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EQUILIBRIUM OF AXIALLY SYMMETRIC TOROIDAL PLASMA WITH A RECTANGULAR CROSS SECTION

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Abstract

The equilibrium problem of toroidal plasma with distributed current in a fat torus with any small toroidicity and rectangular cross section is analyzed.

The analytical solution is deduced by the use of Green's function and the constant-variable method and also the some numerical results are given.

1. Introduction

It is difficult to solve the equilibrium problem of toroidal plasma with any toroidicity and any boundary conditions. The methods usually used are the iterational one under the assumption of small toroidicity and the near-axis expansion around a magnetic axis.[1, 2] However, as the critical beta β_c (β=plasma pressure/magnetic pressure) is inversely proportional to the aspect ratio A (A=major radius/minor radius),[3] the fat torus with small aspect ratio is favorable. In the present paper we give the exact solution for equilibrium of plasma with more general current distribution and with rectangular cross section.[2, 4] The boundary conditions are usually specified for given special cases, but we may regard any magnetic surface given by a solution obtained as a plasma boundary and may study the equilibrium of plasma with various kinds of cross section. We solve the inhomogeneous partial differential equation for plasma equilibrium by applying the constant-variable method to the singular equation in order to obtain Green's function in 2. In 3 the numerical calculation of solution is performed and the conclusions are summarized.

In cylindrical coordinates (r, φ, z) (Fig. 1) we consider the equilibrium equations for axially symmetric plasma which is written in terms of a stream function

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as follows[5];

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\right)+\frac{\partial^2 \Psi}{\partial z^2}=rj_\varphi, \tag{1}$$

$$j_\varphi=r\frac{dp(\Psi)}{d\Psi}+\frac{I(\Psi)}{r}\frac{dI(\Psi)}{d\Psi}, \tag{2}$$

where I and p, the current stream function and the pressure, respectively, are arbitrary functions of Ψ. Expanding the both functions in powers of Ψ, we get

$$-R^4\frac{dp}{d\Psi}=a+a'\Psi+a''\Psi^2+\cdots, \tag{3}$$

$$-R^2 I(\Psi)\frac{dI}{d\Psi}=b+b'\Psi+b''\Psi^2+\cdots, \tag{4}$$

where the prime denotes $d/d\Psi$.

The case when only the first constant term is retained was solved by Zueva et al., but the result is not likely realistic because the z-dependence of current distributions is disregarded.[2]

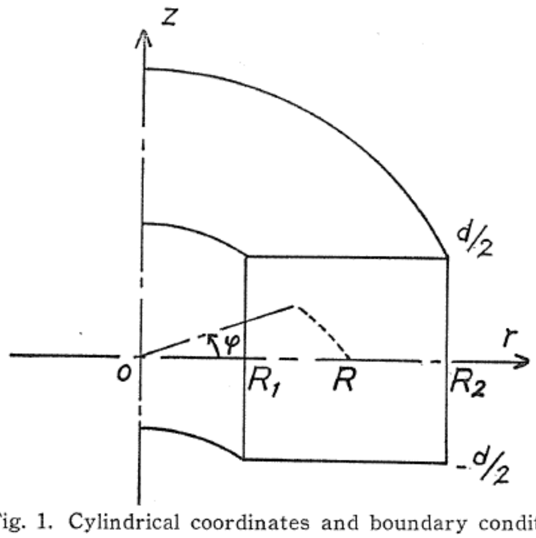


Fig. 1. Cylindrical coordinates and boundary conditions

2. Analytical Solution

In the present paper we consider the equilibrium with current distribution which is dependent on Ψ (r, z), so that the second term a' in eq. (3) is included. Therefore, the equations to be solved have the following forms;

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\right)+\frac{\partial^2\Psi}{\partial z^2}-r^2a'\Psi=r^2a+b, \tag{5}$$

$$j_\varphi=-\frac{1}{R^3}\left(ra+\frac{b}{r}+ra'\Psi\right), \tag{6}$$

with the boundary conditions for rectangular cross section (Fig. 1);

$$\begin{aligned}\Psi(R_1)=\Psi(R_2)=0,\\ \Psi(0)=\Psi(d)=0,\end{aligned} \tag{7}$$

where the arguments r, z are normalized by the major radius of torus R.

The equation for Green's function corresponding to Eq. (5) is

$$\frac{\partial^2 G}{\partial r^2}+\frac{\partial^2 G}{\partial z^2}-\frac{1}{r}\frac{\partial G}{\partial r}-b'G=-\delta(r-\xi)\delta(z-\eta). \tag{8}$$

Expanding Green's function and delta function by the eigen functions which satisfy the boundary conditions, we get the following expressions;

$$G(r,\ z;\xi,\ \eta)=\frac{2}{d}\sum_{n=1}^{\infty}g_{0n}(r,\ \xi)\sin\frac{n\pi z}{d}\cdot\sin\frac{n\pi\eta}{d}, \tag{9}$$

$$\delta(z-y)=\frac{2}{d}\sum_{n=1}^{\infty}\sin\frac{n\pi z}{d}\cdot\sin\frac{n\pi\eta}{d}. \tag{10}$$

Substituting Eqs. (9) and (10) into Eq. (8) and transforming from g_{on} to g_n by the relation $g_{on}=rg_n$, Eq. (8) becomes

$$\frac{d}{dr}\left(r\frac{dg_n}{dr}\right)-\frac{1}{r}g_n-\beta_n^2 rg_n=-\delta(r-\xi), \tag{11}$$

where

$$\beta_n^2=\left(\frac{n\pi}{d}\right)^2+b'.$$

This is formally the Strum-Liouville type equation, but it is not easy to seek for the eigen values for r, so that we solve this problem by applying the constant-variable method to this singular equation as will be described in Appendix.

(1) The case of $\beta_n^2\geqq 0$

Transforming the arguments from $(r,\ \xi)$ to $(x,\ y)$ according to the relations $\beta_n r=x$ and $\beta_n\xi=y$, the equation for Green's function is written by

$$\frac{d^2g_{2n}}{dx^2}+\frac{1}{x}\frac{dg_{2n}}{dx}-\left(1+\frac{1}{x^2}\right)g_{2n}=-\delta(x-y)/x. \tag{12}$$

We introduce two functions g_{2n}^1 and g_{2n}^2 that satisfy respectively the boundary conditions at x_1 and x_2, which are defined as a function of the modified Bessel function $I_1(x)$, $K_1(x)$, *i. e.*, the solutions to the homogeneous equation of Eq. (12) ;

$$\begin{aligned} g_{2n}^1(x) &= D_{2n}(x_1,\ x), \\ g_{2n}^2(x) &= D_{2n}(x_2,\ x), \end{aligned} \tag{13}$$

where

$$D_{2n}(x,\ y) = K_1(x)I_1(y) - I_1(x)K_1(y). \tag{14}$$

The function corresponding to $w_n\ (x_1,\ x_2)$ in Appendix is

$$\begin{aligned} w_{2n}(x_1,\ x_2) &= xW[g_{2n}^1(x),\ g_{2n}^2(x)] \\ &= D_{2n}(x_1,\ x_2), \end{aligned} \tag{15}$$

where W is Wronskian for g_{2n}^1 and g_{2n}^2. Therefore, we obtain the following expressions as the solution to Eq. (12) ;

$$g_{2n}(x) = \begin{cases} -\dfrac{g_{2n}^2(x)}{D_{2n}(x_1,\ x_2)} g_{2n}^1(y), & x_1 \leqq y \leqq x, \\ -\dfrac{g_{2n}^1(x)}{D_{2n}(x_1,\ x_2)} g_{2n}^2(y), & x \leqq y \leqq x_2. \end{cases} \tag{16}$$

(ii) The case of $\beta_n^2 = -b_n^2 = 0$

In this case, transforming the arguments from $(r,\ \xi)$ to $(x,\ y)$ by the relations $b_n r = x$ and $b_n \xi = y$, the equation for Green's function is given by

$$\frac{d^2 g_{1n}}{dx^2} + \frac{1}{x} g_{1n} + \left(1 - \frac{1}{x^2}\right) g_{1n} = -\frac{\delta(x-y)}{x}. \tag{17}$$

As the solutions to the homogeneous equation of Eq. (17) are Bessel functions $J_1(x)$ and $N_1(x)$, we get the solution to Eq. (17) by the same procedures as for the first case. It is

$$g_{1n} = -\begin{cases} -\dfrac{n}{2} \dfrac{g_{1n}^2(x)}{D_{1n}(x_2,\ x_1)} g_{1n}^1(y), & x_1 \leqq y \leqq x, \\ -\dfrac{n}{2} \dfrac{g_{1n}^1(x)}{D_{1n}(x_2,\ x_1)} g_{1n}^2(y), & x \leqq y \leqq x_2, \end{cases} \tag{18}$$

where

$$D_{1n}(x,\ y) = N_1(x)J_1(y) - J_1(x)N_1(y). \tag{19}$$

From Eqs. (9), (16) and (18) the Green's function that satisfies the boundary conditions is obtained as follows ;

$$G(x, z; y, \eta)=\frac{2}{d}r\sum_{n=1}^{\infty}g_n(x, y)\sin\frac{n\pi z}{d}\cdot\sin\frac{n\pi\eta}{d}, \tag{20}$$

where

$$g_n(x)=\begin{cases}g_{2n}(x), & \beta_n^2\geqq 0\\ g_{1n}(x), & \beta_n^2\leqq 0.\end{cases} \tag{21}$$

By using this Green's function the solution to the boundary value problem with the rectangular cross section can be expressed in the following integral form;

$$\Psi(r, z)=-\int_{R1}^{R2}d\xi\int_0^d d\eta G(r, z; \xi, \eta)(a\xi^2+b), \tag{22}$$

and performing this integral we get

$$\Psi(r, z)=\frac{4}{\pi}r\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{nb'D_n(R_1, R_2)}\{D_n(R_1, r)(aR_2+b/R_2)$$

$$-D_n(R_2, r)(aR_1+b/R_1)-(ar+b/r)D_n(R_1, R_2)\}\cos\frac{n\pi z}{d}, \tag{23}$$

where we used the following transformation and the relations for Bessel functions;

$$z\rightarrow z+d/2, \tag{24}$$

$$\left.\begin{aligned}&[y^2z_2(y)]'=y^2z_1(y),\\ &[z_0(y)]'=z_1(y),\\ &I_\nu K_{\nu+1}+I_{\nu+1}K_\nu=\frac{1}{y},\\ &I_\nu K_\nu'-I_\nu'K_\nu=-\frac{1}{y}.\end{aligned}\right\} \tag{25}$$

3. Numerical Calculations and Conclusions

By calculationg the analytical solution given by Eq. (23) for various kinds of parameters we study the equilibrium state of toroidal plasma. The parameters included in the solution are the geometrical parameters R_1, R_2 and d and the physical parameters a, b, and b' that were introduced in the expansions of p and $I(\Psi)$, *i. e.*, in Eqs. (3) and (4).

If $b'=0$, the physical parameters a and b are related to the plasma current I_p and the poloidal beta $\beta_p(\beta_p=2p/B_p^2)$;

$$a=\frac{I_p}{\pi r_0^2}R^3\beta_p, \tag{26}$$

$$b=\frac{I_p}{\pi r_0^2}R^2\frac{R+\sqrt{R^2-r_0^2}}{2}(1-\beta_p), \tag{27}$$

where r_0 is the radius of plasma column. However, the equilibrium is determined by the ratio of these parameters, b/a;

$$\begin{aligned} b/a &= \frac{1+\sqrt{1-\varepsilon^2}}{2}\frac{1-\beta_p}{\beta_p} \\ &\simeq \left(1-\frac{1}{2}\varepsilon\right)\frac{1-\beta_p}{\beta_p}, \end{aligned} \tag{28}$$

where

$$\varepsilon \equiv 1/A = r_0/R,$$

and this parameter becomes negative in the region of $\beta_p \geqq 0$ as shown in Fig. 2.

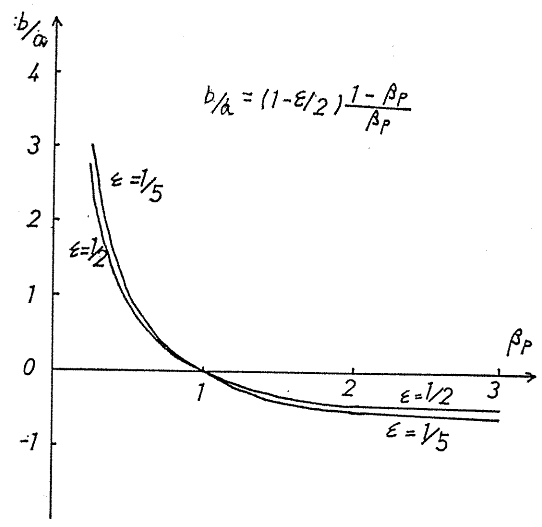


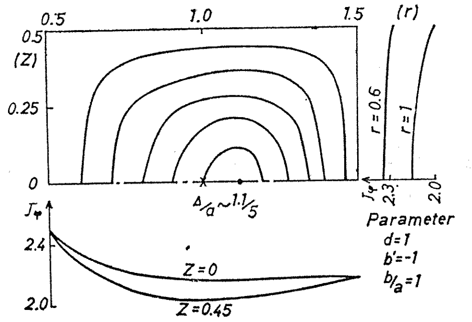
Fig. 2. Parameter b/a v. s. poloidal beta

The parameter b' is related to the diamagneticity or corrective mechanism of plasma that makes the current to coincide the magnetic surface. We then analyze numerically taking the following values of parameters;

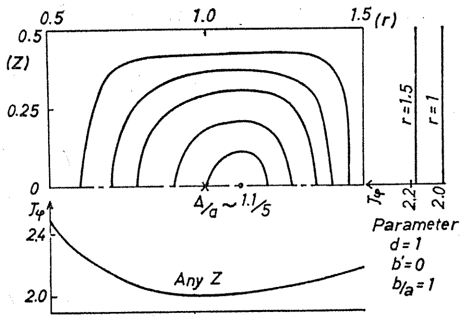
$$\begin{aligned} &R_1=1/2, \quad R_2=3/2, \quad d=1, \\ &b/a=1,\ 0.5,\ 0,\ -0.5,\ -1, \\ &b'=1,\ 0,\ -1,\ -5,\ -60. \end{aligned} \tag{29}$$

In this case the aspect ratio of torus, A, is 2 and the results calculated are shown in Fig. 3-(a) to (k).

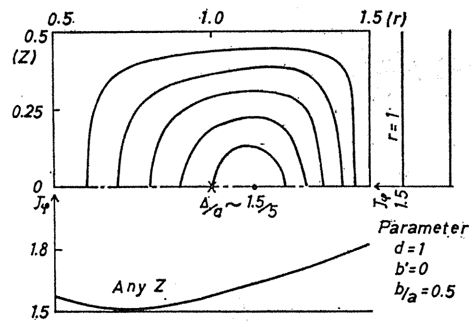
Fig. 3. Equilibrium of toroidal plasma with rectangular cross section; $A=2$, $d=1$



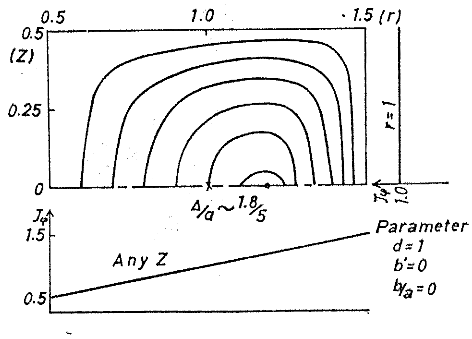
(a) $b'=0$, $b/a=0.5$



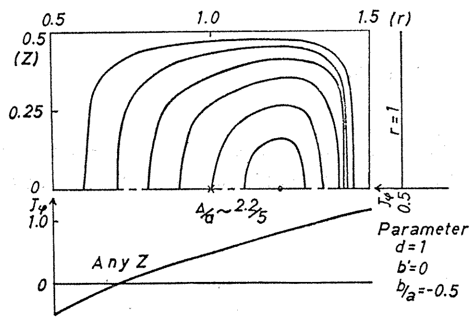
(b) $b'=0$, $b/a=0$



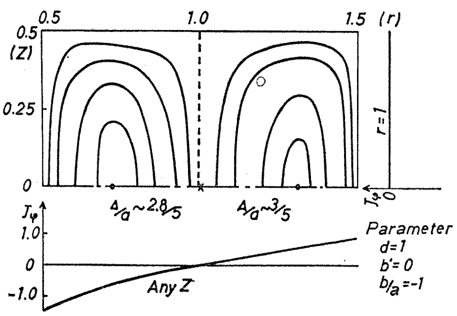
(c) $b'=0$, $b/a=-0.5$



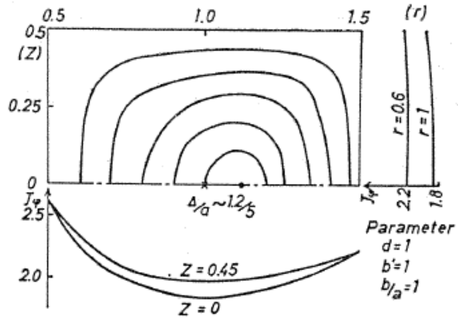
(d) $b'=0$, $b/a=-1$



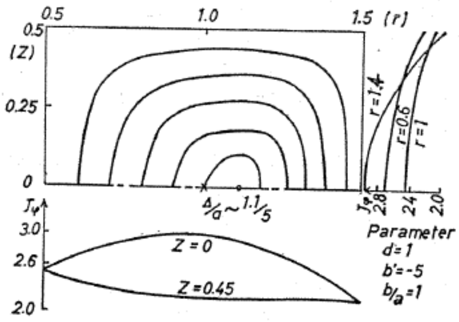
(e) $b'=1$, $b/a=1$



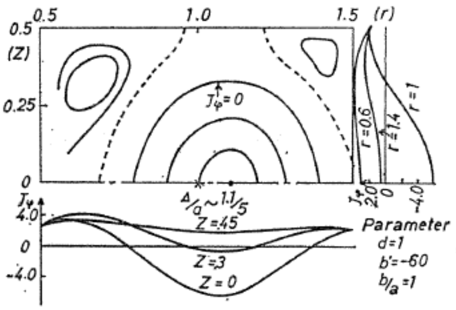
(f) $b'=0$, $b/a=1$



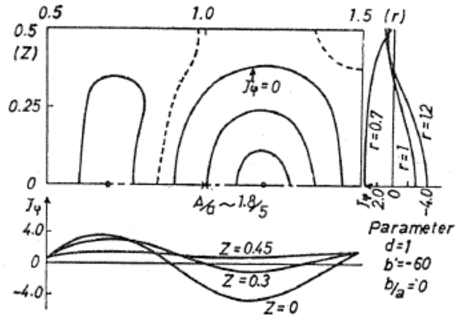
(g) $b'=-1, \quad b/a=1$



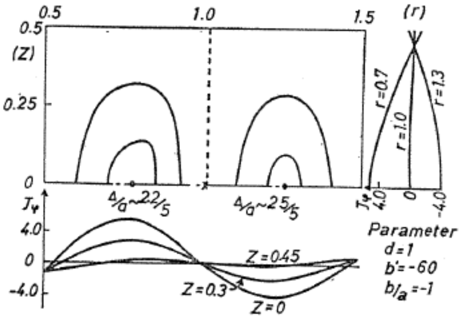
(h) $b'=-5, \quad b/a=1$



(i) $b'=-1, \quad b/a=1$



(j) $b'=-60, \quad b/a=0$



(k) $b'=-60, \; b/a=-1$

We can deduce the following conclusions from the results of numerical calculations; (1) the displacement of plasma Δ is almost determined by b/a, *i. e.*,

$$\Delta/(d/2)=\frac{1}{5}(1.8-0.7b/a) \tag{30}$$

(2) only if $b'>b/a$ the equilibrium with vanishing current density on a certain magnetic surface can be realized, (3) the appearance of multiple magnetic axis is

related to the inversion of plasma current and if $b'=0$ this takes place at $\beta_p \geqq 2$, (4) the configuration shown in Fig. 3-(i) is also realizable by replacing the current at the outside of the surface $j_\varphi=0$ by that of external conductors, (5) the solution of Doublet type corresponds to the configuration appearing around the left-side axis in Fig. 3-(j).

With the help of the solution presented in this paper we can investigate the equilibrium with very elongated shape that is favourable for the fusion reactor from the point of view of the efficiency and so on, and can study the fat torus with any small aspect ratio. The equilibrium obtained by replacing the plasma current outside the surface $j_\varphi=0$ by that flowing in the external conductors such as shown in Fig. 3-(i) has an advantage over the usual Tokamak with mettalic shell. Such configuration is shell-less and may constitute a diverter or magnetic limitter.

We wish to thank Dr. Y. Tanaka for valuable discussion.

Appendix

Here we prove that the constant-variable method is applicable to the singular differential equation of Strum-Liouville type;

$$L_x[g(x)]=\frac{d}{dx}\left(p(x)\frac{dg(x)}{dx}\right)+q(x)g(x)=-\delta(x-\xi). \quad \text{(A—1)}$$

We take the functions, g^1 and g^2, which are solutions to the homogeneous equation of Eq. (A-1) and satisfy respectively the boundary conditions at x_1 and x_2. We define a function $w(x^1, x^2)$ by g^1 and g^2;

$$w(x_1, x_2)=p(x)W[g^1(x), g^2(x)], \quad \text{(A—2)}$$

where W is Wronskian for g^1 and g^2. Therefore, formally applying the constant-variable method we obtain the solution to Eq. (A-1) as follows;

$$g(x, \xi)=-\frac{g^2(x)}{w}\int_{x_1}^{x} g^1(x')\delta(x'-\xi)dx'-\frac{g^1(x)}{w}\int_{x}^{x_2} g^2(x')\delta(x'-\xi)dx'$$

$$=\begin{cases}-g^2(x)g^1(\xi)/w(x_1, x_2) & x_1 \leqq \xi \leqq x, \\ -g^1(x)g^2(\xi)/w(x_1, x_2) & x \leqq \xi \leqq x_2.\end{cases} \quad \text{(A—3)}$$

It is proved that $g(x, \xi)$ in Eq. (A-3) is always the solution to Eq. (A-1), in the following way. The equation (A-1) is to be considered as an equation of distribution and can be expressed by the relation for distribution

$$\int g(x, \xi)L_x\varphi(x)dx=-\varphi(\xi), \quad \text{(A—4)}$$

or

$$-g(x)\Big|_{\xi-0}^{\xi+0} p(\xi)\varphi'(\xi)+g^1(x)\Big|_{\xi-0}^{\xi+0} p(\xi)\varphi(\xi)+L_x[g(x)](\varphi)=-\varphi(\xi). \quad \text{(A—5)}$$

Comparing both sides of Eq. (A-5), we get

$$g(x)\Big|_{\xi-0}^{\xi+0}=0, \tag{A—6}$$

$$L_x[g(x)]=0, \tag{A—7}$$

$$g^1(x)\Big|_{\xi-0}^{\xi+0}=-1/p(\xi). \tag{A—8}$$

Therefore, the function $g(x)$ is continuous as usual function and is a solution to the homogeneous equation of Eq. (A-1) and also has a jump only at point ξ, that is, these are conditions for Green's function. Considering the relation

$$\frac{1}{p(\xi)}=\frac{W[g^1,\ g^2]}{w(x_1,\ x_2)}, \tag{A—9}$$

and substituting Eq. (A-3) into Eqs. (A-6) to (A-8), it is clear that $g(x)$ satisfies these all conditions.

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