 u_\alpha(\mathbf{k}). \end{aligned} \right\} \tag{3.13}$$

We define the characteristic functional as

$$\begin{aligned} \Phi[\mathbf{z}(\mathbf{k}), t] &= \int D[\mathbf{u}(\mathbf{k}), t] \exp\{i(\mathbf{z}, \mathbf{u})^*\} \delta \mathbf{u}(\mathbf{k}), \\ (\mathbf{z}, \mathbf{u}) &= \int \mathbf{z}(\mathbf{k})^* \mathbf{u}(\mathbf{k}) d^3\mathbf{k}. \end{aligned} \tag{3.14}$$

The ensemble mean of velocity product is given by

$$i^l \langle u_{\alpha_1}(\mathbf{k}_1) \dots u_{\alpha_l}(\mathbf{k}_l) \rangle = \left. \frac{\delta^l \Phi}{\delta z_{\alpha_1}(\mathbf{k}_1) \dots \delta z_{\alpha_l}(\mathbf{k}_l)} \right|_{z=0}. \tag{3.15}$$

For the definition of $\Phi[\mathbf{z}, t]$ in (3.14) to be valid,

$$\left. \begin{aligned} \mathbf{z}(-\mathbf{k}) &= \mathbf{z}(\mathbf{k})^*, \\ \mathbf{z}(\mathbf{k}) &\rightarrow 0, \quad \text{for } |\mathbf{k}| \rightarrow \infty, \end{aligned} \right\} \tag{3.16}$$

should be satisfied. From the definition of Φ , we have

$$\left. \begin{aligned} \Phi[0, t] &= 1, & \Phi[\mathbf{z}, t] &\leq 1, \\ \Phi[-\mathbf{z}, t] &= \Phi[\mathbf{z}, t]^*. \end{aligned} \right\} \quad (3.17)$$

Since

$$(\mathbf{z}(\mathbf{k}) + \phi(\mathbf{k})\mathbf{k}, \mathbf{u}(\mathbf{k})) = (\mathbf{z}(\mathbf{k}), \mathbf{u}(\mathbf{k}))$$

for any scalar complex function $\phi(\mathbf{k})$,

$$\Phi[\mathbf{z}(\mathbf{k}) + \phi(\mathbf{k})\mathbf{k}] = \Phi[\mathbf{z}(\mathbf{k})] \quad \text{for any } \phi(\mathbf{k}). \quad (3.18)$$

From (3.5) and (3.10), we have

$$\mathbf{k} \cdot \frac{\partial \Phi}{\partial \mathbf{z}(\mathbf{k})} = 0. \quad (3.18')$$

(3.18) and (3.18') come from the continuity equation. These conditions represent that $\Phi[\mathbf{z}(\mathbf{k})]$ does not depend on the component of $\mathbf{z}(\mathbf{k})$ parallel to \mathbf{k} , but depends only on normal two components to \mathbf{k} .

In the homogeneous turbulence, the vector field $\mathbf{v}(\mathbf{x})$ is in one-to-one correspondence with $\mathbf{u}(\mathbf{k})$ each other. If we put a relation between $\mathbf{y}(\mathbf{x})$ and $\mathbf{z}(\mathbf{k})$ as

$$\left. \begin{aligned} \mathbf{z}(\mathbf{k}) &= \int \mathbf{y}(\mathbf{x}) \exp(-i\mathbf{k} \cdot \mathbf{x}) d^3\mathbf{x}, \\ \mathbf{y}(\mathbf{x}) &= \frac{1}{(2\pi)^3} \int \mathbf{z}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}) b^3\mathbf{k}, \end{aligned} \right\} \quad (3.19)$$

we have

$$\Phi[\mathbf{z}(\mathbf{k}), t] = \Phi[\mathbf{y}(\mathbf{x}), t]. \quad (3.20)$$

In the similar manner to the case with the independent variables $\mathbf{y}(\mathbf{x})$ and t , the Hopf equation with respect to the independent variables $\mathbf{z}(\mathbf{k})$ and t is obtained as

$$\begin{aligned} \frac{\partial \Phi}{\partial t} &= \iint A_{\alpha\beta}(\mathbf{k} + \mathbf{k}') z_{\beta}(\mathbf{k} + \mathbf{k}') k_{\gamma} \frac{\partial^2 \Phi}{\partial z_{\alpha}(\mathbf{k}) \partial z_{\gamma}(\mathbf{k}')} d^3\mathbf{k} d^3\mathbf{k}' \\ &\quad - \nu \int k^2 z_{\alpha}(\mathbf{k}) \frac{\partial \Phi}{\partial z_{\alpha}(\mathbf{k})} d^3\mathbf{k}. \end{aligned} \quad (3.21)$$

If we confine ourselves for $\mathbf{z}(\mathbf{k})$ to be

$$\mathbf{k} \cdot \mathbf{z}(\mathbf{k}) = 0, \quad (3.22)$$

(3.21) reduces to

$$\begin{aligned} \frac{\partial \Phi}{\partial t} &= \iint z_{\alpha}(\mathbf{k} + \mathbf{k}') k_{\beta} \frac{\partial^2 \Phi}{\partial z_{\alpha}(\mathbf{k}) \partial z_{\beta}(\mathbf{k}')} d^3\mathbf{k} d^3\mathbf{k}' \\ &\quad - \nu \int k^2 z_{\alpha}(\mathbf{k}) \frac{\partial \Phi}{\partial z_{\alpha}(\mathbf{k})} d^3\mathbf{k}. \end{aligned} \quad (3.21')$$

Since

$$(\mathbf{y}(\mathbf{x}), \mathbf{v}(\mathbf{x})) = (\mathbf{y}(\mathbf{x}), \mathbf{v}(\mathbf{y}-\mathbf{a})) = (\mathbf{y}(\mathbf{x}+\mathbf{a}), \mathbf{v}(\mathbf{x})),$$

for any \mathbf{a} , in the homogeneous turbulence, we have

$$\Phi[\mathbf{y}(\mathbf{x}+\mathbf{a})] = \Phi[\mathbf{y}(\mathbf{x})] \quad \text{for any } \mathbf{a}. \tag{3.23}$$

From (3.19), the Fourier transform of $\mathbf{y}(\mathbf{x}+\mathbf{a})$ is $\mathbf{z}(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{a})$. Thus, the homogeneity condition (3.23) becomes

$$\Phi[\mathbf{z}(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{a})] = \Phi[\mathbf{z}(\mathbf{k})] \quad \text{for any } \mathbf{a}, \tag{3.23'}$$

by use of (3.20).

When we consider the initial value problem of the Hopf equation (3.22), we should choose the initial value of Φ to satisfy (3.17), the incompressibility condition (3.18) or (3.18'). If these conditions for Φ are satisfied initially, $\Phi[\mathbf{z}(\mathbf{k}), t]$ would satisfy them after the initial instant.

Similarly as (2.19), we have

$$i^l \langle u_{\alpha_1}(\mathbf{k}_1) \dots u_{\alpha_l}(\mathbf{k}_l) \rangle = \frac{\delta^l \Phi}{\delta z_{\alpha_1}(\mathbf{k}_1) \dots \delta z_{\alpha_l}(\mathbf{k}_l)} \Big|_{z=0}. \tag{3.24}$$

4. The Burgers Turbulence

In a one-dimensional incompressible flow, i. e. $u=u(x, t)$, $v=w=0$, and $p=p(x, t)$, (2.1) and (2.2) reduce to

$$\left. \begin{aligned} \frac{\partial u}{\partial x} &= 0, \\ \frac{\partial u}{\partial t} &= -\frac{1}{\rho} \frac{\partial p}{\partial x}. \end{aligned} \right\} \tag{4.1}$$

Thus, we have only a rigid motion $u=u(t)$, which is a very trivial solution. Burgers proposed a model equation, which corresponds to a simplification of the Navier-Stokes equation to the one-dimensional flow, and has the nonlinear and dissipation features:

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \nu \frac{\partial^2 v}{\partial x^2}, \tag{4.2}$$

in order to study a turbulence. The basic equation of the Fourier transform $u(k, t)$ of $v(x, t)$ is obtained by transformation of (4.2),

$$\frac{\partial u}{\partial t} + i \int k' u(k-k') u(k') dk' + \nu k^2 u = 0, \tag{4.3'}$$

where

$$\left. \begin{aligned} u(k, t) &= \int v(x, t) \exp(-ikx) dx, \\ v(x, t) &= \frac{1}{2\pi} \int u(k, t) \exp(ikx) dk. \end{aligned} \right\} \tag{4.3}$$

Correspondingly to $v(x, t)$, we have the phase space $\Omega(v(x))$ and the characteristic functional $\Phi[y(x), t]$ which is subject to

$$\left. \begin{aligned} \Phi[0, t] &= 1, & |\Phi[y(x), t]| &\leq 1, \\ \Phi[-y, t] &= \Phi[y, t]^* \end{aligned} \right\} \quad (4.4)$$

(see 2.20)), and the homogeneity condition:

$$\Phi[y(x+a)] = \Phi[y(x)] \quad \text{for any } a, \quad (4.5)$$

(see (3.23)). The Hopf equation for $\Phi[y(x), t]$ is from (2.28)

$$\frac{\partial \Phi}{\partial t} = \int_{-\infty}^{\infty} y \left[i \frac{\partial}{\partial x} \frac{\delta^2 \Phi}{\delta y^2} + \nu \frac{\partial^2}{\partial x^2} \frac{\delta \Phi}{\delta y} \right] dx. \quad (4.6)$$

Similarly, we have the phase space $\Omega(u(k))$ and the characteristic functional $\Phi[z(k), t]$ for $u(k, t)$. (4.4) ~ (4.6) reform as

$$\left. \begin{aligned} \Phi[0, t] &= 1, & |\Phi[z(k), t]| &\leq 1, \\ \Phi[-z, t] &= \Phi[z, t]^* \end{aligned} \right\} \quad (4.7)$$

(see (3.17)),

$$[z(k) \exp(ika)] = [z(k)] \quad \text{for any } a, \quad (4.8)$$

(see (3.23')),

$$\frac{\partial \Phi}{\partial t} = \iint_{-\infty}^{\infty} z(k+k') k \frac{\delta^2 \Phi}{\delta z(k) \delta z(k')} dk dk' - \nu \int k^2 z(k) \frac{\delta \Phi}{\delta z(k)} dk, \quad (4.9)$$

(see (3.22')).

II. Method for Solving the Hopf Equation by Expanding the Argument Function into the Orthonormal Functions

5. Formulation of the Initial Value Problem of the Hopf Equation

In this and the following two chapters, we propose three methods for solving the initial value problem of the Hopf equation for $\Phi[z(k), t]$:

$$\frac{\partial \Phi}{\partial t} + A\Phi = 0, \quad (5.1)$$

$$A = - \iint_{-\infty}^{\infty} z(k+k') k \frac{\delta^2 \Phi}{\delta z(k) \delta z(k')} dk dk' - \frac{1}{R} \int_{-\infty}^{\infty} k z(k) \frac{\delta \Phi}{\delta z(k)} dk. \quad (5.2)$$

$\Phi[z(k), t]$ should satisfy the following conditions

$$\left. \begin{aligned} \Phi[0] &= 1, & |\Phi[z]| &\leq 1, \\ \Phi[-z] &= \Phi[z]^*, \end{aligned} \right\} \quad (5.3)$$

and

$$\Phi[z \exp(iaz)] = \Phi[z] \quad \text{for any } a. \tag{5.4}$$

Here and in the following lines, the normalization is done:

$$\left. \begin{aligned} v(k)/UL &\longrightarrow v(k), \\ z(k)UL &\longrightarrow z(k), \\ kL &\longrightarrow k, \\ t/(L/U) &\longrightarrow t, \\ UL/\nu &= R, \end{aligned} \right\} \tag{5.5}$$

where the factors in the left hand side are dimensional ones and the non-dimensionalized variables are put as those in the right hand side, which is the same letters as the dimensional ones, and U and L are representative velocity and length.

The definition (3.14) of Φ gives

$$E(k, t)\delta(k+k') = \frac{1}{2} \langle u(k)u(k') \rangle = - \left. \frac{\delta^2 \Phi}{\delta z(k')\delta z(k)} \right|_{z=0}, \tag{5.6}$$

where $E(k, t)$ is a spectrum of the turbulent field.

Now we shall consider the initial value problem of the Burgers turbulence, and examine how the energy spectrum $E(k, t)$ is changed from a given initial spectrum $E(k, 0) = \hat{E}(k)$. We choose the initial condition for Φ as

$$\hat{\Phi}[z] = \exp\left[-\frac{1}{2} \int \hat{E}(k) z z^* dk\right]. \tag{5.7}$$

The reason why we choose (5.7) as the initial condition, is that it is the simplest expression which satisfies the conditions (5.3) and (5.4).

6. Method of Solution by Expanding the Argument Function into Orthonormal Functions

The possibility of the integration in the first of (3.4) demands that

$$\left. \begin{aligned} z(k) &\longrightarrow 0 \text{ for } |k| \longrightarrow \infty, \\ z(k) &\text{ is real,} \\ z(-k) &= z(k)^*. \end{aligned} \right\} \tag{6.1}$$

or equivalently

In general, a function of bounded variation can be expanded into a series of the complete set of continuous functions in the sense of L_2 -norm. We assume that $z(k)$ is of bounded variation and choose the following complete set of functions $\{\phi_l(k)\}$ ($l=0, 1, 2, \dots$), ($-\infty < k < \infty$):

$$\left. \begin{aligned} \phi_l(-k) &= \phi_l(k)^*, \\ \phi_l(k) &\longrightarrow 0 \text{ for } |k| \longrightarrow \infty, \\ (\phi_l, \phi_m) &= \int \phi_l(k)^* \phi_m(k) dk = \delta_{lm}. \end{aligned} \right\} \tag{6.2}$$

Then $z(k)$ may be expanded as

$$z(k) = \sum_{l=0}^{\infty} a_l \phi_l(k), \quad (6.3)$$

where a_l is real variables. The functional $\Phi[z(k), t]$ may be interpreted as a function of infinite number of variables $\{a_l\}$:

$$\Phi[z(k), t] = \Phi(a_0, a_1, a_2, \dots, t). \quad (6.4)$$

From (6.2) and (6.3), we have

$$a_l = \int \phi_l(k) * z(k) dk = (\phi_l, z). \quad (6.5)$$

Thus, the coefficient a_l of the expansion of $z(k)$ is a functional of $z(k)$. From the definition (2.15) and (2.16) of the functional derivatives and (6.5), we have

$$\frac{\delta a_l}{\delta z(k)} = \phi_l^*(k). \quad (6.6)$$

Then, the functional derivative with respect to $z(k)$, which can be expanded into a series of the complete orthonormal functions, can be expressed by an infinite sum of ordinary derivative:

$$\frac{\delta}{\delta z(k)} = \sum_{l=0}^{\infty} \frac{\delta a_l}{\delta z(k)} \frac{\partial}{\partial a_l} = \sum_{l=0}^{\infty} \phi_l^*(k) \frac{\partial}{\partial a_l}, \quad (6.7)$$

where the functional, which is operated, is recognized as the right hand side of (6.4).

Substituting the expansion (6.3) of $z(k)$ and the transformation (6.7) of the functional derivative into (5.2), we have the transformed equation for $\Phi(a_0, a_1, a_2, \dots, t)$:

$$\frac{\partial \Phi}{\partial t} + \sum_{l, m, n=0}^{\infty} A_{lmn} a_n \frac{\partial^2 \Phi}{\partial a_l \partial a_m} + \frac{1}{R} \sum_{l, m=0}^{\infty} A_{lm} a_m \frac{\partial \Phi}{\partial a_l} = 0, \quad (6.8)$$

$$\left. \begin{aligned} A_{lmn} &= \iint_{-\infty}^{\infty} \phi_l(k) * k' \phi_m(k') \phi_n(k+k') dk dk', \\ A_{lm} &= \int_{-\infty}^{\infty} k^2 \phi_l(k) * \phi_m(k) dk. \end{aligned} \right\} \quad (6.9)$$

In this scheme, the energy spectrum expression (5.6) is transformed into

$$E(k, t) = \sum_{l, m=0}^{\infty} \left(\frac{\partial^2 \Phi}{\partial a_l \partial a_m} \right)_{a=0} (1, \phi_m(k)) \phi^*(k). \quad (6.10)$$

7. The Application of the Method by Expanding the Argument Function into Orthonormal Functions to the Burgers Turbulence

The domain of $z(k)$ is $-\infty < k < \infty$ and $z(k)$ should satisfy (6.1). Then, $z(k)$ may be assumed to be expanded into the Hermite functions:

$$z(k) = \sum_{l=0}^{\infty} a_l \phi_l(k), \quad (7.1)$$

$$\phi_l(k) = \left(\frac{2}{\pi}\right)^{1/4} \frac{1}{\sqrt{l!}} i^{\sum_{m=0}^{\infty} \delta_{l, (2m+1)}} H_l(2k) \exp(-k^2).$$

The expansion which is retained the first three terms, i. e. a_0 , a_1 , and a_2 , is substituted into (6.9) to give the transformed equation:

$$\begin{aligned} \frac{\partial \Phi}{\partial t} - i \frac{(2\pi)^{1/4}}{3^{3/2}} & \left[-a_0 \left(\sqrt{2} \frac{\partial^2}{\partial a_0 \partial a_1} - \frac{\partial^2}{\partial a_1 \partial a_2} \right) + a_1 \left(\sqrt{2} \frac{\partial^2}{\partial a_0^2} \right. \right. \\ & \left. \left. + 2 \frac{\partial^2}{\partial a_0 \partial a_2} - \frac{1}{\sqrt{2} 3} \frac{\partial^2}{\partial a_2} \right) - a_2 \left(3 \frac{\partial^2}{\partial a_0 \partial a_1} - \frac{1}{\sqrt{2} 3} \frac{\partial^2}{\partial a_1 \partial a_2} \right) \right] \Phi \\ & - \frac{1}{4R} \left[a_0 \frac{\partial}{\partial a_0} + 3a_1 \frac{\partial}{\partial a_1} + \sqrt{2} a_0 \frac{\partial}{\partial a_2} + \sqrt{3} a_2 \frac{\partial}{\partial a_0} \right. \\ & \left. + 5a_2 \frac{\partial}{\partial a_2} \right] \Phi = 0. \end{aligned} \quad (7.2)$$

The domain of the truncated Φ are $-\infty < a_0, a_1, a_2 < \infty$, $t \geq 0$. The boundary conditions are

$$\Phi = 0 \quad \text{for } a_0, a_1, \text{ and } a_2 \longrightarrow \pm\infty. \quad (7.3)$$

The initial condition is chosen as (5.7). For the initial energy spectra, we consider two cases:

$$\left. \begin{aligned} \text{(i)} \quad \hat{E}(k) &= \frac{1}{\sqrt{\pi}} \exp(-k^2) \quad (\text{bell shape}), \\ \text{(ii)} \quad \hat{E}(k) &= \frac{2}{\sqrt{\pi}} k^2 \exp(-k^2) \quad (\text{double-bell shape}), \end{aligned} \right\} \quad (7.4)$$

which have been normalized

$$\int_{-\infty}^{\infty} \hat{E}(k) dk = 1. \quad (7.5)$$

Substituting (7.1) whose first three terms are retained and (7.4) into (5.7), we have the initial conditions for Φ as

$$\left. \begin{aligned} \text{(i)} \quad \Phi(a_0, a_1, a_2) &= \exp \left[-\frac{1}{\sqrt{6\pi}} \left(a_0^2 + \frac{2}{3} a_1^2 - \frac{\sqrt{2}}{3} a_0 a_2 \right. \right. \\ & \quad \left. \left. + \frac{1}{2} a_2^2 \right) \right], \\ \text{(ii)} \quad \Phi(a_0, a_1, a_2) &= \exp \left[-\sqrt{\frac{2}{3^2\pi}} \left(\frac{1}{2} a_0^2 + a_1^2 \right. \right. \\ & \quad \left. \left. + \frac{1}{\sqrt{2}} a_0 a_2 + \frac{11}{12} a_2^2 \right) \right]. \end{aligned} \right\} \quad (7.6)$$

The total energy density $TE(t)$ is defined by

$$TE(t) = \int_{-\infty}^{\infty} E(k, t) dk. \tag{7.7}$$

Then, the initial total energy density $T\hat{E}$ is

$$T\hat{E} = TE(0) = \int_{-\infty}^{\infty} \hat{E}(k) dk = 1, \tag{7.8}$$

from (7.5).

It is noted that the representative values of length and velocity, i. e. L and U , are related to the initial energy spectrum and the initial total energy density in the dimensional form. If the initial total energy density in the dimensional form is taken as $(1/2)U^2$, and the dimensional initial energy spectra are expressed by

$$\left. \begin{aligned} \text{(i)} \quad & \hat{E}(k) = A \exp(-\alpha k^2), \\ \text{(ii)} \quad & \hat{E}(k) = Bk^2 \exp(-\alpha k^2), \end{aligned} \right\} \tag{7.9}$$

we have

$$\left. \begin{aligned} & L = 2\pi \sqrt{\alpha}, \\ \text{(i)} \quad & U = \left(\frac{\pi}{\alpha}\right)^{1/4} A^{1/2}, \\ \text{(ii)} \quad & U = \left(\frac{\pi}{\alpha^3}\right)^{1/4} B^{1/2}. \end{aligned} \right\} \tag{7.10}$$

By use of (7.1), (6.10) gives

$$\begin{aligned} E(k, t) = & -\sqrt{2} \left[\left(\frac{\partial^2 \Phi}{\partial a_0^2} - \frac{1}{2} \frac{\partial^2 \Phi}{\partial a_2^2} \right)_{a=0} + 2^{3/2} \left(\frac{\partial^2 \Phi}{\partial a_0 \partial a_2} \right. \right. \\ & \left. \left. + \frac{1}{\sqrt{2}} \frac{\partial^2 \Phi}{\partial a_2^2} \right)_{a=0} k^2 \right] \exp(-k^2), \end{aligned} \tag{7.11}$$

which gives the total energy density

$$\begin{aligned} TE(t) = & \int_{-\infty}^{\infty} E(k, t) dk \\ = & -\sqrt{2\pi} \left[\left(\frac{\partial^2 \Phi}{\partial a_0^2} - \frac{1}{2} \frac{\partial^2 \Phi}{\partial a_2^2} \right) + \sqrt{2} \left(\frac{\partial^2 \Phi}{\partial a_0 \partial a_2} \right. \right. \\ & \left. \left. + \frac{1}{\sqrt{2}} \frac{\partial^2 \Phi}{\partial a_2^2} \right) \right]_{a=0}. \end{aligned} \tag{7.12}$$

8. Numerical Calculation

We now proceed to solving the initial value problem of the differential equation (7.2) subject to the boundary conditions (7.3) and the initial condition (7.6). It

is difficult to solve (7.2) analytically and a numerical method, i. e. the finite difference method is used.

The symmetry of the initial and boundary conditions, as well as the basic equation simplifies the problem, so that we may consider only the region $-\infty < a_0 < \infty$, and $0 \leq a_1, a_2 < \infty$. Φ may be expected to decrease rapidly to zero in the region far from the origin. Thus, we may confine ourselves to the region: $-10 \leq a_0 \leq 10$, and $0 \leq a_1, a_2 \leq 10$. This region is divided into a set of equal cubic cells, whose edge is $h=10/N$ (N =integer) long. For the boundary conditions at $a_2=0$ and $a_2=10$ appropriately to be applied, we extend our region to the negative sides of the a_1 and a_2 direction one mesh more, i. e. we have $-h \leq a_1 \leq 10$ and $-h \leq a_2 \leq 10$. The symmetry conditions imply that

$$\left. \begin{aligned} \Phi(mh, -h, 0) &= \Phi(mh, h, 0)^*, \\ \Phi(mh, -h, -h) &= \Phi(-mh, -h, h)^*, \quad m = -N, \dots, N \\ \Phi(mh, 0, -h) &= \Phi(-mh, 0, h). \end{aligned} \right\} \quad (8.1)$$

We have also the boundary conditions corresponding to the vanishing Φ at infinity:

$$\left. \begin{aligned} \Phi(\pm Nh, lh, mh) &= 0, \quad l, m = -1, 0, \dots, N \\ \Phi(lh, Nh, mh) &= 0, \quad l = -N, \dots, N \\ \Phi(lh, mh, Nh) &= 0, \quad m = -1, 0, \dots, N \end{aligned} \right\} \quad (8.2)$$

We take $h=1$ in this calculation.

The characteristic time for the change of Φ may be expected as R for $R \ll 1$ and 1 for $R \gg 1$, respectively. The integration with respect to time is carried out by means of the implicit scheme and the time increment is taken as 0.01 in the (t/R) -scale for $R \leq 1$ and 0.05 in t -scale for $R \geq 1$.

The temporal development of the energy spectrum and the total energy density are carried out for the various Reynolds numbers, and for the initial energy spectrum of two types mentioned in 7.

Fig. 1 shows the temporal development of the energy spectrum with respect to the initial energy spectrum of the "bell shape", (a) for $R=0, 0.01, 0.1$ and 1 and (b) for $R=1, 10, 100$ and ∞ . In Fig. 1 (a) and (b), the total energy density $TE(t)$ are also shown.

Fig. 2 shows the similar results for the initial energy spectrum of the "double-bell shape".

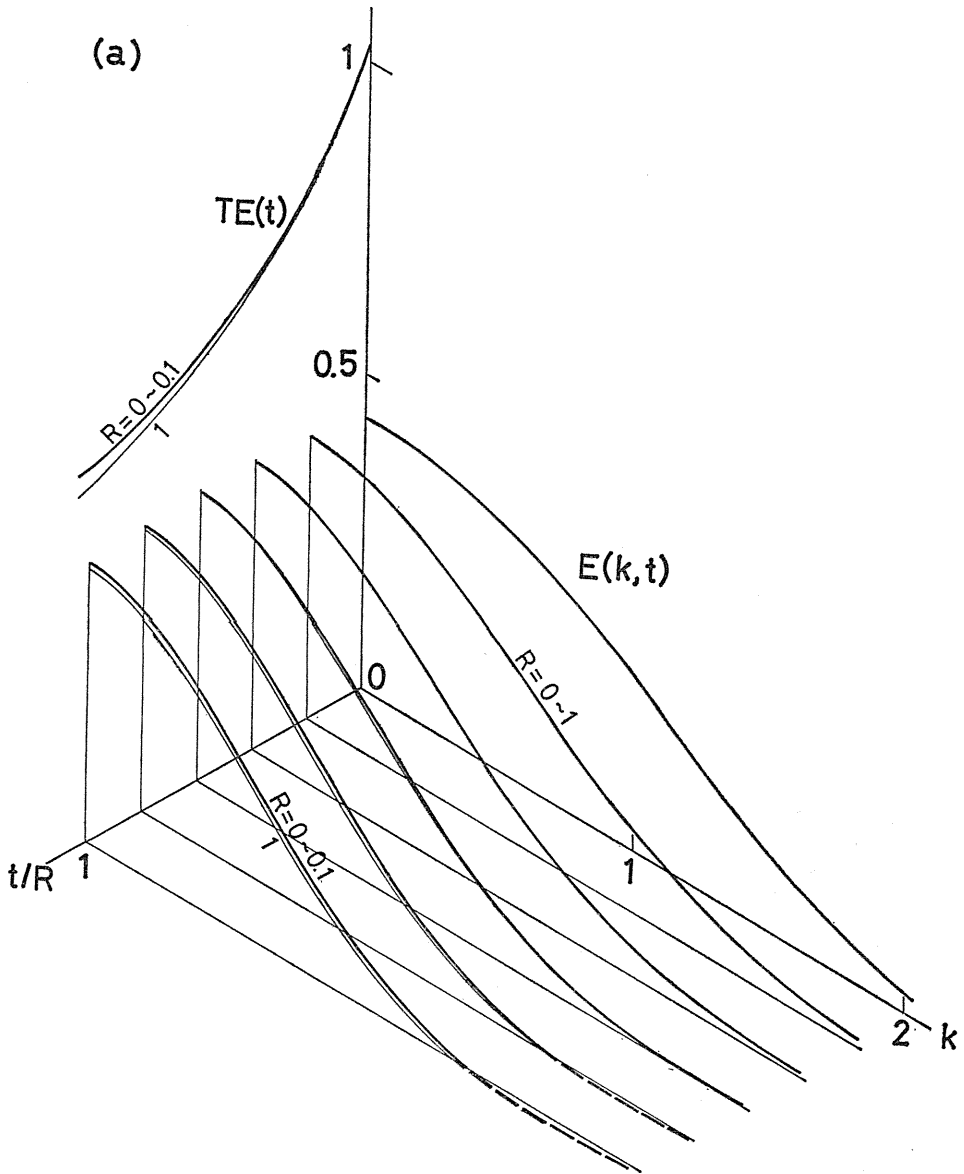
We can summarize the results as follows:

i) This method seems to give reasonable results in $0 \leq t/R < 1$ for $R < 1$ and in $0 \leq t < 1$ for $R > 1$.

ii) The temporal developments of energy spectrum are very similar both for R smaller than 1 and for R larger than 10 , when the changes are measured in the time scale t/R and t , respectively. Between $R=1$ and 10 , the features of the temporal developments of the energy spectrum seem to be abruptly changed.

iii) The paradox of negative values of energy spectrum occurs in the range of large wavenumbers after some time from the start.

iv) The constancy of energy spectrum for small wavenumber, i. e. $E(0, t)$ = the temporal invariance, is not satisfied in this method.



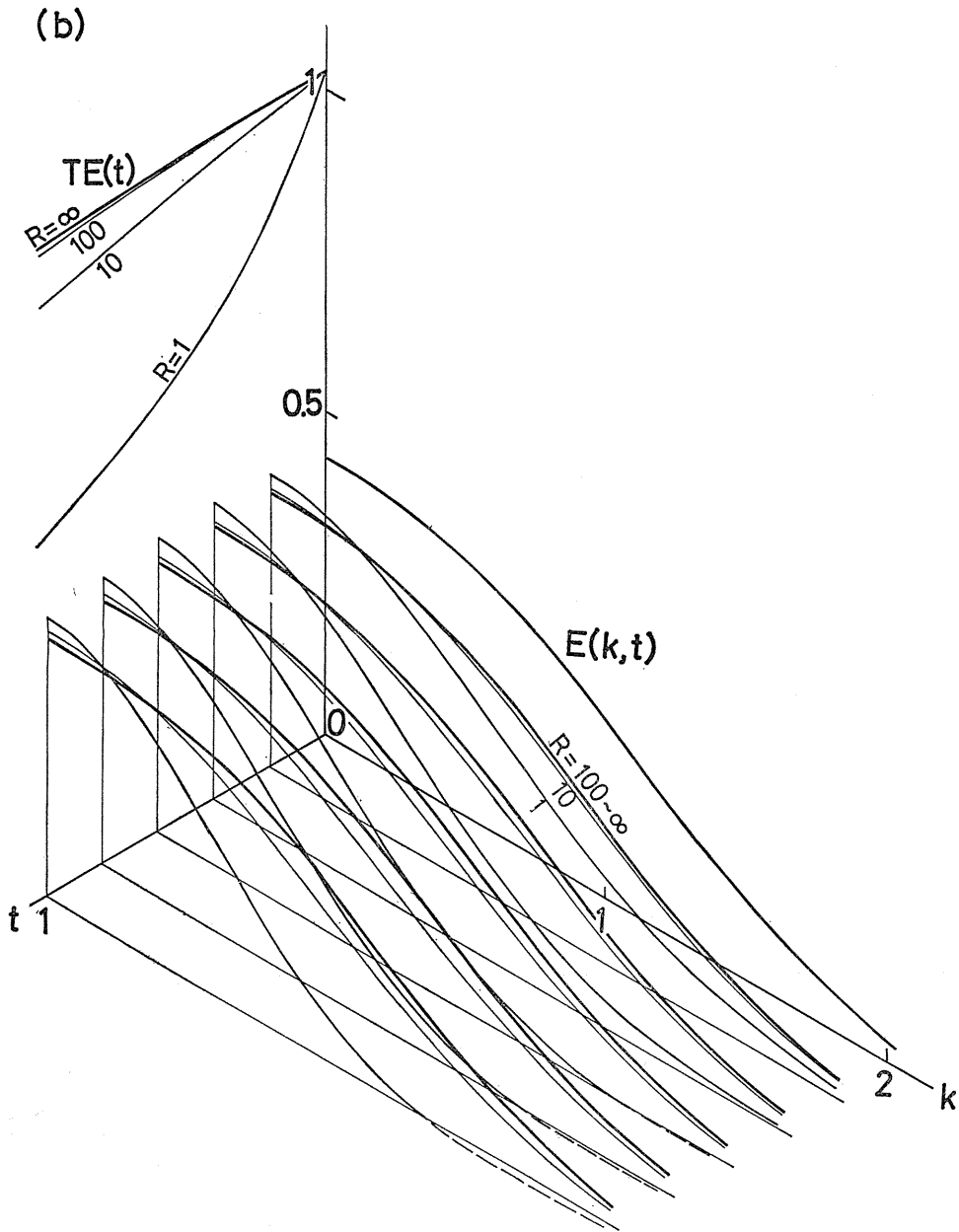
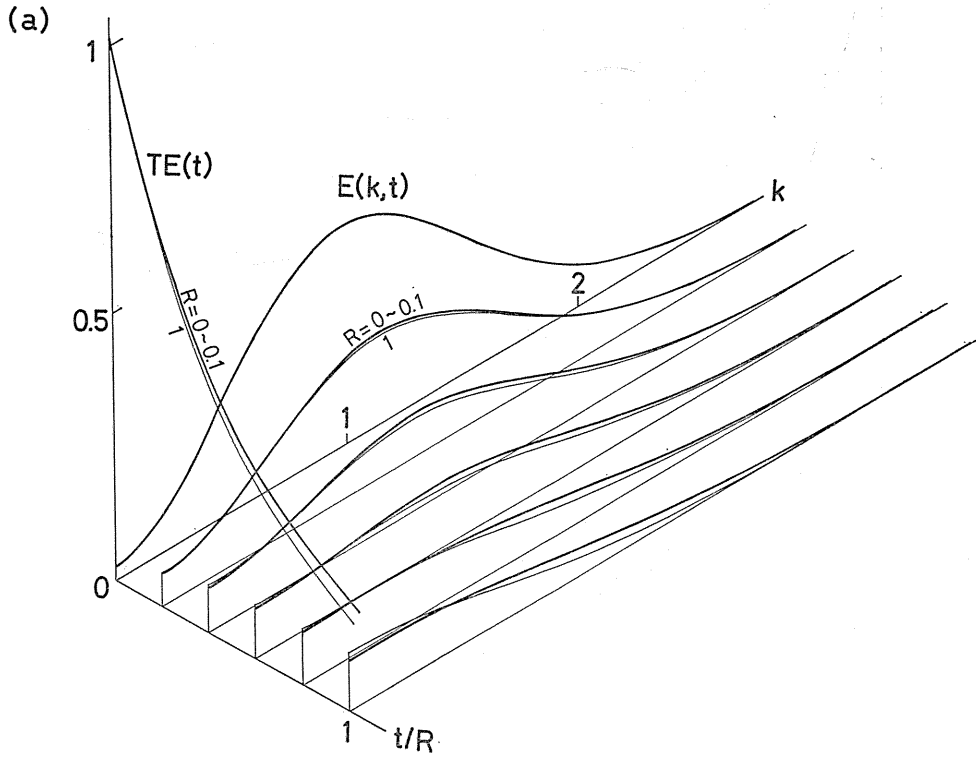


Fig. 1. Temporal developments of energy spectrum $E(k, t)$ and total energy density $TE(t)$ from the initial bell shape spectrum (a) for $R=0, 0.01, 0.1$ and 1 , and (b) for $R=1, 1, 10, 100$ and ∞ , calculated by the first method in II.



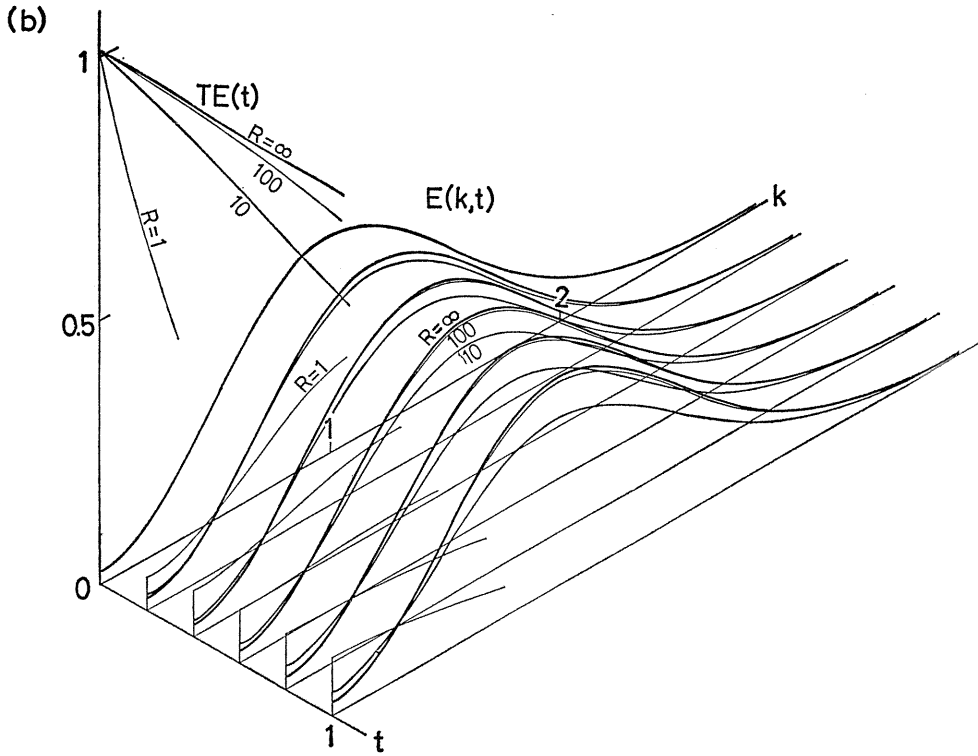


Fig. 2. Temporal developments of energy spectrum $E(k, t)$ and total energy density $TE(t)$ from the initial double-bell shape spectrum (a) for $R=0, 0.0, 0.1$ and 1 , and (b) for $R=1, 10, 100$ and ∞ , calculated by the first method in II.

III. Semigroup-Theoretic Expression for the Solution and its Application to the Perturbation Method

9. Semigroup-Theoretic Expression for the Solution of the Hopf Equation

The Hopf equation (5.1) with (5.2) is rewritten as

$$\frac{\partial \Phi}{\partial t} + A\Phi = 0, \tag{9.1}$$

$$A = -\mathfrak{D}_{21}k_2 + \frac{1}{R}\mathfrak{D}_1k_1^2, \tag{9.2}$$

where

$$\mathfrak{D}_1 = \int dk_1 z(k_1) D_1 \tag{9.3}$$

$$\mathfrak{D}_{21} = \iint dk_2 dk_1 z(k_2 + k_1) D_2 D_1 \tag{9.4}$$

$$D_l = -\frac{\delta}{\delta z(k_l)}, \quad l=1, 2 \tag{9.5}$$

For the initial condition :

$$\Phi[z(k), 0] = \hat{\Phi}[z(k)], \tag{9.6}$$

we put

$$\hat{\Phi}[z(k)] = \exp\left[-\frac{1}{2} \int \hat{E}(k) z(k) * z(k) dk\right]. \tag{9.7}$$

The boundary condition is given by

$$\Phi[z(k), t] = 0 \quad \text{for } z(k) \longrightarrow \infty, t \geq 0. \tag{9.8}$$

Since the operator A of (9.2) does not include t , we may express the formal solution of (9.1) as

$$\begin{aligned} \Phi[z, t] &= \exp(-tA) \hat{\Phi}[z(k)], \\ &= \sum_{l=0}^{\infty} \frac{1}{l!} (-1)^l t^l A^l \hat{\Phi}[z(k)]. \end{aligned} \tag{9.9}$$

Since (9.9) includes infinite-times differentiation, we have to assume that $\hat{\Phi}[z(k)]$ is infinitely differentiable, in order that the expression (9.9) is meaningful. The boundary condition (9.8) is easily proved to be satisfied by $\Phi[z, t]$ given by (9.9).

The operator $\exp(-tA)$ has the following property :

$$\exp\{-(s+t)\} = \exp(-s) \exp(-t). \tag{9.10}$$

The operator which has the property (9.10) is called a semigroup of operator. The perturbation theory for semigroup of operator have been extensively studied⁷⁾. But, only the last expression of (9.9) will be considered here.

10. The Perturbation Method

The coefficients of A^l in the last expression of (9.9) are further expanded into a power series of R^{-1} as

$$A^l = \sum_{m=0}^{\infty} R^{-m} A_m^{(l)}, \tag{10.1}$$

where

$A_m^{(l)}$ = sum of permutations of the term

$$(\mathfrak{D}_{\lambda\mu} k_{\lambda})^{l-m} (\mathfrak{D}_{\nu} k_{\nu}^2)^m, \tag{10.2}$$

for example,

$$\left. \begin{aligned} A_1^{(1)} &= \mathfrak{D}_1 k_1^2, \\ A_1^{(3)} &= \mathfrak{D}_5 k_5^2 \mathfrak{D}_{43} k_4 \mathfrak{D}_{21} k_2 + \mathfrak{D}_{54} k_5 \mathfrak{D}_3 k_3^2 \\ &\quad \mathfrak{D}_{21} k_2 + \mathfrak{D}_{54} k_5 \mathfrak{D}_{32} k_3 \mathfrak{D}_1 k_1^2. \end{aligned} \right\} \quad (10.3)$$

The initial functional (9.7) is also expanded into a power series of square of z :

$$\hat{\Phi}[z(k)] = \sum_{l=0}^{\infty} \hat{\Phi}^{(2l)}, \quad (10.4)$$

where

$$\left. \begin{aligned} \hat{\Phi}^{(0)} &= 1, \\ \hat{\Phi}^{(2)} &= -\frac{1}{2} K_{21} z_2 z_1, \\ \hat{\Phi}^{(4)} &= \frac{1}{2!} \frac{1}{4} K_{43} K_{21} z_4 \dots z_1, \\ \hat{\Phi}^{(6)} &= -\frac{1}{3!} \frac{1}{8} K_{65} K_{43} K_{21} z_6 \dots z_1, \\ &\dots \dots \dots \end{aligned} \right\} \quad (10.5)$$

$$K_{21} = \iint dk_2 dk_1 E(k_2) \delta(k_2 + k_1) \text{ etc.} \quad (10.6)$$

and

$$z_1 = z(k_1) \text{ etc.}$$

The energy spectrum $E(k, t)$ is derived from the relation

$$E(k, t) \delta(k' + k) = -D' D \hat{\Phi}[z(k), t] |_{z=0}, \quad (10.7)$$

where

$$D = \frac{\delta}{\delta z(k)}, \text{ and } D' = \frac{\delta}{\delta z(k')}.$$

Then the temporal development of the energy spectrum is expressed as

$$E(k, t) = \sum_{l=0}^{\infty} \frac{1}{l!} t^l E^{(l)}(k), \quad (10.8)$$

where

$$\left. \begin{aligned} E^{(0)} \delta(k' + k) &= -D' D \hat{\Phi}^{(2)}, \\ E^{(1)} \delta(k' + k) &= -D' D A_1^{(1)} \hat{\Phi}^{(2)}, \\ E^{(2)} \delta(k' + k) &= -D' D (A_0^{(2)} \hat{\Phi}^{(4)} + R^{-1} A_2^{(2)} \hat{\Phi}^{(2)}), \end{aligned} \right\}$$

$$\left. \begin{aligned} E^{(3)}\delta(k'+k) &= -D'D(R^{-1}A_1^{(3)}\hat{\phi}^{(4)} + R^{-3}A_3^{(3)}\hat{\phi}^{(2)}), \\ E^{(4)}\delta(k'+k) &= -D'D(A_0^{(4)}\hat{\phi}^{(6)} + R^{-2}A_2^{(4)}\hat{\phi}^{(4)} + R^{-4}A_4^{(4)}\hat{\phi}^{(2)}), \\ \dots\dots\dots \end{aligned} \right\} \quad (10.9)$$

We have obtained the expressions for the energy spectrum (10.8) with (10.9) in a power series of time. In working out each term of this series, we have to calculate functional derivatives $D' D$ and those included in $A_m^{(l)}$ which operate on the functional $\hat{\phi}^{(2l)}$. From the definition of the functional derivative, we obtain

$$Dz(k_1) = \frac{\delta z(k_1)}{\delta z(k)} = \delta(k - k_1). \quad (10.10)$$

The first term of (10.9) is easily obtained as

$$E^{(0)}(k) = \hat{E}(k). \quad (10.11)$$

In order to work out the second term:

$$\left. \begin{aligned} E^{(1)}(k)\delta(k'+k) &= D'DR^{-1}A_1^{(1)}\hat{\phi}^{(2)}, \\ A_1^{(1)} &= \mathfrak{D}_1 k_1^2, \\ \hat{\phi}^{(2)} &= -\frac{1}{2}K_{21}z_2z_1, \end{aligned} \right\} \quad (10.12)$$

we carry out the operation $A_1^{(1)}$ and $D' D$ as follows:

$$\begin{aligned} A_1^{(1)}\hat{\phi}^{(2)} &= \mathfrak{D}^3 k^3 \left(-\frac{1}{2}K_{21}z_2z_1 \right) \\ &= -\frac{1}{2} \iiint dk_3 dk_2 dk_1 z_3 k_3^2 D_3 \hat{E}(k_2) \delta(k_2 + k_1) z_2 z_1 \\ &= -\iint dk_2 dk_1 k_2^2 \hat{E}(k_2) \delta(k_2 + k_1) z_2 z_1, \end{aligned} \quad (10.13)$$

$$\begin{aligned} D'DA_1^{(1)}\hat{\phi}^{(2)} &= -\iint dk_2 dk_1 k_2^2 \hat{E}(k_2) \delta(k_2 + k_1) D'D(z_2 z_1) \\ &= -2\hat{E}(k)k^2 \delta(k' + k). \end{aligned} \quad (10.14)$$

Substituting (10.14) to (10.12), we have

$$E^{(1)}(k) = -2R^{-1}\hat{E}(k)k^2. \quad (10.15)$$

In the similar manner we can express higher order terms as functions of $\hat{E}(k)$, and we obtain the energy spectrum and the total energy density $TE(t)$ as

$$\begin{aligned} E(k, t) &= \hat{E} - (2R^{-1}t)k^2\hat{E} + \frac{1}{2!}t^2k^2\{-2\hat{E} + \hat{E} * \hat{E} + (2R^{-1})^2k^2\hat{E}\} \\ &\quad + \frac{1}{3!}(2R^{-1})t^3k^2[6(k^2 + (k^2, \hat{E}))\hat{E} - \frac{3}{2}\{k^2(\hat{E} * \hat{E}) + 2(k^2\hat{E}) * \hat{E}\}] \end{aligned}$$

$$\begin{aligned}
& -(2R^{-1})^2 k^4 \hat{E}] + \frac{1}{4!} t^4 k^2 [4k^2 \{3\hat{E} - 3(\hat{E} * \hat{E}) + \hat{E} * (\hat{E} * \hat{E})\} \\
& -(2R^{-1})^2 \{2(6k^4 + 13(k^2, \hat{E})k^2 + 6(k^4, \hat{E}))\hat{E} - 12k^2(k^2\hat{E}) * \hat{E} \\
& - 7(k^2\hat{E}) * (k^2\hat{E})\} + (2R^{-1})^4 k^6 \hat{E}] + \dots, \tag{10.16}
\end{aligned}$$

$$\begin{aligned}
TE(t) &= \int_{-\infty}^{\infty} E(k, t) dk \\
&= 1 - (2R^{-1})(k^2, \hat{E})t + \frac{1}{2!} t^2 \{-2(k^2, \hat{E}) + (k^2, \hat{E} * \hat{E}) \\
&+ (2R^{-1})^2 (k^4, \hat{E})\} + \frac{1}{3!} (2R^{-1}) t^3 [6(k^4, \hat{E}) + 6(k^2, \hat{E})^2 \\
&- \frac{3}{2} (k^4, \hat{E} * \hat{E}) - 3(k^2, (k^2\hat{E}) * \hat{E}) - (2R^{-1})^2 (k^6, \hat{E})] \\
&+ \frac{1}{4!} t^4 [12(k^4, \hat{E}) - 12(k^4, \hat{E} * \hat{E}) + 4(k^4, \hat{E} * (\hat{E} * \hat{E})) \\
&- (2R^{-1})^2 \{12(k^6, \hat{E}) + 38(k^2, \hat{E})(k^4, \hat{E}) - 12(k^4, (k^2\hat{E}) * \hat{E}) \\
&- 7(k^2, (k^2\hat{E}) * (k^2\hat{E}))\} + (2R^{-1})(k^8, \hat{E})] + \dots, \tag{10.17}
\end{aligned}$$

where

$$\begin{aligned}
(k^2\hat{E}) * \hat{E} &= \int_{-\infty}^{\infty} k'^2 \hat{E}(k') \hat{E}(k - k') dk', \\
(k^2, \hat{E}) &= \int_{-\infty}^{\infty} k^2 \hat{E}(k) dk,
\end{aligned}$$

and similar abbreviations have been used. The first is the convolution of $k^2\hat{E}(k)$ and $E(k)$, and the second the scalar product of k^2 and $\hat{E}(k)$.

11. Some Examples

We consider typical three examples, in which the initial energy spectra have the form of the δ -function, the bell shape and the double-bell shape. The last two cases have also been studied in the previous chapter.

(1) The δ -function type spectrum

$$\hat{E}(k) = \frac{1}{2} \{\delta(k-1) + \delta(k+1)\}, \tag{11.1}$$

$$\begin{aligned}
E(k, t) &= \hat{E}(k) - (2R^{-1}t)\hat{E}(k) + \frac{1}{2!} t^2 [-2\hat{E}(k) + \hat{E}(k/2) \\
&+ (2R^{-1})^2 \hat{E}(k)] + \frac{1}{3!} (2R^{-1}) t^3 [12\hat{E}(k) - 9\hat{E}(k/2)
\end{aligned}$$

$$\begin{aligned}
 & - (2R^{-1})^2 \hat{E}(k) \square + \frac{1}{4!} t^4 [15\hat{E}(k) - 48\hat{E}(k/2) + 27\hat{E}(k/3) \\
 & - (2R^{-1})^2 5\{10\hat{E}(k) - 11\hat{E}(k/2)\} + (2R^{-1})^4 \hat{E}(k) \square] \\
 & + \dots, \tag{11.2}
 \end{aligned}$$

$$\begin{aligned}
 TE(t) = & 1 - 2R^{-1}t + \frac{1}{2!} (2R^{-1}t)^2 - \frac{1}{3!} 2R^{-1}t^3 \{6 + (2R^{-1})^2\} \\
 & + \frac{1}{4!} (2R^{-1})^2 t^4 \{60 + (2R^{-1})^2\} + \dots \tag{11.3}
 \end{aligned}$$

In (11.2), the second harmonics $\hat{E}(k/2)$ comes from convolution in (10.15), i. e. $\hat{E} * \hat{E}$, $(k * \hat{E}) * \hat{E}$ and the third harmonics $\hat{E}(k/3)$ from the triple convolution $\hat{E} * (\hat{E} * \hat{E})$.

In following two cases, we show only the initial spectrum, the expressions for convolution and the values of scalar product which appear in (10.16) and (10.17).

(2) The bell shape spectrum

$$\hat{E}(k) = \frac{1}{\sqrt{\pi}} \exp(-k^2), \tag{11.4}$$

$$\left. \begin{aligned}
 \hat{E} * \hat{E} &= \frac{1}{\sqrt{2}} \hat{E}(k/\sqrt{2}) \\
 (k^2 \hat{E}) * \hat{E} &= \frac{1}{4\sqrt{2}} (k^2 + 1) \hat{E}(k/\sqrt{2}), \\
 (k^2 \hat{E}) * (k^2 \hat{E}) &= \frac{1}{16\sqrt{2}} (k^4 - 2k^2 + 3) \hat{E}(k/\sqrt{2}), \\
 \hat{E} * (\hat{E} * \hat{E}) &= \frac{1}{\sqrt{3}} \hat{E}(k/\sqrt{3}),
 \end{aligned} \right\} \tag{11.5}$$

$$\left. \begin{aligned}
 (k^{2l}, \hat{E}) &= (2l-1)!! / 2^l, \quad (l=1, 2, 3, 4) \\
 (k^2, \hat{E} * \hat{E}) &= 1, \quad (k^4, \hat{E} * \hat{E}) = 3, \\
 (k^2, (k^2 \hat{E}) * \hat{E}) &= 1, \quad (k^4, (k^2 \hat{E}) * \hat{E}) = 9/2, \\
 (k^2, (k^2 \hat{E}) * (k^2 \hat{E})) &= 3/4, \quad (k^4, \hat{E} * (\hat{E} * \hat{E})) = 27/4,
 \end{aligned} \right\} \tag{11.6}$$

where $(2l-1)!! = (2l-1)(2l-3) \dots 3.1$ is the double factorial.

(3) The double-bell shape spectrum

$$\hat{E}(k) = \frac{2}{\sqrt{\pi}} k^2 \exp(-k^2), \tag{11.7}$$

$$\left. \begin{aligned}
 \hat{E} * \hat{E} &= \frac{1}{4\sqrt{2}}(k^4 - 2k^2 + 3)k^{-2}\hat{E}(k/\sqrt{2}), \\
 (k^2\hat{E}) * \hat{E} &= \frac{1}{16\sqrt{2}}(k^6 - k^4 - 3k^2 + 15)k^{-2}\hat{E}(k/\sqrt{2}), \\
 (k^2\hat{E}) * (k^2\hat{E}) &= \frac{1}{64\sqrt{2}}(k^8 - 4k^6 + 18k^4 - 60k^2 + 105) \\
 &\quad k^{-2}\hat{E}(k/\sqrt{2}), \\
 \hat{E} * (\hat{E} * \hat{E}) &= \frac{1}{243\sqrt{3}}(2k^6 - 18k^4 + 81k^2 + 81) \\
 &\quad k^{-2}\hat{E}(k/\sqrt{3}),
 \end{aligned} \right\} (11.8)$$

$$\left. \begin{aligned}
 (k^{2l}, \hat{E}) &= (2l+1)!!/2^l, \quad (l=1, 2, \dots) \\
 (k^2, \hat{E} * \hat{E}) &= 3, & (k^4, \hat{E} * \hat{E}) &= 21, \\
 (k^2, (k^2\hat{E}) * \hat{E}) &= 6, & (k^4, (k^2\hat{E}) * \hat{E}) &= 105/2, \\
 (k^2, (k^2\hat{E}) * (k^2\hat{E})) &= 45/4, & (k^4, \hat{E} * (\hat{E} * \hat{E})) &= 207/4.
 \end{aligned} \right\} (11.9)$$

The temporal developments of energy spectrum subject to three kinds of the given initial energy spectrum, i. e. the δ -function type (Fig. 3), the bell shape (Fig. 4), and the double-bell shape (Fig. 5), are calculated (a) for $R=0, 0.01, 0.1$, and (b) $R=1, 10, 100$ and changes of the total energy density are also shown in these diagrams.

In summarizing the results, we have

i) Our theory seems to give reasonable results in $0 \leq s/R \leq 0.8$ for $R \leq 0.1$ and in $0 \leq t \leq 0.4$ for $R \geq 1$, when the expansion is taken up to t^4 .

ii) The temporal developments of the energy spectrum are very similar both for R smaller than 0.1 and for R larger than 10, when the change are measured in the time scale s/R and t , respectively.

iii) The paradox of negative values of energy spectrum occurs in the range of large wavenumbers after some time from the start for the initial energy spectra of the bell shape and double-bell shape. The occurrence of the paradox is much complicated for the initial energy spectrum of the δ -function type.

iv) The constancy of energy spectrum for small wavenumber is very satisfactory.

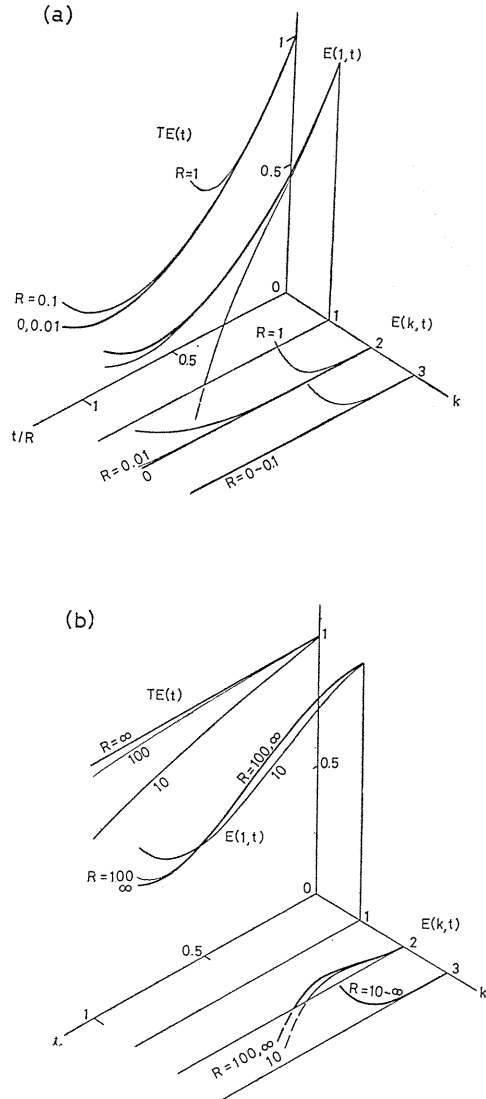
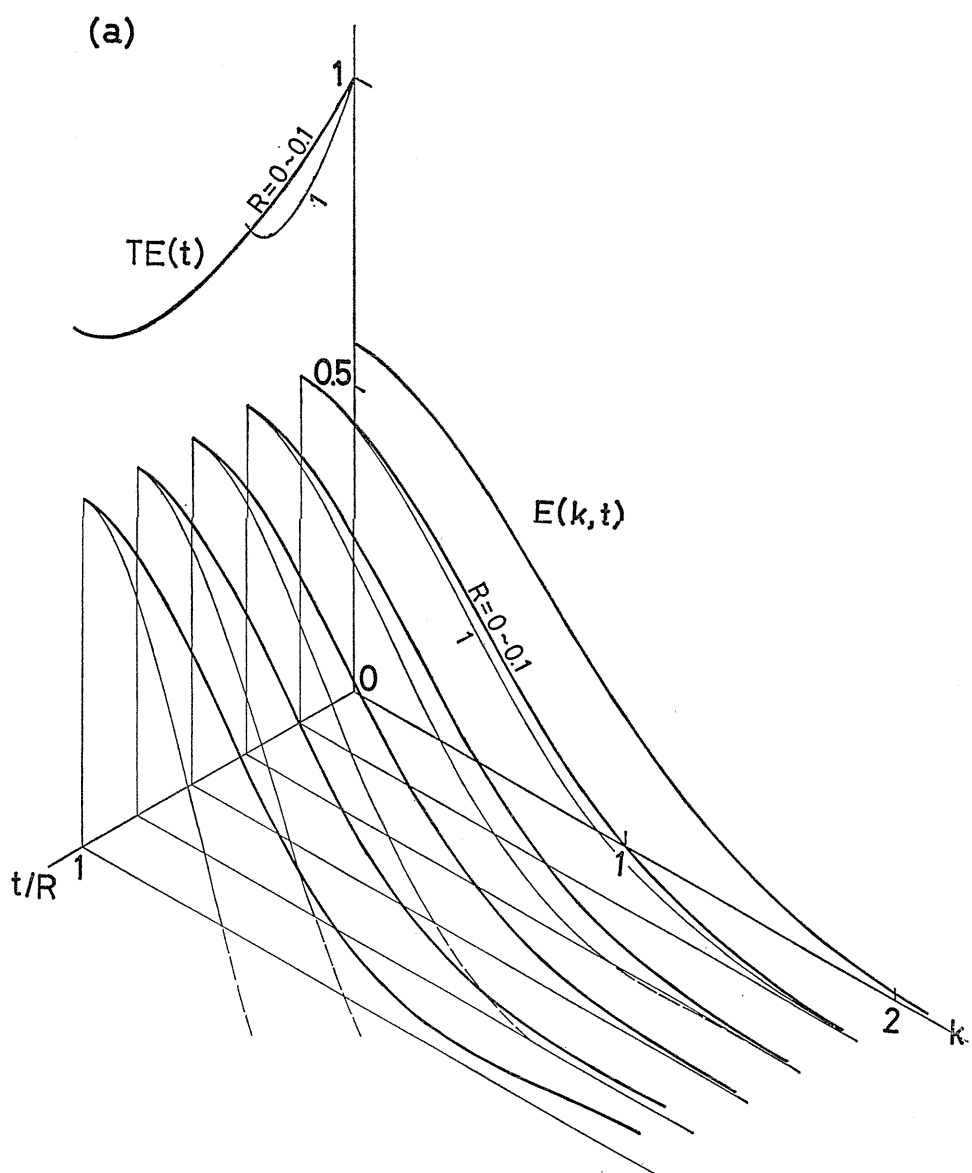


Fig. 3. Temporal developments of energy spectrum $E(k, t)$ and total energy density $TE(t)$ from the initial spectrum of δ -function type, (a) for $R=0, 0.01, 0, 1$ and 1, and (b) for $R=10, 100$ and ∞ , calculated by the second method in III. The energy spectrum is not original one, which is singular at $k=1, 2, \dots$

and vanishes otherwise, but $2 \int_{n-a}^{n+a} E(k, t) dk$ ($0 < a < 1$, and n is an integer).



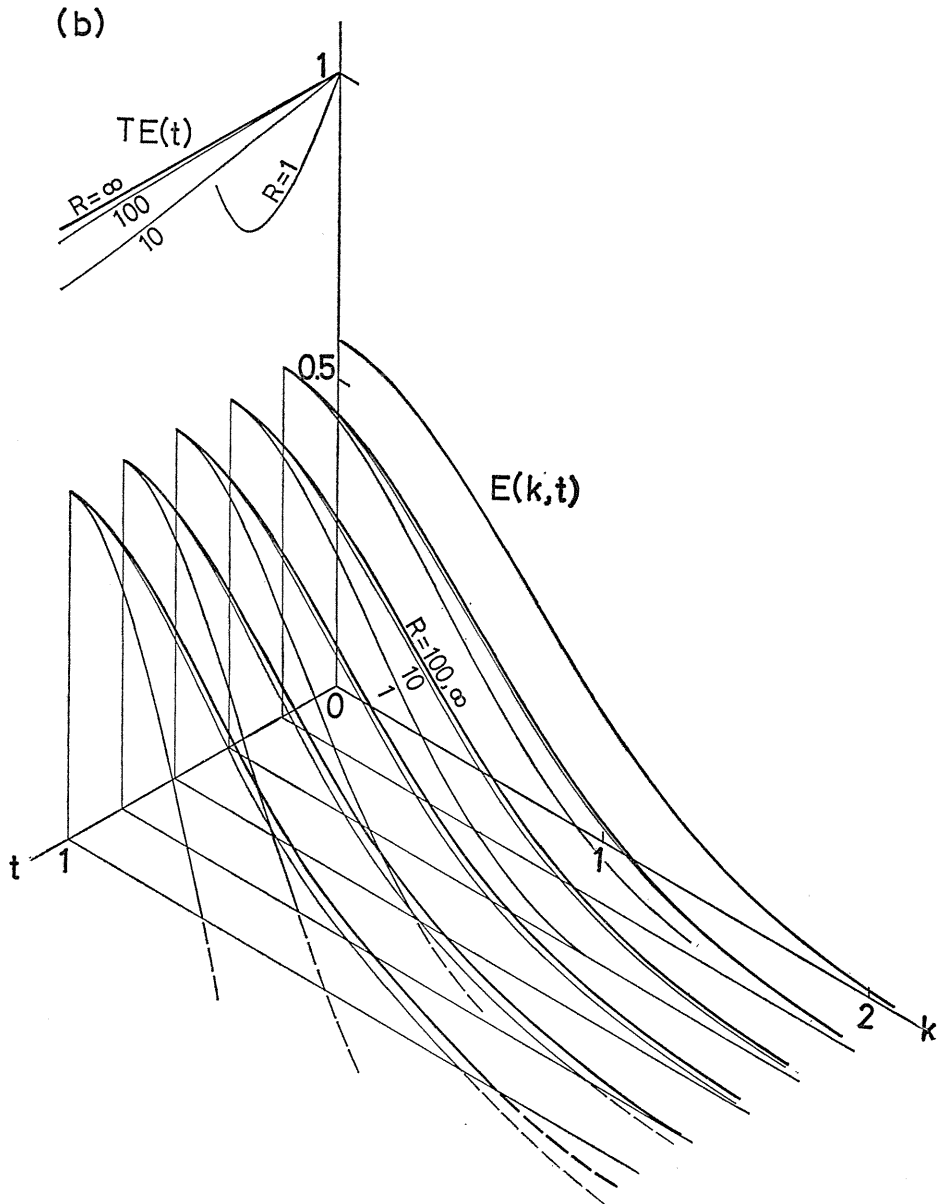
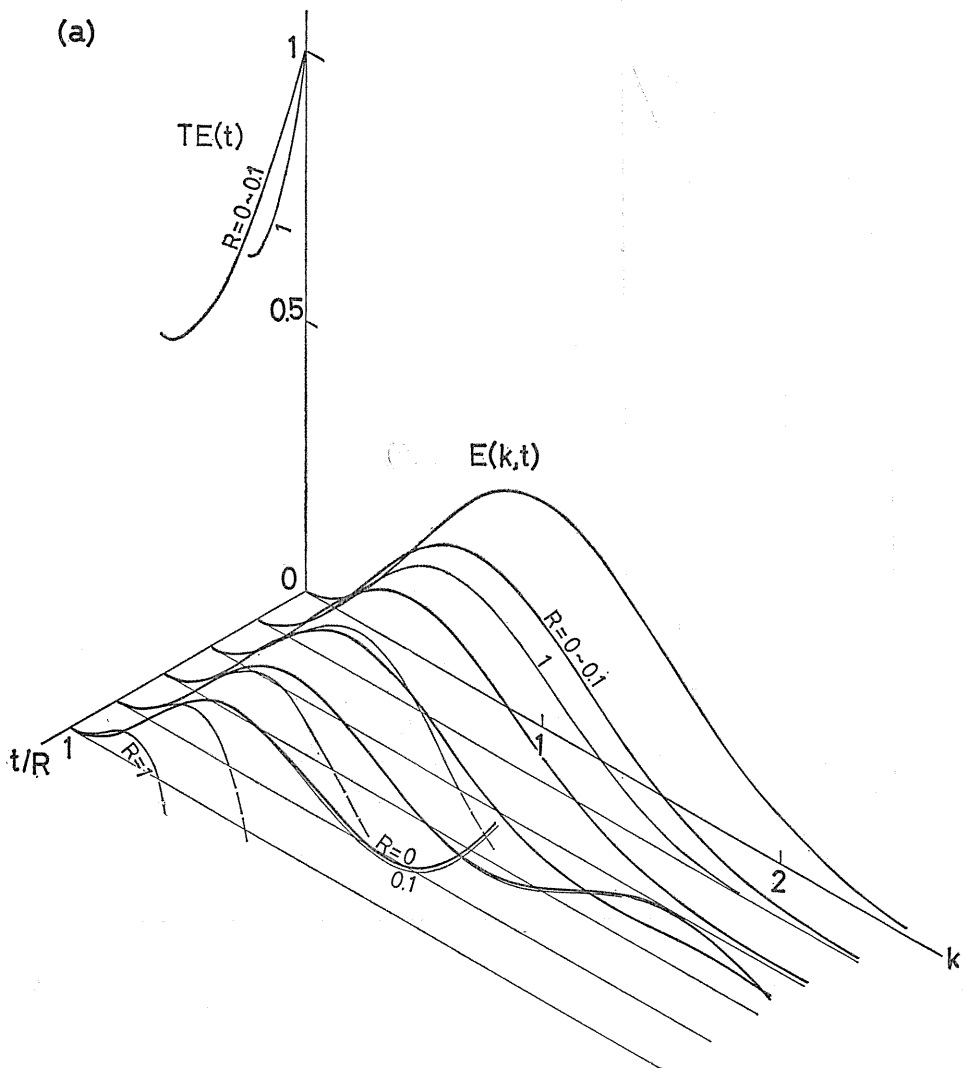


Fig. 4. Temporal developments of energy spectrum and total energy density from the initial bell shape spectrum (a) for $R=0, 0.01, 0.1$ and 1 , and (b) for $R=1, 10, 100$ and ∞ , calculated the second method in III.



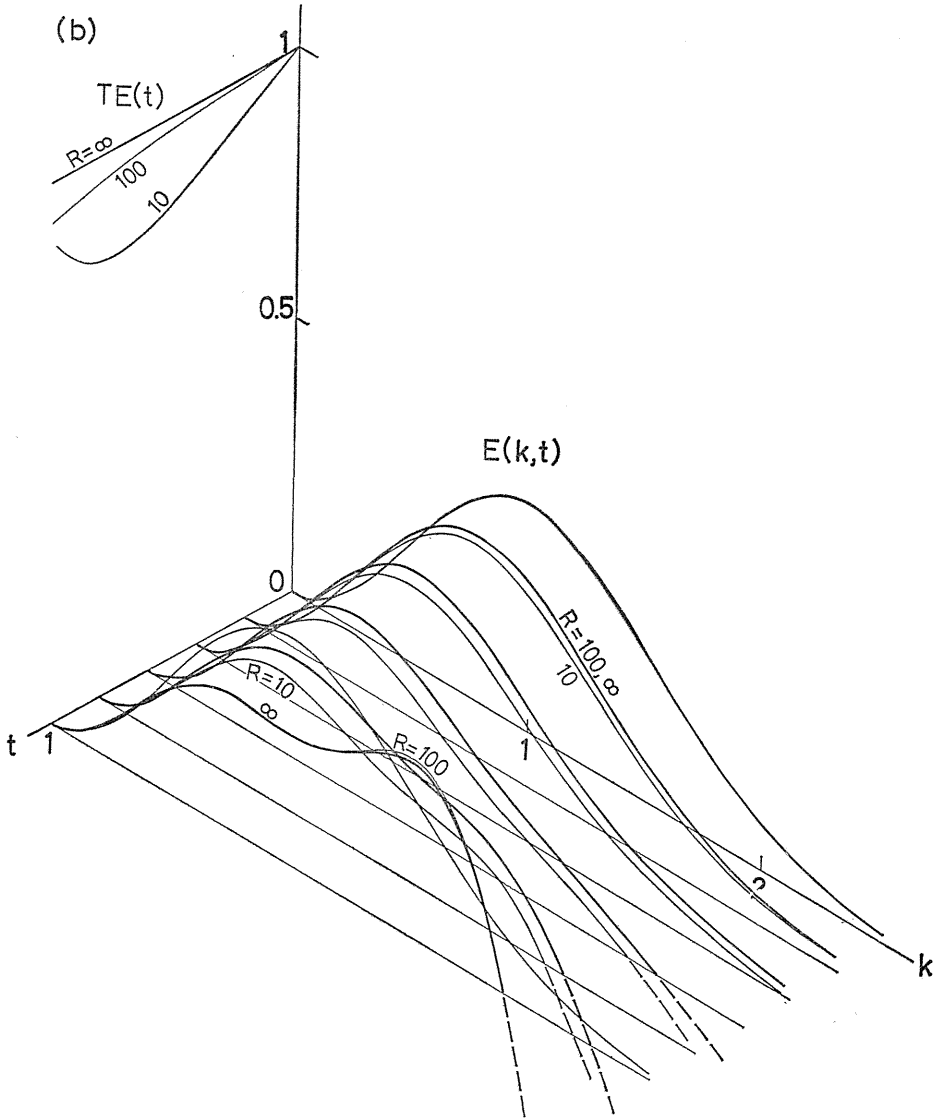


Fig. 5. Temporal developments of energy spectrum and total energy density from the initial double-bell spectrum (a) for $R=0, 0.01, 0.1$ and 1 , and (b) for $R=10, 100$ and ∞ , calculated by the second method in III.

IV. The Interaction Representation of the Hopf Equation and the Perturbation Method Based on this Representation

12. The Interaction Representation for the Hopf Equation

Our formulation of the initial value problem of the Hopf equation for the Burgers turbulence is put as

$$-\frac{\partial \Phi}{\partial t} + A\Phi = 0, \quad (12.1)$$

$$A = - \iint dk_2 dk_1 k_2 z(k_2 - k_1) \frac{\delta^2 \Phi}{\delta z(k_2) \delta z(k_1)} + \bar{\nu} \int dk_1 k_1^2 z(k_1) \frac{\delta \Phi}{\delta z(k_1)}, \quad (12.2)$$

the initial condition is

$$\Phi[z(k), 0] = \hat{\Phi}[z(k)], \quad (12.3)$$

and the boundary condition is

$$\Phi[z(k), t] = 0 \quad \text{for } |z| \longrightarrow \infty, \quad t \geq 0. \quad (12.4)$$

We choose the initial value of Φ as

$$\hat{\Phi}[z(k)] = \exp\left\{-\frac{1}{2} K_{21} z(k_2) z(k_1)\right\}, \quad (12.5)$$

$$K_{21} = \iint dk_2 dk_1 \hat{E}(k_2) \delta(k_2 + k_1), \quad (12.6)$$

where $\hat{E}(k)$ is the initial energy spectrum and $\bar{\nu} = 1/R$. (12.5) with (12.6) satisfies the imposed condition on Φ :

$$\left. \begin{aligned} \Phi[0] &= 1, \quad \Phi[z] \leq 1, \\ \Phi[z] &= \Phi[-z]^* \end{aligned} \right\} \quad (12.7)$$

and $\Phi[z \exp(iaz)] = \Phi[z]$ for any a .

Here we express the Hopf equation in the interaction representation¹⁶⁾, in which the independent variables $z(k)$ and t are transformed into new independent variables $\zeta(k)$ and s ,

$$\left. \begin{aligned} \zeta(k) &= z(k) \exp(\bar{\nu} k^2 t), \\ s &= t. \end{aligned} \right\} \quad (12.8)$$

We use the time s for t , for keeping away from the confusion of the differentiations, i. e. $\partial/\partial t = \partial/\partial t|_{z=\text{const.}}$ and $\partial/\partial s = \partial/\partial t|_{\zeta=\text{const.}}$. By use of the transformation, we have

$$\left. \begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial}{\partial s} - \int dk k_2 \zeta D, \\ \frac{\delta}{\delta z(k)} &= \exp(-\bar{\nu} k^2 s) D, \end{aligned} \right\} \quad (12.9)$$

where

$$D = \frac{\delta}{\delta \zeta(k)}. \quad (12.10)$$

Substituting (12.9) into (12.1), we have

$$\frac{\partial \Phi}{\partial s} + B\Phi = 0, \quad (12.11)$$

$$B = - \iint dk_2 dk_1 k_1 \zeta(k_2 + k_1) \exp(2\bar{\nu} k_2 k_1 s) D_2 D_1, \quad (12.12)$$

where

$$D_l = \frac{\delta}{\delta \zeta(k_l)} \quad l = 1, 2. \quad (12.13)$$

This is the Hopf equation in the interaction representation. From (12.13), we see that the viscous term disappears formally, and B includes the nonlinear interaction term only. B depends on the time in this representation.

The initial condition is

$$\Phi[\zeta(k), 0] = \hat{\Phi}[\zeta(k)], \quad (12.15)$$

and the boundary condition is

$$\Phi[\zeta(k), t] = 0 \quad \text{for } |\zeta(k)| \rightarrow \infty, \quad \text{and } t \geq 0. \quad (12.16)$$

We assume the following form of $\hat{\Phi}$,

$$\left. \begin{aligned} \hat{\Phi}[\zeta] &= \exp\left(-\frac{1}{2} K_{21} \zeta_2 \zeta_1\right), \\ K_{21} &= \iint dk_2 dk_1 \hat{E}(k_2) \delta(k_2 + k_1). \end{aligned} \right\} \quad (12.17)$$

The energy spectrum can be obtained by

$$\begin{aligned} E(k, t) \delta(k' + k) &= - \left. \frac{\delta^2 \Phi}{\delta z(k') \delta z(k)} \right|_{z=0} \\ &= - \left. \frac{\delta^2 \Phi}{\delta \zeta(k') \delta \zeta(k)} \right|_{\zeta=0}. \end{aligned} \quad (12.18)$$

The total energy density is obtained by

$$TE(t) = \int E(k, t) dk. \quad (12.19)$$

13. The Perturbation Method Based on the Interaction Representation

We assume that Φ can be expanded into a power series of time s :

$$\Phi[\zeta(k), s] = \sum_{l=0}^{\infty} \frac{1}{l!} s^l \Phi^{(l)}. \tag{13.1}$$

Substituting (13.1) into (12.11) with (12.13), we have the recurrence formulas for $\Phi^{(m)}$;

$$\left. \begin{aligned} \Phi^{(0)} &= \hat{\Phi}, \\ \Phi^{(m+1)} &= \sum_{l=0}^{\infty} \frac{m!}{l!(m-l)!} (2\bar{\nu})^{m-l} \mathfrak{D}_{21} k_2^{m-l+1} \\ &\quad k_1^{m-l} \Phi^{(l)} \quad m=0, 1, 2, \dots \\ \mathfrak{D}_{21} &= \iiint dk_2 dk_1 \zeta(k_2 + k_1) D_2 D_1. \end{aligned} \right\} \tag{13.2}$$

$\Phi[\zeta]$ can be expanded into a power series of square of ζ :

$$\Phi[\zeta] = \sum_{l=0}^{\infty} \hat{\Phi}^{(2l)}, \tag{13.3}$$

where

$$\left. \begin{aligned} \hat{\Phi}^{(0)} &= 1, \\ \hat{\Phi}^{(2)} &= -\frac{1}{2} K_{21} \zeta_2 \zeta_1, \\ \hat{\Phi}^{(4)} &= \frac{1}{8} K_{43} K_{21} \zeta_4 \dots \zeta_1, \\ \hat{\Phi}^{(6)} &= -\frac{1}{48} K_{65} K_{43} K_{21} \zeta_6 \dots \zeta_1, \\ \zeta_l &= \zeta(k_l), \quad l=1, 2. \end{aligned} \right\} \tag{13.4}$$

In order to get $\Phi^{(l)}$ successively, we should make the operation \mathfrak{D}_{21} in (13.2) to a polynomial of ζ with the k -dependent coefficient like (13.3). We should make also the functional differentiation to get the energy spectrum by (12.18). These operation can be done by use of

$$D' \zeta = \frac{\delta \zeta(k)}{\delta \zeta(k')} = \delta(k - k') \text{ etc.} \tag{13.5}$$

The total energy density of turbulence can be obtained by (12.19).

Thus we have the expressions for the energy spectrum and the total energy density as

$$E(k, t) = \exp(-2\bar{\nu}k^2t) \left[\hat{E} + \frac{1}{2!} t^2 k^2 (-2\hat{E} + \hat{E} * \hat{E}) + \frac{1}{3!} 3(2\bar{\nu})t^3 k^2 \right]$$

$$\begin{aligned} & \{2(k^2, \hat{E})\hat{E} + \frac{1}{2}k^2(\hat{E} * \hat{E}) - (k^2\hat{E}) * \hat{E}\} + \frac{1}{4!}t^4k^2\{4k^2(3\hat{E} \\ & - 3(\hat{E} * \hat{E}) + \hat{E} * (\hat{E} * \hat{E})) - (2\nu)^2(2\hat{E}(k^2, \hat{E})k^2 + 6(k^4, \hat{E})) \\ & - 7(k^2\hat{E}) * (k^2\hat{E})\} + \dots \}, \end{aligned} \tag{13.6}$$

$$\begin{aligned} TE(t) = & (1, \hat{E} \exp(-2\nu k^2 t)) + \frac{1}{2!}t^2\{-2(k^2, \hat{E} \exp(-2\nu k^2 t)) \\ & + (k^2, \hat{E} * E \exp(-2\nu k^2 t))\} + \frac{1}{3!}t^33(2\nu)\{2(k^2, \hat{E}) \\ & (k^2, \hat{E} \exp(-2\nu k^2 t)) + \frac{1}{2}(k^4, \hat{E} * \hat{E} \exp(-2\nu k^2 t)) - (k^2, (k^2\hat{E}) \\ & * \hat{E} \exp(-2\nu k^2 t))\} + \frac{1}{4!}t^4[12(k^4, \hat{E} \exp(-2\nu k^2 t)) - 12(k^4, \\ & \hat{E} * \hat{E} \exp(-2\nu k^2 t)) + 4(k^4, \hat{E} * (\hat{E} * \hat{E}))\exp(-2\nu k^2 t) - (2\nu)^2 \\ & \{2(k^2, \hat{E})(k^4, \hat{E} \exp(-2\nu k^2 t))12(k^4, \hat{E})(k^2, \hat{E} \exp(-2\nu k^2 t)) \\ & - 77(k^2, (k^2\hat{E}) * (k^2\hat{E}))\exp(-2\nu k^2 t)\}] + \dots \dots \end{aligned} \tag{13.7}$$

In (13.6) and (13.7), we have changed s for t , because the meaning of time would be clearly expressed, and the confusion due to the differentiation does not occur in this stage.

If we expand the exponential factor $\exp(-2\nu k^2 t)$ in (13.6) and (13.7) into a power series of time for $E(k, t)$ and $TE(t)$. These expressions agree with (10.15) and (10.16).

Relation between this theory and Kawahara's should be mentioned. Kawahara dealt with the same problem, i. e. the initial value problem of the Burgers turbulence based on the Hopf equation, using the expansion of the logarithm of the characteristic functional, which, if truncated at a finite state, is equivalent to the cumulant discarding approximation. His expression for the energy spectrum is also of the form of $\exp(-2\nu k^2 t) F(k, t)$ like ours. We can expand $F(k, t)$ into a power series of t . Comparing his series with ours, we find that they are in agreement up to t^2 but the higher order terms are different. This may be because the initial conditions are different; although all initial energy transfer functions (of 3rd, 4th, ...) are put to vanish in his theory, those of odd order vanish but those of even order do not vanish in our theory. In our theory, the initial transfer functions $F^{(l)}$ of the l -th order can be obtained as

$$\begin{aligned} & F^{(l)}(k_1, \dots, k_l)\delta(k_1 + \dots + k_l) \\ & = \langle u(k_1) \dots u(k_l) \rangle \\ & = i^{-l} D_1 \dots D_l \Phi[\zeta(k)]|_{\zeta=0}. \end{aligned} \tag{13.8}$$

We have given only the initial energy spectrum, but have not given the initial energy transfer functions higher than 3rd explicitly. But the form (5.7) of the initial characteristic functional results in prescribing automatically the initial energy transfer functions.

14. Some Examples

as in 11, we consider three examples, in which the initial energy spectra are of the δ -function type, of the bell shape and of the double-bell shape.

(1) The δ -function type spectrum

$$\hat{E}(k) = \frac{1}{2} \{ \delta(k-1) + \delta(k+1) \}, \quad (14.1)$$

$$\begin{aligned} E(k, t) = & \hat{E}(k) \exp(-2\bar{\nu}t) + \frac{1}{2!} t^2 \{ -2\hat{E}(k) \exp(-2\bar{\nu}t) \\ & + \hat{E}(k/2) \exp(-8\bar{\nu}t) \} + \frac{1}{3!} 3(2\bar{\nu}) t^3 \{ 2\hat{E}(k) \exp(-2\bar{\nu}t) \\ & + \hat{E}(k/2) \exp(-8\bar{\nu}t) \} + \frac{1}{4!} t^4 \{ 15\hat{E}(k) \exp(-2\bar{\nu}t) \\ & - 48\hat{E}(k/2) \exp(-8\bar{\nu}t) + 27\hat{E}(k/3) \exp(-18\bar{\nu}t) \\ & - 7(2\bar{\nu}) (2\hat{E}(k) \exp(-2\bar{\nu}t) - \hat{E}(k/2) \exp(-8\bar{\nu}t)) \} \\ & + \dots, \end{aligned} \quad (14.2)$$

$$\begin{aligned} TE(t) = & [1 - t^2 + 2\bar{\nu}t^3 + \frac{1}{4!} t^4 (15 - 14(2\bar{\nu})^2) + \dots] \\ & \exp(-2\bar{\nu}t) + [t^2 + (2\bar{\nu})t^3 + \frac{1}{4!} 2t^4 (-48 + 7(2\bar{\nu})^2) \\ & + \dots] \exp(-8\bar{\nu}t) + ((81/4!)t^4 + \dots) \exp(-18\bar{\nu}t) \\ & + \dots \end{aligned} \quad (14.3)$$

In following two cases, we show only the initial spectrum, and the functional forms of scalar product included in (13.6) and (13.7). The expressions for the convolution and the values of scalar product are also necessary for obtaining the spectrum and the total energy density, but these are given in (11.5)~(11.9) and we do not write these here.

(2) The bell shape spectrum

$$\hat{E}(k) = \frac{1}{\sqrt{\pi}} \exp(-k^2), \quad (14.4)$$

$$\left. \begin{aligned}
 (1, \hat{E} \exp(-2\bar{\nu}k^2t)) &= (1+2\bar{\nu}t)^{-1/2}, \\
 (k^2, \hat{E} \exp(-2\bar{\nu}k^2t)) &= \frac{1}{2}(1+2\bar{\nu}t)^{-3/2}, \\
 (k^4, \hat{E} \exp(-2\bar{\nu}k^2t)) &= \frac{3}{4}(1+2\bar{\nu}t)^{-5/2}, \\
 (k^2, \hat{E} * \hat{E} \exp(-2\bar{\nu}k^2t)) &= (1+4\bar{\nu}t)^{-3/2}, \\
 (k^4, \hat{E} * \hat{E} \exp(-2\bar{\nu}k^2t)) &= 3(1+4\bar{\nu}t)^{-5/2}, \\
 (k^2, (k^2\hat{E}) * \hat{E} \exp(-2\bar{\nu}k^2t)) &= (1+\bar{\nu}t)(1+4\bar{\nu}t)^{-5/2}, \\
 (k^2, (k^2\hat{E}) * (k^2\hat{E}) \exp(-2\bar{\nu}k^2t)) &= \frac{3}{4}(1+(2\bar{\nu}t)^2) \\
 &\quad (1+4\bar{\nu}t)^{-7/2}, \\
 (k^4, \hat{E} * (\hat{E} * \hat{E}) \exp(-2\bar{\nu}k^2t)) &= \frac{27}{4}(1+6\bar{\nu}t)^{-5/2}.
 \end{aligned} \right\} \quad (14.5)$$

(3) The double-bell shape spectrum

$$\hat{E}(k) = \frac{2}{\sqrt{\pi}} k^2 \exp(-k^2), \quad (14.6)$$

$$\left. \begin{aligned}
 (1, \hat{E} \exp(-2\bar{\nu}k^2t)) &= (1+2\bar{\nu}t)^{-3/2}, \\
 (k^2, \hat{E} \exp(-2\bar{\nu}k^2t)) &= \frac{3}{4}(1+2\bar{\nu}t)^{-5/2}, \\
 (k^4, \hat{E} \exp(-2\bar{\nu}k^2t)) &= \frac{15}{4}(1+2\bar{\nu}t)^{-7/2}, \\
 (k^2, \hat{E} * \hat{E} \exp(-2\bar{\nu}k^2t)) &= 3(1+(2\bar{\nu}t)^2)(1+4\bar{\nu}t)^{-7/2}, \\
 (k^4, \hat{E} * \hat{E} \exp(-2\bar{\nu}k^2t)) &= 3\{7-2(2\bar{\nu}t)+3(2\bar{\nu}t)^2\} \\
 &\quad (1+4\bar{\nu}t)^{-7/2}, \\
 (k^2, (k^2\hat{E}) * \hat{E} \exp(-2\bar{\nu}k^2t)) &= \frac{3}{2}\{4+2\bar{\nu}t+6(2\bar{\nu}t)^2 \\
 &\quad +5(2\bar{\nu}t)^3\}(1+4\bar{\nu}t)^{-9/2}, \\
 (k^2, (k^2\hat{E}) * (k^2\hat{E}) \exp(-2\bar{\nu}k^2t)) &= \frac{15}{4}\{3+6(2\bar{\nu}t)^2 \\
 &\quad -8(2\bar{\nu}t)^3-5(2\bar{\nu}t)^4\}(1+4\bar{\nu}t)^{-11/2}, \\
 (k^4, \hat{E} * (\hat{E} * \hat{E}) \exp(-2\bar{\nu}k^2t)) &= \frac{9}{4}\{23+2(2\bar{\nu}t) \\
 &\quad +126(2\bar{\nu}t)^2+36(2\bar{\nu}t)^3\}(1+6\bar{\nu}t)^{-11/2}.
 \end{aligned} \right\} \quad (14.7)$$

The developments of the energy spectrum and the total energy density subject to given initial spectrum, i. e. the δ -function type (Fig. 6), the bell shape (Fig. 7) and the double-bell shape (Fig. 8) are calculated for $R=0, 0.01, 0.1, 1, 10, 100$ and ∞ as in 8 and 11.

We summarize the results:

i) Our theory seems to give reasonable results in $0 \leq t/R \leq 1$ for $R \leq 1$, and in $0 \leq t \leq 0.5$ for $R \leq 10$ when the expansion is taken up to t .

ii) The temporal developments of the energy spectrum and the total energy density is almost similar for $R \leq 0.1$ and in the time interval $0 \leq t/R \leq 1$, for all types of the initial spectrum. The similarity of those for larger R is not so satisfactory.

iii) The paradox of the negative energy spectrum seems not to appear for $R \leq 1$ for all types of the initial spectrum. But it appears for $t \leq 1, R \geq 10$ and all types of initial spectrum.

iv) The constancy of the energy spectrum for small wavenumber is very satisfactory.

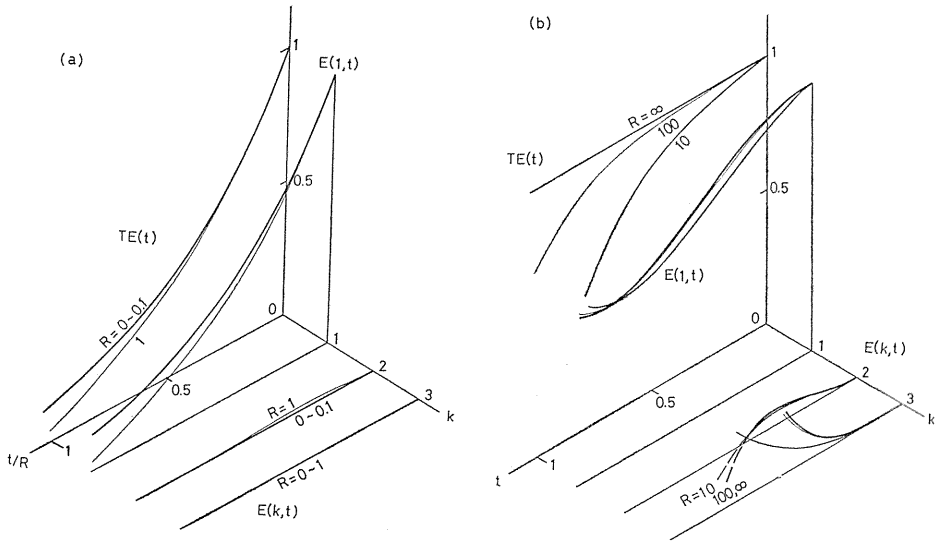
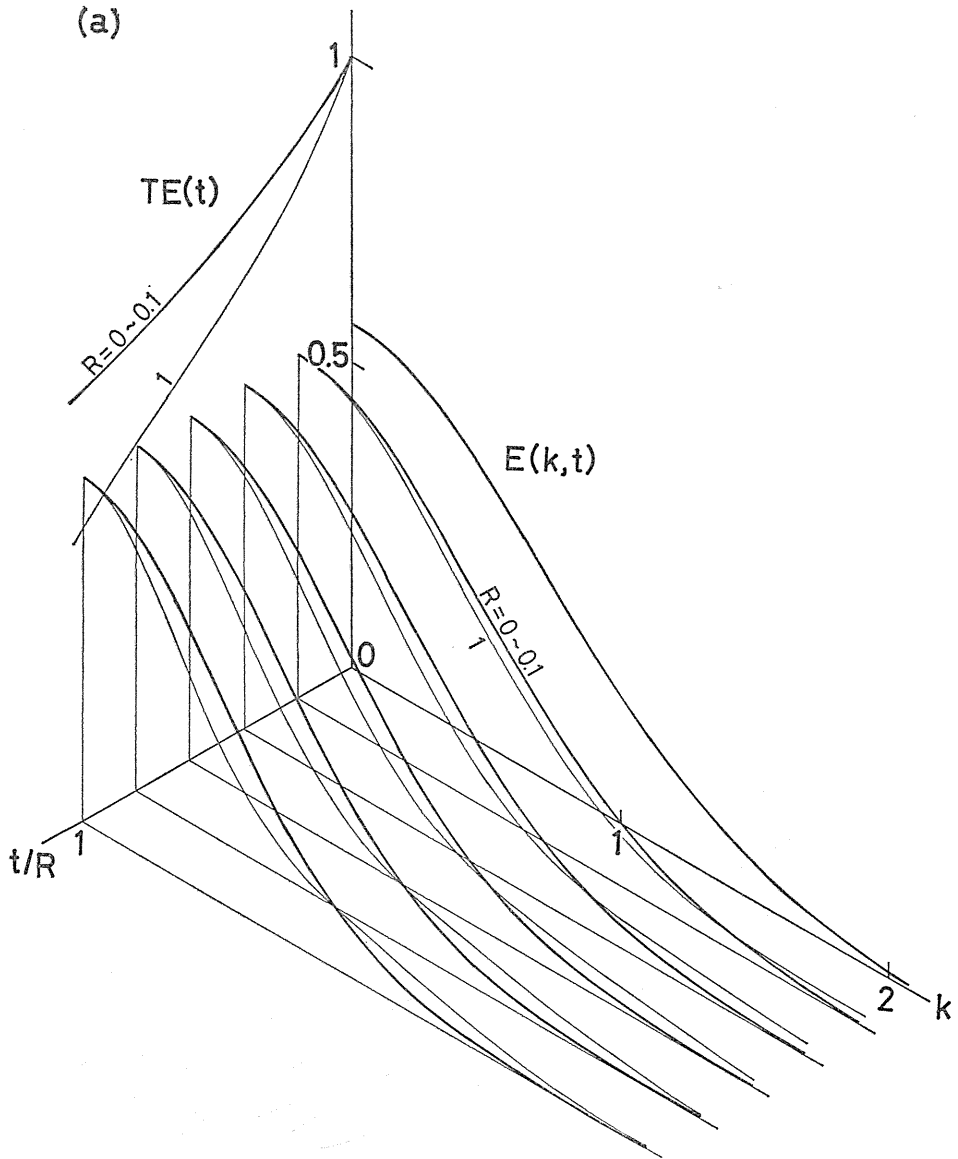


Fig. 6. Temporal developments of energy spectrum and total energy density from the initial spectrum of δ -function type, (a) for $R=0, 0.01, 0.1$ and 1 , and (b) for $R=10, 100$ and ∞ , calculated by the third method in IV. For the energy spectrum, see the note in the last paragraph of Fig. 3.



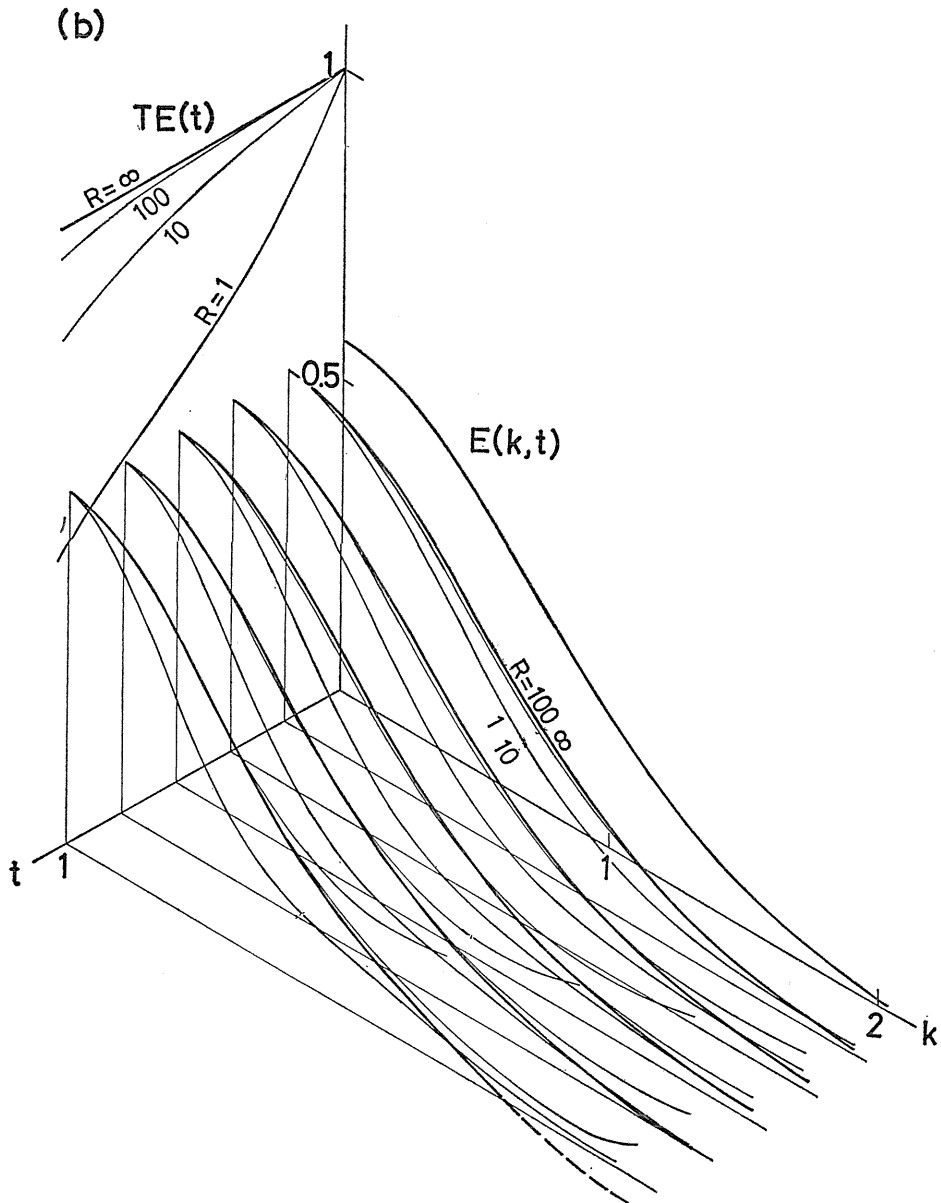
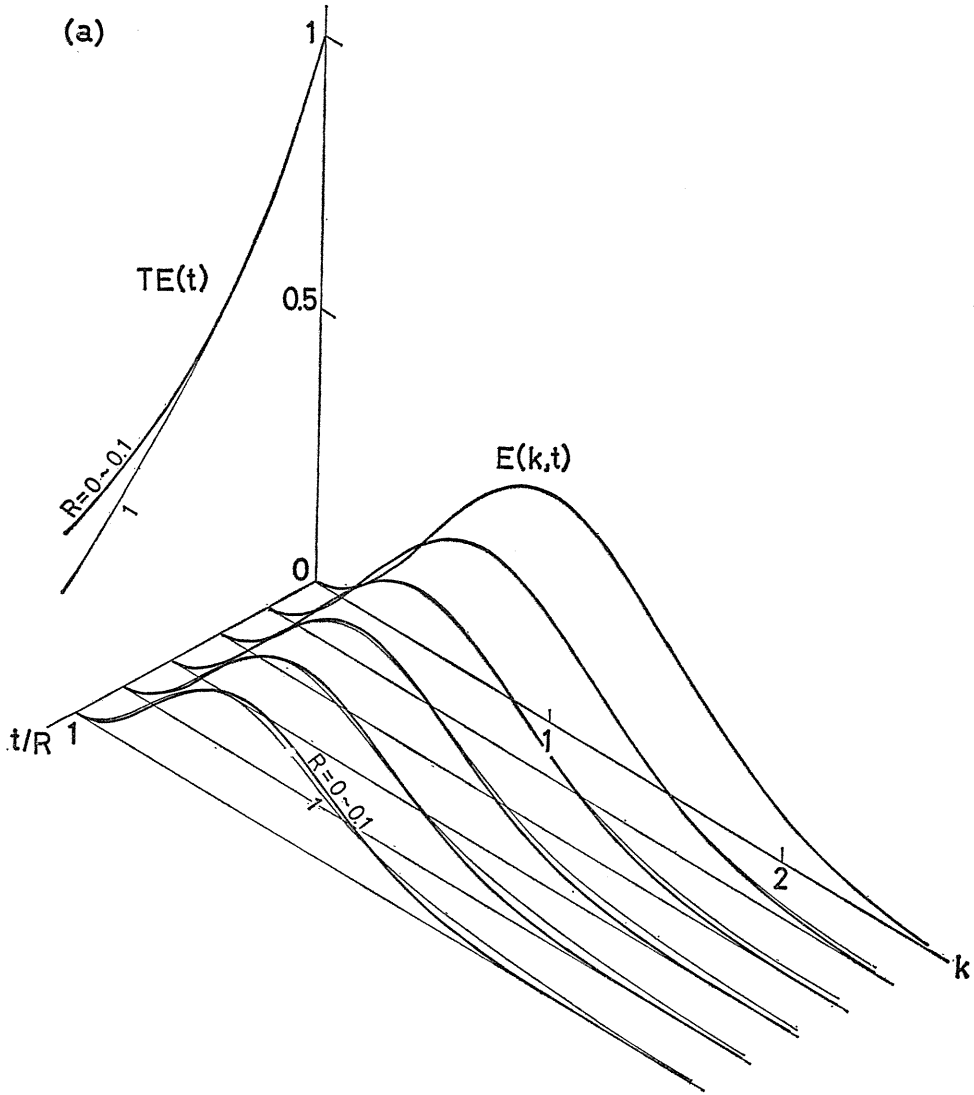


Fig. 7. Temporal developments of energy spectrum and total energy density from the initial bell shape spectrum (a) for $R=0, 0.01, 0.1$ and 1 , and (b) for $R=10, 100$ and ∞ , calculated by the third method in IV.



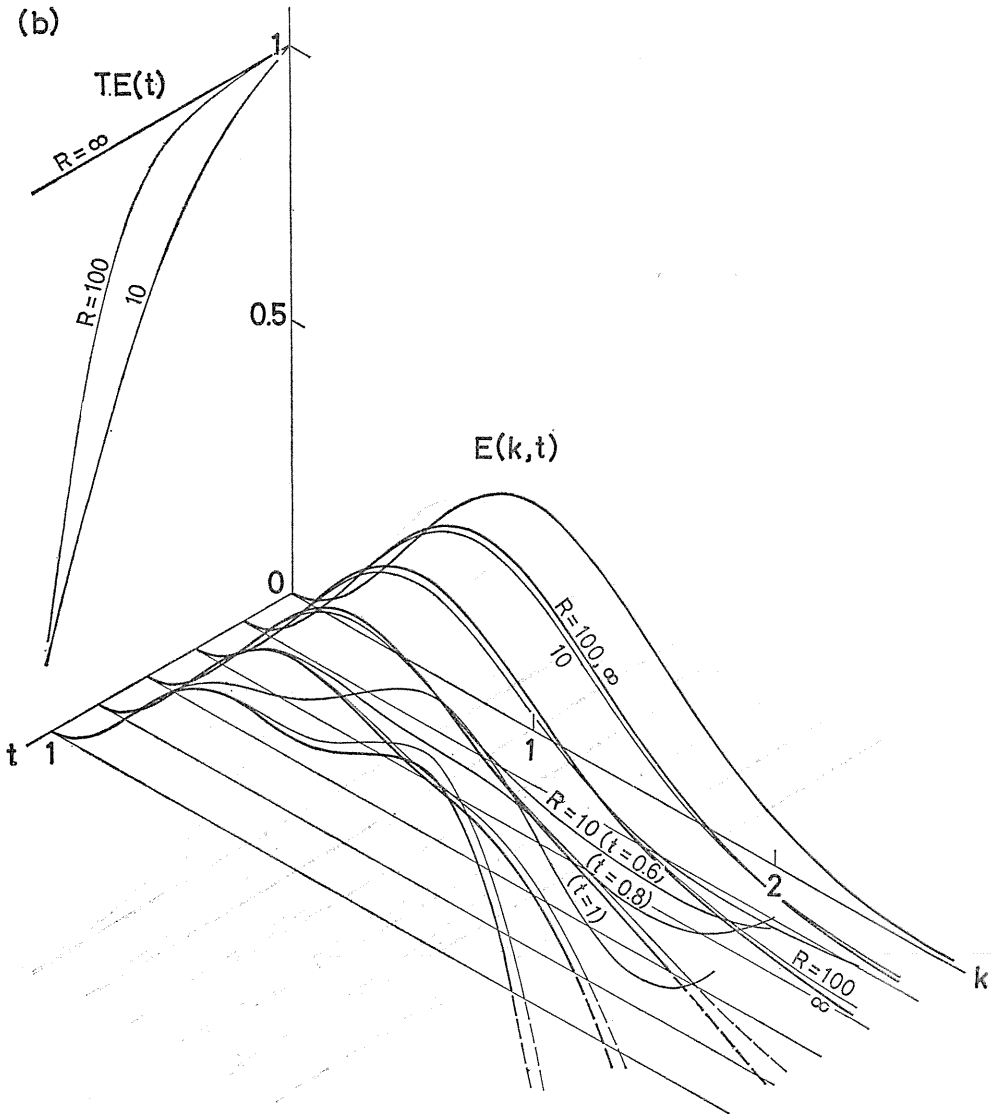


Fig. 8. Temporal developments of energy spectrum and total energy density from the initial double-bell shape spectrum (a) for $R=0, 0.01, 0.1$ and 1 , and (b) for $R=10, 100$ and ∞ , calculated by the third method in IV.

V. Conclusions

15. Comparison of Three Perturbation Methods

The generalized Liouville equation or the Hopf equation whose derivation and physical meanings have been shown in 2 and 3, give the complete mathematical tool to investigate the statistical features of turbulence. We have investigated the methods for solving the initial value problem of the Hopf equation for the Burgers turbulence, which is a simplified model of real turbulence to the one-dimensional coordinate space.

From the mathematical point of view, the Hopf equation is the functional-differential equation of evolution type in the function space, which is the dual $z(k)$ -space to the velocity $u(k)$ -function space in the sense of the Fourier transform $\int \delta u(k) \exp(i(z, u))$. The first method, which is described in II, is based on the assumption that $z(k)$ can be expanded into the complete orthonormal functions. In this case, the function $z(k)$ is equivalent to the infinite set of the expansion coefficients. The functional-differential equation reduces to the conventional differential equation in the space of the expansion coefficients. We have retained only 3 coefficients. Thus we considered the equation of evolution type in 3-dimensional space. The calculation after this stage is purely numerical. The energy spectrum can be calculated from the curvatures of the solution at the origin (see (7.11)).

The second method is based on the expansion of the semigroup-theoretic expression for the solution of the Hopf equation into a power series of the exponential argument. We have carried out the perturbation up to the 4th order. The energy spectrum and total energy density can be obtained both in an analytical form of a pure power series of time.

In the third method, the Hopf equation is written in the interaction representation, in which the viscous term disappears formally, and the nonlinear interaction term, as well as the unsteady term, remains. The perturbation method in this case is similar to the second method, i. e. the expansion into a power series of time. But the expression of the energy spectrum has always the factor $\exp(-2\nu k^2 t)$ and another factor is expanded into a power series of time. If we expand $\exp(-2\nu k^2 t)$ into a power series of time and combine it with the other factor, the resultant expression agrees with that of the second method.

The constancy of the energy spectrum for the small wavenumber, i. e. the temporal invariance of $E(0, t)$, is not satisfied in the first method but is satisfied in the second and third methods.

We have estimated that the characteristic time of the change is R for small R and 1 for large R . The conjecture seems to be valid, because the similarity of the temporal developments of the energy spectrum and total energy density with respect to the same initial energy spectrum is rather satisfactory for $0 \leq R \leq 0.1$ and $10 \leq R \leq \infty$, when the changes are measured in the time scale t/R and t , respectively.

The paradox of the negative energy spectrum occurs all in the three methods. But, it seems to be improved in the third method, i. e. reduction of the range of occurrence of negative energy in the wavenumber and time.

Judging from the numerical calculation, i. e. the constancy of the negative energy spectrum for the small wavenumber, the occurrence of negative energy, the convergence of power series etc., degrees of approximation seem to be improved step by step from the first to third method.

References

- 1) J. M. Burgers : Proc. Roy. Neth. Acad. Sci. (Amsterdam) **43** (1940) 2.
- 2) W. Heisenberg : Z. Phys. **124** (1948) 628.
- 3) E. Hopf : J. Rat. Mech. Analysis **1** (1952) 87.
- 4) E. Hopf and E. W. Titt : J. Rat. Mech. Analysis **2** (1953) 587.
- 5) D. T. Jeng, R. Foerster, S. H. Haaland and W. C. Meecham : Phys. Fluids **9** (1966) 2114.
- 6) T. V. Kármán and L. Howarth : Proc. Roy. Soc. **A164** (1938) 192.
- 7) T. Kato : *Perturbation Theory for Linear Operators*, Springer Verlag (1966) 477.
- 8) T. Kawahara : J. Phys. Soc. Japan **25** (1968) 892.
- 9) A. N. Kolmogoroff : C. R. Acad. Sci. U. S. S. R. **30** (1940) 301.
- 10) S. Kuwabara : *Statistical Hydrodynamics* (in Japanese), Ryutairikigaku no Tenbo vol. 1, Ryuriki-Kondankai (1976) 49.
- 11) I. Proudman and W. H. Reid : Phil. Trans. Roy. Soc. **A247** (1954) 163.
- 12) O. Reynolds : Phil. Trans. Roy. Soc. **A174** (1883) 935.
- 13) T. Tatsumi : Proc. Roy. Soc. **A239** (1957) 16.
- 14) T. Tatsumi : Butsuri **30** (1975) 106 (in Japanese).
- 15) G. I. Taylor : Proc. Roy. Soc. **A164** (1938) 476.
- 16) A. S. Monin and A. M. Yaglom : *Statistical Fluid Mechanics* Vol. 2 (translated from Russian), The MIT Press (1975) 757.