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THEORY OF SECONDARY FLOW IN CASCADES

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Abstract

It was in 1951 that Squire & Winter[21] published a new conception that the secondary flow could occur in a bend through which a perfect fluid is flowing, as a result of a non-uniform distribution of velocity at entrance to the bend. Their idea was reformed by Hawthorne[4] and qualitative successes were gained on the flow in duct bend and the flow around strut placed in an approaching stream with a nonuniform velocity (such as a bridge pier in a river).

Although the first attempt of Squire & Winter is supposed to take aim at the solution of cascade secondary flows, the application of this idea to blade rows (turbo-machine cascade and linear cascade) has almost eintirely been unsuccessful. The reason of this can be divided into two parts. The first is the assumption of inviscid perfect fluid assumed by Squire & Winter and viscosity neglected, and the second is that the theory itself had some imperfectness. The opinion, that the reason why the secondary flow theory is not useable to the problem of cascade simply only because of the assumption of perfect fluid without viscosity, is considered by the author to be unsatisfactory. The author wants to consider the imperfectness of theory itself prior to the effect of viscosity. After mending the imperfectness of theory there comes a step of saying the effect of viscosity.

Contributors to the secondary flow theory other than Squire & winter and Hawthorne mentioned above must be L. H. Smith Jr.[20] and Wislicenus.[19][20] Smith rendered a distinguished service in verifying clearly that the vortex components in the exit flow of cascade are consisted of the trailing vortices which are formed by a component concerning the vorticity in the upstream and a component corresponding to the variation of blade circulation, in addition to the passage vortex (vorticity) which is formed by the deformation of the upstream vorticity through the blade passage. If we rearrange his results, we have

〔vortex component normal to the cascade exit flow〕

① normal component to the flow of the passage vortex (vorticity).

〔vortex component parallel to the cascade exit flow〕

①′ parallel component to the flow of the passage vortex.
②′ trailing filament vortex (component of trailing vortex concerning the vorticity in the upstream).
③′ trailing shed vortex (component of trailing vortex corresponding the variation of blade circulation).

In spite of his important contribution mentioned above he made a regrettable mistake in the henceforth calculation of secondary vortex. The mistake made by him is a point of problem itself of secondary flow theory and will be considered later. And we need some elucidation about the method of solution prior to the explanation of the mistake.

At first we must define what is the secondary flow. Generally the secondary flow is defined as the difference of the ideal flow to the actual flow which differs from the former by the existence of boundary layer in the flow. But this definition is inconvenient and ambiguous since the solution differs in accordance with the definition of "ideal" flow. But the situation is left as it is, and this might be the root of the mistake such as Smith's.

The most intelligible definition of secondary flow may be the one starting from the secondary vortex. If the vortex is contained in the flow, the streamwise component of this vortex is called the secondary vortex, and the flow induced by the secondary vortex is the secondary flow. The ideal flow mentioned above, therefore, is the one containing no streamwise vortex, and the irrotational flow will be conveniently accepted.

This idea is especially expedient for obtaining the first approximation of secondary flow. Namely, the method of solution is to find what attitude will be taken by the vortex which was originally contained in the upstream and drifted with the fluid into the downstream. This is the means often used in the treatment of secondary flow in linear cascade. (see Fig. 7-1)

But his idea is not convenient for the treatment of secondary flows in turbomachine. When the flow in a turbomachine is irrotational, it is of so-called free-vortex type. But the flow in turbomachine is often different from the free-vortex type, and therefore it may not be advisable to take the free-vortex type of flow as the zeroth approximation. Generally we choose the axisymmetric flow as the zeroth approximation in this occasion. (The flow around two-dimensional cascade is employed so as to approximate the flow near blade row.) But we must notice here that in ordinary cases the axisymmetric flow has vorticity in it. Therefore, we should think that the secondary flow is contained in this flow, which means we should have sufficient knowledges of the behavior of vortex in this flow.

There are two understandings on this axisymmetric flow corresponding to the zeroth approximation, that it is a flow of ideal fluid and has no boundary layer in the inlet side, and then the effect of boundary layer and finite blade spacing are caught as the secondary flow, ..., or it is an axisymmetric vortex flow containing the boundary layer in it and the variation of flow caused by the finiteness of blade pitch is caught as the secondary flow ...... Both will be of use, and it is easy to understand that the knowledge of vortices in this axisymmetric flow should be indispensable for the solution. But the situation was not so. Smith's mistake mentioned above is that the axisymmetric flow considered by him was not the true one, and if sufficient considerations had been done his miscarriage would be avoided.

The examination of axisymmetric flow had some curious difficulties in spite of its simple appearances. Namely, the ordinary method of solution of axisymmetric flow has no information of the stream-wise component of vortex (secondary vortex) in the exit flow of cascade. This was pointed out by Wislicenus(19)(25) for the first time. In fine the answer cannot be obtained from the ordinary axisymmetric theory when we want the solution from the standpoint of secondary flows.

Even in the axisymmetric theory the streamwise component of passage vortex and the trailing filament vortex can be easily obtained by assuming the flow in blade passage (of infinitesimal spacing). (The sum of the both can be obtained without assuming its flow(15)(16).) The problem is the trailing shed vortex, and a few trials were done without any success.

Because the flow pattern of the exit flow of blade row can be obtained from the axisymmetric theory, the author tried to calculate the vorticity in it (which means he calculates the secondary vorticity) and found that it could be easily obtainable(17). The fact which is most important and curious in the results is that when the exit angle of blade row is of free vortex type i. e. $\tan \gamma_{2e}=C/r$ (where γ_{2e} : flow angle at the exit of blade row, C : a constant, r : radial position), there is no streamwise vorticity or secondary vorticity in the downstream whatever vortices are contained in the upstream. Although this fact was pointed out by Preston(18), this phenomenon which can be named as vortex rectification may be interesting and important.

This fact has a great meaning when we employ an axisymetric flow as a flow of the zeroth approximation (base flow). In other words, because there is no secondary flow in the downstream of base flow of free vortex type no matter how the condition of the upstream may be, we can get the three-dimensional flow of the blade row of finite blade spacing if we add the secondary flow correspoing to the finite blade spacing to this base flow. The data of the secondary flow of linear cascades will be useful for this process. If the flow is other than the free vortex type the method of treatment is not clear, but the ideas mentioned above may become good references.

In the next place, the calculation of secondary velocities from secondary vortices will be done by Hawthorne's propositions(4). Hawthorne set the following assumptions,

[1] the secondary flow occurs in planes which are normal to the average stream direction, (Trefftz plane),

[2] the secondary vorticity is normal to these planes, and

[3] the secondary flows may be treated as a two-dimensional plane flow superimposed on the main flow.

If we represent these about the linear cascade, we have Fig. 7-2.

The treatment in regard to Trefftz plane has a great defect that although the solution as two-dimensional flows is possible if this Trefftz plane can be considered being a plane, but nothing can be done when the flow in a turbomachine should be treated. The Hawthorne's propositions, therefore, can be effectively applicable only to the linear cascade.

When we want to get secondary velocities in linear cascade such as illustrated in Fig. 7-2, the shape of boundary of Trefftz plane should be a problem. In the figure the boundary is illustrated as a rectangle, but even though *AB* is straight we must examine whether *CD* and other vortex sheets are straight or not. These have been proved to be straight(11). In fine the secondary flow in exit flow of linear cascade can be obtained from the calculation of flow in a rectangle *ABDC* which has vorticities ω_{2ps} in it. Induced velocities in x-direction induced at *AB* or *CD* form the trailing vortex sheet which is the sum of the trailing filament vortex and the trailing shed vortex.

Now, we must be careful that there exists an assumption of great importance in the above ideas. Consideration of the secondary flow in a rectangle means that we accept the idea that there is no flow in *AC*-direction (y-direction) at the trailing edge *AB*, and there remains some doubt that the Kutta's condition of blade in the flow containing secondary flows can be expressed by the above or not. Probably it may be accepted when the trailing edge is very thin or of cusped form. And if we accept the above assumption, we can reach a noteworthy conclusion that the direction of the wake (vortex sheet) will show the exit flow direction of two-dimensional cascade (i. e. the cascade containing no secondary flow), because sides *AB* and *CD* are not deformed by the secondary flow. The exit flow angle of two-dimensional cascade should be obtained immediately from the direction of wake without a trouble-some method such as the boundary layer suction! This is going to be proved experimentally. (not yet published)

Now let us consider again on the component of vortices in the downstream of linear cascade.

① corresponds to the boundary layer in the downstream and we can explain the phenomenon such as the development of boundary layers in the decelerating cascade.

②′+③′ forms the trailing vortex and can be expressed by spanwise velocities along upper and lower sides of the rectangle.

①′+②′ can be obtained from the idea that the passage vortex is connected through the wake (its streamwise component). This is not related to the cascade configurations but related only to inflow and outflow directions(15)(16). In fine ①′+②′ is the secondary vortex appearing from the phenomenon that the flow is turned.

③′ is the trailing vortex component corresponding to the variation of blade circulation which has close relation to the cascade configuration, as against ①′+②′ being not related to those. We can recognize what characterizes the secondary flow in cascade (that is, what characterizes the turning of flow under the existence of blades) is the variation of blade circulation and the trailing vortex accompanied by it.

As aforesaid the solution of secondary flow in linear cascade seems to be obtained from the calculation of the flow in the rectangle, but even though the boundary is maintained rectangular the averaged flow angle of flow is changed from the direction of wake (may be the directoin of two-dimensional case) by the flow inside. We can find the turning angle is smaller (under turning) at the center of span, larger (over turning) at side wall, and smallest at the border between boundary layer and main flow. These explain qualitatively the result of cascade experiment to some extent. Further studies must be needed on the effect of fluid viscosity.

1. Introduction

The existence of secondary flow in bends in pipes and rivers has been known for some time. James Thomson[24] showed experimentally that a spiral flow could be obtained in a curved stream of water, the secondary motion at the bottom being inward and that the top outward. The secondary flow was attributed to the effect of the centrifugal pressure gradient in the main flow acting on the relatively stagnant fluid in the wall boundary layer.

Theoretical analysis of secondary flows had almost eintirely been confined to the work of Dean[1] for laminar flow in pipe bends of large ratio of bend radius to pipe diameter, before Squire & Winter[21] showed that secondary flow could occur in a bend through which a perfect fluid is flowing, as a result of a non-uniform distribution of velocity at entrance to the bend. Squire & Winter's work had its freshness in the suggestion that a more general theoretical investigation of the rotational flow of a perfect fluid in three dimensions might yield useful results, if attention was concentrated on the secondary circulation, that is the component of vorticity in the direction of flow. (From Hawthorne's paper[4]).

The author intend in this report to show how to take this Squire & Winter's idea into the clarification of the flow in (axial-flow) turbomachinery. The application of Squire & Winter's theory is not so difficult provided the flow is confined in a single domain, but its application to the flow in turbomachinery or cascade is so difficult that simple idea cannot explain the phenomena, the situation of which will be discussed in detail in the report, but the difficulty will be able to be understood by the following comments.

It is well known that there exists the axisymmetric theory for the treatment of flows in turbo-machinery, which, in fine, is to solve the problem by neglecting the tangential variation caused by the finite number of blades ......, or under an assumption that the number of blades is infinite. There is the case, of course, that the inflow to blade row contains vorticity, and we can get the solution although the practical process has much difficulties. In the next place, because the theory of secondary flows treats the flow in the passage of blade row (in fine, the case of finite number of blades), the problem which was not treated in the axisymmetric theory is transacted. We can imagine, therefore, that to sum up both solutions is the final goal of three-dimensional treatment of flows in the turbomachinery......? The author thinks that we cannot say so. The reason of which is that the theory of secondary flows is a perfect theory in which the complete flow field around blade row is treated, and the combination with the axisymmetric theory has the fear calculating the phenomena twice.

Because of difficulties around the circumstances mentioned above, the theory had been hazy for more than 20 years, and even the correction of outlet flow angle of linear cascade experiment was not succeeded. But the author feels the haze has become cleared up.

In this report, only stationary blade rows including linear cascade are treated. Moving blade rows are omitted because of difficulties included, and also blade tip clearance problems.

2. Secondary Flows

When a passage of flow has a curvature, the centrifugal force of fluid must be balanced with static pressure. The solid line in Fig. 2-1 shows this condition. This is achieved in the main flow where the effect of boundary layer or the like is not reached. Let us now consider a case in which the boundary layer parallel to the surface of this paper exists. According to the idea that the static pressure of main flow passes into the boundary layer, the static pressure gradient of this main flow is the pressure gradient in the boundary layer. But the boundary layer velocity is smaller, thus the centrifugal force is smaller, the balance of static pressure and centrifugal force is destroyed, and finally the flow is turned by the static pressure much more than the main flow. The broken line in Fig. 2-1 showes this situation. This is the reason why the secondary flow occures.

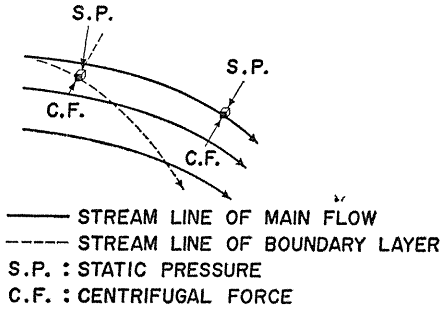


Fig. 2—1

2. 1. Secondary Flows in Linear Cascade

The problem in a single domain such as the secondary flow in a curved pipe or the flow in a bend of river is rather simple provided the story is confind in the limit of assumptions mentioned above, and can be considered being the technique of solution. But, the applicability of solution to the practical phenomenon is another problem .... We can find examples in a few reports(1)(4)(21).

Since in the case of linear cascade the domain is no longer single, the phenomenon becomes rather complex. The secondary flow in cascades is originated by the spanwise nonuniformity of incoming flow which is caused by the existence of side walls at the inlet of cascade experiment or the like. (see Fig. 2-2). Because the particle in the side wall boundary layer is turned strongly as explained in the above, we recognize spiral motions of fluid as illustrated (A) in the figure. This mode of phenomenon is quite same as that in a single domain, but the spiral motion (A) is repeated at every passage in cascade, which results in the vortex surface having opposite velocities at each side of the wake just like a needle bearing as illustrated (B) in the figure. This vortex surface is unstable, and rolls up in a pair of vortices before long as illustreated (C) in the figure. The pair of spiral motions (A) is also

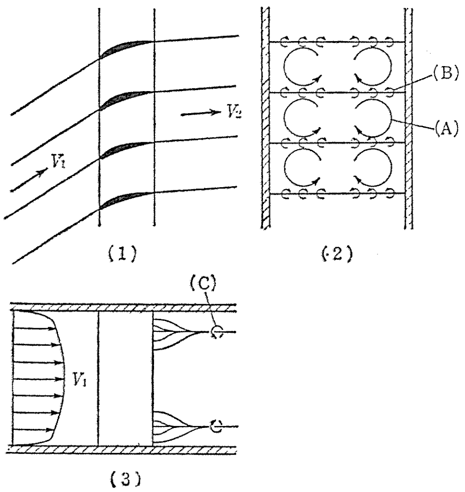


Fig. 2—2

the vortices which have also a tendency to roll up into a pair of vortex, and these locate at corners of the suction surface of blade and side walls.

This is the outline of the secondary flow in linear cascades, and the fact that the secondary flows are consisted from two vortices, which are the passage vortex (A) and the trailing vortex (B), makes it difficult to treat the phenomenon in a simple manner as in the single domain. Moreover, because the roll-up of these vortices into two pairs of vortices, and the process of diffusion of the latters have not yet been clarified at all, the analytical treatment is difficult. At the present state, the theory is treated with assumptions that the roll-up and the diffusion is entirely neglected or the roll-up is eintirely finished, and we cannot tell whether these assumptions are suitable or not.

2. 2. Secondary Flows in Axial-flow Turbomachinery

The flow in the blade row of axial-flow turbomachinery, for example the annular cascade of the stationary blade of axial-flow turbine (Fig. 2-3)[7], is much more complex and difficult to be understood. In the annular cascade, not only the main flow and side wall boundary layers are turned by the cascade (this is same as the linear cascade) but the tangential flow which is accompanied with boundary layers on blade surfaces is turned by outer and inner casings. The latter causes the secondary flow toward the inner casing through the blade boundary layer or wake. This phenomenon is known experimentally but has not yet treated analytically.

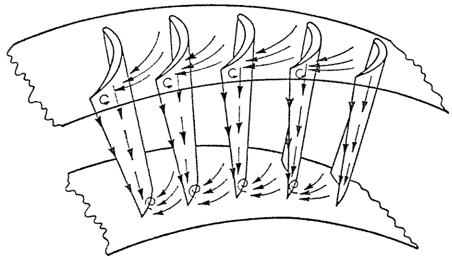


Fig. 2—3

3. Theory of Secondary Flows

Although the analysis of Squire & Winter[21] should be noticed because of its new concept, the handling was not so smart. On the contrary the Hawthorne's analysis[4] was much more worthy to learn, in which an exact solution was obtained in the limit of assumptions. In this report only the outline of Hawthorne's anlaysis will be written, and a simple first approximation theory will be stated.

3. 1. *Hawthorne's Analysis*(4)

Hawthorne's theory is presented for a steady, inviscid incompressible fluid in motion in the absence of body forces. Representing the velocity vector by **V** and its scalar by q, the vorticity vector

$$\Omega = \mathrm{rot}\,V \tag{1}$$

The component of the vorticity resolved in the direction of flow, whose scalar will be presented by ω, gives rise to a secondary circulation which, when measured around a stream tube of cross-sectional area dA, has a magnitude ωdA. Since qdA is the constant volume flow along a stream tube, the secondary circulation around any given stream tube will be proportional to ω/q.

Let us consider the special type of flow in which the streamlines and vortex lines lie in a surface of constant total pressure p_0 or a Bernoulli surface. (For example the inlet boundary layer flow of side walls of a linear cascade windtunnel can be regarded the shear flow satisfying this condition.)

Abridging the modification process of equation (1), the final result given by Hawthorne is

$$\left(\frac{\omega}{q}\right)_2 - \left(\frac{\omega}{q}\right)_1 = -2\int_1^2 \frac{1}{q^2}\left|\mathrm{grad}\left(\frac{p_0}{\rho}\right)\right|\frac{\sin\phi}{R}ds \tag{2}$$

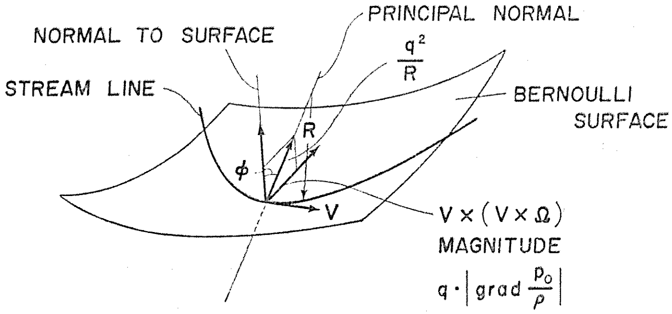


Fig. 3—1

where ds : an elementary arc of the streamline.

ϕ : the angle between the direction of principal normal and the normal to the Bernoulli surface (see Fig. 3-1).

This is expressed in the form of integration along a streamline.

Now $(\sin\phi/R) = (1/R_g)$ is the geodesic curvature of the streamlines on the Bernoulli surface. By definition

$$\frac{1}{R} = \frac{d\theta}{ds}$$

where $d\theta$: the angle between tangents to the streamline at points arc length ds apart.

Hence equation (2) may also be written

$$\left(\frac{\omega}{q}\right)_2 - \left(\frac{\omega}{q}\right)_1 = -2\int_1^2 \left|\mathrm{grad}\left(\frac{p_0}{\rho}\right)\right| \sin\phi\frac{d\theta}{q^2} \tag{3}$$

An important result of this analysis is that if $\phi=0$ or the direction of acceleration (or pressure gradient) lies in the plane containing the velocity vector and the normal to the Bernoulli surface, there is no change in secondary circulation along the streamline. Hence streamlines along which the secondary circulation remains unchanged are geodesics on the Bernoulli surface.

(From Hawthorne's paper)

3. 2. Simple Examples(4)(5)

The simplest examples to which this analytical result may be applied are those in which the initial flow has a uniform pressure and a velocity varying in only one direction. Such a flow may exist in the boundary layer of a large straight duct, in an open channel whose width is large compared to its depth or in a linear cascade windtunnel. The Bernoulli surfaces are planes and the total pressure varies in one direction only. If the stream enters a bend whose plane is parallel to the Bernoulli surfaces, the angle ϕ is initially $\pi/2$, so that a secondary circulation is created in the bend. Since each particle of perfect fluid retains its original total pressure and the particles are carried with the secondary flow, the Bernoulli surfaces are distorted as the fluid passes downstream so that the original unidirectional feature of the total pressure variation is lost.

Equation (3) is not sufficient to determine the flow, but Hawthorne proposed that if certain assumptions are made, approximate solutions may be obtained by estimating the secondary vorticity. These assumptions are, [1] the secondary flow occurs in planes which are normal to the average direction, [2] the secondary vorticity is normal to these planes, and [3] the secondary flow may be treated as a two-dimensional plane flow superimposed on the main flow. To obtain the secondary vorticity from equation (3) the behaviour of the Bernoulli surfaces and streamlines must either be estimated or calculated by some step-by-step process.

In the flow of a boundary layer described above, the Bernoulli surfaces are initially planes and are distorted as the flow proceeds round the bend. Under certain conditions the distortion of the Bernoulli surfaces may be small and $|\mathrm{grad}(p_0/\rho)|$ and ϕ will be considered to retain their initial values. If the variation in q along a streamline is also small, and noting that initially

$$\mathrm{grad}\left(\frac{p_0}{\rho}\right)=\mathrm{grad}\left(\frac{p}{\rho}+\frac{1}{2}q^2\right)=q\cdot\mathrm{grad}\ q=q\Omega_0$$

where Ω_0 is the vorticity in the upstream which lies on the Bernoulli surface and is perpendicular to the stream line. Equation (3) gives for the vorticity downstream ($\omega_1=0$),

$$\omega_2=-2\Omega_0\varepsilon \tag{4}$$

where ε is the angle of turn in the bend. These are the assumptions and the result obtained by Squire & Winter(21) using a different analytical approach, which is not adopted in this report because of its prolixity.

A similar methd of superimposition may be used for the flow around an obstacle, such as an aerofoil, when the velocity varies only in the direction of the span. Then equation (3) yields for the downstream vorticity

$$\left(\frac{\omega}{q}\right)_2=-2\Omega_0 q_0\int_1^2\frac{d\theta}{q^2} \tag{5}$$

Equation (5) shows that a symmetrical obstacle such as a cylinder or strut will create a secondary vorticity due to its thickness alone. The secondary vorticity will result in induced drag effects which may be reduced by minimizing the integral in equation (5). This is a possible basis for a method of designing thick struts for use in boundary layers. (From Hawthorne's paper(5))

3. 3. Determination of Secondary Velocities

Let the secondary velocity components be expressed by V_x, V_y, V_z. z-direction is chosen to correspond to the direction of main stream, and V_z is the velocity parallel to the main stream. V_x, V_y are velocities in a plane normal to the main stream. This plane is called Trefftz plane.

The continuity equation of secondary velocities are

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}=0 \tag{1}$$

If we consider a state in which the secondary flow is fully settled, we can put

$$\frac{\partial V_z}{\partial z}=0$$

and we have

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}=0 \tag{2}$$

Since we considered (assumed) the secondary vorticity ω_2 is perpendicular to Trefftz plane, we have

$$\omega_2=\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y} \tag{3}$$

Equation (2) allows us to define a stream function Ψ, such that

$$\left.\begin{aligned} V_x&=\frac{\partial \Psi}{\partial y} \\ V_y&=-\frac{\partial \Psi}{\partial x} \end{aligned}\right\} \tag{4}$$

Substituting these into equation (3), we obtain Poisson's equation

$$\frac{\partial^2 \Psi}{\partial x^2}+\frac{\partial^2 \Psi}{\partial y^2}=-\omega_2 \tag{5}$$

which is to be solved for Ψ with the boundary condition given. We can get V_x and V_y from equation (4).

These process is along the line of assumptions [1]～[3] stated in the last paragraph.

3. 4. Simple Method to obtain the First Approximation of Secondary Vorticity

Hawthorne's analysis mentioned in 3. 1 and 3. 2 is pretty difficult to understand because of the vector method used. Here the author will show that the same result can be reached by using the vortex law and simple calculations.

Let us consider the (base) flow which was possibly originally two-dimensional and irrotational, but in reality it has the vorticity and secondary flows are induced which make the flow three-dimensional. Assuming the vorticity contained being small, we can get the final three-dimensional flow by the addition of the induced flow by the vorticity to the basic two-dimensional flow. Accordingly the result is the first approximation.

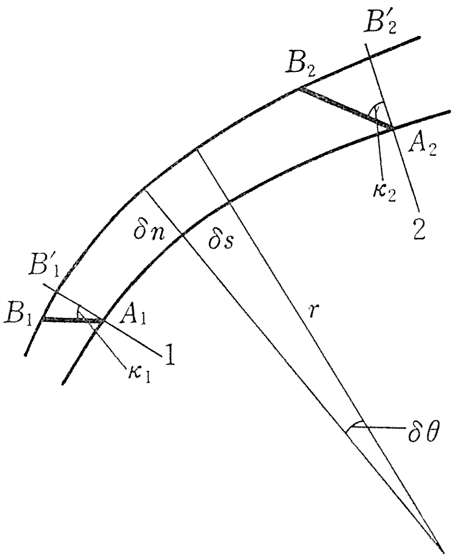


Fig 3—2

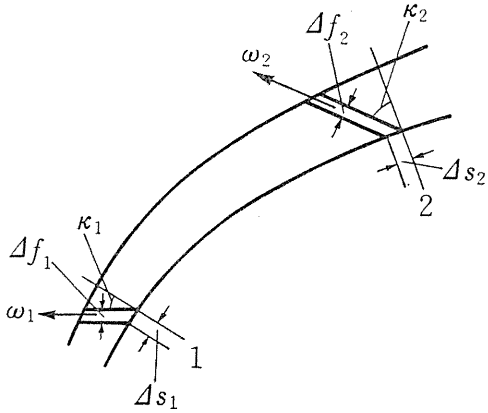


Fig 3—3

Let us consider the streamline $\widehat{A_1A_2}$ in Fig. 3-2. The vortex A_1B_1 at A_1 on this steamline drifts with the fluid to A_2 in t second. The attitude of vortex becomes A_2B_2. $\widehat{B_1B_2}$ in the figure is the adjoining streamline of $\widehat{A_1A_2}$. The distance of the both is δ_n. Time required by the fluid to flow from B_1 to B_2 is t second.

Subscript $_1$ and $_2$ correspond to positions A_1 and A_2 respectively.

Let the radius of curvature at a point on the stream line be r, we have a equation of equilibrium of centrifugal force, which is

$$\frac{\partial p}{\partial n}=\frac{\rho V^2}{r} \tag{1}$$

where p : static pressure
n : distance normal to streamline
ρ : density
V : velocity

The Bernoulli's equation is

$$p+\frac{1}{2}\rho V^2=C$$

Assuming the base flow is irrotational, C is constant throughout the flow field. Hence the differentiation of the above equation by n becomes,

$$\frac{\partial p}{\partial n}+\rho V\frac{\partial V}{\partial n}=0 \tag{2}$$

From (1) and (2), we get

$$\frac{\partial V}{\partial n}=-\frac{V}{r} \tag{3}$$

In the next place, we have

$$t=\int_{A_1}^{A_2}\frac{ds}{V} \tag{4}$$

where s : distance along the streamline
and

$$t=\int_{B_1}^{B_2}\frac{ds_b}{V_b} \tag{5}$$

where subscript $_b$ represents the value along $\widehat{B_1B_2}$. Assuming δ_n is small, we get from (5)

$$t\doteqdot\frac{\delta_{n1}\tan\kappa_1}{V_1}+\int_{B_1}^{B_2}\frac{ds_b}{V_b}-\frac{\delta_{n2}\tan\kappa_2}{V_2} \tag{5'}$$

On the other hand,

$$V_b=V+\frac{\partial V}{\partial n}\delta_n$$

and from (3),

$$V_b=V\left(1-\frac{\delta_n}{r}\right)$$

and

$$ds_b=ds\left(1+\frac{\delta_n}{r}\right)$$

Substituting these relations into (5′), and using (4), we have at last

$$\frac{\delta_{n2}\tan\kappa_2}{V_2}-\frac{\delta_{n1}\tan\kappa_1}{V_1}\doteqdot 2\int_{A_1}^{A_2}\frac{\delta_n}{r}\frac{ds}{V} \tag{6}$$

Expressing the flow quantity between two streamlines by δQ, we have,

$$\delta Q=V\delta_n=V_1\delta_{n1}=V_2\delta_{n2}$$

And

$$\frac{ds}{r}=-d\theta$$

where θ : inclination angle of the tangent to streamline. Substituting these into (6), we get the following relation on the inclination of the vortex.

$$\frac{\tan \kappa_2}{V_2^2} - \frac{\tan \kappa_1}{V_1^2} \doteqdot -2\int_{A_1}^{A_2} \frac{d\theta}{V^2} \tag{7}$$

In the next place, let us examine the change of vortex strength at stations *1* and *2*. (see Fig. 3-3). Vorticity is expressed by ω, and the vortex tube whose largeness is Δf_1 at station *1* changes to Δf_2 at station *2*. Because the strength of vortex tube is invariable, we have

$$\omega_1 \Delta f_1 = \omega_2 \Delta f_2 \tag{8}$$

Since Δs in the figure drifts with the fluid, we get

$$\frac{\Delta s_1}{\Delta s_2} = \frac{V_1}{V_2} \tag{9}$$

and

$$\left.\begin{aligned} \Delta f_1 &= \Delta s_1 \cos \kappa_1 \\ \Delta f_2 &= \Delta s_2 \cos \kappa_2 \end{aligned}\right\} \tag{10}$$

From (8), (9) and (10), we obtain

$$\frac{\omega_2}{\omega_1} = \frac{V_1 \cos \kappa_1}{V_2 \cos \kappa_2} \tag{11}$$

The secondary circulation of flow is the streamwise component of ω, and let us denote it ω_s,

$$\left.\begin{aligned} \omega_{s1} &= \omega_1 \sin \kappa_1 \\ \omega_{s2} &= \omega_2 \sin \kappa_2 \end{aligned}\right\} \tag{12}$$

From (11) and (12)

$$\frac{\omega_{s2}}{\omega_{s1}} = \frac{V_1}{V_2} \frac{\tan \kappa_2}{\tan \kappa_1} \tag{13}$$

From (9), (13) and (7), (12), we get

$$\frac{\omega_{s2}}{V_2} - \frac{\omega_{s1}}{V_1} \doteqdot -2\frac{\omega_1}{V_1} \cos \kappa_1 \int_{A_1}^{A_2} \frac{d\theta}{\left(\frac{V}{V_1}\right)^2} \tag{14}$$

If we consider the case $\kappa_1 = 0$, which means that there is no secondary circulation in the upstream as we experience in the linear cascade experiment, we have,

$$\frac{\omega_{s2}}{V_2} = -2\frac{\omega_1}{V_1} \int_{A_1}^{A_2} \frac{d\theta}{\left(\frac{V}{V_1}\right)^2} \tag{15}$$

This is same as the equation derived by Hawthorne. [equation (5) in 3. 2.]

4. Axisymmetric Theory of Turbomachinery

Although the secondary flow theory stated above is thought to be an useful method to approach the three-dimensional flow in axial-flow turbomachinery, the axisymmetric theory is also useful as an intermediate method (which is called the actuator disc theory when the axial length of blade row is infinitesimally small). The latter is understood to be a treatment of the case in which the tangential variation of three-dimensional flow is ignored or the number of blades is infinite, and the former is thought to be a method to deal with the flow in blade passage and the change of outlet flow caused by it. To clarify the connection of the both is important for the understanding of three-dimensional flow in turbomachinery and the establishment of the fundamental conception of design, and this is one of the object of the report and will be explained later. In this article the axisymmetric theory will be explained at first.

The axisymmetric theory has been studied by many researchers, but let us follow the method developed by Wislicenus(25), because in this method the physical meaning of the introduction is clear, the treatment is possible in the case in which meridional streamline is not axial, and the calculation by computer will be easier even in the case of compressible fluid.

Only the flow through the stationary blade row is treated in this report. Although treatments about the moving blade row may be possible by uses of the idea of relative energy(19)(25) (in axisymmetric theory) and the consideration of secondary flow in moving blade, they are problems to be solved in the future.

4. 1. Theory for Stationary Blades

Let the total enthalpy of the flow be H which is,

$$H=\frac{V^2}{2g}+h \tag{1}$$

where V is absolute velocity of flow, and h is static enthalpy. If we assume isentropic flow, H is constant along a streamline provided it does not pass moving blades. Let us consider two closely adjacent meridional streamlines as shown in Fig. 4-1, and let the total enthalpy along these two streamlines be H and $H+dH$. Then we have

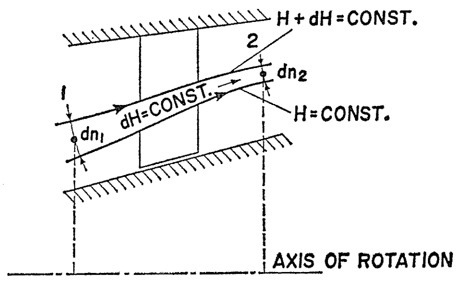


Fig. 4—1

$$dH=\left(\frac{\partial H}{\partial n}\right)_1 dn_1=\left(\frac{\partial H}{\partial n}\right)_2 dn_2=\text{const.} \tag{2}$$

Therefore

$$\left(\frac{\partial H}{\partial n}\right)_2=\left(\frac{\partial H}{\partial n}\right)_1 \frac{dn_1}{dn_2} \tag{3}$$

where n is the co-ordinate normal to meridional stream surface. From the condition of continuity, we get

$$\frac{dn_1}{dn_2}=\frac{\rho_2 w_{m2} r_2}{\rho_1 w_{m1} r_1} \tag{4}$$

where ρ is density and w_m is component of velocity V along meridional streamline. Differentiating equation (1) with respect to n,

$$g\frac{\partial H}{\partial n}=\frac{1}{2}\frac{\partial V^2}{\partial n}+g\frac{\partial h}{\partial n} \tag{5}$$

From the differentiation of equation of adiabatic change,

$$\frac{p}{\rho}=\text{const.}\cdot p^{\left(1-\frac{1}{k}\right)}=gRT$$

we get

$$\frac{1}{\rho}\frac{\partial p}{\partial n}=gC_p\frac{\partial T}{\partial n}=g\frac{\partial h}{\partial n} \tag{6}$$

where C_p is specific heat at constant pressure, and T is absolute temperature.

For any curved flow the pressure gradient normal to the flow is related to the velocity V and radius of curvature R of the streamlines by the condition of radial equilibrium

$$\frac{1}{\rho}\frac{\partial p}{\partial n}=\frac{V^2}{R} \tag{7}$$

The application of this equation to such flows as illustrated in Fig. 4-2 and 4-3,

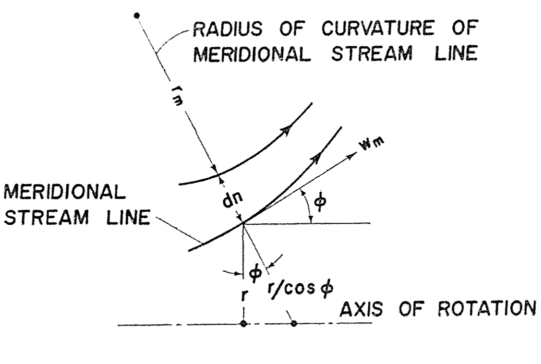


Fig. 4—2

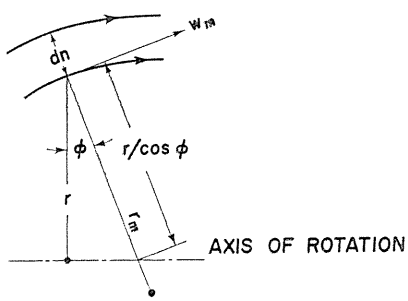


Fig. 4—3

in which a peripheral and meridional components of velocity are w_θ and w_m respectively, leads us to

$$\frac{1}{\rho}\frac{\partial p}{\partial n}=\frac{w_\theta^2}{r/\cos\varphi}\pm\frac{w_m^2}{r_m} \tag{8}$$

where r_m is the radius of curvature of meridional streamline. The plus sign applies to the case shown in Fig. 4-3, the minus sign to that shown in Fig. 4-2, referring to the direction of curvature of the meridional streamlines.

Considering that the total velocity V can be expressed by its peripheral and meridional components,

$$V^2=w_\theta^2+w_m^2$$

and differentiating it with respect to n,

$$\frac{1}{2}\frac{\partial}{\partial n}(V^2)=w_\theta\frac{\partial w_\theta}{\partial n}+w_m\frac{\partial w_m}{\partial n} \tag{9}$$

Substituting from equations (6), (8) and (9) into equation (5), the following relation is obtained.

$$g\frac{\partial H}{\partial n}=w_\theta\left(\frac{\partial w_\theta}{\partial n}+\frac{w_\theta}{r/\cos\varphi}\right)+w_m\left(\frac{\partial w_m}{\partial n}\pm\frac{w_m}{r_m}\right) \tag{10}$$

The components of vorticity parallel to w_m and parallel to w_θ are denoted ω_m and ω_θ respectively, we have

$$\omega_m=\frac{\partial w_\theta}{\partial n}+\frac{w_\theta}{r/\cos\varphi} \tag{11}$$

$$w_\theta=-\left(\frac{\partial w_m}{\partial n}\pm\frac{w_m}{r_m}\right) \tag{12}$$

Equation (10) may thus be written in the form

$$g\frac{\partial H}{\partial n}=w_\theta\omega_m-w_m\omega_\theta \tag{13}$$

Considering that the velocity and vorticity components normal to the meridional stream surfaces are zero*, equation may also be given by the vector form

$$g\frac{\partial H}{\partial n}=\left|\boldsymbol{V}\times\omega\right| \tag{14}$$

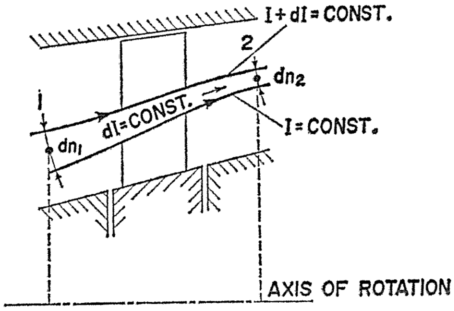
It will be understood that the last equation could have been derived more directly from the laws of vortex flow of an ideal fluid. However, the physical background of this equation might have been greatly clarified by the method mentioned above.

In order to derive a corresponding relation to equations (10), (13) or (14) for the flow through rotating blade systems it is customary to introduce the concept of "relative energy" I, by the definition,

* Meridional stream surface is Bernoulli surface. (see 3.1)

$$I=H-\frac{uw_\theta}{g}=\frac{V^2}{2g}+h-\frac{uw_\theta}{g} \tag{15}$$

where u is the tangential velocity of moving blade at the radial position we are now considering. (see Fig. 4-4) Because the analysis of secondary flows of moving blade system is not under way, further considerations of the moving blade system will be not performed in this report.



FLOW THROUGH MOVING BLADE ROW

Fig. 4—4

4. 2. Equation correlating Conditions before and behind Blade Row

Combining equations (3), (4), (10) and (14), we obtain for the relation between two stations 1 and 2 on the same meridional streamline before and behind a stationary blade system.

$$\left[w_{\theta 1}\left(\frac{\partial w_\theta}{\partial n}+\frac{w_\theta}{r/\cos\varphi}\right)_1+w_{m1}\left(\frac{\partial w_m}{\partial n}\pm\frac{w_m}{r_m}\right)_1\right]\frac{1}{\rho_1 r_1 w_{m1}}$$

$$=\left[w_{\theta 2}\left(\frac{\partial w_\theta}{\partial n}+\frac{w_\theta}{r/\cos\varphi}\right)_2+w_{m2}\left(\frac{\partial w_m}{\partial n}\pm\frac{w_m}{r_m}\right)_2\right]\frac{1}{\rho_2 r_2 w_{m2}} \tag{16}$$

Vectorially,

$$\frac{|\boldsymbol{V}_1\times\boldsymbol{\omega}_1|}{\rho_1 r_1 w_{m1}}=\frac{|\boldsymbol{V}_2\times\boldsymbol{\omega}_2|}{\rho_2 r_2 w_{m2}} \tag{17}$$

or

$$\frac{w_{\theta 1}\omega_{m1}-w_{m1}\omega_{\theta 1}}{\rho_1 r_1 w_{m1}}=\frac{w_{\theta 2}\omega_{m2}-w_{m2}\omega_{\theta 2}}{\rho_2 r_2 w_{m2}} \tag{18}$$

We must be careful about the fact that we cannot get the component of ω_2 parallel to V_2 from equation (17) or (18). This is because the vector product of two vectors which are parallel to each other is zero. Therefore, the vortex shed from a system such as the trailing vortex cannot be obtained from this axisymmetric equation. Since the secondary flow theory is the consideration of vortex parallel to the flow, there exists a solicitude that the axisymmetric theory is impotent to the secondary flow problem. We shall discuss about it later.

For the convenience of comparison with the secondary flow theory, we take the incompressible flow assumption and a case in which the meridional streamline can be regarded to be parallel to the axis. From (18) we have,

$$\frac{w_{\theta 1}\omega_{a1}-w_{a1}\omega_{\theta 1}}{r_1 w_{a1}}=\frac{w_{\theta 2}\omega_{a2}-w_{a2}\omega_{\theta 2}}{r_2 w_{a2}}$$

(where subscript $_a$ indicates axial direction.)
and using next relation,

$$\frac{w_{\theta 1}}{w_{a1}}=-\tan\gamma_1, \quad \frac{w_{\theta 2}}{w_{a2}}=-\tan\gamma_2 \tag{19}$$

(where γ_1 and γ_2 represent the inflow and out flow angles to and from the blade row respectively.) we get the following equation after a readjustment.

$$\frac{\omega_{\theta 1}}{r_1}+\frac{\omega_{a1}}{r_1}\tan\gamma_1=\frac{\omega_{\theta 2}}{r_2}+\frac{\omega_{a2}}{r_2}\tan\gamma_2 \tag{20}$$

4. 3. Method of Solution

Equations (16) through (18) may be used in the following manner to predict flow with vorticity through a stationary blade row of turbomachinery. The flow through a rotating blade system can be obtained in the same manner, which was explained in the Wislicenus' report(19)(25) and is not discussced in this report because of the reason said above.

The flow may be assumed as given at station 1. This determines completely one side of the equation with which we are concerned.

For the other side of the same equation one may choose one of the two components of the flow and then numerically calculate the other. To carry out this operation it is necessary to approximate in advance the local inclination φ, as well as the local radius of curvature r_m of the meridional streamlines at the two stations considered. The required approximation must be derived from a lower-order approximation of the entire flow. The potential (free vortex) pattern of the meridional flow may be used as first approximations for φ and r_m. With such an assumption the above-mentioned numerical calculation of the missing flow component can be completed except for a constant of integration.

If the missing velocity component is the meridional one (which is usually the case) then the constant of integration is to be determined from the condition of continuity. It is usually possible to estimate the average density of the gas in the cross section considered, and thus calculate the average meridional velocity from the given mass flow. There may be difficulties when the velocities are near the acoustic.

If the peripheral component of the flow is to be determined, the constant of integration is usually given by certain requirements regarding the angular momentum of the flow in the cross section considered which follow from the prescribed performance of the stage of machine.

In the case of application to the axial-flow machine, the potential flow pattern, which has straight meridional streamlines parallel to the axis of the machine, is to be used as the first approximation. Consequently, everywhere $\varphi=0$ and $r_m=\infty$. If we also assume that $\rho_2 r_2 w_{m2}=\rho_1 r_1 w_{m1}$ (or $dn_2=dn_1=dn=dr$), we can easily obtain the first approximation at the station *2* (and stations farther behind in the same way).

We can now plot the approximate meridional streamlines, and get the local radius of curvature r_m and the inclination φ for a second approximation. Repeating this process we can get the final converged value. The detailed process must be referenced to the report written by Wislicenus et al(19).

The important point of this paragraph is that we can obtain the flow field (velocity) from the axisymmetric theory, although the vortex shed from a system such as the trailing vortex could not be obtained from this theory.

4. 4. Vortex along stream Line

Because we could obtain the flow field from the axisymmetric theory, there may be no doubt that we shall be able to get the vortex in the flow. The practice to get streamwise vortex will be explained later. This has an important meaning in connection with the secondary flow mentioned in the next chapter.

5. Theory of Secondary Flows in Axial-flow Turbo-machinery

The theory of secondary flows got a new development by Squire and Winter(21). They regarded the shear flow of the side wall boundary layer of incoming flow such a vortex as

$$\omega_1 = \frac{\partial V_1}{\partial y}$$

which is illustrated in Fig. 5-1, and investigated the movement of this vortex on an

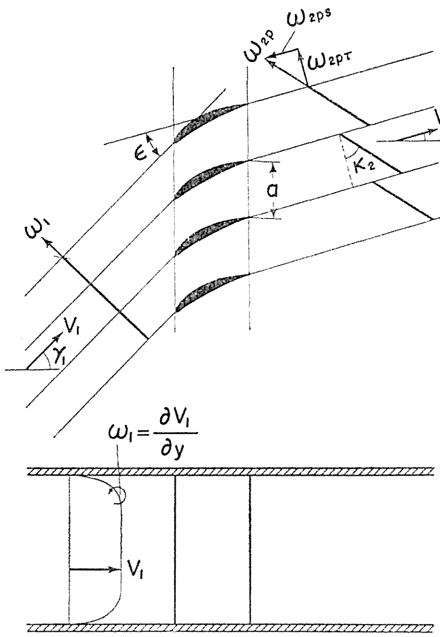


Fig. 5—1

assumption of inviscid fluid. Although the theory was on the linear cascade in this case and ω_1 was perpendicular to the inflow, the theory we are now to consider is the first approximation theory of secondary flows caused by casing boundary layers of axial-flow turbo-machinery and the condition is somewhat generalized. The phenomenon such as the boundary layer growth on casing walls together with blade surfaces which is supposed to have severe influences on the cascade performance is neglected in the theory.

5. 1. *Symbols*

a : blade pitch
C : constant
n : number of blades
r : radial position (Fig. 5-2)
t : time
u : peripheral velocity
V : relative velocity
w : absolute velocity
z : axial position (Fig. 5-2)
Δf : breadth of vortex (Fig. 5-4)
Δs : breadth of vortex in the direction of stream (Fig. 5-4)
Γ : blade circulation
Γ_T : strength of trailing vortex per unit blade height
γ : stream angle from axial direction (Fig. 5-4)
θ : angular position (Fig. 5-2)
κ : angle between vortex and stream (Fig. 5-4)
ω : vorticity

Subscripts

a : axial or z direction
B : base flow
e : exit of blade row
p : passage vortex
r : radial direction
s : stream direction
T : trailing vortex
θ : peripheral or tangential direction
τ : perpendicular direction to the stream line on the meridional stream surface (Fig. 5-4)
1 : upstream of blade row
2 : downstream of blade row

5. 2. *Fundamental Equations*
Vorticity Written by Cylindrical Coordinates

We determine the coordinate, velocity and vorticity as shown in Fig. 5-2. The vorticity and velocity are connected by the law of right-turn screw.

$$\omega_r=\frac{1}{r}\frac{\partial w_a}{\partial\theta}-\frac{\partial w_\theta}{\partial z} \tag{1}$$

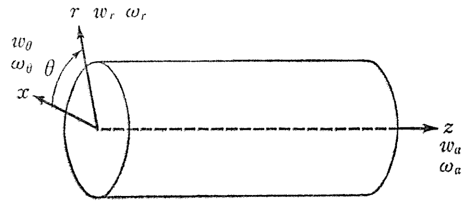


Fig. 5—2

$$\omega_\theta = \frac{\partial w_r}{\partial z} - \frac{\partial w_a}{\partial r} \tag{2}$$

$$\omega_a = \frac{\partial w_\theta}{\partial r} + \frac{w_\theta}{r} - \frac{1}{r}\frac{\partial w_r}{\partial \theta} \tag{3}$$

If we neglect the effect caused by the finite number of blades by averaging it in θ-direction, and think the flow at the position far downstream of the blade row where the stream is recognized to be invariable in z-direction, we have

$$\omega_r = 0 \tag{1'}$$

$$\omega_\theta = -\frac{\partial w_a}{\partial r} \tag{2'}$$

$$\omega_a = \frac{\partial w_\theta}{\partial r} + \frac{w_\theta}{r} \tag{3'}$$

These equations can be easily solved, and we get

$$w_a = -\int \omega_\theta dr + C_a \tag{4}$$

where C_a : integration constant

$$w_\theta = \frac{1}{r}\int_0^r \omega_a r\, dr \tag{5}$$

5. 3. Vortex Motion

We consider here only stationary blade system (annular cascade). The moving blade system can be treated by adding or substracting the peripheral velocity to or from the flow field as illustrated in Fig. 5-3, provided the radial deviation of streamlines being small (solid lines in the figure are relative streamlines, broken lines are absolute streamlines, and the arrangement of vortices is applicable to the both). But since there should be difficulties when the radial deviation is large, we would postpone the treatment till another day. Because the theory of secondary flows ought to be developed generally under the assumption that the radial deviation of streamlines is sufficiently small, it can be applied to the case of moving blade system without any modification.

The following theory is a first approximation (perturbation) theory, and we think that the vortex drifts with the basic ideal flow (primary flow). (Basic flow

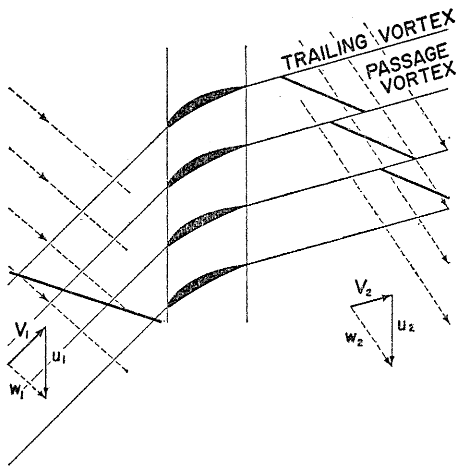


Fig. 5—3

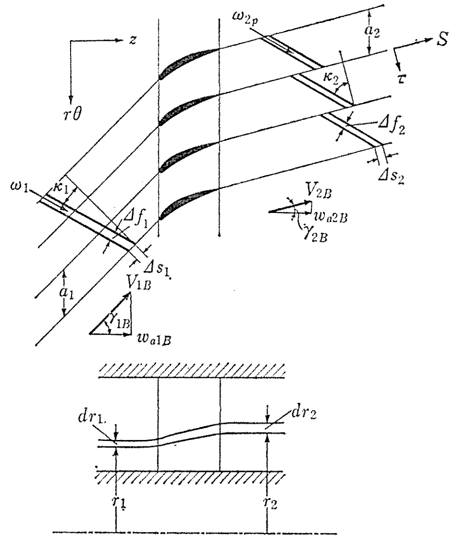


Fig. 5—4

or base flow must be selected to be a flow which can be theoretically in existence.) In the case of a linear cascade, we can choose the irrotational flow of two-dimensional cascade as the base flow. But when we consider the turbo-machine, the choice of free vortex type of flow, which corresponds to the irrotational flow of the above, as the base flow has some defect in general cases because of its severe difference from practical flow patterns (unless we consider the free vortex type of machine), and we shall be unable to get high accuracy of approximation. We choose, therefore, the axisymmetric flow as a base flow in the following treatment. Because the flow field in blade passage cannot be represented in good accuracy with such a base flow, we employ the treatment of linear cascade for the flow field near the blade element. Practically, we treat the problem by the extension of the flow field as illustrated in Fig. 5-4. Because of the radial deviation of streamlines, the extension of meridional stream surface into a plane is impossible, and the extended figures shown in Fig. 5-4 etc. must be considered to be conventional ones and not strict.

The axisymmetric flow contains generally the vortex in it. We also assume the gradient of radial deviation of streamline being small (or radial flow velocity being small), although the deviation itself is not confined to be small. This assumption was not employed in the previous axisymmetric theory, but we need it in the secondary flow theory because we feel the difficulty in the development of theory without this assumption.

The situation of flow is like Fig. 5-4. In this case, ω_1 is not perpendicular to the flow, and the circumferential (θ-wise) distance of two streamlines which pass through adjacent two blades are different before and behind the blade row.

Let the number of blades be n, we get

$$\left.\begin{aligned} a_1 n &= 2\pi r_1 \\ a_2 n &= 2\pi r_2 \end{aligned}\right\} \tag{6}$$

and

$$\left.\begin{aligned}\omega_{1\tau}&=\omega_1\cos\kappa_1\\ \omega_{1s}&=\omega_1\sin\kappa_1\end{aligned}\right\}\tag{7}$$

$$\left.\begin{aligned}\omega_{2p\tau}&=\omega_{2p}\cos\kappa_2\\ \omega_{2ps}&=\omega_{2p}\sin\kappa_2\end{aligned}\right\}\tag{8}$$

The condition of continuity for base flow is

$$V_{1B}\cos\gamma_{1B}\cdot a_1dr_1=V_{2B}\cos\gamma_{2B}\cdot a_2dr_2\tag{9}$$

or

$$\frac{w_{a1B}}{w_{a2B}}=\frac{r_2dr_2}{r_1dr_1}\tag{9'}$$

and concerning vortices we have

$$\left.\begin{aligned}\Delta f_1&=\Delta s_1\cos\kappa_1\\ \Delta f_2&=\Delta s_2\cos\kappa_2\end{aligned}\right\}\tag{10}$$

From the law of vortex

$$\omega_1\Delta f_1dr_1=\omega_{2p}\Delta f_2dr_2\tag{11}$$

and, because the vortex drifts with the fluid, we get

$$\frac{\Delta s_1}{\Delta s_2}=\frac{V_{1B}}{V_{2B}}\tag{12}$$

Calculating from equations (7) ~ (12)

$$\frac{\omega_{2p\tau}}{\omega_{1\tau}}=\frac{\cos\gamma_{2B}}{\cos\gamma_{1B}}\cdot\frac{r_2}{r_1}\tag{13}$$

$$\frac{\omega_{2ps}}{\omega_{1\tau}}=\frac{\cos\gamma_{2B}}{\cos\gamma_{1B}}\cdot\tan\kappa_2\cdot\frac{r_2}{r_1}\tag{14}$$

If we know the value of κ_2, we can get the passage vorticity ω_{2p}. (Notice, we can get $\omega_{2p\tau}$ without the knowledge of value of κ_2.) κ_2 cannot be obtained if the flow between blades is not known. This will be clarified later.

5. 3. 1. Determination of the Inclination of Passage Vortex

ω_{2p} makes an inclination angle $\left(\frac{\pi}{2}-\kappa_2\right)$ to the flow direction as illustrated in Fig. 5-4, but this should not be a straight line in a correct sense. If we allow to assume this to be straight, the inclination will be obtained from the calculation of $\overline{A_2'A_2}$ in Fig. 5-5.

$\overline{A_2'A_2}$ can be obtained from the calculation of difference of times required by a particle on the incoming stagnation streamline travelling upper or lower surfaces of the blade. We refer here to the work of Smith[20].

Let a particle of fluid at A_1 in Fig. 5-6 be reached to A_2 in t second flowing

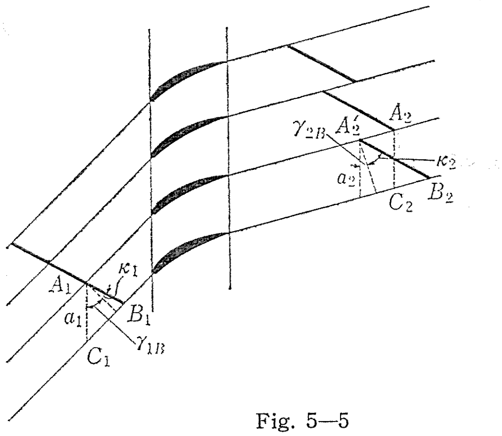


Fig. 5—5

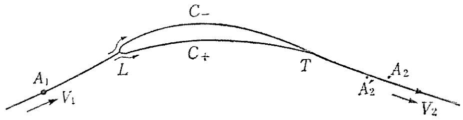


Fig. 5—6

along the upper surface of blade, and A_2' in same t second along the lower surface. The time Δt required by a particle reached at A_2' to flow from A_2' to A_2 is

$$\Delta t = \frac{\overline{A_2'A_2}}{V_2} \tag{15}$$

Δt is equal to the difference of times required by a particle to flow along upper or lower surfaces of blade. That is

$$\Delta t = \int_{L\,+}^{T} \frac{d\delta}{V} - \int_{L\,-}^{T} \frac{d\delta}{V}$$

where $+, -$ represent lower, upper surfaces and
L, T leading, trailing edges respectively.
δ is the length along blade surface.

The above equation becomes

$$\fallingdotseq \frac{\widehat{LT}_+}{\overline{V}_+} - \frac{\widehat{LT}_-}{\overline{V}_-}$$

where $\widehat{LT}$ is the distance between leading and trailing edges,
$\overline{V}$ is the averaged flow velocity.

and, furthermore

$$\fallingdotseq \overline{C}\left(\frac{1}{\overline{V}_+} - \frac{1}{\overline{V}_-}\right) = \overline{C}\frac{\overline{V}_- - \overline{V}_+}{\overline{V}_+ \cdot \overline{V}_-}$$

where $\overline{C}$ is the mean value of $\widehat{LT}_+$ and $\widehat{LT}_-$.

If we represent

$$\overline{C}(\overline{V}_- - \overline{V}_+) \fallingdotseq \Gamma$$

$$\overline{V}_+ \cdot \overline{V}_- = V_\infty^2$$

We have finally

$$\Delta t \doteqdot \frac{\Gamma}{V_{\infty}^2} \tag{16}$$

If we think that Γ is the blade circulation and V_∞ is the uniform flow velocity in the case of monoplane or the geometrical mean velocity of inflow and exit-flow in the case of cascade, this expression is known to be a good approximation[20].

There exist many problems in exact consideration of Δt. The introduction of equation (16) was wild, and furthermore the time required by a particle to reach to A_2 or A_2' from A_1 must be infinite when there exists a stagnation point at the leading or trailing edge. But, fortunately, the values of Δt, $\overline{A_2'A_2}$ or κ_2 have no serious meaning if we consider the averaged secondary flow as will be explained later, which will allow us to recognize equation (16) being sufficient for the present purpose.

Because times required by the fluid to pass $\overline{C_1B_1}$ and $\overline{C_2B_2}$ in Fig. 5-5 are equal, we have

$$\frac{\overline{C_1B_1}}{V_{1B}} = \frac{\overline{C_2B_2}}{V_{2B}} \tag{17}$$

At the same time, we have

$$\overline{C_1B_1} = a_1 \sin \gamma_{1B} + a_1 \cos \gamma_{1B} \cdot \tan \kappa_1 \tag{18}$$

and

$$\overline{A_2'A_2} = a_2 \sin \gamma_{2B} + a_2 \cos \gamma_{2B} \cdot \tan \kappa_2 - \overline{C_2B_2}$$

Equations (9), (17), (18) and the above yield

$$\overline{A_2'A_2} = a_1\left[\frac{r_2}{r_1} \sin \gamma_{2B} + \frac{r_2}{r_1} \cos \gamma_{2B} \cdot \tan \kappa_2 - \frac{\sin \gamma_{1B} \cdot \cos \gamma_{1B}}{\cos \gamma_{2B}} \frac{r_1 dr_1}{r_2 dr_2} - \frac{\cos^2 \gamma_{1B}}{\cos^2 \gamma_{2B}} \tan \kappa_1 \cdot \frac{r_1 dr_1}{r_2 dr_2}\right] \tag{19}$$

From (15) and (16)

$$\overline{A_2'A_2} = \frac{\Gamma_B}{V_\infty^2} V_{2B} \tag{20}$$

and

$$\Gamma_B = a_2 V_{2B} \sin \gamma_{2B} - a_1 V_{1B} \sin \gamma_{1B} \tag{21}$$

From (19), (20) and (21) we can get κ_2.

5. 4. Trailing Vortex

The author said in 2.1 and 2.2 that there exists the trailing vortex in the wake of cascade. We must now examine the character of this trailing vortex.

Although ω_{2p} which is illustrated in Fig. 5-4 or 5-5 is cut by the wake, this was originally a continuous vortex ω_1, and therefore ω_{2p} must be continuous according to the vortex law. This means that ω_{2p} makes a train through the wake, and we can think that the part in the wake of vortex train makes the trailing vortex.

Another idea on the trailing vortex is that there exists the trailing vortex corresponding the variation of circulation along blade span which is shown in the theory of three-dimensional mono-plane.

Although both ideas are correct and we can get the trailing vortex along the line mentioned above, the explanation which was developed by Smith[20] and will be reproduced here is more exact and clear.

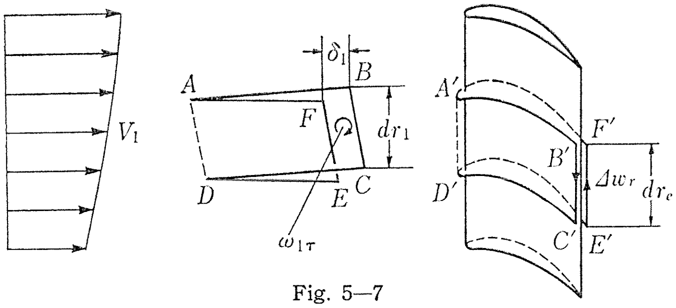


Fig. 5—7

We focus attention on one of the blade in a cascade as illustrated in Fig. 5-7. The oncoming fluid has spanwise distributed vorticity. Consider the circuit $ABCDEFA$ in the oncoming fluid which is chosen so that at some time later it will wrap itself around the blade in the position $A'B'D'E'F'A'$. The line $A'B'$, $D'C'$, $A'F'$ and $D'E'$ are selected to lie in the axisymmetric stream surfaces of the base flow. The line AF does not coincide in direction with the line AB because it is desired to have B' next to F' at the trailing edge of the blade. Therefore, due to the spanwise secondary motion, particle F and B must come from different streamlines. We assume this incoincidence being small.

Now let us examine the circulation $\Delta\Gamma$ around $ABCDEFA$. If we assume the incoincidences between AF and AB, or DE and DC being small, we have

$$\Delta\Gamma=\omega_{1T}\cdot\square EFBC=\omega_{1T}\cdot\delta_1\cdot dr_1 \tag{22}$$

Because the circulation around $A'B'C'D'E'F'A'$ is equal to the around $ABCDEFA$, we get

$$\Delta\Gamma=\Gamma_{C'D'E'}-\Gamma_{B'A'F'}+2\Delta w_T dr_e$$

where Δw_T is the spanwise velocity which appears as the secondary flow and can be recognized being same in magnitude but opposite in direction on upper and lower surfaces of blade. This may be approved according to ideas explained in Fig. 2-2.

Noting that the first two of the right side of above equation are actually blade circulations, we can write

$$\Delta\Gamma=-\frac{d\Gamma}{dr_e}dr_e+2\Delta w_T dr_e \tag{23}$$

We must notice that Γ is the actual circulation, which means the circulation in the actual flow having boundary layer, and not that in the base flow.

From (22) and (23)

$$2\Delta w_T=\left[\omega_{1T}\cdot\delta_1+\frac{d\Gamma}{dr_1}\right]\frac{dr_1}{dr_e} \tag{24}$$

δ_1 can be calculated using equation (16) as follows,

$$\delta_1 = V_1 \Delta t \doteqdot V_1 \frac{\Gamma}{V_\infty^2} \tag{25}$$

Equation (24) gives the secondary velocity which creates the trailing vortex sheet. It is seen to be made up of two parts, one being proportional to the spanwise gradient in blade circulation as in the Prandtl wing theory, and the other being a function of the vorticity in the oncoming flow. It should be noted that only the component of the vorticity which is perpendicular to the flow enters into the relation, and that the magnitude of the effect is proportional to the blade circulation as can be seen from equation (25). The Prandtl wing theory is applied only in the case of irrotational oncoming flow, and equation (24) must be used when the oncoming flow is rotational.

The strength of trailing vortex Γ_T (which is the vortex contained in unit span) is

$$\Gamma_T = -2\Delta w_T \tag{26}$$

and let us express as

$$\Gamma_T = \Gamma_{T(F)} + \Gamma_{T(S)}$$

where $\Gamma_{T(F)}$ and $\Gamma_{T(s)}$ are named the trailing filament vortex and the trailing shed vortex respectively, and

$$\Gamma_{T(F)} = -\omega_{1\tau} \cdot \delta_1 \frac{dr_1}{dr_e} \tag{27}$$

$$\Gamma_{T(S)} = -\frac{d\Gamma}{dr_e} \tag{28}$$

We put $\overline{A_2' A_2} = \delta_2$ in Fig. 5-6, and consider the immediate downstream of the exit of blade row. Then we have

$$\delta_2 = V_{2e} \Delta t \tag{29}$$

From equation (9) (we abridge subscript B in the exit flow.),

$$V_{1B} \cos\gamma_{1B} r_1 dr_1 = V_{2e} \cos\gamma_{2e} r_e dr_e \tag{30}$$

From equation (13)

$$\frac{\omega_{2p\tau}}{\omega_{1\tau}} = \frac{\cos\gamma_{2e}}{\cos\gamma_{1B}} \frac{r_e}{r_1} \tag{31}$$

Using (25), (29), (30) and (31), equation (27) becomes

$$\Gamma_{T(F)} = -\omega_{2p\tau} \cdot \delta_2 \tag{32}$$

From the idea that ω_{2p} makes a train through the wake, which was mentioned at the start of this paragraph, we get easily the same result as (32). Therefore, the above idea can be regarded being correct.

From equations (32), (31) and (19) $\Gamma_{T(F)}$ can be expressed as follows. (Subscript

e is abridged assuming the distortion of flow downstream the exit blade row being samll.)

$$\Gamma_{T(F)} = -\omega_{1T} a_2 \cos\gamma_{2B}\left[\frac{\sin\gamma_{2B}}{\cos\gamma_{1B}}\frac{r_2}{r_1} - \frac{\sin\gamma_{1B}}{\cos\gamma_{2B}}\frac{r_1}{r_2}\frac{dr_1}{dr_2} + \frac{\omega_{2ps}}{\omega_{1T}} - \frac{\omega_{1s}}{\omega_{1T}}\frac{\cos\gamma_{1B}}{\cos\gamma_{2B}}\frac{r_1}{r_2}\frac{dr_1}{dr_2}\right] \tag{33}$$

5. 5. Vortices in the Downstream of Blade Row

From the discussions above, vortices in the downstream of blade row are consisted of,

(1) the passage vortex ω_{2p}
its component parallel to the stream ω_{2ps}
its component normal to the stream on the meridional stream surface ω_{2pT}
(2) the trailing vortex Γ_T
which is consisted of trailing filament vortex $\Gamma_{T(F)}$
and trailing shed vortex $\Gamma_{T(s)}$

We must call attention to the facts that the passage vortex is the distributed vorticity, makes an angle $\frac{\pi}{2} - \kappa_2$ with the stream direction and, therefore, has components parallel and normal to the stream, on the other hand the trailing vortex is one contained in the wake and has only component in the direction of stream.

The first approximation of passage vortex (vorticity) can be calculated as shown in 5.3., when the blade row and the base flow has been given. The trailing filament vortex can be also calculated as in 5.4.. But the trailing shed vortex cannot be obtained if the slope of blade circulation is not known. Blade circulation must be the actual circulation affected not only the base flow but the secondary flows (see 5.4.). Its solution, therefore, must be done from the other point of view and will be explained later. (In Prandtl wing theory the actual circulation is calculated by considerations in which the induced velocity at the wing position and attack angle-lift characteristics of the wing of ∞ aspect ratio are used, but in the case of cascade other ideas are used which will be discussed later.)

5. 5. 1. To take the Average of Trailing Vortex and the Vorticities in the Downstream of Blade Row

The passage vortex is a distributed vortex, and the trailing vortex is a confined vortex, and these mean that they are different kind of vortices which must be treated in different ways. But when we treat the flow in turbo-machine, we often desire to take the peripheral average of flow, and considering the axisymmetric theory is one example of averaging, let us also try to do so on the secondary flow. To get the peripheral mean value is not equal to get the axisymmetric flow. To get the flow through blade row of finite blade pitch at first, then to calculate the peripheral mean value, are the fundamental ideas, which are somewhat different from the axisymmetric considerations (finite pitch is different from infinitesimal pitch). In this paragraph we consider the averaging of trailing vortices, and some different result from the axisymmetric consideration was expected, but we have get the coincidence of results provided the treatment is confined in the following.

The trailing vortices are

$$\Gamma_T = \Gamma_{T(F)} + \Gamma_{T(S)} \tag{34}$$

Peripheral average of these are

$$\omega_{2T} = \frac{\Gamma_T}{a_2 \cos \gamma_2} \quad \text{etc.} \tag{35}$$

where ω_{2T} is the vorticity as the result of averaging the vortex, and we have

$$\omega_{2T} = \omega_{2T(F)} + \omega_{2T(S)} \tag{36}$$

We had better now refer again to the vortices in the downstream of blade row, which are, as explained in 5. 5.,

(1) The component of vortex perpendicular to flow on the meridional stream surface $\omega_{2\tau}$ is consisted only of the normal component of passage vortex ω_{2p}, which is from equation (13)

$$\omega_{2\tau} = \omega_{2p\tau} = \omega_{1\tau} \frac{\cos \gamma_{2B}}{\cos \gamma_{1B}} \frac{r_2}{r_1} \tag{37}$$

(2) The streamwise component of vortex ω_{2s} is consisted of the streamwise component of passage vortex and two kinds of trailing vortices. If we take averaged values we have

$$\omega_{2S} = \omega_{2ps} + \omega_{2T} = \omega_{2ps} + \omega_{2T(F)} + \omega_{2T(S)} \tag{38}$$

5. 5. 2. Quasi Vortex

Considering the trailing vortex as vorticity distributed we have from (33) and (35)

$$\omega_{2ps} + \omega_{2T(F)} = -\omega_{1\tau} \left[\frac{\sin \gamma_{2B}}{\cos \gamma_{1B}} \frac{r_2}{r_1} - \frac{\sin \gamma_{1B}}{\cos \gamma_{2B}} \frac{r_1}{r_2} \frac{dr_1}{dr_2} \right] + \omega_{1s} \frac{\cos \gamma_{1B}}{\cos \gamma_{2B}} \frac{r_1}{r_2} \frac{dr_1}{dr_2} \tag{39}$$

This relation shows that the sum of the streamwise component of passage vorticity and the trailing filament vorticity is got without the knowledge of blade profile etc. (the knowledge of κ_2). (We must remember the trailing shed vorticity $\omega_{2T(s)}$ is got from other points of view.) In reality, because the fact mentioned above has some meaning which will be explained in the following, the author has named the sum of both vorticities "quasi vortex".

We regard that the base flow is deflected merely mechanically at the actuator disc, and vortices are carried mechanically by this flow (see Fig. 5-8). Let us now think the blade row being a disc, in which an axially stretched disc is permitted. The important point is that the flow is turned at the disc. A vortex at A_1B_1 is carried to the position A_2B_2. The time required is

$$t_{A1\sim A2} = t_{B1\sim B2}$$

and the time required by the flow between $A_1 \sim A_2'$ is equal to that between $B_1' \sim B_2$. That is

$$t_{A1\sim A2'} = t_{B1'\sim B2}$$

Therefore

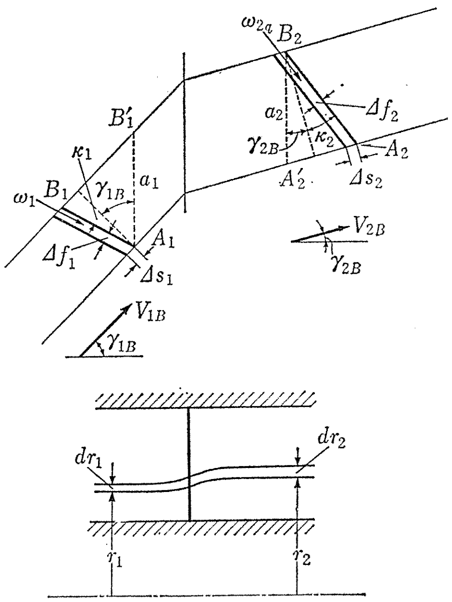


Fig. 5—8

$$t_{A2'\sim A2}=t_{B1\sim B1'}$$

we rewrite this as

$$\frac{a_1\sin\gamma_{1B}+a_1\cos\gamma_{1B}\cdot\tan\kappa_1}{V_{1B}}=\frac{a_2\sin\gamma_{2B}+a_2\cos\gamma_{2B}\cdot\tan\kappa_2}{V_{2B}} \tag{40}$$

From the condition of continuity, we get

$$V_{1B}\cos\gamma_{1B}\cdot a_1 dr_1=V_{2B}\cos\gamma_{2B}\cdot a_2 dr_2 \tag{41}$$

and

$$\frac{a_1}{a_2}=\frac{r_1}{r_2} \tag{42}$$

From (40), (41) and (42), we have after calculations

$$\tan\kappa_2=\frac{1}{\cos^2\gamma_{2B}}\left\{\cos\gamma_{1B}\cdot\sin\gamma_{1B}\cdot\left(\frac{r_1}{r_2}\right)^2\frac{dr_1}{dr_2}-\cos\gamma_{2B}\cdot\sin\gamma_{2B}\right.$$

$$\left.+\cos^2\gamma_{1B}\cdot\tan\kappa_1\cdot\left(\frac{r_1}{r_2}\right)^2\frac{dr_1}{dr_2}\right\} \tag{43}$$

Since the vortex drifts with the fluid, we have

$$\frac{\Delta s_1}{V_{1B}}=\frac{\Delta s_2}{V_{2B}} \tag{44}$$

and since the strength of vortex does not change, we get

$$\omega_1 \Delta f_1 dr_1 = \omega_2 \Delta f_2 dr_2 \tag{45}$$

and

$$\Delta f_1 = \Delta s_1 \cos\kappa_1 \tag{46}$$

$$\Delta f_2 = \Delta s_2 \cos\kappa_2 \tag{47}$$

From (44) ~ (47), we get after calculations

$$\omega_{2q\tau} = \omega_{2q}\cos\kappa_2 = \omega_{1\tau}\frac{\cos\gamma_{2B}}{\cos\gamma_{1B}}\frac{r_2}{r_1} \tag{48}$$

And from

$$\omega_{2qs} = \omega_{2q}\sin\kappa_2 = \omega_{1\tau}\frac{\cos\gamma_{2B}}{\cos\gamma_{1B}}\frac{r_2}{r_1}\tan\kappa_2 \tag{49}$$

we obtain

$$\omega_{2qs} = -\omega_{1\tau}\left[\frac{\sin\gamma_{2B}}{\cos\gamma_{1B}}\frac{r_2}{r_1} - \frac{\sin\gamma_{1B}}{\cos\gamma_{2B}}\frac{r_1}{r_2}\frac{dr_1}{dr_2}\right] + \omega_{1s}\frac{\cos\gamma_{1B}}{\cos\gamma_{2B}}\frac{r_1}{r_2}\frac{dr_1}{dr_2} \tag{50}$$

Equation (48) coincides with (37), and (50) with (39).

The theory treated in the above is started from the idea that the base flow was turned at the actuator disc and the vortex was carried mechanically by it. The coincidence of equations (50) and (39) reveals that the trailing shed vortex is not contained in this theory, namely this theory is the quasi vortex theory. This situation is the important point of question existing in this theory ……, nay, in the whole secondary flow theory. The trailing shed vortex must be calculated from another idea as mentioned formerly.

The author(15) once regarded the trailing shed vortex to be 0. This is, of course, misunderstanding because of the ignorance of conditions which decide the shed vortex.

Smith(20) considered this quasi vortex being the axisymmetric flow, and thought the difference of this quasi vortex and passage vortex being the secondary vortex (because the difference of axisymmetric flow and passage vortex flow is the secondary flow). But we have already recognized that this difference is no more than the trailing filament vortex as shown by equation (39). Here is also the problem of forgetting the trailing shed vortex.

The idea that the flow is deflected at the blade disc and the vortex is carried merely mechanically by it is nothing but the treatment of the flow at bend which was explained at the beginning of the consideration of secondary flows, and there is no cascade or blade row. The question what is the difference between the existence and non-existence of cascade is answered by the existence or non-existence of the trailing shed vortex. Therefore, the method of taking the existence of cascade or the trailing shed vortex into calculation becomes an important point of the secondary flow theory of cascades. This will be explained in another chapter.

5. 5. 3. The Correlating Equations before and behind Cascades

From equations (39) and (37),

$$\omega_{2s}=-\omega_{1\tau}\left[\frac{\sin\gamma_{2B}}{\cos\gamma_{1B}}\frac{r_2}{r_1}-\frac{\sin\gamma_{1B}}{\cos\gamma_{2B}}\frac{r_1}{r_2}\frac{dr_1}{dr_2}\right]$$

$$+\omega_{1s}\frac{\cos\gamma_{1B}}{\cos\gamma_{2B}}\frac{r_1}{r_2}\frac{dr_1}{dr_2}+\omega_{2T(s)} \tag{51}$$

We write equation (37) again,

$$\omega_{2\tau}=\omega_{1\tau}\frac{\cos\gamma_{2B}}{\cos\gamma_{1B}}\frac{r_2}{r_1} \tag{52}$$

Now we change expressions ω_s, ω_τ into ω_a, ω_θ because of convenience. (We abridge subscript B.)

$$\left.\begin{aligned}\omega_{1s}&=\omega_{a1}\cos\gamma_1-\omega_{\theta1}\sin\gamma_1\\ \omega_{1\tau}&=\omega_{a1}\sin\gamma_1+\omega_{\theta1}\cos\gamma_1\end{aligned}\right\} \tag{53}$$

$$\left.\begin{aligned}\omega_{a2}&=\omega_{2s}\cos\gamma_2+\omega_{2\tau}\sin\gamma_2\\ \omega_{\theta2}&=-\omega_{2s}\sin\gamma_2+\omega_{2\tau}\cos\gamma_2\end{aligned}\right\} \tag{54}$$

Using these relations, we reform equations (51) and (52),

$$\omega_{a2}=\omega_{a1}\frac{w_{a2B}}{w_{a1B}}+\omega_{2T(s)}\cos\gamma_{2B} \tag{55}$$

$$\omega_{\theta2}=\omega_{a1}\left[\frac{r_2}{r_1}\tan\gamma_{1B}-\frac{w_{a2B}}{w_{a1B}}\tan\gamma_{2B}\right]+\omega_{\theta1}\frac{r_2}{r_1}-\omega_{2T(s)}\sin\gamma_{2B} \tag{56}$$

Eliminating w_{a2B}/w_{a1B} from (55) and (56), we have

$$\frac{\omega_{\theta1}}{r_1}+\frac{\omega_{a1}}{r_1}\tan\gamma_{1B}=\frac{\omega_{\theta2}}{r_2}+\frac{\omega_{a2}}{r_2}\tan\gamma_{2B} \tag{57}$$

The trailing shed vortex has disappeared in the process of introduction of this equation. We can understand from this situation that the trailing shed vortex cannot be obtained unless other ideas are introduced. This equation can be recognized being identical to equation (20) in 4.2.. (The difference of γ_B or γ depends on ideas that the vortex drifts with the base flow or the resulting flow including the induced flow caused by the vortex itself, and is the difference of first approximation or exact solution.)

5. 5. 4. *Induced Velocities by Averaged Vortices*

From the discussion mentioned above we have understood that we can get same results as the axisymmetric theory by averaging vortices. Let us now try to get induced velocities which correspond to that of quasi vortex, where the trailing shed vortex is omitted since it has not yet been solved. We assume the radial velocity component being small.

Let the variation from base flow (induced velocity) be expressed by Δw. We consider the peripheral induced velocity at first. From equation (5)

$$\Delta w_{\theta1}=\frac{1}{r_1}\int_0^{r_1}\omega_{a1}r_1dr_1 \tag{58}$$

$$\Delta w_{\theta 2}=\frac{1}{r_2}\int_0^{r_2}\omega_{a2}r_2dr_2 \tag{59}$$

Substituting (55) into (59), we get using (58) (where we neglect $\omega_{2T(s)}$ which is put to be 0),

$$\Delta w_{\theta 2}=\frac{r_1}{r_2}\Delta w_{\theta 1} \tag{60}$$

On axial induced velocities, we have from (2′), (3′) and (4)

$$\left.\begin{aligned}\omega_{a1}&=\frac{\partial\Delta w_{\theta 1}}{\partial r_1}+\frac{\Delta w_{\theta 1}}{r_1}\\ \omega_{\theta 1}&=-\frac{\partial w_{a1}}{\partial r_1}\end{aligned}\right\} \tag{61}$$

$$\Delta w_{a2}=-\int\omega_{\theta 2}dr_2+C_{a2} \tag{62}$$

Substituting (56) into (62) (where we put $\omega_{2T(s)}=0$), we get using (61) and (9′)

$$\begin{aligned}\Delta w_{a2}=&-\int\left[\tan\gamma_{1B}-\tan\gamma_{2B}\cdot\left(\frac{r_1}{r_2}\right)^2\frac{dr_1}{dr_2}\right]\frac{w_{a1B}}{w_{a2B}}\frac{1}{r_1}d(r_1\cdot\Delta w_{\theta 1})\\ &+\int\frac{w_{a1B}}{w_{a2B}}d(\Delta w_{a1})+C_{a2}\end{aligned} \tag{63}$$

C_{a2} is decided from the condition of continuity of flow quantity. (This cannot be immediately obtained since we have not yet get the induced velocity by $\omega_{2T(s)}$)

Blade circulation is,

$$\Gamma=-a_1w_{\theta 1}+a_2w_{\theta 2} \tag{64}$$

therefore,

$$\Delta\Gamma=-a_1\Delta w_{\theta 1}+a_2\Delta w_{\theta 2} \tag{64′}$$

Because of equation (60) and $a_1/a_2=r_1/r_2$, we get

$$\Delta\Gamma=0 \tag{65}$$

We can say, therefore, that the blade circulation does not vary by the flow induced by the passage vortex and the trailing filament vortex. We must notice that the actual blade circulation is not invariable, and the trailing shed vortex which corresponds to this variation exists in the downstream besides the vortex of base flow and the quasi vortex (passage vortex plus trailing filament vortex).

Because of the complexity of equation (63) let us simplify it under some assumptions. We assume at first the radial deviation of streamline being small, that is

$$\frac{r_1}{r_2}\doteqdot 1 \tag{66}$$

and

$$\frac{w_{a1B}}{w_{a2B}} \doteqdot \frac{dr_2}{dr_1} \tag{67}$$

Under these assumptions, equations (60) and (63) become

$$\Delta w_{\theta 2} = \Delta w_{\theta 1} \tag{68}$$

$$\Delta w_{a2} = -\int \left[\tan\gamma_{1B} - \tan\gamma_{2B} \cdot \frac{dr_1}{dr_2} \right] \frac{w_{a1B}}{w_{a2B}} \frac{1}{r} d(r \cdot \Delta w_{\theta 1})$$
$$+ \int \frac{w_{a1B}}{w_{a2B}} d(\Delta w_{a1}) + C_{a2} \tag{69}$$

Let the blade circulation in the base flow be Γ_B. We have under the assumption mentioned above,

$$\Gamma_B = -aw_{\theta 1B} + aw_{\theta 2B} = aw_{a1B} \left\{ \tan\gamma_{1B} - \tan\gamma_{2B} \cdot \frac{dr_1}{dr_2} \right\} \tag{70}$$

Substituting this into (69),

$$\Delta w_{a2} = -\frac{n}{2\pi} \int \frac{\Gamma_B}{w_{a1B}} \cdot \frac{w_{a1B}}{w_{a2B}} \frac{1}{r} d(r \cdot \Delta w_{\theta 1}) + \int \frac{w_{a1B}}{w_{a2B}} d(\Delta w_{a1}) + C_{a2} \tag{71}$$

6. Determination of Vortices in the Downstream (Determination of the Trailing Shed Vortex)

Neither axisymmetric theory nor secondary flow theory could lead us to the trailing (shed) vortex.

In the secondary flow theory, the trailing vortex was determined in the past from the consideration that the trailing vortex is consisted of spanwise velocities induced at the upper and lower surfaces of blade trailing edge. This is the idea already explained in many reports and is to be calculated by the method stated in 2. 3..

In the axisymmetric theory, various ideas appeared without sufficient successes(6)(10). But this can be solved easily by the consideration of the out-going flow of blade row as explained in the following.

The reader should be careful that the vortex does not drift with the base flow but with the final flow including induced velocities by the vortex itself, which is just same as was explained in 3.. There is no subscript B, therefore, on the velocity or the angle.

6. 1. Determination of the Trailing Vortex in the Axisymmetric Theory

If we regard the axisymmetric theory being a case of infinitesimally small blade spacing, we can apply the idea of secondary flow theory to it, although the latter is originally applied to the case of finite spacing.

Vortices in the flow just behind the blade row are consisted of passage vorticity ω_{2p} and trailing vortex $\Gamma_T (= \Gamma_{T(F)} + \Gamma_{T(s)})$ (if we consider the trailing vortex being distributed, it is trailing vorticity ω_{2T}), and the direction of flow is the exit direction of blade row which is given by γ_{2e}. Vorticities parallel to the flow (s-direction) and normal to both the flow and the span (τ-direction) are respectively

$$\omega_{2s}=\omega_{2ps}+\omega_{2T} \tag{1}$$

$$\omega_{2\tau}=\omega_{2p\tau} \tag{2}$$

where

$$\omega_{2T}=\omega_{2T(F)}+\omega_{2T(s)}$$

For the sake simplifying explanation, let us consider a case in which radial component of velocity can be neglected (where the change of radial position of streamline is arbitrary), then we have

$$\left.\begin{aligned}\omega_s&=\omega_a\cos\gamma-\omega_\theta\sin\gamma\\ \omega_\tau&=\omega_a\sin\gamma+\omega_\theta\cos\gamma\end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned}\omega_a&=\frac{\partial w_\theta}{\partial r}+\frac{w_\theta}{r}\\ \omega_\theta&=-\frac{\partial w_a}{\partial r}\end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned}w_a&=V\cos\gamma\\ w_\theta&=-V\sin\gamma\end{aligned}\right\} \tag{5}$$

From these we get following equations just behind the exit of blade row,

$$\omega_{2s}=-V_{2e}\frac{\partial\gamma_{2e}}{\partial r_2}-\frac{V_{2e}}{r_2}\cos\gamma_{2e}\cdot\sin\gamma_{2e} \tag{6}$$

$$\omega_{2\tau}=-\frac{\partial V_{2e}}{\partial r_2}-\frac{V_{2e}}{r_2}\sin^2\gamma_{2e} \tag{7}$$

(where subscript e which must be attached to ω was omitted.) From equations (1), (2), (6) and (7) we get

$$-V_{2e}\frac{\partial\gamma_{2e}}{\partial r_2}-\frac{V_{2e}}{r_2}\cos\gamma_{2e}\cdot\sin\gamma_{2e}=\omega_{2ps}+\omega_{2T} \tag{8}$$

$$-\frac{\partial V_{2e}}{\partial r_2}-\frac{V_{2e}}{r_2}\sin^2\gamma_{2e}=\omega_{2p\tau} \tag{9}$$

Since if we assume $r_2 \doteqdot r_1$, we can get $\omega_{2p\tau}$ from equation (13) in chapter 5 and V_{2e} from equation (9). We can have, therefore, $\omega_{2ps}+\omega_{2T}$ from equation (8). If we want to get ω_{2ps} and ω_{2T} separately, we get at first ω_{2ps} with the aid of knowledge of flow in blade passage and then get ω_{2T}, but provided that we know $\omega_{2ps}+\omega_{2T}$ we can get vorticity components just behind blade row from equations

$$\left.\begin{aligned}\omega_{a2}&=(\omega_{2ps}+\omega_{2T})\cos\gamma_{2e}+\omega_{2p\tau}\sin\gamma_{2e}\\ \omega_{\theta 2}&=-(\omega_{2ps}+\omega_{2T})\sin\gamma_{2e}+\omega_{2p\tau}\cos\gamma_{2e}\end{aligned}\right\} \tag{10}$$

If there exists an assumption that the distortion of flow is small, these can be regarded immediately as the vorticity component in the downstream.

Thus we have been able to get the vorticity (or trailing vortex) in the downstream by the consideration of condition just behind the exit of blade row. (see 6. 2.)

When the spacing of blades is infinitesimal, the velocity component perpendicular to blade surfaces at the exit of blade passage is negligiblly small [this component exists as the secondary flow when the spacing is finite (see Fig. 2-2)]. Accordingly we can say that the component of secondary velocities perpendicular to the main flow and the span (τ-component) is *0*. In this case, therefore, the flow angle γ_{2e} just behind the blade row can be considered being the exit angle of the blade row (which may be equal to the efflux angle of two-dimensional cascade of the blade element, because there is no secondary velocity in the two-dimensional cascade and τ-component of it is *0*). This means that γ_{2e} can be fixed when the cascade is given. This is the reason why we used the values just behind the blade row exit (subscript e) in the expression mentioned above. We have the similar equation at the far downstream of blade row, but we cannot fix γ_2 beforehand in this case.

6. 2. Considerations

We could find in previous considerations that we can get the vorticity in the downstream including the trailing shed vortex, but it had been unsucceeded to get it in a neat form. The author will, therefore, enumerate what he did in hopeing to be the information for the future study.

At first we summarize various equations for the convenience of considerations, where we assume,

◎ radial component of velocity is negligible,
◎ where the radial position (variation) of streamline is arbitrary,
◎ we consider arbitrary axial positions in upstream and downstream of blade row,
◎ it is axisymmetric flow, and
◎ we treat the finally built up flow incluing induced velocities by vortices (theoretically real flow, not the first approximation).

From equation (8),

$$-V_2\frac{\partial\gamma_2}{\partial r_2}-\frac{V_2}{r_2}\cos\gamma_2\cdot\sin\gamma_2=\omega_{2ps}+\omega_{2T(F)}+\omega_{2T(s)} \tag{11}$$

From equation (9),

$$-\frac{\partial V_2}{\partial r_2}-\frac{V_2}{r_2}\sin^2\gamma_2=\omega_{2p\tau} \tag{12}$$

From equation (13) in chapter 5.,

$$\omega_{2p\tau}=\frac{\cos\gamma_2}{\cos\gamma_1}\cdot\frac{r_2}{r_1}\omega_{1\tau} \tag{13}$$

The equation of circulation of blade,

$$\Gamma=\frac{2\pi}{n}(-r_1V_1\sin\gamma_1+r_2V_2\sin\gamma_2)=\frac{2\pi}{n}(r_1w_{\theta 1}-r_2w_{\theta 2}) \tag{14}$$

where $n \longrightarrow \infty$

The equation of trailing shed vortex (vorticity) is given from equations (28) and (35) in chapter 5.,

$$\omega_{2T(s)}=-\frac{n}{2\pi}\frac{1}{r_2\cos\gamma_2}\frac{d\Gamma}{dr_2} \tag{15}$$

From equation (39) in chpater 5.,

$$\omega_{2ps}+\omega_{2T(F)}=-\omega_{1\tau}\left[\frac{\sin\gamma_2}{\cos\gamma_1}\frac{r_2}{r_1}-\frac{\sin\gamma_1}{\cos\gamma_2}\frac{r_1dr_1}{r_2dr_2}\right]+\omega_{1s}\frac{\cos\gamma_1}{\cos\gamma_2}\frac{r_1dr_1}{r_2dr_2} \tag{16}$$

Equation of continuity,

$$\frac{r_1}{r_2}\frac{dr_1}{dr_2}=\frac{V_2\cos\gamma_2}{V_1\cos\gamma_1}=\frac{w_{a2}}{w_{a1}} \tag{17}$$

Equation of vorticity,

$$\left.\begin{aligned}\omega_\theta&=-\frac{\partial w_a}{\partial r}\\ \omega_a&=\frac{\partial w_\theta}{\partial r}+\frac{w_\theta}{r}=\frac{1}{r}\frac{\partial(rw_\theta)}{\partial r}\end{aligned}\right\} \tag{18}$$

Relations of vorticity components

$$\left.\begin{aligned}\omega_s&=\omega_a\cos\gamma-\omega_\theta\sin\gamma\\ \omega_\tau&=\omega_a\sin\gamma+\omega_\theta\cos\gamma\end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned}\omega_a&=\omega_s\cos\gamma+\omega_\tau\sin\gamma\\ \omega_\theta&=-\omega_s\sin\gamma+\omega_\tau\cos\gamma\end{aligned}\right\} \tag{20}$$

Equation (16) can be written in the following form, using (13) and (17),

$$\omega_{2ps}+\omega_{2T(F)}=-\omega_{2p\tau}\tan\gamma_2+\omega_{1\tau}\frac{V_2}{V_1}\tan\gamma_1+\omega_{1s}\frac{V_2}{V_1} \tag{21}$$

Substituting (14) into (15) and using (18),

$$\omega_{2T(s)}=\frac{1}{\cos\gamma_2}\left\{\omega_{a1}\frac{w_{a2}}{w_{a1}}-\omega_{a2}\right\} \tag{22}$$

From equation (10),

$$\omega_{a2}=(\omega_{2ps}+\omega_{2T(F)}+\omega_{2T(s)})\cos\gamma_2+\omega_{2p\tau}\sin\gamma_2 \tag{23}$$

$$\omega_{\theta 2}=-(\omega_{2ps}+\omega_{2T(F)}+\omega_{2T(s)})\sin\gamma_2+\omega_{2p\tau}\cos\gamma_2 \tag{24}$$

We get $\omega_{2p\tau}$ from (13), and $(\omega_{2ps}+\omega_{2T(F)}+\omega_{2T(s)})$ from (11) and (12), furthermore

ω_{a2} and $\omega_{\theta 2}$ from (23) and (24), we can obtain, therefore, $\omega_{2T(s)}$ from (22).

6. 2. 1. The Case having no Vortex in upstream of Blade Row

In this case, we have from (13) and (16),

$$\omega_{2pT}=0$$

$$\omega_{2ps}+\omega_{2T(F)}=0$$

we get, therefore, from (11)

$$\omega_{2T(s)}=-V_2\frac{\partial\gamma_2}{\partial r_2}-\frac{V_2}{r_2}\cos\gamma_2\cdot\sin\gamma_2 \qquad (25)$$

On the other hand , substituting (14) into (15), we have (abridging lengthy calculations in which we use the fact that the flow in upstream is of free vortex type),

$$\omega_{2T(s)}=-V_2\frac{\partial\gamma_2}{\partial r_2}-\frac{V_2}{r_2}\cos\gamma_2\cdot\sin\gamma_2$$

This is perfectly same as (25), which means that there exists only the trailing shed vortex in the downstream corresponding the variation of blade circulation (as a matter of course).

6. 2. 2. The Case in which Exit Flow is of Free Vortex Type

The meaning of *free vortex type* is that γ_{2e} has the following character,

$$\tan\gamma_{2e}=\frac{K}{r_2} \qquad (26)$$

where K is a constant. (In this case we don't think that the axial velocity is constant which is a special feature of free vortex type. This type of flow is one where the true free vortex is realized when the flow is ideal in which no secondary flow exists. From equation (11) we have directly,

$$\omega_{2ps}+\omega_{2T(F)}+\omega_{2T(s)}=0 \qquad (27)$$

If the blade row is a linear cascade, γ_{2e} is constant which is a special case of the above, and naturally we get the same result. Accordingly, we recognize that in axisymmetric condition the free vortex type of blading or the linear cascade has no streamwise or so-called secondary vortex in the downstream of blade row. This result coincides with the result obtained by Preston[18]. The author wants to suggest to say this as the vortex rectification of cascade.

6. 2. 3. Axial and tangential Component of Vortices

In many occasions in the treatment of axial flow machine we find that axial and peripheral components of vortices are more convenient to handle than expressions of streamwise component and perpendicular one to it. Let us consider here the components of quasi-vorticity ω_q.

From (20), (13) and (21), we have

$$\omega_{qa}=\frac{w_{a2}}{w_{a1}}\omega_{a1} \qquad (28)$$

and from (20), (13), (21) and (19),

$$\omega_{q\theta}=\left[\frac{r_2}{r_1}\tan\gamma_1-\frac{w_{a2}}{w_{a1}}\tan\gamma_2\right]\omega_{a1}+\frac{r_2}{r_1}\omega_{\theta 1} \tag{29}$$

When the inflow is axial and there exist boundary layers along the casing, which is the case that ω_1 has only tangential component, $\omega_{a1}=0$, we have from equations (28) and (29),

$$\left.\begin{aligned}\omega_{qa}&=0\\ \omega_{q\theta}&=\frac{r_2}{r_1}\omega_{\theta 1}\end{aligned}\right\} \tag{30}$$

We find that the quasi-vortex has only tangential component, and this can be easily understood from the fact explained in 5. 5. 4. that the blade circulation does not change. It is clear that the blade circulation should be changed if the quasi-vortex has the axial component. We must not forget that the actual flow has another vortex, i. e. the trailing shed vortex.

When ω_1 has only axial component, ω_q has axial and tangential components which are from (28) and (29),

$$\omega_{qa}=\frac{w_{a2}}{w_{a1}}\omega_{a1} \tag{31}$$

$$\omega_{q\theta}=\left[\frac{r_2}{r_1}\tan\gamma_1-\frac{w_{a2}}{w_{a1}}\tan\gamma_2\right]\omega_{a1} \tag{32}$$

Let us change here our consideration to another problem, from equation (14)

$$\frac{\Gamma}{a_1 w_{a1}}=\left[-\tan\gamma_1+\frac{r_2}{r_1}\frac{w_{a2}}{w_{a1}}\tan\gamma_2\right] \tag{33}$$

where

$$a_1=\frac{2\pi r_1}{n}$$

The similarity of this to the coefficient of ω_{a1} in equation (29) is interesting. Assuming

$$\frac{r_2}{r_1}\doteqdot 1$$

and substituting this into (29), we get

$$\omega_{q\theta}\doteqdot\omega_{\theta 1}-\frac{\Gamma}{a w_{a1}}\omega_{a1} \tag{34}$$

And from (14)

$$w_{\theta 2}\doteqdot w_{\theta 1}-\frac{\Gamma}{a w_{a1}}w_{a1} \tag{35}$$

If we express ω_{a1} and w_{a1} in a same magnitude, ω_{qa} and w_{a2} become equal and the tangential induced vorticity $\frac{\Gamma}{aw_{a1}}\omega_{a1}$ and velocity $\frac{\Gamma}{aw_{a1}}w_{a1}$ become also equal. These conditions are illustrated as in Fig, 6-1. But the significance of this fact is not clear because this is the consideration only of the quasi vortex in which the trailing shed vortex was omitted.

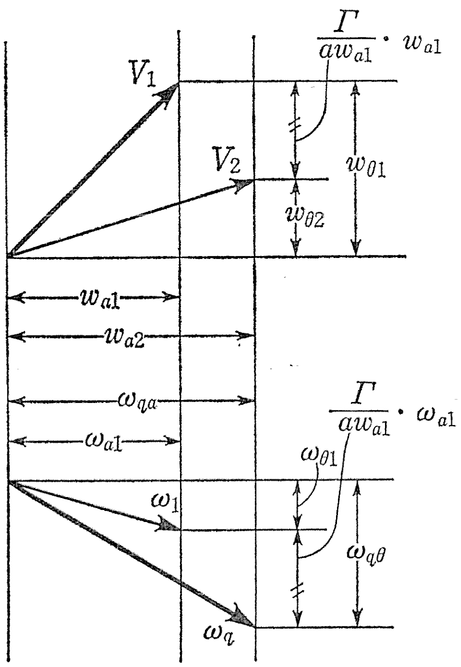


Fig. 6—1

6. 2. 4. The Case in which ω_1 has only component Perpendicular to Flow (τ-component)

This corresponds to the linear cascade experiment.
Since

$$\omega_{1s}=0$$

we have from equations (13), (16) and (17) [see also equation (37) in chapter 5.],

$$\omega_{2\tau}=\omega_{2p\tau}=\frac{\cos\gamma_2}{\cos\gamma_1}\frac{r_2}{r_1}\omega_{1\tau} \tag{36}$$

$$\omega_{qs}=\omega_{2ps}+\omega_{2T(F)}=-\left[\frac{\sin\gamma_2}{\cos\gamma_1}\frac{r_2}{r_1}-\frac{\sin\gamma_1}{\cos\gamma_2}\frac{w_{a2}}{w_{a1}}\right]\omega_{1\tau} \tag{37}$$

The trailing shed vortex must be added to these.

6. 2. 5. The Case in which ω_1 has only Streamwise Component

This may be a rare case. Since

$$\omega_{1T}=0$$

we have from equations (13), (16) and (17)

$$\omega_{2T}=\omega_{2pT}=0 \tag{38}$$

$$\omega_{qs}=\omega_{2ps}+\omega_{2T(F)}=\frac{V_2}{V_1}\omega_{1s} \tag{39}$$

The trailing shed vortex is added in the downstream of the cascade, and anyhow there exist only streamwise vortices in the flow.

6. 3. Determination of Trailing vortices by the Secondary Flow theory

As aforesaid the trailing vortex in the axisymmetric theory was obtained from the equation of vortex at the cascade exit side. Then how is in the secondary flow theory? On the linear cascade, this can be calculated from the consideration of flow in the Trefftz plane at cascade exit, and this process is regarded being correct (see 3. 3.). This method will be explained in the next chapter.

If we want to apply the same idea to the blade row of turbomachine, we should treat the problem from the consideration of such a surface $B_1C_1C_5B_5B_1$ illustrated in Fig. 6-2[25] as Trefftz plane, but there was no attempt of doing it and is no ample hope for success[6]. The author, therefore, cannot do further explanations.

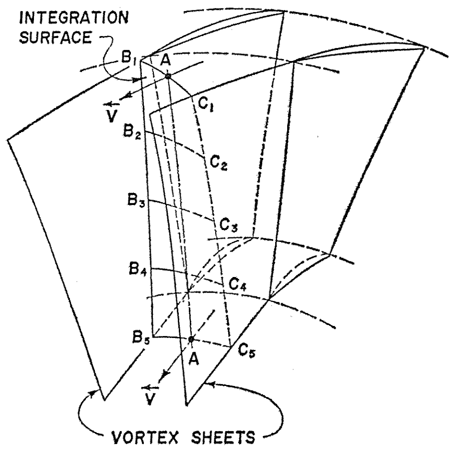


Fig. 6—2

7. Theory of Secondary Flow in Linear Cascades

The theory of secondary flow in linear cascades corresponds to the special case of the theory of secondary flow in turbomachines which was stated in chapters 5 and 6. The situation, that the theory of secondary flow in turbomachines has not yet progressed at present beyond the stage of axisymmetric theory and the treatment on cascade of finite spacing is pending[26], must be compared to the one that the linear cascade can be anyhow advanced to the latter stage. The adaption of two-dimensional cascade flow as a base flow is a further good point in approximation in comparison with the axisymmetric flow used in turbomachines. After all, since the examination of the secondary flow in the form of linear cascades is supposed to be useful for the clarification of merits or defects of secondary flow theory, the theory will be explained again in this new chapter.

7. 1. Vortex Motions and Attitudes

Vortices in the flow field are thought to be as illustrated in Fig. 7-1. When the side wall boundary layer of cascade windtunnel is considered, ω_1 is perpendicular to the flow and $\kappa_1 = 0$, but let us consider now a more generalized case.

For the sake of comparison with the theory of secondary flows in axial-flow turbo-machinery (chapter 5.) equation numbers are matched to that of the former, which means that the reader will recognize some missing numbers.

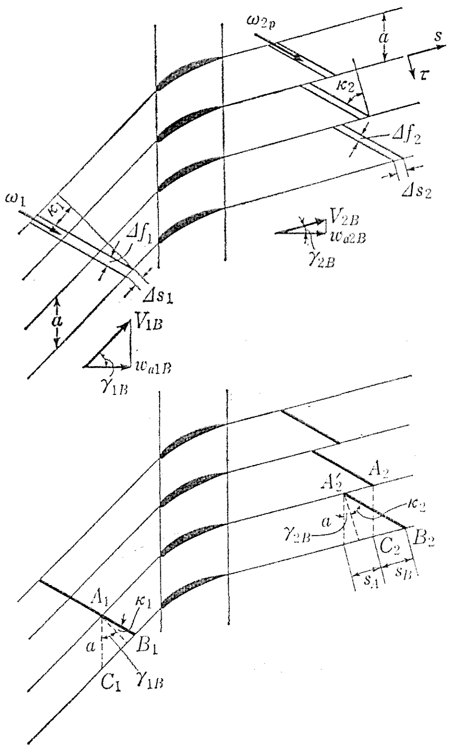


Fig. 7—1

$$\left.\begin{aligned}\omega_{1\tau} &= \omega_1 \cos\kappa_1 \\ \omega_{1s} &= \omega_1 \sin\kappa_1\end{aligned}\right\} \qquad (7)$$

$$\left.\begin{aligned}\omega_{2p\tau} &= \omega_{2p} \cos\kappa_2 \\ \omega_{2ps} &= \omega_{2p} \sin\kappa_2\end{aligned}\right\} \qquad (8)$$

The condition of continuity of base flow is

$$V_{1B}\cos\gamma_{1B}=V_{2B}\cos\gamma_{2B} \tag{9}$$

or

$$w_{a1B}=w_{a2B} \tag{9'}$$

On the vortex we have,

$$\left.\begin{aligned}\Delta f_1&=\Delta s_1\cos\kappa_1\\ \Delta f_2&=\Delta s_2\cos\kappa_2\end{aligned}\right\} \tag{10}$$

From the vortex law,

$$\omega_1\Delta f_1=\omega_{2p}\Delta f_2 \tag{11}$$

and

$$\frac{\Delta s_1}{\Delta s_2}=\frac{V_{1B}}{V_{2B}} \tag{12}$$

Calculating from relations of equations (7) ～ (12)

$$\frac{\omega_{2p\tau}}{\omega_{1\tau}}=\frac{\cos\gamma_{2B}}{\cos\gamma_{1B}} \tag{13}$$

$$\frac{\omega_{2ps}}{\omega_{1\tau}}=\frac{\cos\gamma_{2B}}{\cos\gamma_{1B}}\tan\kappa_2 \tag{14}$$

κ_2 can be obtained when the flow between blades is known. ω_{2p} is thought not to be straight in general cases, but if we allow to assume this to be straight, the inclination κ_2 will be obtained from the calculation of $\overline{A'_2A_2}$ in Fig. 7-1. What we did in 5. 3. 1. can be directly applied to this calculation. Let a particle of fluid at A_1 be reached to A_2 in t second flowing along the upper surface of blade, and A'_2 in same t second along the lower surface. The time Δt required by a particle reached at A'_2 to flow from A'_2 to A_2 is

$$\Delta t=\frac{\overline{A'_2A_2}}{V_2} \tag{15}$$

This Δt can be expressed by the following relation which was explained in 5. 3. 1.,

$$\Delta t\doteqdot\frac{\Gamma_B}{V_\infty^2} \tag{16}$$

where Γ_B is the blade circulation originated from the base flow, and V_∞ is the geometrical mean velocity of cascade inflow and exit flow.

Because times required by the fluid to pass $\overline{C_1B_1}$ and $\overline{C_2B_2}$ in Fig. 7-1 are equal, we have

$$\frac{\overline{C_1B_1}}{V_{1B}}=\frac{\overline{C_2B_2}}{V_{2B}} \tag{17}$$

At the same time, we have

$$\overline{C_1B_1}=a\sin\gamma_{1B}+a\cos\gamma_{1B}\cdot\tan\kappa_1 \tag{18}$$

and

$$\overline{A_2'A_2}=a\sin\gamma_{2B}+a\cos\gamma_{2B}\cdot\tan\kappa_2-\overline{C_2B_2}$$

Equations (9), (17), (18) and the above yield

$$\overline{A_2'A_2}=a\left[\sin\gamma_{2B}+\cos\gamma_{2B}\cdot\tan\kappa_2-\frac{\sin\gamma_{1B}\cdot\cos\gamma_{1B}}{\cos\gamma_{2B}}-\frac{\cos^2\gamma_{1B}}{\cos\gamma_{2B}}\tan\kappa_1\right] \tag{19}$$

From (15) and (16)

$$\overline{A_2'A_2}=\left(\frac{\Gamma_B}{V_\infty^2}\right)V_{2B} \tag{20}$$

and

$$\Gamma_B=a(V_{2B}\sin\gamma_{2B}-V_{1B}\sin\gamma_{1B}) \tag{21}$$

From (19), (20) and (21) we can get κ_2.

7. 2. Trailing Vortex

Same things as explained in article 5. 4. can be applied to the trailing vortex in this case. Namely, the spanwise velocity Δw_T which appears at the blade trailing edge is,

$$2\Delta w_T=\omega_{1T}\cdot\delta+\frac{d\Gamma}{dx} \tag{24}$$

where r is replaced by x. And,

$$\delta_1=V_1\Delta t\doteqdot V_1\frac{\Gamma}{V_\infty^2} \tag{25}$$

Where Γ is an actual blade circulation including the consideration of secondary flows (see 5. 4.).

The strength of trailing vortex (the strength of vortex contained in unit blade span) Γ_T is,

$$\Gamma_T=-2\Delta w_T \tag{26}$$

which can be put as,

$$\Gamma_T=\Gamma_{T(F)}+\Gamma_{T(s)}$$

where

$$\Gamma_{T(F)}=-\omega_{1T}\cdot\delta_1 \qquad \text{trailing filament vortex} \tag{27}$$

$$\Gamma_{T(s)}=-\frac{d\Gamma}{dx} \qquad \text{trailing shed vortex} \tag{28}$$

And by the same consideration as in 5. 4.

$$\Gamma_{T(F)}=-\omega_{2p_T}\cdot\delta_2 \tag{32}$$

where

$$\delta_2=\overline{A_2'A_2}=V_2\Delta t$$

From equations (32), (13) and (19)

$$\Gamma_{T(F)}=-\omega_{1T}a\cos\gamma_{2B}\left[\frac{\sin\gamma_{2B}}{\cos\gamma_{1B}}-\frac{\sin\gamma_{1B}}{\cos\gamma_{2B}}+\frac{\omega_{2ps}}{\omega_{1T}}-\frac{\omega_{1s}}{\omega_{1T}}\frac{\cos\gamma_{1B}}{\cos\gamma_{2B}}\right] \tag{33}$$

7. 3. Determination of Δw_T in Linear Cascade

Δw_T is given by equation (24) as explained, but the problem has not yet been solved since $d\Gamma/dx$ is not obtained. Δw_T is thought to be obtained by the consideration of Trefftz plane at the exit of cascade, which was explained in articles 3. 2. and 3. 3.. The followings are the practical method of solution along this idea.

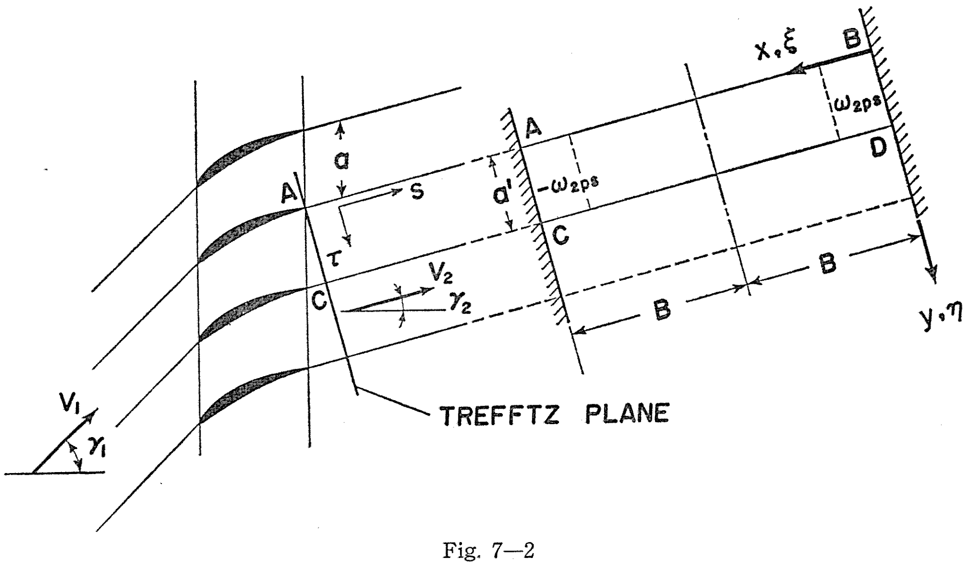


Fig. 7—2

Let us consider a plane perpendicular to the flow at the exit of linear cascade (Trefftz plane) as illustrated in Fig. 7-2. *ABDC* in the figure is this plane viewed from the downstream concerning one passage of the cascade, and forms a rectangle. *2B* corresponds to the span of cascade and a' is the distance between two adjacent stagnation streamlines. The vortex perpendicular to this plane is the passage vorticity ω_{2ps}. When there exists such a vorticity in this rectangle, the calculation what velocities will appear at the boundary of rectangle from the standpoint of two-dimensional flows is the solution for Δw_T.

Fig. 7-3 (a) is another illustration of the rectangle. ω_{2ps} can be generally considered being constant in y-direction and a function of x. If the flow is bilaterally symmetrical in spanwise as in the ordinary experiment of linear cascade, the half of rectangle illustrated in (b) of the figure is sufficient to be considered.

Before we shall do following mathematical treatments, we must be careful to a point. In Fig. 7-2 we assumed that $ABDC$ in Trefftz plane is a rectangle, but, although the side AB may be assumed to be straight because it is a trailing edge of blade, the straightness of the side CD is not assured. There were some discussions on this problem(2)(12)(22). The conclusion of the problem, although will be shown in the later article, shows that CD can be regarded being straight which differs from the result of the discussions. Since we are now in a state without the conclusion, we should assume that the distance from the trailing edge to CD is quite small, namely $\gamma_2 \doteqdot 0$, or the blade spacing is small. Under the above assumptions, we shall consider the flow in $ABDC$ which is a rectangle.

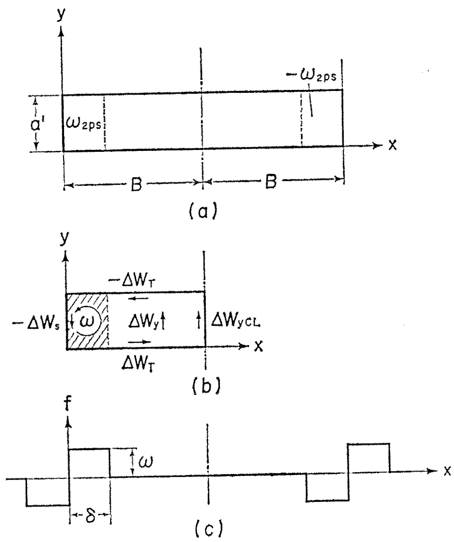


Fig. 7—3

7. 3. 1. Flow in a Rectangle having Vorticity in it

From article 3. 3. we define the following stream function Ψ for the solution of two-dimensional flow in Trefftz plane.

$$V_x = \frac{\partial \Psi}{\partial y}$$
$$V_y = -\frac{\partial \Psi}{\partial x} \tag{1}$$

where V_x, V_y are x, y components of the flow. Ψ satisfies next equation,

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = -\omega \tag{2}$$

The flow in a rectangle having vorticity ω in it, as illustrated in Fig. 7-4, has already been solved as follows(23),

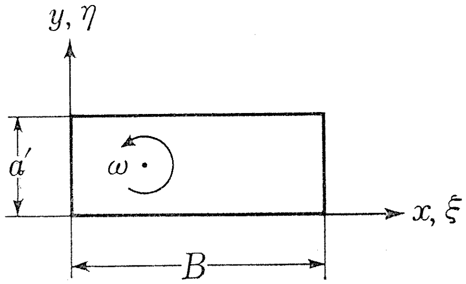


Fig. 7—4

$$\Psi=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(n\pi\frac{x}{B}\right)}{n\sinh\left(n\pi\frac{a'}{B}\right)}\int_0^B\sin\left(n\pi\frac{\xi}{B}\right)d\xi$$

$$\times\left[\int_0^y\sinh\left(n\pi\frac{\eta}{B}\right)\cdot\sinh\left(n\pi\frac{a'-y}{B}\right)\cdot\omega(\xi,\ \eta)d\eta\right.$$

$$\left.+\int_y^{a'}\sinh\left(n\pi\frac{y}{B}\right)\cdot\sinh\left(n\pi\frac{a'-\eta}{B}\right)\cdot\omega(\xi,\ \eta)d\eta\right] \qquad (3)$$

As aforesaid $\omega(=\omega_{2ps})$ is generally constant in y-direction and a function of x. Being constant in y-direction comes from the assumption that the inclination of ω_{2p} is constant (see 5. 3. 1), which may be approved as the first approximation. If we think ω being of a step-like distribution as illustrated in Fig. 7-5, ω can be regarded combinations of cases in which ω is constant in a range of δ. The treatment of ω in which ω is regarded as the above, therefore, doesn't lose the generality.

Assuming ω is constant in y-direction and a function only of x, we have

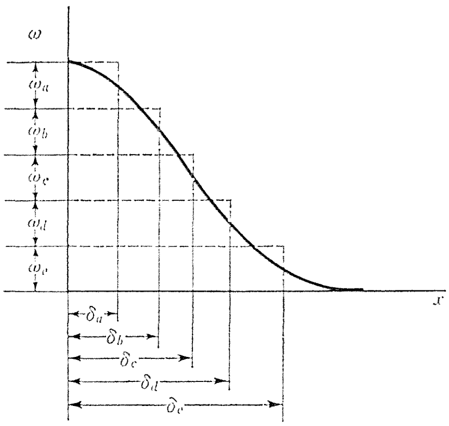


Fig. 7—5

$$\Psi=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(n\pi\frac{x}{B}\right)}{n\sinh\left(n\pi\frac{a'}{B}\right)}\int_0^B\omega(\xi)\sin\left(n\pi\frac{\xi}{B}\right)d\xi$$

$$\times\left[\sinh\left(n\pi\frac{a'-y}{B}\right)\int_0^y\sinh\left(n\pi\frac{\eta}{B}\right)d\eta\right.$$

$$\left.+\sinh\left(n\pi\frac{y}{B}\right)\int_y^{a'}\sinh\left(n\pi\frac{a'-\eta}{B}\right)d\eta\right] \qquad (4)$$

Integrations in [ ] can be performed easily. In conditions as expressed in Fig. 7-3, we notice for the integration with respect to ξ, that

$$\omega(\xi)=\omega \quad (constant) \qquad 0<\xi<\delta$$
$$\omega(\xi)=0 \qquad \delta<\xi\leqq B$$

Therefore, we get after calculations,

$$\Psi=\frac{2B^2}{\pi^3}\omega\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right]$$
$$\times\left[1-\frac{\sinh\left(n\pi\frac{y}{B}\right)+\sinh\left(n\pi\frac{a'-y}{B}\right)}{\sinh\left(n\pi\frac{a'}{B}\right)}\right] \tag{5}$$

The spanwise velocity $\varDelta w_T$ at the blade trailing edge, which corresponds to the trailing vortex, is

$$\varDelta w_T=\left(\frac{\partial\Psi}{\partial y}\right)_{y=0}=\frac{2B}{\pi^2}\omega\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(n\pi\frac{x}{B}\right)$$
$$\times\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right]\cdot\tanh\left(\frac{n\pi}{2}\frac{a'}{B}\right) \tag{6}$$

The induced velocity in y-direction $\varDelta w_y$ at any place is

$$\varDelta w_y=-\frac{\partial\Psi}{\partial x}=-\frac{2B}{\pi^2}\omega\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right]$$
$$\times\left[1-\frac{\sinh\left(n\pi\frac{y}{B}\right)+\sinh\left(n\pi\frac{a'-y}{B}\right)}{\sinh\left(n\pi\frac{a'}{B}\right)}\right] \tag{7}$$

7. 3. 2. The case of Infinitesimal Blade Spacing

The flow in a linear cascade of infinitesimal blade spacing which corresponds to the axisymmetric flow is given by putting $a' \longrightarrow 0$. From equation (6) we have

$$\varDelta w_T=\omega\frac{a'}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right] \tag{8}$$

On the other hand, the relation $\omega\sim x$ which is illustrated in Fig. 7-3 (c) can be expressed by Fourier series as,

$$f=\omega\frac{2}{\pi}\sum_{n=1}\frac{1}{n}\sin\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right] \tag{9}$$

Therefore we get,

$$\left.\begin{aligned}\varDelta w_T&=\frac{a'}{2}f=\frac{a'}{2}\omega \qquad (0<x<\delta)\\ &=0 \qquad (\delta<x\leqq B)\end{aligned}\right\} \tag{10}$$

The strength of trailing vortex at unit of span is,

$$\Gamma_T = -2\Delta w_T \tag{11}$$

Let us now take the average value of this in tangential direction (cascade direction) as was done in 5. 5. 1.. Because we are now thinking about a case a′ ⟶ 0, there may be no problem which existed in the case of a' being finite (the doubt if it is allowed or not to take such an average). Expressing the vorticity in such a case by

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The proof of (13) can be done also in the following manner. Let us examine the circulation along the path $ABCDA$ in Fig. 7-6.

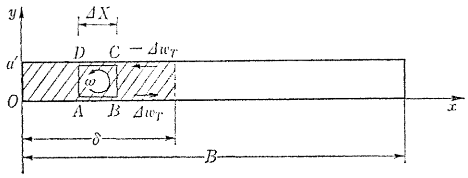


Fig. 7—6

$$2\Delta w_T \cdot \Delta x + \int_0^{a'} \Delta w_y(x+\Delta x)\,dy + \int_{a'}^{0} \Delta w_y(x)\,dy = \omega \cdot a' \cdot \Delta x \tag{A}$$

Now we examine the limiting case of $a' \to O$. First term becomes from equation (6),

$$\lim_{a'\to 0} \Delta w_T = \frac{2B}{\pi^2}\omega \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right] \times \frac{n\pi}{2}\frac{a'}{B} = O\left[\frac{a'}{B}\right] \tag{B}$$

where $O[\ \ ]$ represents "order".

For examinations of second and third terms, we use equation (7) in which the contents of last [ ] becomes under the condition $a' \to O$,

$$\left[1-\frac{\sinh\left(n\pi\frac{y}{B}\right)+\sinh\left(n\pi\frac{a'-y}{B}\right)}{\sinh\left(n\pi\frac{a'}{B}\right)}\right] = \left[1-\frac{\cosh\left(n\pi\frac{2y-a'}{2B}\right)}{\cosh\left(n\pi\frac{a'}{2B}\right)}\right]$$

$$= \frac{2\sinh\left(n\pi\frac{y}{2B}\right)\cdot\sinh\left(n\pi\frac{a'-y}{2B}\right)}{\cosh\left(n\pi\frac{a'}{2B}\right)} \fallingdotseq 2\sinh\left(n\pi\frac{y}{2B}\right)\cdot\sinh\left(n\pi\frac{a'-y}{2B}\right)$$

$$\fallingdotseq 2\left(n\pi\frac{y}{2B}\right)\left(n\pi\frac{a'-y}{2B}\right) < \frac{1}{2}n^2\pi^2\frac{a'\cdot a'}{B^2} = \frac{1}{2}n^2\pi^2\left(\frac{a'}{B}\right)^2$$

where

$$0 < y < a'$$

We have, therefore,

$$\lim_{a'\to 0}\left|\Delta w_y\right| < \left(\frac{a'}{B}\right)^2 \omega B \sum_{n=1}^{\infty} \cos\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right] = O\left[\left(\frac{a'}{B}\right)^2\right] \tag{C}$$

Second and third terms in equation (A) is, therefore, of higher order of infinitesimal than the first, and we have from equation (A),

$$2\Delta w_T \cdot \Delta x = \omega \cdot a' \cdot \Delta x$$

or

$$\Delta w_T = \frac{a'}{2}\omega \tag{D}$$

This is completely same as equation (10).

ω_{2T}, we get using (10) and (11)

$$\left.\begin{aligned} \omega_{2T} &= \frac{\Gamma_T}{a'} = -\omega_{2ps} && (0<x<\delta) \\ &= 0 && (\delta<x\leqq B) \end{aligned}\right\} \tag{12}$$

where

$$\omega_{2ps} = \omega$$

Therefore, we have

$$\omega_{2ps} + \omega_{2T} = 0 \tag{13}$$

at every x-position. This is completely same as equation (27) in 6. 2. 2. which means that there is no streamwise vortex i. e. secondary flows in the outgoing flow of linear cascade having infinitesimal blade spacing from the consideration of Trefftz plane.

7. 4. Outgoing Flow from the Linear Cascade

7. 4. 1. Discussions about the Boundary of Trefftz Plane

There was some question in the idea that the boundary of Trefftz plane is a rectangle as said in 7. 3.. Stephenson[22] once published a paper in which he said that the result of Squire & Winer can be obtained by much more simple method originated from the law of conservation of circulation. In this paper he applied this idea to the passage vortex and pointed out that the trailing vortex must be added to the former. Unfortunately his theory should be said to be erroneous, but he may be the first man who pointed out that both the passage vortex and the trailing vortex must be considered on the secondary flow in cascades. Furthermore, he took the average of the trailing vortex in cascade direction. This is the idea same as the author's which was adopted with some doubt in the paragraph of axisymmetric treatment. Because there were such points of questions in his paper several discussions were held in the Reader's Forum in Journal of the Aeronautical Science.

Eichenberger[2] said that there is no need of consideration of both the passage vortex and the trailing vortex, and only the secondary vortex (passage vortex) of Squire and Winter is sufficient to be considered. Namely, because the trailing vortex can be replaced by the boundary, the flow is regarded as the one in a duct and the trailing vortex can be disregarded.

Yeh[27] said that the algebraic summation of the passage vortex and the trailing vortex done by Stephenson is not good, since locations of both vortices are different.

Loos and Zwaaneveld[12] commented on Eichenberger's conclusion that thinking the trailing vortex as a boundary and taking no account in the calculation are correct, but there remains a question that the shape of boundary is not yet obtained and, therefore, the problem will not be solved. To get the solution of the flow field, we must calculate both the passage vortex and the trailing vortex.

Interpretations of Eichenberger or Loos and Zwaaneveld are both not erroneous. The problem is what is the shape of boundary which is formed by the trailing vortex, and let us examine this in the following articles.

7. 4. 2. Flows in Trefftz Plane (Rectangular Boundary)

The flow in the Trefftz plane having rectangular boundary was treated in 7. 3. 1.. The assumption of rectangular boundary may be adopted as the zeroth approximation. This is the same idea as an assumption used in the monoplane wing theory in which the trailing vortex sheet is assumed to be flat. Assumptions that, $\omega(=\omega_{2ps})$ is constant in y-direction, and constant in the range of δ, are also allowed for the sake of simplification.

Induced velocities at any point (x, y) in the rectangle can be obtained from equation (5) in 7. 3..

$$\Delta w_x(x,\ y)=\frac{2B}{\pi^2}\omega\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right]$$

$$\times\frac{\sinh\left(n\pi\frac{a'-2y}{2B}\right)}{\cosh\left(n\pi\frac{a'}{2B}\right)} \tag{14}$$

$$\Delta w_y(x,\ y)=-\frac{2B}{\pi^2}\omega\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right]$$

$$\times\left[1-\frac{\cosh\left(n\pi\frac{a'-2y}{2B}\right)}{\cosh\left(n\pi\frac{a'}{2B}\right)}\right] \tag{15}$$

The spanwise velocity Δw_T at the blade trailing edge, which corresponds to the trailing vortex, is expressed by equation (6) which is rewritten as,

$$\Delta w_T=\frac{2B}{\pi^2}\omega\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right]\times\tanh\left(n\pi\frac{a'}{2B}\right) \tag{16}$$

The induced velocity Δw_{yCL} in y-direction at the center of span is obtained from (15) by putting $x=B$.

$$\Delta w_{yCL}=-\frac{2B}{\pi^2}\omega\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right]$$

$$\times\left[1-\frac{\cosh\left(n\pi\frac{a'-2y}{2B}\right)}{\cosh\left(n\pi\frac{a'}{2B}\right)}\right] \tag{17}$$

The mean induced velocity Δw_{ym} in y-direction is given from equation (15)

$$\Delta w_{ym}=\frac{1}{a'}\int_0^{a'}\Delta w_y dy=-\frac{2B}{\pi^2}\omega\left[\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(n\pi\frac{x}{B}\right)\cdot\left\{1-\cos\left(n\pi\frac{\delta}{B}\right)\right\}\right.$$

$$\left.-\frac{1}{\pi}\frac{2B}{a'}\sum_{n=1}^{\infty}\frac{1}{n^3}\cos\left(n\pi\frac{x}{B}\right)\cdot\left\{1-\cos\left(n\pi\frac{\delta}{B}\right)\right\}\cdot\tanh\left(n\pi\frac{a'}{2B}\right)\right] \tag{18}$$

7. 4. 3. Considerations about the Trefftz Plane

The above was the consideration on Trefftz plane having rectangular boundary. On the other hand it is a problem in what location the Trefftz plane should be placed along the stream, and let us place it as shown in Fig. 7-2.. In this case the Trefftz plane becomes semi-infinite rectangle as illustrated in Fig. 7-7.. AB in the figure corresponds to the trailing edge of blade shown in Fig. 7-2, and is regarded to be straight. ACE..... and BDF...... are side walls, and regarded also to be straight. CD, EF, ... indicate wakes of blades where trailing vortices exist, but it is not clear if these are straight or not. This is the problem we are now going to examine.

7. 4. 3. 1. Strength of Trailing Vortex Sheet

To fulfill the straightness of boundaries (AB, ACE.... and BDF.....) shown in Fig. 7-7, we consider mirror images as shown in Fig. 7-8. The upper side of LM-line is the fold-back of the lower, and we can easily see the arrangement of vortices in each domain is same as the one shown in Fig. 7-7.

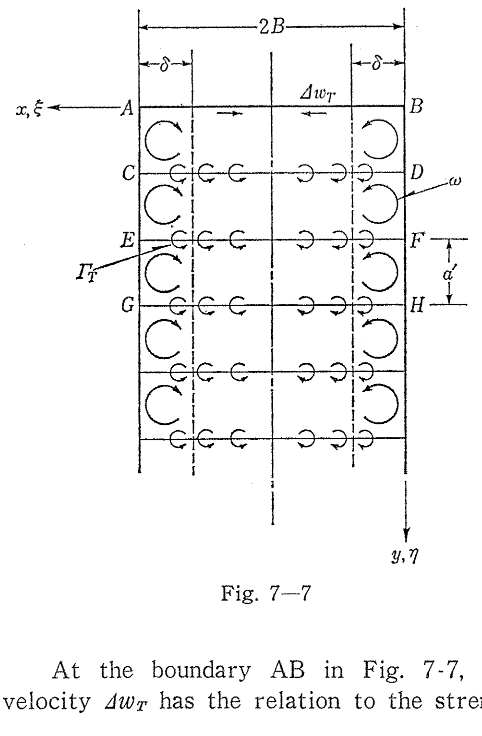


Fig. 7—7

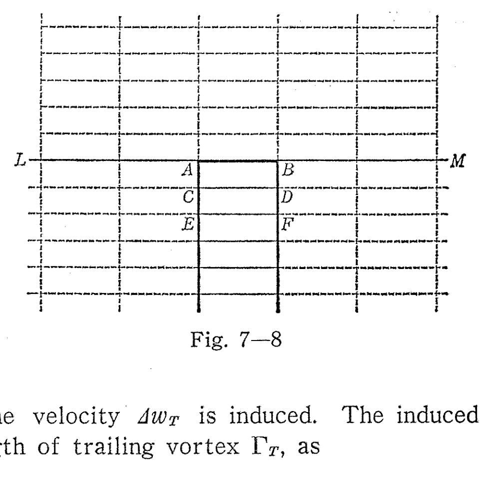


Fig. 7—8

At the boundary AB in Fig. 7-7, the velocity Δw_T is induced. The induced velocity Δw_T has the relation to the strength of trailing vortex Γ_T, as

$$\Gamma_T = -2\Delta w_T \tag{19}$$

Δw_T or Γ_T is thought to be unknown and will be calculated hereafter. The passage vorticity ω is regarded being given (which was obtained as the secondary vorticity ω_{2ps} in 7. 1.).

Let the induced velocity at AB by passage vorticity ω be Δw_{Tp}, and the one by trailing vortices CD, EF, ... be Δw_{TT}, then we have

$$\Delta w_T = \Delta w_{TP} + \Delta w_{TT} \tag{20}$$

Δw_{TT} contains Γ_T or Δw_T in it, and equation (20) becomes finally an integral equation concerning Δw_T.

Δw_{Tp} can be obtained from equation (16), which is the expression of the flow in the rectangle, by putting $a' \longrightarrow \infty$ without the consideration of mirror images.

$$\Delta w_{TP} = \frac{2B}{\pi^2}\omega \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right] \tag{21}$$

Δw_{TT} is obtained with the aid of Biot-Savart law considering mirror images.

$$\Delta w_{TT}(x) = \frac{1}{\pi}\int_0^B \Gamma_T(\xi) \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\left[\frac{n\frac{a'}{B}}{\left(2m+\frac{\xi}{B}-\frac{x}{B}\right)^2+\left(n\frac{a'}{B}\right)^2}\right.$$
$$\left. - \frac{n\frac{a'}{B}}{\left(2m-\frac{\xi}{B}+\frac{x}{B}\right)^2+\left(n\frac{a'}{B}\right)^2}\right]\frac{d\xi}{B} \tag{22}$$

(where n is the number of trailing vortex sheet, for examples CD is $n=1$, EF is $n=2$, ... etc. and m is the number of mirror images on right and left sides. Detailed calculations are quite complex and may be seen in the reference (11).) Abridging detailed calculation, the final result is,

$$\Delta w_{TT}(x) = \int_0^B 2\Delta w_T(\xi) \sum_{m=1}^{\infty}\left[1-\coth\left(\frac{m\pi}{2}\frac{a'}{B}\right)\right]$$
$$\times \sin\left(m\pi\frac{x}{B}\right)\cdot\sin\left(m\pi\frac{\xi}{B}\right)\cdot\frac{d\xi}{B} \tag{23}$$

Substituting (21) and (23) into (30), we have

$$\frac{\Delta w_T(x)}{\omega\delta} = \frac{2}{\pi^2}\frac{1}{\frac{\delta}{B}}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{x}{B}\right)\right]$$
$$+\int_0^B 2\frac{\Delta w_T(\xi)}{\omega\delta}\sum_{m=1}^{\infty}\left[1-\coth\left(\frac{m\pi}{2}\frac{a'}{B}\right)\right]\cdot\sin\left(m\pi\frac{x}{B}\right)$$
$$\times\sin\left(m\pi\frac{\xi}{B}\right)\cdot\frac{d\xi}{B} \tag{24}$$

This is a Fredholm type of integral equation of second kind with respect to Δw_T as an unknown function, and the solution can be obtained by the method of successive substitution(8). The result is,

$$\frac{\Delta w_T(x)}{\omega\delta} = \frac{2}{\pi^2}\frac{1}{\frac{\delta}{B}}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right]$$

$$\times \tanh\left(\frac{n\pi}{2}\frac{a'}{B}\right) \tag{25}$$

This coincides with equation (16). Namely, $\varDelta w_T$ from the consideration of Trefftz plane mentioned above coincides perfectly with the one from the case of rectangular boundary.

7. 4. 3. 2. Form of Trailing Vortex Sheet

The treatment mentioned above was done under an assumption that the section of trailing vortex sheet at Trefftz plane is straight. The straightness of trailing vortex sheet must be confirmed by the examination of y-directional induced velocity at that position. If this is zero, the assumption of straigtness becomes correct.

Let us examine the y-directional induced velocity at the trailing vortex sheet $y = Na'$ (where N is positive integer) in Fig. 7-7.. This can be divided into two parts. The one is that caused by the passage vortex and denoted as $\varDelta w_{yp}$. Another is by the trailing vortex and denoted as $\varDelta w_{yT}$.

$$\varDelta w_y(y=Na') = \varDelta w_{yP} + \varDelta w_{yT} \tag{26}$$

$\varDelta w_{yp}$ caused by the passage vortex can be obtained from equation (15) by putting $a' \longrightarrow \infty$, where the semi-infinite domain in Fig. 7-7 is considered being a rectangle of $a' \longrightarrow \infty$. We have (putting $y = Na'$)

$$\varDelta w_{yP} = \frac{2B}{\pi^2}\omega \sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right]$$

$$\times\left[1-\exp\left(-n\pi\frac{a'}{B}N\right)\right] \tag{27}$$

To obtain $\varDelta w_{yT}$ caused by the trailing vortex Biot-Savart law is used. For the satisfaction of boundary conditions we must consider mirror images as shown in Fig. 7-8, where we must be careful that the trailing vortex sheet along the boundary AB or $n = 0$ must be treated as a boundary and not a vortex sheet. The distribution of trailing vortices can be divided into two groups, which are

(1) symmetrical distribution with respect to $y = Na'$

(2) additional distribution which is added to (1) to return to the original condition.

The y-directional induced velocity at $y = Na'$ by (1) is easily understood to be zero. Distributions of trailing vortex sheet of (2) become,

Γ_T at $y = 0$

Γ_T at $y = Na'$

$2\Gamma_T$ at $y = la'$ where $l = 1, 2, \ldots\ldots 9, N\text{-}1$

0 at any y except the above

Induced velocities by these vortices are expressed respectively as $\varDelta w_{yTO}$, $\varDelta w_{yTN}$ and $\varDelta w_{yTl}$. We have,

$$\varDelta w_{yT} = \varDelta w_{yTO} + \varDelta w_{yTN} + \varDelta w_{yTl} \tag{28}$$

Using Biot-Savart law, y-directional induced velocity at (x, Na') caused by the vortex sheet existing at $y = Na'$ is,

$$\Delta w_{yT}(x,\ Na') = -\frac{1}{2\pi}\int_0^B \Gamma_T(\xi) \sum_{m=-\infty}^{\infty} \left[\frac{2m+\dfrac{\xi}{B}-\dfrac{x}{B}}{\left\{2m+\dfrac{\xi}{B}-\dfrac{x}{B}\right\}^2+\left\{\dfrac{a'}{B}(n-N)\right\}^2} - \frac{2m-\dfrac{\xi}{B}-\dfrac{x}{B}}{\left\{2m-\dfrac{\xi}{B}-\dfrac{x}{B}\right\}^2+\left\{\dfrac{a'}{B}(n-N)\right\}^2} \right] \frac{d\xi}{B} \tag{29}$$

Δw_{yTO} is obtained from equation (29) by putting $n=0$ and substituting (19) and (25) into it.

Δw_{yTN} is obtained from equation (29) by putting $n=N$.

Δw_{yTl} is obtained from equation (29) where we put $\Gamma_T \longrightarrow 2\Gamma_T$ and $n=l$ and take the sum of $l=1, 2, \ldots\ldots, N\text{-}1$.

Substituting these results into (28), we get after calculations.

$$\frac{\Delta w_{yT}}{\omega\delta} = -\frac{2}{\pi^2}\frac{1}{\dfrac{\delta}{B}}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right] \times\left[1-\exp\left(-n\pi\frac{a'}{B}N\right)\right] \tag{30}$$

where we used a relation that

$$\tanh\left(n\pi\frac{a'}{2B}\right)\cdot\left[2\sum_{l=1}^{N-1}\exp\left(-n\pi\frac{a'}{B}l\right)+\exp\left(-n\pi\frac{a'}{B}N\right)+1\right] = 1-\exp\left(-n\pi\frac{a'}{B}N\right) \tag{31}$$

Now we substitute equations (27) and (30) into equation (26) getting the result that

$$\Delta w_y(y=Na')=0 \tag{32}$$

Lengthy explanations yield a very simple result which states that y-directional induced velocity at the position of trailing vortex sheet is zero. This means that the vortex sheet is flat, and therefore the boundary of Trefftz plane can be considered being rectangle. From the result we have obtained, we can say that the flow in Trefftz plane haveing rectangular boundary treated in 7. 4. 2. can be directly applicable to the consideration of secondary flows in linear cascades.

7. 4. 4. Results of Calculations of Flow in Trefftz Plane

Spanwise velocities at the trailing edge Δw_T, which corresponds to the trailing vortex, are shown in Fig. 7-9 (a) ~ (d) calculated from equation (16).

y-directional induced velocities at the center of blade span Δw_{yCL} are shown in Fig. 7-10 (a) ~ (d) calculated from equation (17).

y-directional mean induced velocities Δw_{ym} are shown in Fig. 7-11 (a) ~ (d) calculated from equation (18).

y-directional mean induced velocities at the center of blade span, which are most indispensable for the correction of exit flow direction of cascade experiments, are values at $x/B = 1.0$ in Fig. 7-11 and reproduced collectively in Fig. 7-12.

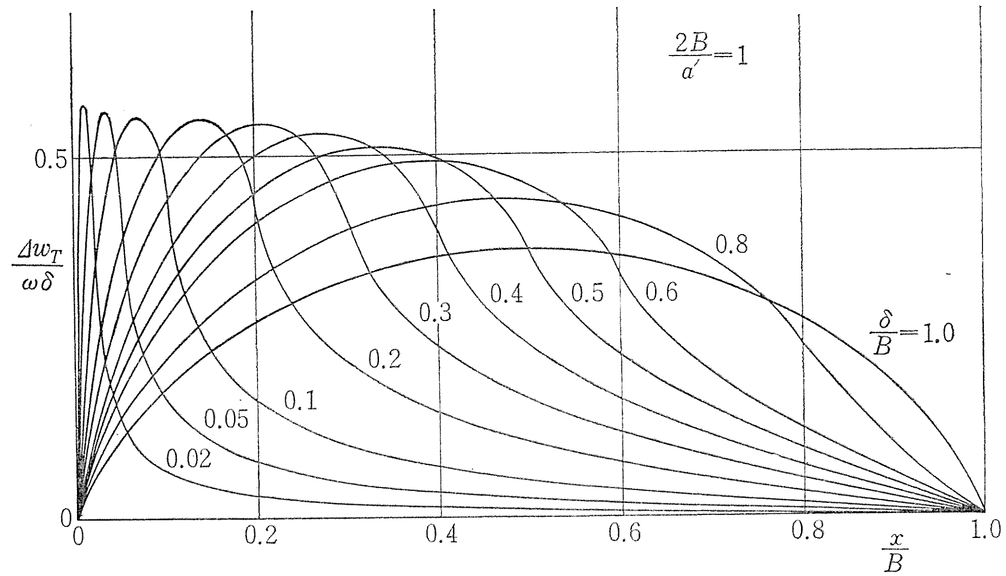


Fig. 7—9 (a)

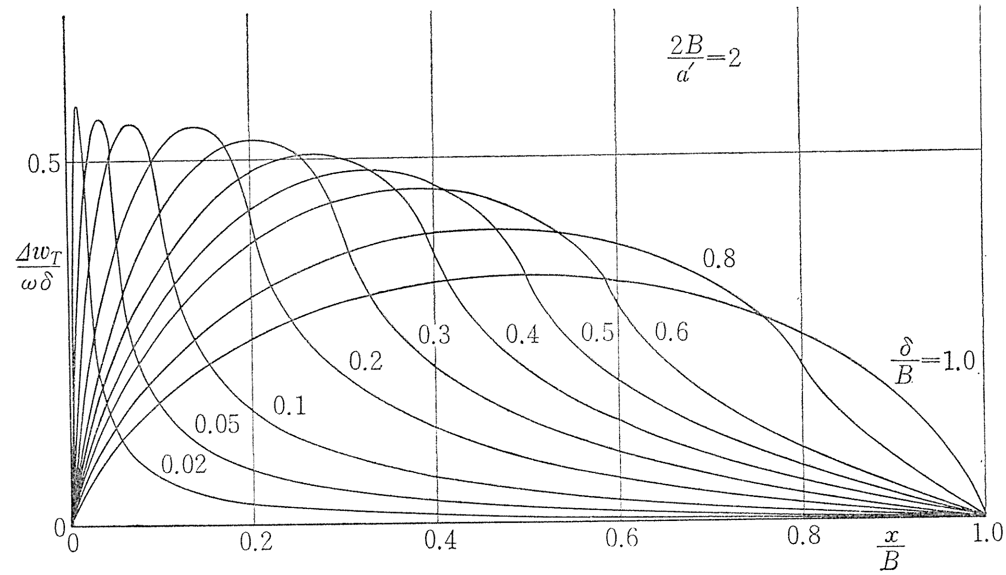


Fig. 7—9 (b)

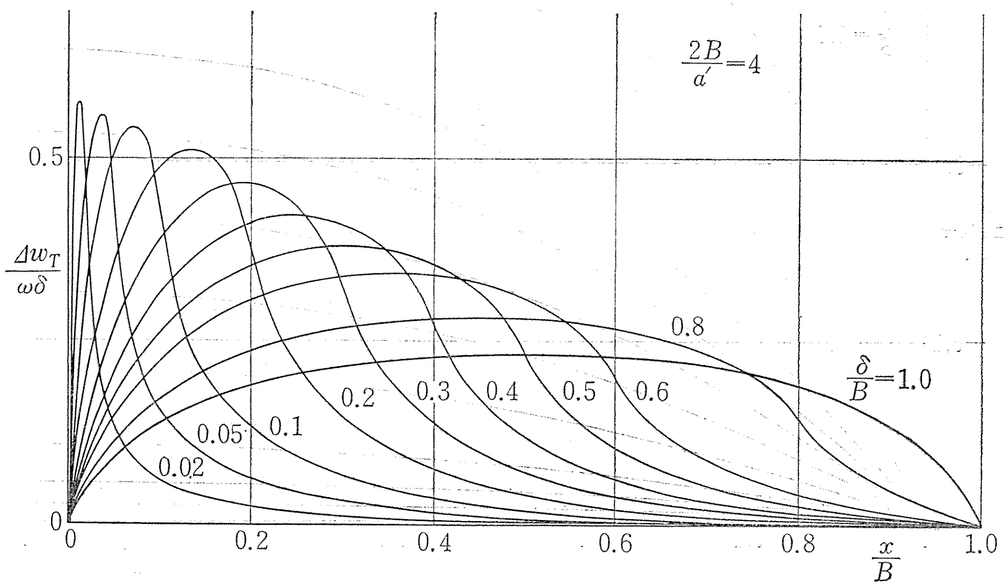


Fig. 7—9 (c)

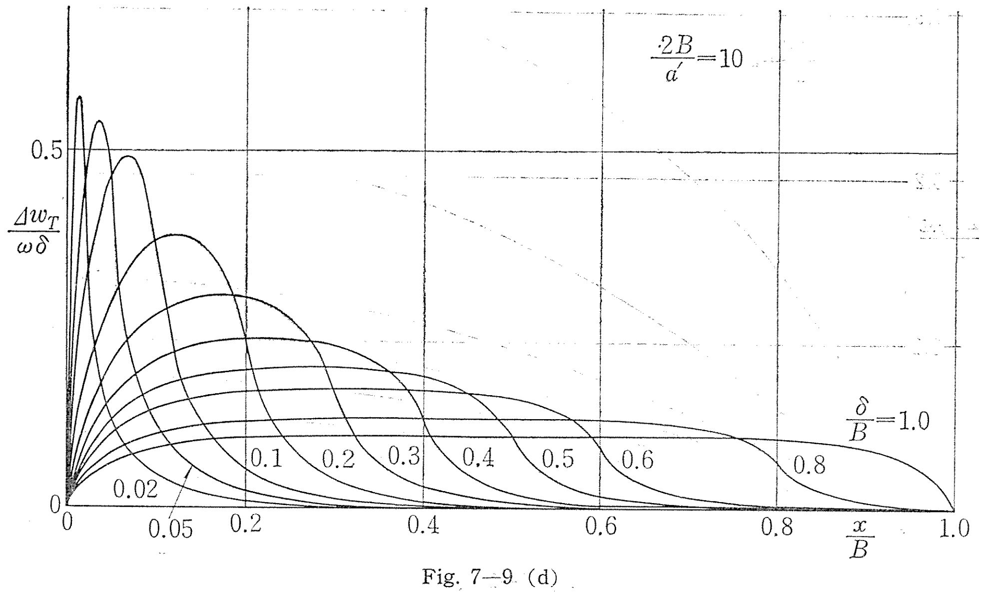


Fig. 7—9 (d)

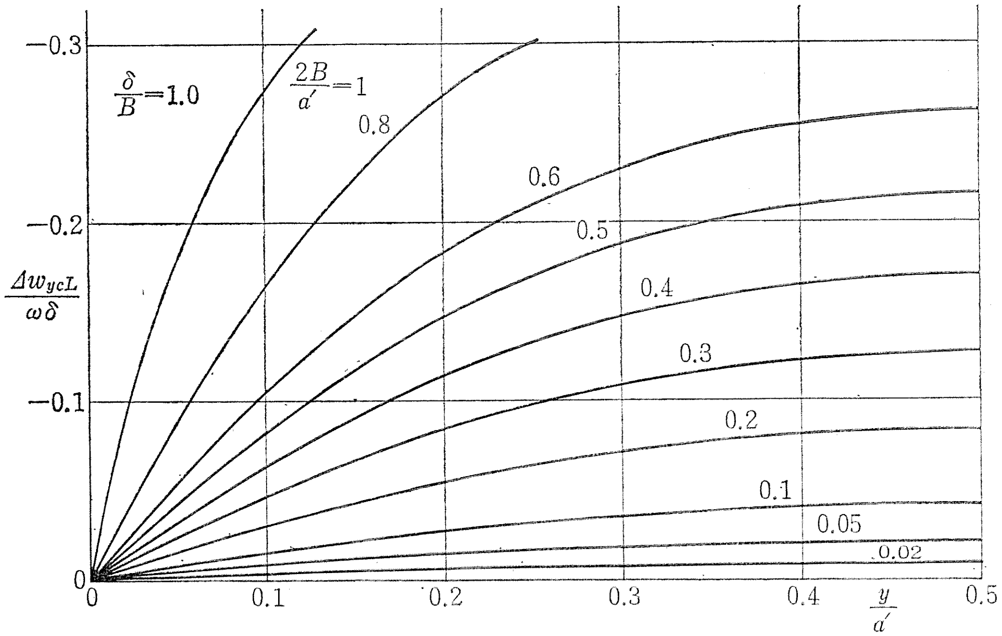


Fig. 7—10 (a)

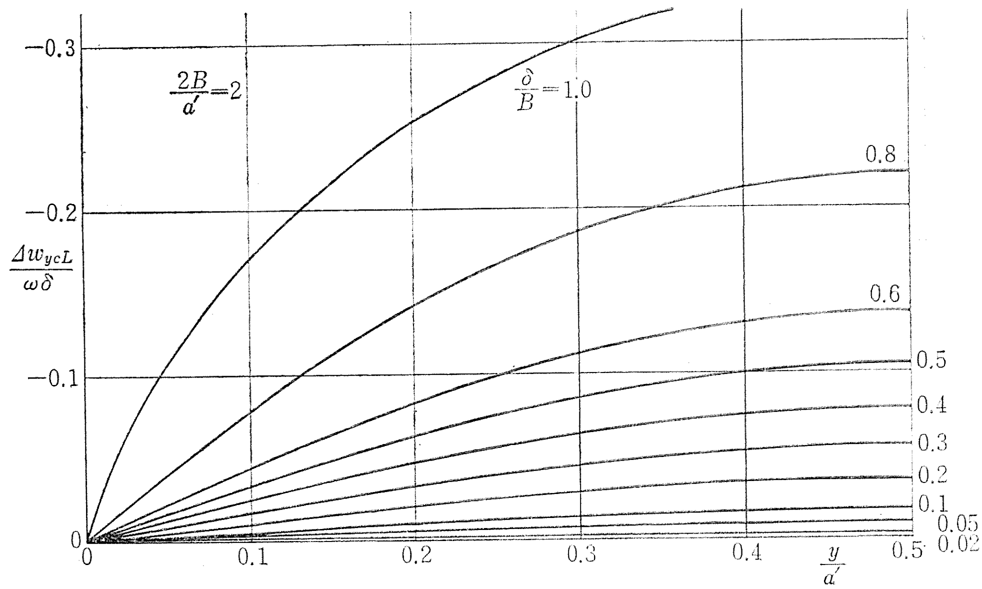


Fig. 7—10 (b)

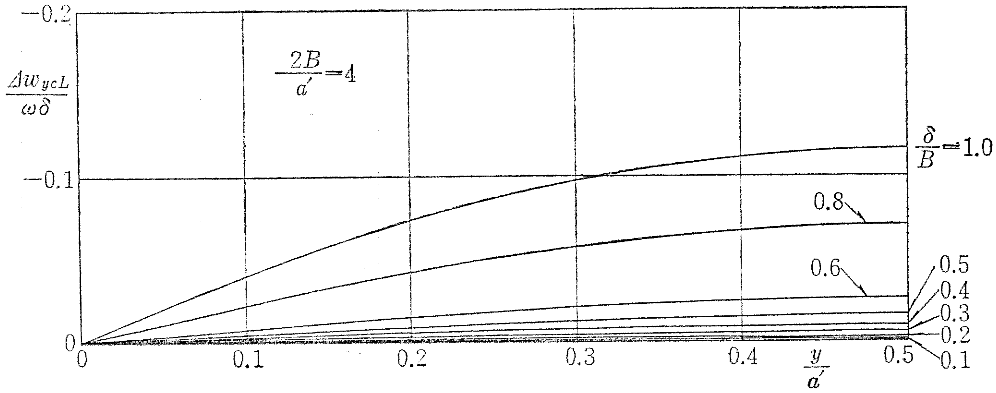


Fig. 7—10 (c)

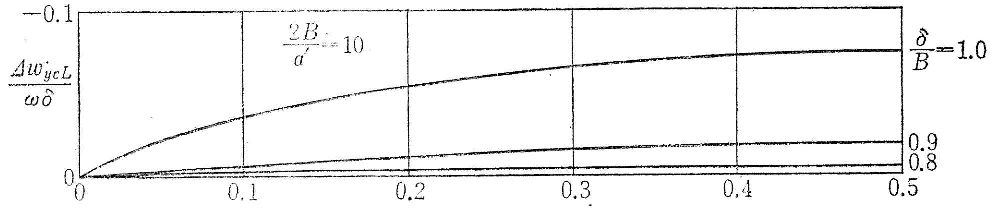


Fig. 7—10 (d)

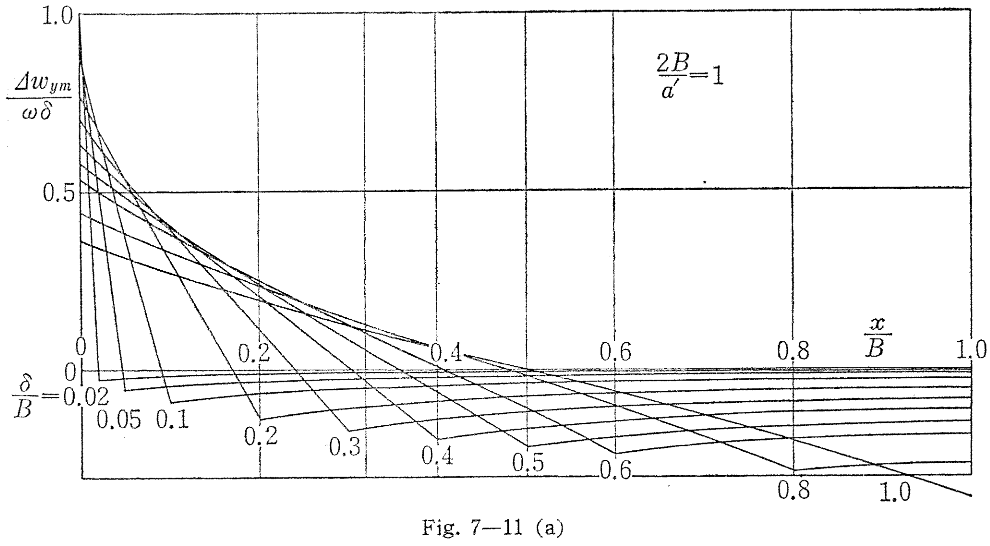


Fig. 7—11 (a)

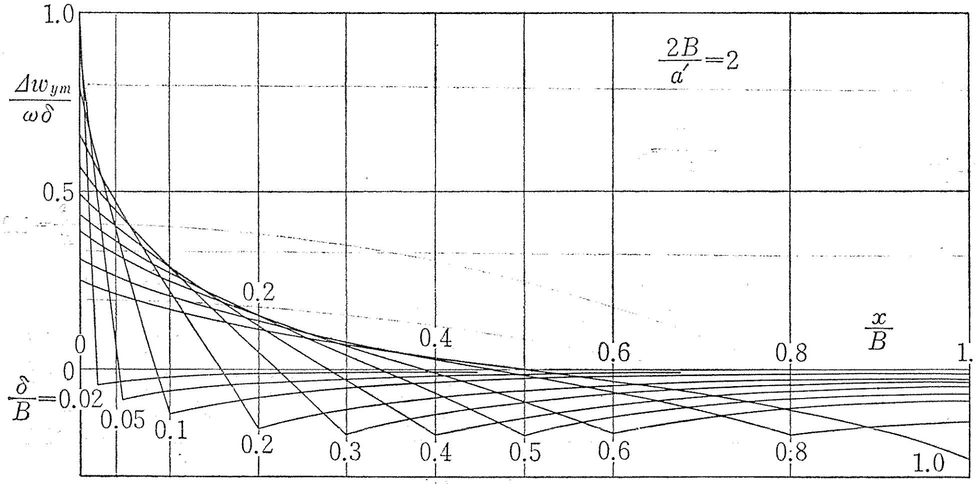
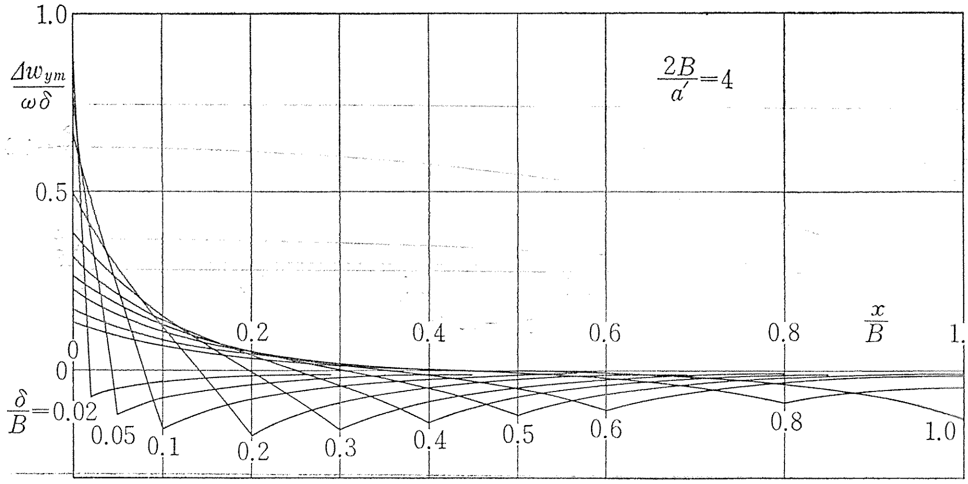
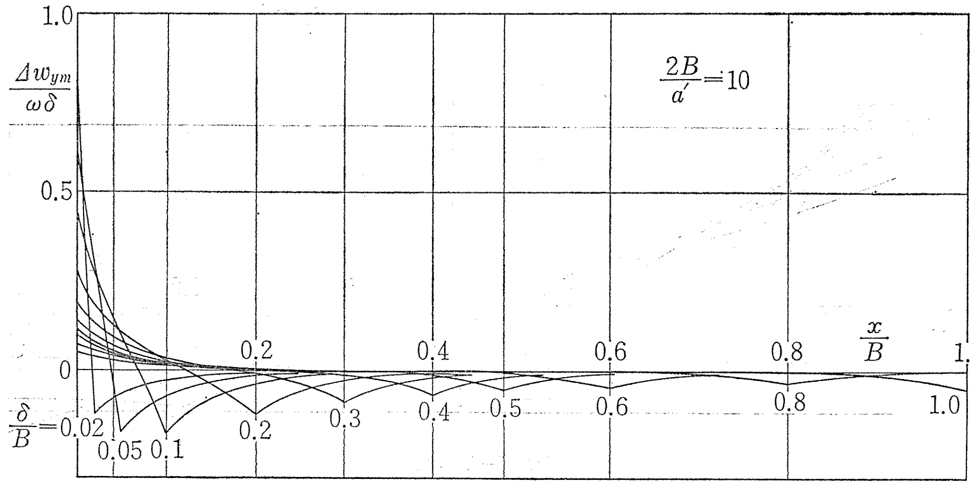


Fig. 7—11 (b)



↑ Fig. 7—11 (c) ↓ Fig. 7—11 (d)



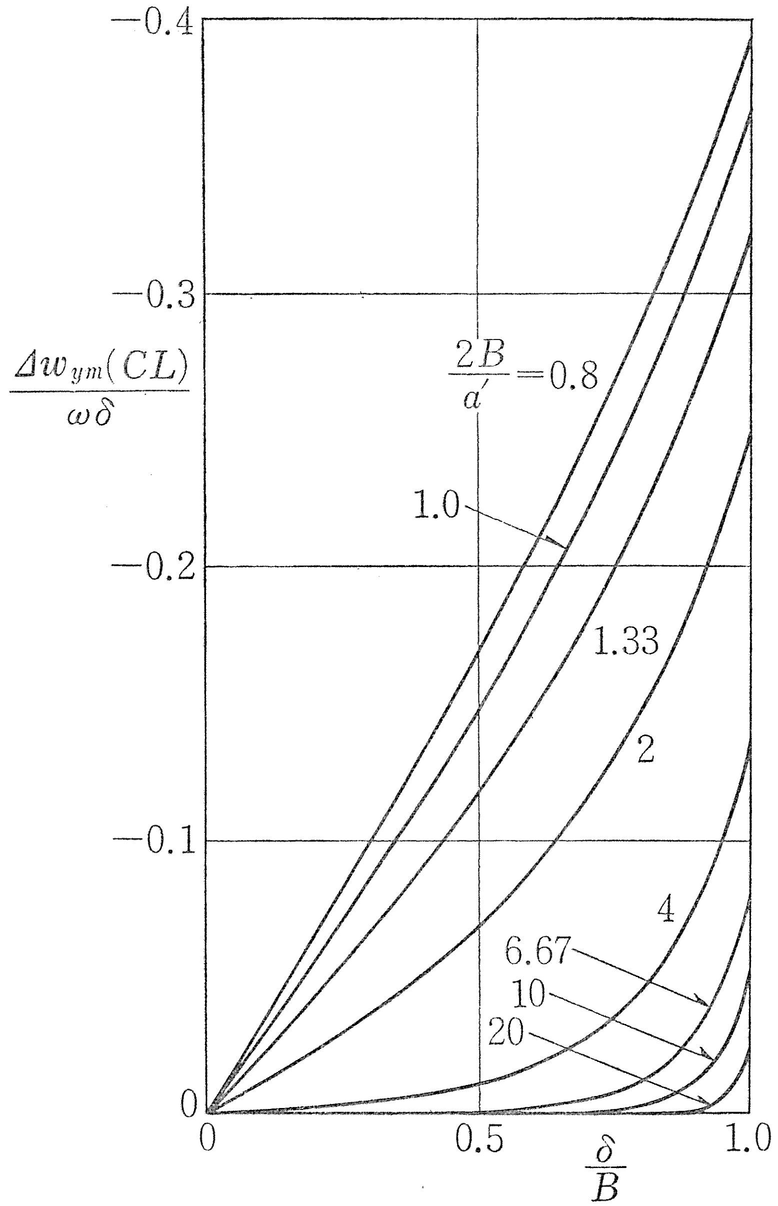


Fig. 7—12

7. 4. 5. Streamwise Velocity Distribution of Exit Flow

The first approximation of the component of vorticity normal to exit flow in linear cascade is given by equation (13) in 7. 1. (see Fig. 7-1).

$$\frac{\omega_{2p\tau}}{\omega_{1\tau}}=\frac{\cos\gamma_{2B}}{\cos\gamma_{1B}} \tag{33}$$

Because there is no other normal component (see 5. 5.), we can get exit streamwise velocity distribution.

Let the velocity deviation from the base flow (induced velocity) be denoted ΔV, then we have in the upstream of cascade,

$$\Delta V_1 = \int \omega_{1\gamma} dx \tag{34}$$

where we note the condition of continuity which is

$$\int_0^{2B} \Delta V_1 dx = 0 \tag{35}$$

In the downstream of cascade we have also,

$$\Delta V_2 = \int \omega_{2p\gamma} dx \tag{36}$$

The condition of continuity will be expressed as follows with the aid of result obtained in 7. 4. 3. which states that the boundary of flow through the blade passage stays rectangular still in the downstream,

$$\int_0^{2B} \Delta V_2 dx = 0 \tag{37}$$

From the condition of continuity of base flow,

$$V_{1B} \cos \gamma_{1B} = V_{2B} \cos \gamma_{2B} \tag{38}$$

After calculations using equations (33), (34), (36) and (38), we have

$$\frac{\Delta V_2}{V_{2B}} = \left(\frac{\cos \gamma_{2B}}{\cos \gamma_{1B}} \right)^2 \frac{\Delta V_1}{V_{1B}} \tag{39}$$

Because $|\gamma_1| < |\gamma_2|$ in the accelerating cascade, we get

$$\frac{|\Delta V_2|}{V_{2B}} < \frac{|\Delta V_1|}{V_{1B}} \tag{40}$$

and because $|\gamma_1| > |\gamma_2|$ in the decelerating cascade, we get

$$\frac{|\Delta V_2|}{V_{2B}} > \frac{|\Delta V_1|}{V_{1B}} \tag{41}$$

Since the velocity of base flow is considered being averaged velocity, the velocity deviation from the average in the downstream looks becoming smaller than in the upstream in the accelerating cascade, and this gives us an impression that the side wall boundary layer becomes suppressed. And conversely the one in the decelerating cascade seems growing. It is worth noticing that we have got the result above stated without considerations of the boundary layer growth caused by the viscosity of fluid.

7. 4. 6. Exit Flow Angles

Provided the spanwise velocity distribution of entrance flow of cascade is given, we can obtain the exit flow velocity distribution from equation (39) as,

$$V_2 = V_{2B} + \Delta V_2 \tag{42}$$

The averaged y-directional (normal to span and flow) flow velocity is obtainable from equation (18) or Fig. 7-12 (a) $\sim$ (d), where ω is passage vorticity ω_{2ps}, and from equation (4) in 3. 2. we have,

$$\omega_{2ps} = -2\omega_1 \varepsilon$$

ε is the turning angle and we assume the vortex in upstream of cascade is normal to the flow or $\kappa_1 = 0$. Namely $\omega_1 = \omega_{1\tau}$. (When $\kappa_1 \neq 0$ we employ the way mentioned in 7. 1., and can get ω_{2ps}.) In the next place, because

$$\omega_{1\tau} = -\frac{\partial V_1}{\partial x} = -\frac{\partial \Delta V_1}{\partial x}$$

we get

$$\omega = \omega_{2ps} = 2\varepsilon \frac{\partial \Delta V_1}{\partial x} \tag{43}$$

Expressing the deviation angle of exit flow by $\Delta\varepsilon$ we have

$$\Delta\varepsilon \doteqdot \frac{\Delta w_{ym}}{V_2} \tag{44}$$

where Δw_{ym} is the y-directional mean induced velocity.

From the above consideration we can find that the turning angle is smaller (under turning) at the center of span, larger (over turning) at side walls, and smallest at the border between boundary layer and main flow. This explains qualitatively the result of cascade experiment to some extent, but sufficient explanation of the latter is not accomplished in which the turning angle at the center of span is not always smaller than the ones at side walls. Further studies must be needed on these points, and we feel some limit of theory which does not consider the viscosity of fluid.

Now, we must be careful that there exists an assumption of great importance in the above ideas. Consideration of the secondary flow in a rectangle means that we accept the idea that there is no flow in AC-direction (y-direction) at the trailing edge AB, and there remains some doubt that the Kutta's condition of blade in the flow containing secondary flows can be expressed by the above or not. Probably it may be accepted when the trailing edge is very thin or of cusped form. And if we accept the above assumption, we can reach a noteworthy conclusion that the direction of the wake (vortex sheet) will show the exit flow direction of two-dimensional cascade (i. e. the cascade containing no secondary flow), because sides AB and CD are not deformed by the secondary flow. The exit flow angle of two-dimensional cascade should be obtained immediately from the direction of wake without a troublesome method such as the boundary layer suction! This is going to be proved experimentally. (not yet published).

8. Conclusions

Because this report on the secondary flow theory was lengthy, and many inaccurate points were involved, and furthermore explanations were along the development of the author's idea, the author is afraid it has defects that the story is diffuse, redundant and difficult to seize the essence. Therefore let us rearrange the story.

(The method of approach to the secondary flow is not confined to the theory explained in this report. Honda, Gomi and Namba have developed other methods(9)(3)(14).)

The fundamentals of secondary flow theory are clear and sufficient by the methods developed by Squire & Winter or Hawthorne. When we apply these method to the cascade problem the consideration of Trefftz plane at the exit of cascade is probably not entirely erroneous, although it cannot be said being strict. The objection in considering the boundary of Trefftz plane being rectangle was eliminated in Chapter 7. The idea that the spanwise velocity induced at the boundary of rectangle corresponds to the trailing vortex is not doubtful, and therefore the treatment of secondary flow theory customary used should be sufficiently usable for the purpose.

On the other hand the axisymmetric theory is the one to solve the flow in axial machine under an assumption of axisymmetry and has no room for doubt in itself.

If so, where was the point of problem? It was in the process connecting the secondary flow theory to the axisymmetric theory. The author thinks that the employment of the axisymmetric flow as a base flow to get the first approximation of secondary flow did especially make the problem complicated. Furthermore, to intend to introduce the idea of trailing vortex of secondary flow into the axisymmetric flow is supposed to be another source of troublesome complexity. It is easy to understand that the strength of the trailing vortex in the axisymmetric theory can be obtained by the analysis of axisymmetric flow itself or by the examination of secondary flow at the limiting case of infinitesimal blade spacing of cascade, but strangely neither of them were attempted until the author has tried (for twenty years!). This might show, however, the want of assiduity of the author himself who has engaged in the secondary flow theory almost all the time during this period!

8. 1. Summary of Results

Explanations on the secondary flow were done under an assumption of the theory of ideal fluid flow with vortex. The foundation was the theory created by Squire & Winter and improved by Hawthorne. The first approximation of secondary flows could be easily obtained from the law of vortex (3. 4.).

Brief explanations were presented on the axisymmetric theory, which has the meaning as the zeroth approximation for the use of secondary flow theory and, therefore, is important. It was clarified that the equation introduced in the axisymmetric theory gives us no information on the trailing vortex. But because the flow in the downstream of cascade can be obtained from the equation, it was hoped to get the vorticity from this flow, and the trial was succeeded (6. 1.). But the expression of trailing vortex in a neat form has not yet been obtained. One of the important present results is that, if the exit flow of blade row is of free vortex type, namely $\tan\gamma_{2e}$ is inversely proportional to radius, there is no streamwise vortex in the exit flow even if any vortex is contained in the inlet flow, which means there is no secondary circulation in the exit flow. This produces an important meaning when the axisymmetric flow is employed as a base flow. Namely, when the flow is of free vortex type, there is no secondary flow in the base flow, and therefore in the case of blade row of finite spacing the final three-dimensional flow can be obtained by adding the secondary flow corresponding to the cascade of finite spacing to this base flow. In this instance, the data of linear cascade will be useful. (Since there is no secondary flow also in the exit flow of linear cascade of infinitesimal blade spacing, this flow is to be used as the base flow of linear cascade.) (Practical method has not yet be presented.) One of the questions mentioned in *Introduction* has been answered. When the flow is not of free vortex type, the treatment has not yet solved and must be examined in the future, but considering the case of free vortex type good results may be expected by adding the correction of finite spacing to the axisymmetric flow (although the method of finding this value for correction is unknown). The practical method of treatment is left to be the future problem.

In the explanation of secondary flow theory stated in Chapter 5., the movement of vortex in the upstream of cascade into the downstream was examined and the fact that the trailing vortex is consisted of the trailing filament vortex and the trailing shed vortex was clarified. The former can be explained as a connecting vortex of vortex in a passage with neighboring one when the vortex in the upstream is deformed in passing through cascade passages, and the latter as a shed vortex corresponding to the variation of circulation of blade. The sum of the both corresponds to the spanwise flow induced at the boundary of Trefftz plane (or the blade trailing edge). (3. 3 and 7. 3).

Vortices existing in the downstream of blade row are the passage vortex, the trailing filament vortex and the trailing shed vortex, and the author tried to name the sum of the former two "quasi vortex". In contrast to the fact that the trailing shed vortex corresponds to the variation of blade circulation, the quasi vortex, when regarded in an averaged vlaue, is related only to inlet and outlet conditions and not to the blade profile etc.. Also there is no change of blade circulation by the quasi vortex. These facts perceived the author that the quasi vortex flow is to be understandable to correspond to the flow in the bend of single domain, and affairs which characterize the secondary flow in cascades (characterize the existence of blades) are the variation of blade circulation and the accompanying trailing shed vortex.

Since the treatment of the Trefftz plane of the circular cascade is impossible, considerations were limited to the Trefftz plane of linear cascade (Chapter 7.). The Trefftz plane is thought to be rectangular at the exit of cascade, and the examination of subsequent modification of the shape of boundary in the downstream showed us that the rectangular form must be retained, and calculations of flow in rectangle were guaranteed to be useful for the purpose. The fact that the rectangle of boundary is retained showed us also that the direction of wake of blade is expected to be the direction of exit flow of two-dimensional cascade, and this is expected to be a quite easier way to find the two-dimensional exit direction without using troublesome methods such as the boundary layer control in cascade experiments.

Calculation results of induced velocity in y-direction were also shown, but sufficient agreements to experimental data are not expected. The phenomena of decay of vortex and such should be adopted to get the final agreement, and the author has deeply felt the way to the goal being quite far despite of the haze has been cleared up.

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