ON THE POTENTIAL THEORY OF DISTRIBUTED SINGULARITIES AND ITS EDGE CONDITION FOR A LIFTING FLOW OF THREE-DIMENSIONAL BODY

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Abstract

Basing on the Green's theorem the potential flow around an arbitrary three-dimensional body can be expressed by the surface distributions of singularities as soures and doublets. When the surface of body has a discontinuity of tangent across the edge line, infinite velocities are possibly induced at the edge point by the self-induction, which is produced by the distributions of source and doublet on the surface element containing the concerning edge point. Considering physical characteristics of flow such an infinite velocity should be eliminated in some cases, for instance, at the trailing edge. In the present paper some fundamental characters of the theory are first introduced, and eliminating the singular terms of induced velocity at the trailing edge an expression of the Kutta's edge condition is obtaind, which is useful to determine the circulations around a three-dimensional lifting body. This trailing edge condition containes some continuities of components of vortex vector parallel and perpendicular to the trailing edge, through the body and wake surfaces. It is found that these relations can be used as a matching condition between body and wake, giving the starting value of vortex vector at the wake initiation. Similar relations can also be applied to the separation edges, from which vortex sheet issues.

CONTENTS

1. Introduction

The theory of distributed singularity is based on the Green's theorem of potential flow. The velocity potential at an inner point of the space can be expressed by distributions of source and doublet along the boundary surface of this space. When the surface is continuous no singular characteristics of velocity is induced in any point, excepting the finite amount of jump of velocity at the inner and outer point on the boundary surface. This discontinuity is produced by the self-induction due to source and doublet or vortex distributions on the surface element containing the concerning surface point in its center.

When the boundary surface has a discontinuity of tangent across the edge line as like the trailing edge, infinite velocities are possibly produced by the self-induction from the surface element surrounding an edge point, i. e. the edge element, which is divided into some parts of different inclinations. The self-induction to a trailing edge point T is originated from distributions on the trailing edge element, which consists of infinitesimal parts of upper and lower surfaces of body and wake adjoining and surrounding the point T. The purpose of the present paper is to find an edge condition, which satisfies the Kutta condition, by eliminating these singular terms of self-induction which may cause infinite velocities. This condition can also be applied to the edge, where flows along upper and lower surfaces of the body join on the edge line and form a separated sheet issuing into the fluid, such as the leading edge of a delta wing at high attack angle and the tip of an lifting wing of ordinary form.

In the original form of Green's theorem the velocity potential is expressed by source and doublet distributions along the boundary surface. As was suggested by Maxwell[1] the surface distributions of doublet can be transformed to those of vortex sheet and vortex filament. Introducing the formulations given by Hess[2] and Ebihara[3], a physical interpretation is derived in the present paper. This theorem provides the foundation of a three-dimensional wing theory, where solutions are expressed by source and by doublet or vortex distributions. In order to make the space simply connected region, the boundary surface of the body is extended to the infinity covering the wake surface, and the space is divided into two regions, i. e. the inner region, where the flow is considered, and the outer one.

The solution of flow around a specified body is obtained by determining the strengths of source and vortex distributions as to satisfy the boundary conditions. To calculate the surface value of functions, there are two different ideas. In the first method only the concerning flow field is solved by setting the potential of the body to a constant value, which means that no fluid motion appears in the inside region of body. In the second method the whole field of flow containing the inside of body is treated. Analysing self-induced velocities we discuss the relations between these two methods. As it is more suitable for the present purpose, the second method is used through the remainder of the present paper.

The distributed singularity technique is first developed for the flow around an arbitrary two-dimensional thick body,[4),5),6)] where only the vortex distributions are considered, and the Kutta condition is satisfied by putting the strength of vortex sheet zero at the trailing edge.[5] This result depends on the fact that in the two-dimensional flow no vortex sheet is shed into the wake. It is noted that the result can be reduced as a special case of the present analysis.

Respecting to the three-dimensional case the theory of distributed singularity is successfully applied to solve the non-lifting flow around an arbitrary body,[7] where no edge conditions are needed.

Analyses of the three-dimensional lifting flows are early developed by the lifting surface theories, in which the effect of a lifting surface is represented by the vortex distributons over surfaces of thin wing and wake. The kutta condition is simply satisfied by the continuity of vortex at the trailing edge, which is derived as a special case of the present calculation. The lifting surface theory with vortex sheet is recently extended to solve the flow around a thin wing with large angle of attack.[8], [9], [10]

The lifting flow around a thick body is early solved by the combination of source and vortex distributions[11], [12] in which the Kutta condition for the two-dimensional flow is conventionally used. In a theory of distributed doublet[13] for the lifting flow around a body of general configuration the continuity of doublet strength between body and wake is used for the Kutta condition. Considering the equivalency of surface doublet and vortex distributions, such an edge condition should contain some relations for the vortex strength. The present results consist of continuity relations of doublet and vortex strength between body and wake, as shown later.

A trailing edge condition presented by Mangler and Smith[14] states that the wake surface is tangential to the upper or lower surface of body according to the combination of direction of the lifting force and of velocity component along the trailing edge. Since the direction or sign are easily changed by an infinitesimal change of these quantities across zero value, an abrupt change of the inclination of wake surface can happen on the trailing edge, which is an unnatural circumstance of the flow. In some numerical computations[2], [15] this flip-flop condition is disregarded, resulting in a well convergent solution. Mangler and Smith's conclusion is derived only by the wake conditions, in which a vortex line is parallel to the mean stream line. This condition is also considered in the following calculations. On Kutta condition Legendre[16] gives a general survey with some qualitative descriptions.

In the present paper terms of infinite velocities due to the self-induction from the trailing edge element, which is a composite surface element surrounding the trailing edge point T, are calculated by assuming piecewise continuity of surfaces and of strengths of source and vortex distributions. Eliminating these terms the Kutta's edge condition is formulated with terminal values at T of source and vortex distributions on the four parts of infinitesimal surface elements. The principle of the theory is based on the idea given by Ebihara[3], while it is extended and improved in the present calculation. The present results on the trailing edge condition are expressed in some forms of continuities of doublet and of vortex strengths through the body and wake surfaces.

2. Fundamental Relations

2. 1. Characteristics of Distributed Singularities

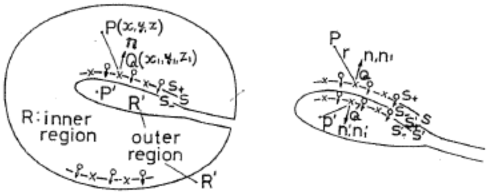
The steady potential flow of an incompressible fluid is considered. According to the Green's theorem the velocity potential ϕ at a point $P(x, y, z)$ in the simply connected region is expressed by a system of distributed source and doublet on the boundary surface, which contains the wake surface extended to the infinity for making the region simply connected one, as shown in Fig. 1. Introducing a surface point $Q(x_1, y_1, z_1)$, the distance $\overline{PQ}=r=\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}$, the unit inward normal $n(l, m, n)$, and a differential operator $\partial/\partial n_1=l(\partial/\partial x_1)+m(\partial/\partial y_1)+n(\partial/\partial z_1)=-\partial/\partial n$, the original form of Green's theorem[17] is given by

$$\phi_P=\iint\left(\frac{\partial\phi}{\partial n}\right)_S\frac{-1}{4\pi r}dS+\iint(-\phi_S)\frac{\partial}{\partial n_1}\left(\frac{1}{4\pi r}\right)dS \tag{1}$$

where $-1/4\pi r$ and $-\partial/\partial n_1(-1/4\pi r)$ represent unit potentials of source and doublet directed to $-n$, respectively. Eq. (1) states that the velocity potential at a space point is expressed by the source distributions of strength $(\partial\phi/\partial n)_S=w_S$ and by the doublet distributions of strength $(-\phi_S)$ along the surface S. We may refer the two different ways of analysis according to the condition of outer surface, i. e. of inside of the body.

a) The method of inner space analysis

Denoting the inner and outer spaces by R and R', respectively, the velocity potential valid for the inner space is expressed by Eq. (1), where the surface S is



a) Method of inner space analysis

b) Method of whole space analysis

Fig. 1. Surface distributions of source and doublet

placed along the inside of boundary surface as shown in Fig. 1 a). Taking the limit process of $r\to 0$, we have the surface value $\phi_\pm$, where $\pm$ sign refers to the upper and the lower point on the surface.[18] $\phi_\pm$ consists of the self-induced discontinuous term $\pm(1/2)(-\phi_S)$ and of the remainder continuous term ϕ_C, which is represented by the sign of Cauchy's principal value of integral, as follows:

$$\phi_\pm=\pm(1/2)(-\phi_S)+\phi_C,$$
$$\phi_C=\oint\!\!\!\oint(w_S)\frac{-1}{4\pi r}dS+\oint\!\!\!\oint(-\phi_S)\frac{\partial}{\partial n_1}\left(\frac{1}{4\pi r}\right)dS. \tag{2}$$

we have, therefore,

$$-\phi_S=\phi_+-\phi_-,\quad \phi_C=(1/2)(\phi_++\phi_-) \tag{3}$$

The n component of velocity is calculated by

$$w_P=\iint(w_S)\frac{\partial}{\partial n}\left(\frac{-1}{4\pi r}\right)dS+\iint(-\phi_S)\frac{\partial}{\partial n}\left[\frac{\partial}{\partial n_1}\left(\frac{1}{4\pi r}\right)\right]dS \tag{4}$$

Taking the limit process of $r \to 0$, the surface value $w_{\pm}$ is obtained[18] by

$$w_{\pm} = \pm(1/2)w_S + w_C,$$

$$w_C = \oiint (w_S)\frac{\partial}{\partial n}\left(\frac{-1}{4\pi r}\right)dS + \oiint (-\phi_S)\frac{\partial}{\partial n}\left[\frac{\partial}{\partial n_1}\left(\frac{1}{4\pi r}\right)\right]dS, \tag{5}$$

where $\pm(1/2)w_S$ is the term of self-induction and w_C is the remainder term. Eq. (5) gives

$$w_S = w_+ - w_-, \quad w_C = (1/2)(w_+ + w_-). \tag{6}$$

Substituting Eqs. (5) and (6) into Eq. (1) we have

$$\phi_P = \iint (w_+ - w_-)\frac{-1}{4\pi r}dS + \iint (\phi_+ - \phi_-)\frac{\partial}{\partial n_1}\left(\frac{1}{4\pi r}\right)dS. \tag{7}$$

From the Green's theorem for an outer space point P', we have

$$\phi_{P'} = 0 = \iint \left(\frac{\partial \phi}{\partial n}\right)_S \frac{-1}{4\pi r}dS + \iint (-\phi_S)\frac{\partial}{\partial n_1}\left(\frac{1}{4\pi r}\right)dS, \tag{8}$$

and, therefore, there is no flow in the outer space R', resulting in

$$w_- = 0, \quad \phi_- = 0 \tag{9}$$

When w_+ and ϕ_+ on the surface S_+ are known, ϕ_P can be calculated by

$$\phi_P = \iint w_+ \frac{-1}{4\pi r}dS + \iint \phi_+ \frac{\partial}{\partial n_1}\left(\frac{1}{4\pi r}\right)dS. \tag{10}$$

The strengths of source and doublet distributions are found to be

$$(w_S) = w_+, \quad (-\phi_S) = \phi_+. \tag{11}$$

It is noticed that the sign of ϕ is changed. Values of ϕ and w close to the boundary are schematically shown in Fig. 2.

b) The method of whole space analysis

To get the expression of ϕ valid for the whole field, sources and doublets are distributed on two layers in both sides of boundary, i. e. on the inner surface S and on the outer surface S', as shown in Fig. 1 b). Applying the Green's theorem to these two layers the velocity potential at the inner space point P are given, respectively:

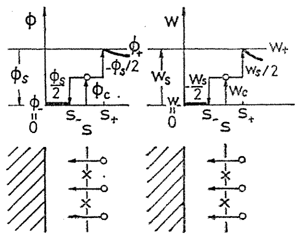


Fig. 2. Potential and normal velocity near the boundary (Method of inner space analysis)

$$\phi_P = \iint \left(\frac{\partial \phi}{\partial n}\right)_S \frac{-1}{4\pi r}dS + \iint (-\phi_S)\frac{\partial}{\partial n_1}\left(\frac{1}{4\pi r}\right)dS$$

$$0 = \iint \left(\frac{\partial \phi'}{\partial n'}\right)_{S'} \frac{-1}{4\pi r'}dS' + \iint (-\phi'_{S'})\frac{\partial}{\partial n'_1}\left(\frac{1}{4\pi r'}\right)dS' \tag{12}$$

In the limit of $S'\to S$, n' and r' approach $n'\to n$ and $r'\to r$. Adding these two equations we have

$$\phi_P=\iint\left(\frac{\partial\phi}{\partial n}-\frac{\partial\phi'}{\partial n}\right)_S\frac{-1}{4\pi r}dS+\iint(-\phi_S+\phi'_S)\frac{\partial}{\partial n_1}\left(\frac{1}{4\pi r}\right)dS \tag{13}$$

The velocity potential at the outer space point P' induced by those two layers are, respectively:

$$\begin{aligned}0&=\iint\left(\frac{\partial\phi}{\partial n}\right)_S\frac{-1}{4\pi r}dS+\iint(-\phi_S)\frac{\partial}{\partial n_1}\left(\frac{1}{4\pi r}\right)dS\\ \phi_{P'}&=\iint\left(\frac{\partial\phi'}{\partial n'}\right)_{S'}\frac{-1}{4\pi r'}dS'+\iint(-\phi'_{S'})\frac{\partial}{\partial n'_1}\left(\frac{1}{4\pi r'}\right)dS'\end{aligned} \tag{14}$$

Approaching $S'\to S$, we have $n'\to n$ and $r'\to r$. The addition of these two equations gives

$$\phi_{P'}=\iint\left(\frac{\partial\phi}{\partial n}-\frac{\partial\phi'}{\partial n}\right)_S\frac{-1}{4\pi r}dS+\iint(-\phi_S+\phi'_S)\frac{\partial}{\partial n_1}\left(\frac{1}{4\pi r}\right)dS \tag{15}$$

We have the same expressions for ϕ in both inner and outer spaces, and, therefore, getting together these two layers into one, the formula of the velocity potential valid through the whole space can be obtained in the form of Eq. (13). Denoting the strength of distributed source and that of doublet directed to $-\boldsymbol{n}$ by σ and μ, respectively, they are defined by

$$\sigma=\left(\frac{\partial\phi}{\partial n}-\frac{\partial\phi'}{\partial n}\right)_S=w_S-w'_S,\quad \mu=-(\phi_S-\phi'_S). \tag{16}$$

In order to get the physical meaning of σ and μ, surface values of velocity potential $\phi_\pm$ and of normal velocity $w_\pm$ at the surface point Q is calculated by the limiting process $\overline{PQ}\to 0$, i. e. $r\to 0$. $\phi_\pm$ and $w_\pm$ consist of self-induced term which has a discontinuous jump of value across the surface S, and of the remainder continuous term which is shown by using the sign of Cauchy's principal value of integral. The velocity potential at Q is expressed by

$$\begin{aligned}\phi_\pm&=\pm(1/2)(-\phi_S+\phi'_S)+\phi_C,\\ \phi_C&=\oiint(w_S-w'_S)\frac{-1}{4\pi r}dS+\oiint(-\phi_S+\phi'_S)\frac{\partial}{\partial n_1}\left(\frac{1}{4\pi r}\right)dS,\end{aligned} \tag{17}$$

which gives

$$\mu=-\phi_S+\phi'_S=\phi_+-\phi_-,\quad \phi_C=(1/2)(\phi_++\phi_-). \tag{18}$$

The velocity component along inner normal at Q is calculated in similar form with Eq. (5):

$$\begin{aligned}w_\pm&=\pm(1/2)(w_S-w'_S)+w_C,\\ w_C&=\oiint(w_S-w'_S)\frac{\partial}{\partial n}\left(\frac{-1}{4\pi r}\right)dS+\oiint(-\phi_S+\phi'_S)\frac{\partial}{\partial n}\left[\frac{\partial}{\partial n_1}\left(\frac{1}{4\pi r}\right)\right]dS,\end{aligned} \tag{19}$$

which gives

$$\sigma = w_S - w'_S = w_+ - w_-, \quad w_C = (1/2)(w_+ + w_-) \tag{20}$$

Values of ϕ and w near the boundary region are scnematically shown in Fig. 3. It is noticed that the sign of boundary value of ϕ is changed similar to the previous case.

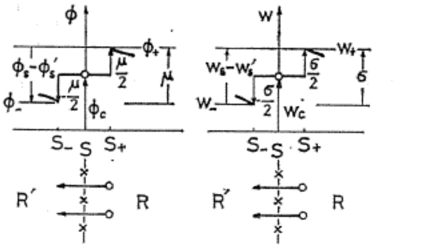


Fig. 3. Potential and normal velocity near the boundary (Method of whole space analysis)

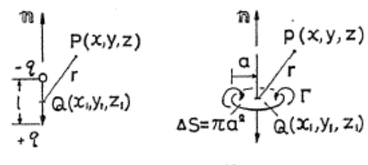


Fig. 4. Equivalency of doublet and infinitesimal vortex ring

2. 2. The Equivalency Theorem of Doublet and Vortex Distributions

It is easily proved that a doublet and an infinitesimal vortex produce the same form of dipole potential, as those shown in Fig. 4. Setting a couple of source q and sink $-q$ apart by a small distance l at Q, the velocity potential at P induced by a doublet is given by the limit process of $l \to 0$.

$$\phi = \lim_{l \to 0,\ q \to \infty} \frac{lq}{4\pi} \frac{\partial}{\partial n_1}\left(\frac{1}{r}\right) = \frac{m}{4\pi} \frac{\partial}{\partial n_1}\left(\frac{1}{r}\right), \tag{21}$$

where the strength is defined by $m = \lim_{l \to 0,\, q \to \infty} lq$. An infinitesimal ring vortex of circulation Γ with a vanishingly small diameter a placed at Q induces the velocity potential at P:

$$\phi = \lim_{a \to 0,\ \Gamma \to \infty} \left[\frac{\pi a^2 \Gamma}{4\pi} \frac{\partial}{\partial n_1}\left(\frac{1}{r}\right) + 0(a^4) + \cdots\right] = \frac{m}{4\pi} \frac{\partial}{\partial n_1}\left(\frac{1}{r}\right), \tag{22}$$

where the strength is defined by $m = \lim_{a \to 0,\, \Gamma \to \infty} \pi a^2 \Gamma$.

When the doublet is distributed along a surface S and has variant strength μ, its velocity potential can be replaced by the distribution of vortex, i. e. by a vortex sheet, whose strength and direction is given by

$$\begin{aligned} &\boldsymbol{\gamma} = \boldsymbol{n} \times \nabla_1 \mu, \\ &\therefore \quad \operatorname{div} \boldsymbol{\gamma} = 0 \end{aligned} \tag{23}$$

where $\nabla_1 = \boldsymbol{i}(\partial/\partial x_1) + \boldsymbol{j}(\partial/\partial y_1) + \boldsymbol{k}(\partial/\partial z_1)$. This relation is easily known by the schematic illustration in Fig. 5. The strength of dipole distribution on S changes a finite amount across the boundary curve C and, therefore, along the curve C there remain a concentrated vortex filament with circulation $(-\mu)_C$, which is also known by Fig. 5.

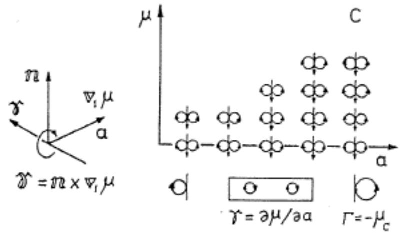


Fig. 5. Relation between dipole (doublet) and vortex distributions

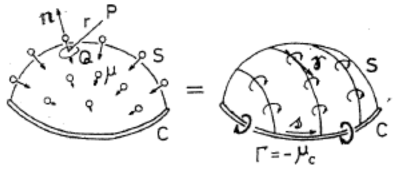


Fig. 6. Equivalency of dipole (doublet) and vortex distributions

The equivalency theorem between doublet and vortex distributions is derived by Hess[2)] and Ebihara[3)] as shown in the followings. Considering a concentrated vortex filament with circulation $-\mu$ along the closed boundary curve C of a surface S as shown in Fig. 6, the velocity induced at P is first calculated by the Biot-Savart's law. When it is transformed by the formula[19)] deduced from the Stokes' theorem, we have

$$\frac{1}{4\pi}\oint(-\mu)\,d\boldsymbol{s}\times\left(\nabla_1\frac{1}{r}\right)=\frac{1}{4\pi}\iint(\boldsymbol{n}\times\nabla_1)\times\left(-\mu\cdot\nabla_1\frac{1}{r}\right).$$

Using the formula of vector triple product and $\nabla_1^2 r^{-1}=0$, we have

$$\begin{aligned}
&(\boldsymbol{n}\times\nabla_1)\times(-\mu\cdot\nabla_1 r^{-1})\\
&=\boldsymbol{n}[\nabla_1(\mu\cdot\nabla_1 r^{-1})]-\nabla_1[\boldsymbol{n}(\mu\cdot\nabla_1 r^{-1})]\\
&=\boldsymbol{n}(\nabla_1\mu)(\nabla_1 r^{-1})-\nabla_1\mu\cdot\boldsymbol{n}\cdot\nabla_1 r^{-1}-\mu\nabla_1(\boldsymbol{n}\cdot\nabla_1 r^{-1})\\
&=\nabla_1 r^{-1}\times(\boldsymbol{n}\times\nabla_1\mu)-\mu\nabla_1(\boldsymbol{n}\cdot\nabla_1 r^{-1})
\end{aligned}$$

Applying the nabla $\nabla=\boldsymbol{i}(\partial/\partial x)+\boldsymbol{j}(\partial/\partial y)+\boldsymbol{k}(\partial/\partial z)$ to a function of $r=[(x-x_1)^2+(y-y_1)^2+(z-z_1)^2]^{1/2}$, we have $\nabla[\mu(\boldsymbol{n}\cdot\nabla_1 r^{-1})]=\mu\nabla(\boldsymbol{n}\cdot\nabla_1 r^{-1})=-\mu\nabla_1(\boldsymbol{n}\cdot\nabla_1 r^{-1})$ and, therefore, the velocity induced by a surface distribution of dipole, whose axis directed to $-\boldsymbol{n}$, is given by

$$\begin{aligned}
&\frac{1}{4\pi}\iint\mu\nabla\left(\boldsymbol{n}\cdot\nabla_1\frac{1}{r}\right)dS\\
&=\frac{1}{4\pi}\iint(\boldsymbol{n}\times\nabla_1\mu)\times\left(\nabla_1\frac{1}{r}\right)dS+\frac{1}{4\pi}\oint(-\mu)\,d\boldsymbol{s}\times\left(\nabla_1\frac{1}{r}\right) \qquad (24)\\
&=\frac{1}{4\pi}\iint\boldsymbol{\gamma}\times\left(\nabla_1\frac{1}{r}\right)dS+\frac{1}{4\pi}\oint(-\mu)\,d\boldsymbol{s}\times\left(\nabla_1\frac{1}{r}\right)
\end{aligned}$$

The first and second terms are velocities induced by the vortex distribution $\boldsymbol{\gamma}$ on S and by a concentrated vortex filament $(-\mu)$ along C. The equivalency theorem between doublet and vortex distributions is thus proved.[2)]

When the surface is separated by some closed curves into several subsurface as shown in Fig. 7, concentrated vortex filaments are added along all dividing curves, across which the strengths of dipole are discontinuously changed.

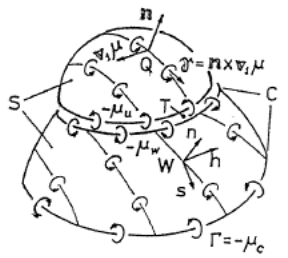


Fig. 7. Vortex sheets and filaments on a multi-surface system

2. 3. Fundamental Theory of the Analysis of Flow by Distributed Singularities

The steady potential flow around an arbitrary lifting body is considered in a simply connected inner space, which is divided by the body and wake surface from the outer space, as shown in Fig. 8. According to the Green's theorem the velocity potential can be expressed by a system of distributed source and doublet on these boundary surfaces, which are represented by ϕ_1 and ϕ_2, respectively. Superposing a known basic flow denoted by ϕ_0, it is given in the form of

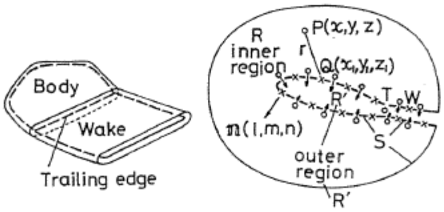


Fig. 8. Surface distributions of source and doublet on a lifting body

$$\phi=\phi_0+\phi_1+\phi_2 \tag{25}$$

Potentials for a space point P are

$$\phi_{1P}=\iint_S \sigma\left(\frac{-1}{4\pi r}\right)dS, \quad \phi_{2P}=\iint_S \mu\frac{\partial}{\partial n_1}\left(\frac{1}{4\pi r}\right)dS, \tag{26}$$

where σ and μ are strengths of distributed source and doulet directed to $-\boldsymbol{n}$, respectively. Potentials for a surface point Q, which is taken on a continuous surface, consists of the self-induced term and of the remainder term, which is conveniently represented by Cauchy's principal value of integral, as follows:

$$\phi_{1Q}=\iint_S \sigma\left(\frac{-1}{4\pi r}\right)dS, \quad \phi_{2Q}=\pm\frac{\mu_Q}{2}+\iint_S \mu\frac{\partial}{\partial n_1}\left(\frac{1}{4\pi r}\right)dS. \tag{27}$$

The velocity is calculated by $\boldsymbol{V}=\nabla\phi$, and is given in the form of

$$\boldsymbol{V}=\boldsymbol{V}_0+\boldsymbol{V}_1+\boldsymbol{V}_2, \tag{28}$$

where $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are induced velocities by source and doublet distributions, respectively. $\boldsymbol{V}_1$ at P is given by

$$\boldsymbol{V}_{1P}=\nabla\iint_S \sigma\frac{-1}{4\pi r}dS=\iint_S \sigma\nabla\left(\frac{-1}{4\pi r}\right)dS, \tag{29}$$

and $\boldsymbol{V}_1$ at Q is

$$\boldsymbol{V}_{1Q}=\pm\frac{\sigma_Q}{2}\boldsymbol{n}+\oiint_S \sigma\nabla\left(\frac{-1}{4\pi r}\right)dS, \tag{30}$$

where the first term represents the self-induced velocity from the infinitesimal surface element containing the point Q.

The induced velocities by a doublet distribution or by the equivalent system of vortex sheet and filament are given for a space point P by

$$\begin{aligned}\boldsymbol{V}_{2P}&=\nabla\iint_S \mu\frac{\partial}{\partial n_1}\left(\frac{1}{4\pi r}\right)dS\\&=\frac{1}{4\pi}\iint_S \boldsymbol{\gamma}\times\left(\nabla_1\frac{1}{r}\right)dS+\frac{1}{4\pi}\oint_C(-\mu)d\boldsymbol{s}\times\left(\nabla_1\frac{1}{r}\right),\end{aligned} \tag{31}$$

where $\boldsymbol{\gamma}$ should hold $\operatorname{div}\boldsymbol{\gamma}=0$ on the surface S. For an ordinary surface point Q we have

$$\begin{aligned}\boldsymbol{V}_{2Q}&=\pm\frac{1}{2}(\nabla_1\mu)_Q+\oiint_S \mu\nabla\left[\frac{\partial}{\partial n_1}\left(\frac{1}{4\pi r}\right)\right]dS\\&=\pm\frac{1}{2}(\boldsymbol{\gamma}\times\boldsymbol{n})_Q+\frac{1}{4\pi}\oiint_S \boldsymbol{\gamma}\times\left(\nabla_1\frac{1}{r}\right)dS\\&\qquad+\frac{1}{4\pi}\oint_C(-\mu)d\boldsymbol{s}\times\left(\nabla_1\frac{1}{r}\right),\end{aligned} \tag{32}$$

where the first term with $\pm$ sign represents the self-induced velocity.

The solution should satisfies the boundary condition at surfaces excepting infinitesimally narrow strips along the trailing edge, where the Kutta's edge condition should be satisfied.

The boundary condition along a body surface is simply given by

$$\boldsymbol{V}_Q\cdot\boldsymbol{n}=(\boldsymbol{V}_{0Q}+\boldsymbol{V}_{1Q}+\boldsymbol{V}_{2Q})\cdot\boldsymbol{n}=0. \tag{33}$$

The wake surface is a vortex sheet flowing with the mean velocity $\boldsymbol{V}_W$ of upper and lower surfaces. In the mean velocity self-induced terms of both sides are cancelled and $\boldsymbol{V}_W$ is given by

$$\begin{aligned}\boldsymbol{V}_W&=\boldsymbol{V}_{OW}+\oiint_S \sigma\nabla\left(\frac{-1}{4\pi r}\right)dS\\&\quad+\frac{1}{4\pi}\oiint_S \boldsymbol{\gamma}\times\left(\nabla_1\frac{1}{r}\right)dS+\frac{1}{4\pi}\oint_C(-\mu)d\boldsymbol{s}\times\left(\nabla_1\frac{1}{r}\right).\end{aligned} \tag{34}$$

The flow should be tangential to the wake surface W and vortex lines are parallel to stream lines. Introducing vorticity components γ_s, γ_h and velocity components v_s, v_h, v_n in the orthogonal co-ordinates s, h, n, where s and h are taken tangential and n is normal to W, the boundary conditions along the wake surface are given by

$$\boldsymbol{V}_W \cdot \boldsymbol{n} = v_n = 0, \qquad \gamma_s/\gamma_h = v_s/v_h, \tag{35}$$

which expresses that $\boldsymbol{\gamma}$ is parallel to V_W. Therefore, this relation can also be derived from[13]

$$\frac{D\mu}{Dt} = v_s \frac{\partial \mu}{\partial s} + v_h \frac{\partial \mu}{\partial h} = v_s \gamma_h - v_h \gamma_s = 0. \tag{36}$$

As a principle unknown parameters σ and μ or $\boldsymbol{\gamma}$ can be determined by solving Fredholm's integral equations, which are obtained by the substitution of Eqs. (28), (30), (32) and (34) into Eqs. (33) and (35). To determine the vector $\boldsymbol{\gamma}$ tangential to S, an additional equation $\operatorname{div} \boldsymbol{\gamma} = 0$ should be used. As the form of wake surface is not known *a priori*, the analysis is started by assuming the form of wake surface. Calculating the velocity V_W, and then, the vortex line of wake is corrected to lie along the mean velocity vector V_W. Some iterative procedures will be necessary.

3. Singularities of the Induced Velocity at the Trailing Edge

To derive the Kutta's edge condition singular terms of the induced velocity at the trailing edge point T is now calculated. When the solution of a lifting flow is expressed by source and vortex distributions along the body and wake surfaces, possible infinite velocities induced at T are caused by distributions on the infinitesimally narrow strips of surfaces containing the trailing edge, i. e. by the self-induction terms of velocity at T.

The surface element of trailing edge, which contains T as the mid point, is a composite one consists of upper and lower surfaces, each of which has the fold line of the trailing edge between body and wake as shown in Fig. 9. If the piecewise continuity of these surfaces and of the strengths of source and vortex distributions are assumed, the self-induced velocity at T can be calculated by integrating inductions of source and vortex distributions along the projected plane elements, which is tangential to the individual part of surface element at T, in the similar way to the method[18] for a continuous surface element given in Appendix 1. The self-induced velocity is finally obtained by taking the infinitesimal limit of the surface element, which in that case approaches to any extent to the tangential plane element at T.

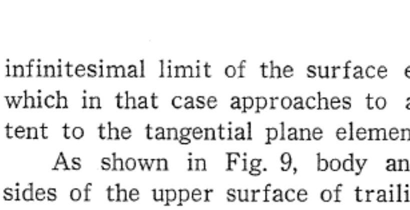


Fig. 9. Trailing edge element

As shown in Fig. 9, body and wake sides of the upper surface of trailing edge element are denoted by u and $\underline{w}$, respectively, and those of the lower surface are

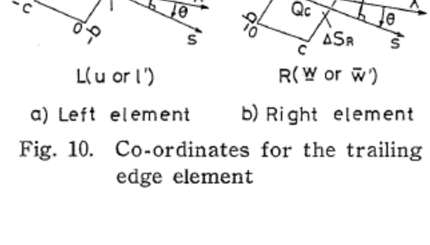


Fig. 10. Co-ordinates for the trailing edge element

denoted by l and $\bar{w}$, respectively, where $\underline{w}$ and $\bar{w}$ are finally got to-gether to a single surface of w. In order to simplify the later calculations on the lower surface by using the same formula for the upper one, the back sides of l and $\bar{w}$ denoted by l' and $\bar{w}'$, respectively, are considered. Regarding to the trailing edge element two systems of cartesian co-ordinates are introduced, as shown in Fig. 10. In X, Y, Z co-ordinates, axis Y is taken along the trailing edge, Z is vertical and X is normal to them. Co-ordinates s, h, n are fixed to the individual part of surface element with different inclinations, where axis h is taken parallel to the trailing edge, axis n is normal to the individual part of projected plane element and s is laid on it perpendicularly to h and n.

Velocity components at P due to source and vortex distributions on the upper and lower surfaces of trailing edge element are calculated by Eqs. (29) and (31), resulting in

$$
\begin{aligned}
(v_X)_P &= (v_{X1})_P + (v_{X2})_P, \\
(v_{X1})_P &= \sum \iint_{\Delta S} \frac{\sigma}{4\pi} \frac{\partial}{\partial X}\left(\frac{-1}{r}\right) dS, \\
(v_{X2})_P &= \sum \left\{ \iint_{\Delta S} \frac{1}{4\pi}\left[\gamma_Y \frac{\partial}{\partial Z_1}\left(\frac{1}{r}\right) - \gamma_Z \frac{\partial}{\partial Y_1}\left(\frac{1}{r}\right)\right] dS \right. \\
&\quad \left. + \oint_{\Delta C} \frac{-\mu}{4\pi}\left[dY_1 \frac{\partial}{\partial Z_1}\left(\frac{1}{r}\right) - dZ_1 \frac{\partial}{\partial Y_1}\left(\frac{1}{r}\right)\right]\right\},
\end{aligned}
\tag{37}
$$

where $r=[(X-X_1)^2+(Y-Y_1)^2+(Z-Z_1)^2]^{1/2}$ and summation is made for four parts, u, $\underline{w}$, l' and $\bar{w}'$, of the trailing edge element, which has T as the mid point. $(v_Y)_P$ and $(v_Z)_P$ are also calculated by cyclic changes of variables of Eq. (37). The self-induced velocity can be calculated by the limit process $\overline{PT}\to 0$, which means $P\to T$, in a higher order than making both area of elements ΔS and length of edge line ΔC infinitesimal, i. e. $\Delta S\to 0$ and $\Delta C\to 0$. This is the same process as in the ordinary surface element, as given in Appendix 1. According to the assumption of piecewise continuity the strengths of source and vortex in each parts approach continuously to their individual values at T, which are denoted by σ_T and $(\gamma)_T$. Components of self-induced velocities at T are expressed by

$$
\begin{aligned}
(v_X)_T &= (v_{X1})_T + (v_{X2})_T, \\
(v_{X1})_T &= \sum \sigma_T \lim_{\substack{\overline{PT}\to 0 \\ (\Delta S\to 0)}} \iint_{\Delta S} \frac{X-X_1}{4\pi r^3} dS, \\
(v_{X2})_T &= \sum \left\{ \left[(\gamma_Y)_T \lim_{\substack{\overline{PT}\to 0 \\ (\Delta S\to 0)}} \iint_{\Delta S} \frac{Z-Z_1}{4\pi r^3} dS - (\gamma_Z)_T \lim_{\substack{\overline{PT}\to 0 \\ (\Delta S\to 0)}} \iint_{\Delta S} \frac{Y-Y_1}{4\pi r^3} dS \right] \right. \\
&\quad \left. + \mu_T \left[-\lim_{\substack{\overline{PT}\to 0 \\ (\Delta C\to 0)}} \oint_{\Delta C} \frac{Z-Z_1}{4\pi r^3} dY_1 + \lim_{\substack{\overline{PT}\to 0 \\ (\Delta C\to 0)}} \oint_{\Delta C} \frac{Y-Y_1}{4\pi r^3} dZ_1 \right] \right\},
\end{aligned}
\tag{38}
$$

and their cyclic changes of variables. By referring to the back side of lower parts of surface element denoted by l' and $\bar{w}'$, the single formulations for the upper surface element can be used in common in these calculations. Denoting the inclination angle of each part of surface element by θ, the transformation of co-ordinates from X, Y, Z to s, h, n system by

$$X=s\cos\theta-n\sin\theta,\quad Y=h,\quad Z=s\sin\theta+n\cos\theta,$$
$$X_1=s_1\cos\theta,\quad Y_1=h_1,\quad Z_1=s_1\sin\theta, \tag{39}$$

is applied, where we have $r=[(s-s_1)^2+(h-h_1)^2+n^2]^{1/2}$. Introducing the limit of integrals defined by

$$I=\lim_{\substack{\overline{PT}\to 0\\(\Delta S\to 0)}}\iint_{\Delta S}\frac{s-s_1}{4\pi r^3}dS,\quad J=\lim_{\substack{\overline{PT}\to 0\\(\Delta S\to 0)}}\iint_{\Delta S}\frac{h-h_1}{4\pi r^3}dS,$$
$$K=\lim_{\substack{\overline{PT}\to 0\\(\Delta S\to 0)}}\iint_{\Delta S}\frac{n}{4\pi r^3}dS, \tag{40}$$

limits of surface integrals contained in Eq. (38) can be calculated by

$$\lim_{\substack{\overline{PT}\to 0\\(\Delta S\to 0)}}\iint_{\Delta S}\frac{X-X_1}{4\pi r^3}dS=I\cos\theta-K\sin\theta,\quad \lim_{\substack{\overline{PT}\to 0\\(\Delta S\to 0)}}\iint_{\Delta S}\frac{Y-Y_1}{4\pi r^3}dS=J,$$
$$\lim_{\substack{\overline{PT}\to 0\\(\Delta S\to 0)}}\iint_{\Delta S}\frac{Z-Z_1}{4\pi r^3}dS=I\sin\theta+K\cos\theta. \tag{41}$$

The limits of line integrals for an infinitesimal part of trailing edge, where $X_1=Z_1=0$, are introduced defining,

$$L=\lim_{\substack{\overline{PT}\to 0\\(\Delta S\to 0)}}\oint_{\Delta C}\frac{Z}{4\pi r^3}dY_1,\quad M=\lim_{\substack{\overline{PT}\to 0\\(\Delta S\to 0)}}\oint_{\Delta C}\frac{X}{4\pi r^3}dY_1. \tag{42}$$

Folded rectangular surface elements having T as the mid point, which consists of parts of $\Delta S=2b\times c$, and line element of trailing edge of $\Delta C=2b$ are considered, and those limit calculations are performed as shown in Appendix 2 and 3, in which limit of $\overline{PT}\to 0$ is first applied and then limit of b, $c\to 0$ is taken. It is found that the results contain pole and logarithmic singularities and an indefinite function:

$$\tau=\frac{1}{2\pi}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon},\quad \varkappa=\frac{1}{4\pi}\lim_{\delta\to 0}\ln\frac{1}{\delta^2},$$
$$\lambda=\frac{1}{4\pi}\ln\frac{1-\sin\omega}{1+\sin\omega},\quad \omega=\tan^{-1}\frac{b}{c} \tag{43}$$

Limits of integrals I through M have different signs according to the position of parts of surface element relative to the trailing edge. Values for left- and right-hand parts are expressed by subscripts L and R. We have for the left-hand parts u and l',

$$I_L=\varkappa+\lambda,\quad J_L=0,\quad K_L=(1/4)+(\Psi-\theta)/2\pi,$$
$$L_L=\tau\cos\Psi,\quad M_L=-\tau\sin\Psi, \tag{44}$$

and for the right-hand parts $\underline{w}$ and $\bar{w}'$,

$$I_R = -(\varkappa+\lambda),\quad J_R=0,\quad K_R=(1/4)-(\Psi-\theta)/2\pi,$$
$$L_R=-\tau\cos\Psi,\quad M_R=\tau\sin\Psi, \tag{45}$$

where Ψ is the projected angle of path of P approaching to T, as shown in Fig. 10.

Substituting Eqs. (41) and (42) into Eq. (38) components of self-induced velocity at T can be calculated by

$$\begin{aligned}
(v_X)_T &= \sum\{[\sigma_T\cos\theta+(\gamma_Y)_T\sin\theta]I \\
&\qquad +[-\sigma_T\sin\theta+(\gamma_Y)_T\cos\theta]K-\mu_T L\},\\
(v_Z)_T &= \sum\{[\sigma_T\sin\theta-(\gamma_Y)_T\cos\theta]I \\
&\qquad +[\sigma_T\cos\theta+(\gamma_Y)_T\sin\theta]K+\mu_T M\},\\
(v_Y)_T &= \sum\{[(\gamma_Z)_T\cos\theta-(\gamma_X)_T\sin\theta]I \\
&\qquad -[(\gamma_Z)_T\sin\theta+(\gamma_X)_T\cos\theta]K \\
&= \sum\{(\gamma_n)_T I-(\gamma_S)_T K\},
\end{aligned} \tag{46}$$

where $\gamma_s=\gamma_Z\sin\theta+\gamma_X\cos\theta$, $\gamma_n=\gamma_Z\cos\theta-\gamma_X\sin\theta=0$, and $\sum$ is made for four parts of surface elements at the trailing edge, i. e. u, $\underline{w}$, l' and $\bar{w}'$. Removing the subscript T for simplicity's sake, strengths of source and vortex at T are briefly denoted by subscripts u, $\underline{w}$, l' and $\bar{w}'$, hereafter. Components of self-induced velocities are calculated by substituting Eqs. (44) and (45) into Eq. (46), and then, quantities for the back side of lower elements, l' and $\bar{w}'$, are replaced by those for the original ones, l and $\bar{w}$, and finally, two sheets of wake surface, $\underline{w}$ and $\bar{w}$ are put to-gether into one sheet w, by using

$$\begin{aligned}
&\theta_{l'}=\theta_l,\quad \theta_{\bar{w}'}=\theta_{\bar{w}}=\theta_{\underline{w}}=\theta_w,\\
&(\gamma_Y)_{l'}=(\gamma_Y)_l,\quad (\gamma_Y)_{\bar{w}'}=(\gamma_Y)_{\bar{w}},\quad (\gamma_Y)_{\underline{w}}+(\gamma_Y)_{\bar{w}}=(\gamma_Y)_w,\\
&(\gamma_s)_{l'}=(\gamma_s)_l,\quad (\gamma_s)_{\bar{w}'}=(\gamma_s)_{\bar{w}},\quad (\gamma_s)_{\underline{w}}+(\gamma_s)_{\bar{w}}=(\gamma_s)_w,\\
&\mu_{l'}=-\mu_l,\quad \mu_{\bar{w}'}=-\mu_{\bar{w}},\quad \mu_{\underline{w}}-\mu_{\bar{w}}=\mu_w.
\end{aligned} \tag{47}$$

Components of self-induced velocities at T are then obtained by

$$\begin{aligned}
(v_X)_T &= A(\varkappa+\lambda)+B(\Psi/2\pi)-C\tau\cos\Psi\\
&\quad +[(1/4)-(\theta_u/2\pi)][-\sigma_u\sin\theta_u+(\gamma_Y)_u\cos\theta_u]\\
&\quad +[(1/4)-(\theta_l/2\pi)][-\sigma_l\sin\theta_l+(\gamma_Y)_l\cos\theta_l]\\
&\quad +[(1/4)+(\theta_w/2\pi)](\gamma_Y)_w\cos\theta_w,\\
(v_Z)_T &= -B(\varkappa+\lambda)+A(\Psi/2\pi)-C\tau\sin\Psi\\
&\quad +[(1/4)-(\theta_u/2\pi)][\sigma_u\cos\theta_u+(\gamma_Y)_u\sin\theta_u]\\
&\quad +[(1/4)-(\theta_l/2\pi)][\sigma_l\cos\theta_l+(\gamma_Y)_l\sin\theta_l]\\
&\quad +[(1/4)+(\theta_w/2\pi)](\gamma_Y)_w\sin\theta_w,
\end{aligned} \tag{48}$$

$$(v_Y)_T = -D(\Psi/2\pi) - \{[(1/4)-(\theta_u/2\pi)](\gamma_s)_u + [(1/4)-(\theta_l/2\pi)](\gamma_s)_l + [(1/4)+(\theta_w/2\pi)](\gamma_s)_w\},$$

where

$$\begin{aligned} A &= \sigma_u \cos\theta_u + \sigma_l \cos\theta_l + (\gamma_Y)_u \sin\theta_u + (\gamma_Y)_l \sin\theta_l - (\gamma_Y)_w \sin\theta_w, \\ B &= -\sigma_u \sin\theta_u - \sigma_l \sin\theta_l + (\gamma_Y)_u \cos\theta_u + (\gamma_Y)_l \cos\theta_l - (\gamma_Y)_w \cos\theta_w, \\ C &= \mu_u - \mu_l - \mu_w, \\ D &= (\gamma_s)_u + (\gamma_s)_l - (\gamma_s)_w = -dC/dY. \end{aligned} \tag{49}$$

4. The Kutta's Edge Condition

The Kutta's edge condition states that the flow is controlled not to induce an infinite velocity at the trailing edge. Eliminating singular terms of induced velocity in Eq. (48), this condition is satisfied by

$$A = B = C = D = 0. \tag{50}$$

Rearranging Eq. (50) the Kutta's edge condition can be obtained by

$$\sigma_u \cos\theta_u + \sigma_l \cos\theta_l + (\gamma_Y)_u \sin\theta_u + (\gamma_Y)_l \sin\theta_l = (\gamma_Y)_w \sin\theta_w, \tag{51}$$

$$-\sigma_u \sin\theta_u - \sigma_l \sin\theta_l + (\gamma_Y)_u \cos\theta_u + (\gamma_Y)_l \cos\theta_l = (\gamma_Y)_w \cos\theta_w, \tag{52}$$

$$(\gamma_s)_u + (\gamma_s)_l = (\gamma_s)_w, \tag{53}$$

$$\mu_u - \mu_l = \mu_w, \tag{54}$$

where σ, γ and μ are specified values at the trailing edge of each part of the surface element.

From Eqs. (51) and (52) we can calculate $(\gamma_Y)_w$ and θ_w.

$$\begin{aligned} (\gamma_Y)_w = \{&\sigma_u^2 + \sigma_l^2 + 2\sigma_u\sigma_l \cos(\theta_l - \theta_u) \\ &+ [\sigma_u(\gamma_Y)_l - \sigma_l(\gamma_Y)_u] \sin(\theta_l - \theta_u) \\ &+ (\gamma_Y)_u^2 + (\gamma_Y)_l^2 + 2(\gamma_Y)_u(\gamma_Y)_l \cos(\theta_l - \theta_u)\}^{1/2}, \end{aligned} \tag{55}$$

$$\theta_w = \tan^{-1} \frac{\sigma_u \cos\theta_u + \sigma_l \cos\theta_l + (\gamma_Y)_u \sin\theta_u + (\gamma_Y)_l \sin\theta_l}{-\sigma_u \sin\theta_u - \sigma_l \sin\theta_l + (\gamma_Y)_u \cos\theta_u + (\gamma_Y)_l \cos\theta_l}. \tag{56}$$

In some analyses by the doublet distributions Eq. (54) is used. By these Eqs. (51) through (56) the present result states that some continuity relations should be held between the strengths of vortex sheet γ, i. e. the derivative of μ. These relations provide the condition of wake initiation, μ_w, $(\gamma_s)_w$, $(\gamma_Y)_w$ and θ_w as functions of quantities at the end of body surfaces, σ_u, σ_l, $(\gamma_s)_u$, $(\gamma_s)_l$, $(\gamma_Y)_u$ and $(\gamma_Y)_l$.

In the method of two-step analysis the lifting flow around a general configuration can be obtained by dividing the solution in two components, i. e. the non-lifting flow and the circulating flow. The first component of non-lifting flow is calculated by the source distributions superposed on the basic flow. Satisfying the boundary condition on body surface

$$(v_n)_0+(v_n)_1=0 \tag{57}$$

we can determine the strength σ on the whole surface point, which contains σ_u and σ_l at the trailing edge.

The second component of circulating flow is calculated by the vortex distributions superposed on the non-lifting solution. Satisfying the boundary condition on body surface

$$v_n=0 \quad or \quad (v_n)_2=0 \tag{58}$$

and those on wake surface

$$\gamma_s/\gamma_h=v_s/v_h \quad \text{and} \quad v_n=0 \tag{59}$$

with the Kutta's trailing edge condition, we can determine the strength γ_s and γ_h $(=\gamma_Y)$ for suitablly chosen co-ordinates. In these analysis the velocity on the body surface is calculated by Eqs. (28), (30) and (32), and that on the wake surface by Eq. (34). The boundary condition on wake surface states that the vortex vector should be parallel to the mean velocity vector on wake surface. In the iterative procedure to determine the form of wake surface the use of this condition will be effective, by putting the vortex line along the velocity vector which obtained in the previous step of iteration.[8), 10)]

At the trailing edge the Kutta condition given by Eqs. (51) through (56) formulates relations between μ_w, $(\gamma_s)_w$, $(\gamma_Y)_w$ and θ_w at the wake initiation and σ_u, σ_l, $(\gamma_s)_u$, $(\gamma_s)_l$, $(\gamma_Y)_u$ and $(\gamma_Y)_l$ at the body end, in which σ_u and σ_l are already known by the non-lifting analysis. In addition to the Kutta condition, these quantities should satisfy the boundary conditions at the body end and at the wake initiation:

$$(v_n)_u=0, \qquad (v_n)_l=0 \tag{60}$$

on the body, and with

$$(\gamma_s)_w/(\gamma_Y)_w=(v_s)_w/(v_h)_w, \qquad (v_n)_w=0 \tag{61}$$

on the wake, where $(v_n)_u$ and $(v_n)_l$ are calculated by Eqs. (28), (30) and (32), and $(v_s)_w$, $(v_Y)_w$ and $(v_n)_w$ by Eq. (34). In principle these four additional relations and four previous equations of edge condition can determine the eight parameters: $(\gamma_s)_u$, $(\gamma_s)_l$, $(\gamma_Y)_u$, $(\gamma_Y)_l$, μ_w, $(\gamma_s)_w$, $(\gamma_Y)_w$ and θ_w, at the trailing edge.

Vectorial representation

The present edge condition consists of vector type continuities, Eqs. (51) and (52), and of scalar type continuities, Eqs. (53) and (54). The former can be transformed to a vector equation, by rearranging Eqs. (51) and (52) in the following form:

$$\begin{aligned}\sigma_u\cos\theta_u+\sigma_l\cos\theta_l+(\gamma_Y)_u\cos(\theta_u-\pi/2)+(\gamma_Y)_l\cos(\theta_l-\pi/2)\\ +(\gamma_Y)_w\cos(\theta_w+\pi/2)=0,\end{aligned} \tag{62}$$

$$\sigma_u\sin\theta_u+\sigma_l\sin\theta_l+(\gamma_Y)_u\sin(\theta_u-\pi/2)+(\gamma_Y)_l\sin(\theta_l-\pi/2)$$

$$+(\gamma_Y)_w \sin(\theta_w+\pi/2)=0. \tag{63}$$

Considering some rules artificial vectors $\vec{\sigma}$ and $\vec{\gamma_Y}$ are introduced. The artificial vector $\vec{\sigma}$ is defined by its magnitude σ and direction $\pm\theta$, and $\vec{\gamma_Y}$ by its magnitude γ_Y and direction $\theta\mp\pi/2$, where the double sign represents values for left and right part of surface element of the trailing edge, i. e. for u, l and for w, respectively. It is noticed that these arguments, $\pm\theta$ and $\theta\mp\pi/2$, indicate directions of infinite velocity induced at the cut of the fold line of each part of the surface element. Eqs. (62) and (63) can be replaced by

$$\vec{\sigma}_u+\vec{\sigma}_l+(\vec{\gamma}_Y)_u+(\vec{\gamma}_Y)_l+(\vec{\gamma}_Y)_w=0, \tag{64}$$

which represents the balance of artificial vectors as shown in some qualitative illustrations in Fig. 11. As mentioned before the final values of these quantities are determined with the finish of computation for the whole field. In special cases of symmetry or anti-symmetry some parameters can be estimated qualitatively and the relation of Eq. (64) can be examined as shown in Fig. 11.

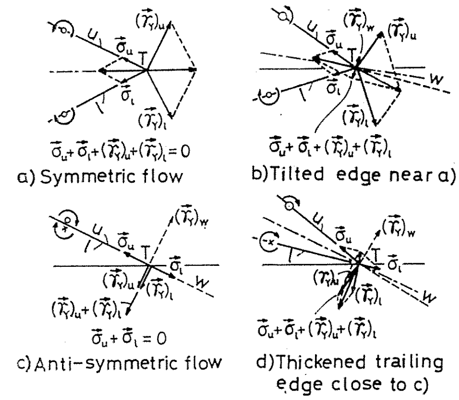


Fig. 11. Schematic illustration of balance of strengths at the trailing edge

In the case of symmetrical flow represented by

$$\sigma_u=\sigma_l, \quad (\gamma_Y)_u=-(\gamma_Y)_l \quad \text{for} \quad \theta_u=-\theta_l \tag{65}$$

Eq. (64) is reduced to

$$(\vec{\gamma}_Y)_w=0 \qquad \therefore \quad (\gamma_Y)_w=0 \tag{66}$$

as shown in Fig. 11 a). It states that the wake has no component of vortex parallel to the trailing edge.

When a little incidence angle is applied to this symmetrical body in a uniform flow, $\vec{\sigma}_u+\vec{\sigma}_l+(\vec{\gamma}_Y)_u+(\vec{\gamma}_Y)_l$ takes small value near zero, and produces a small $(\vec{\gamma}_Y)_w$ which is nearly perpendicular to the bisector of upper and lower surfaces as shown in Fig. 11 b). This means that the symmetrical body with a small angle of attack issues a wake of weak $(\gamma_Y)_w$ lying close to the bisector of upper and lower surfaces of the trailing edge.

Another special case is the antisymmetrical flow along a body of thin trailing edge with $\theta_u=\theta_l$, which is represented by

$$\sigma_u=-\sigma_l, \quad (\gamma_Y)_u=(\gamma_Y)_l \quad \text{for} \quad \theta_u=\theta_l. \tag{67}$$

Applying $\vec{\sigma}_u+\vec{\sigma}_l=0$, Eq. (64) is reduced to

$$(\vec{\gamma}_Y)_u+(\vec{\gamma}_Y)_l+(\vec{\gamma}_Y)_w=0, \tag{68}$$

which gives

$$\theta_w=\theta_u=\theta_l, \quad (\gamma_Y)_w=(\gamma_Y)_u+(\gamma_Y)_l, \tag{69}$$

as shown in Fig. 11 c). This states that the wake surface is tangential to the upper and lower surface of zero trailing edge angle at T, and that the vortex component is preserved from body to wake surface.

If a small thickness is added to this configuration of body, $\vec{\sigma}_u+\vec{\sigma}_l$ is changed a little from zero and the value of $(\vec{\gamma}_Y)_w$ is slightly changed from the previous case of c) as shown in Fig. 11 d). This means that a thinly made body issues a wake nearly tangential to the bisector of upper and lower surfaces at the trailing edge.

5. Conclusion

Some fundamental characteristics of the theory of distributed singularities are investigated for the application to the steady incompressible flow around an arbitrary lifting body, and the trailing edge condition for it is obtained in a suitable form for the present type of analysis. Introducing the mutual relations between analytical methods treating the inner and the whole field, characteristics of source and doublet distributions along the boundary surface are elucidated from the basis of potential theory. The transformation of the potential due to the distributions of doublet to those of vortex is introduced and some insight into its physical situation is presented.

Assuming the piecewise continuities of body and wake surfaces and those of source and doublet strengths, singularities of induced velocities at the trailing edge are investigated. For the calculation of self-induced terms parts of surface element, which have different tangent at the trailing edge, are assumed to be continuous in their individual regions, and usual limit calculations of surface integrals on the projected plane elements are applied to the individual parts of surface elements.

Eliminating the occurrence of these singular velocities the Kutta condition is obtained in the form of some continuities suitable for the theory of distributed singularities. These conditions contain the continuity of doublet strength from body side to wake side, and the continuity of normal component of vortex to the trailing edge. It is easily shown that the latter relation can be obtained by differentiate the former along the trailing edge line. The Kutta condition contains, further, the vectorial continuity of the parallel component of vortex to the trailing edge, which is a continuity relation of normal derivative of doublet strength to the trailing edge line.

It is noticed that the present results can be used as a matching condition between strengths of body end and of wake initiation. The final value of these strengths are determined to-gether with strengths on the whole surface at the final stage of all calculations. The present investigation is performed under the assumption of piecewise continuity of surfaces at the trailing edge, and the problem of the edge condition for a discontinuous surface of higher degree may be left to the future investigations.

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Nomenclature

a: radius of a ring vortex
b: half span of infinitesimal surface element
C: closed circuit of the boundary curve of S
c: half chord of infinitesimal surface element
h: arc length of a co-ordinate axis on the surface
I, J, K: limit of surface integrals
$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$: unit vectors
L, M: limit of line integrals
l: infinitesimal distance of source and sink
m: strength of dipole (doublet or infinitesimal ring vortex)
$n, \boldsymbol{n}$: arc length and unit vector of a normal to the surface
l, m, n: direction consines of vector
P: space point
Q: surface point
q: volume flow from a source
R, R': inner and outer space
$r, \boldsymbol{r}$: distance and vector from Q to P
S: boundary surface of space R
s: arc length of a co-ordinate axis on the surface
$\boldsymbol{s}$: vector along the closed curve C
s, h, n: orthogonal co-ordinates referring to parts of surface element
T: trailing edge point
t: time
$v, \boldsymbol{V}$: magnitude and vector of velocity
W: wake surface point
w: normal component of velocity at the surface
X, Y, Z: cartesian co-ordinates for the trailing edge
x, y, z: cartesian co-ordinates
Γ: circulation of a vortex filament
$\gamma, \boldsymbol{\gamma}$: strength and vector of vortex sheet
δ, ε: infinitesimal quantities
θ: inclination angle of part of surface element in the section perpendicular to the trailing edge line
λ: logarithmic singularity
μ: strength of doublet distributions and circulation of a vortex filament
σ: strength of source distributions
τ: pole singularity
ϕ: velocity potential
φ, Ψ: projected path angle of approching P to T in the section perpendicular to the trailing edge line
ω: angle or solid angle
∇, ∇_1: nabla defined in the above sentence of Eq. (24) and at Eq. (23)
$\pm$: values for inner and outer side of the surface

Subscripts

0: quantities for a known basic flow
1: quantities induced by source distributions or co-ordinates of a surface point

2:	quantities induced by doublet or vortex distributions
+, −:	values of inner and outer side of the surface
C:	mean values of inner and outer side of S
L:	left part of surface element, i. e. substitute of u and l'
l, l':	lower part of body surface in the trailing edge element
P:	quantities at a space point P
Q:	quantities at a surface point Q
R:	right part of surface element, i. e. substitute of $\underline{w}$ and $\bar{w}'$
S, S':	values at the surface S and S'
s, h, n:	components of vector in co-ordinate axis s, h and n
T:	quantities at a trailing edge point T
u:	upper part of body in the trailing edge element
W:	quantities at a wake point W
w:	part of wake surface in the trailing edge element
$\underline{w}$:	upper part of wake surface
$\bar{w}$, $\bar{w}'$:	lower part of wake surface in the trailing edge element

Appendix

Appendix 1. Limit of Integration for the Self-Induction at an Ordinary Surface Point

Consider a surface element dS, which is an infinitesimal part of the continuous surface S, and its project ΔS to the tangent plane at the mid point Q as shown in Fig. 12. Continuous strengths of source and doublet distributions along this conti-

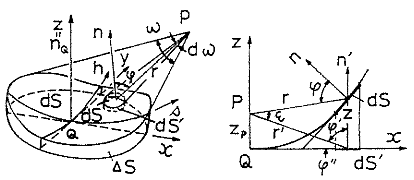

Fig. 12. Surface element

nuous surface are assumed. Denoting the distance from a local surface point to a space point P by r, the local normal to the surface by n and the angle between r and n by φ, the self-induced normal velocity of source distributions and the self-induced potential of doublet distributions at Q are both contain the following limit calculation of surface integration:

$$\lim_{\substack{PQ\to 0\\(dS\to 0)}}\iint_{dS}\frac{n}{4\pi r^3}dS=\lim_{\substack{PQ\to 0\\(dS\to 0)}}\iint_{dS}\frac{\cos\varphi}{4\pi r^2}dS$$

$$=\lim_{\overline{PQ}\to 0}\int\frac{d\omega}{4\pi}=\lim_{\overline{PQ}\to 0}\frac{\omega}{4\pi}=\pm\frac{1}{2}, \tag{A1}$$

where ω is a solid angle and $\pm$ sign is taken for the inner (upper) and the outer (lower) side of point Q. The self-induced normal velocity of source distributions and the self-induced potential of doublet distribution at the ordinary surface point Q are calculated by taking the limit $\overline{PQ}\to 0$ with a higher order than taking the limit $dS\to 0$ as follows:

$$\lim_{\substack{\overline{PQ}\to 0\\(dS\to 0)}}\iint_{dS}\sigma\frac{\partial}{\partial n}\left(\frac{-1}{4\pi r}\right)dS=\sigma_Q\lim_{\substack{\overline{PQ}\to 0\\(dS\to 0)}}\iint_{dS}\frac{n}{4\pi r^3}dS=\pm\frac{\sigma_Q}{2}, \tag{A2}$$

$$\lim_{\substack{\overline{PQ}\to 0\\(dS\to 0)}}\iint_{dS}\mu\frac{\partial}{\partial n_1}\left(\frac{1}{4\pi r}\right)dS=\mu_Q\lim_{\substack{\overline{PQ}\to 0\\(dS\to 0)}}\iint_{dS}\frac{n}{4\pi r^3}dS=\pm\frac{\mu_Q}{2}. \tag{A3}$$

These are appeared in Eqs. (2), (5), (17), (19), (27) and (30).

For the case of continuous source and doublet distributions along a continuous surface, it is shown that the previous limit of integration in Eq. (A1) can be calculated on the projected surface element ΔS to the tangential plane at Q.[18]

For the sake of simplicity an example of the two-dimensional surface element, whose section is shown in xz plane of Fig. 12, is considered, in which point P approaches along z axis to point Q. By the Taylor expansion the ordinate z of the surface can be expressed by

$$\begin{aligned} &z=z_Q+z'_Qx+(1/2!)z''_Qx^2+\cdots,\\ &dz/dx=z'_Q+z''_Qx+\cdots, \end{aligned} \tag{A4}$$

where $'$ represents derivative in x and $z_Q=z'_Q=0$ in this case.

Angles in Fig. 12 are calculated by

$$\begin{aligned} &\delta=\tan^{-1}(dz/dx)=z''_Qx+\cdots,\\ &\varepsilon=\tan^{-1}\frac{z-z_P}{x}+\tan^{-1}\frac{z_P}{x}=\frac{1}{2}\frac{1}{1+(z_P/x)^2}z''_Qx+\cdots,\\ &\because\quad \tan\varepsilon=\frac{(z-z_P)/x+z_P/x}{1-(z-z_P)z_P/x^2}=\frac{1}{2!}\frac{1}{1+(z_P/x)^2}z''_Qx+\cdots. \end{aligned} \tag{A5}$$

Substituting Eq. (A5) into

$$\varphi=\varphi'-(\delta-\varepsilon)=\varphi'-Az''_Qx+\cdots,\quad A=\frac{(1/2)+(z_P/x)^2}{1+(z_P/x)^2}, \tag{A6}$$

we have

$$\begin{aligned} \cos\varphi&=\cos\varphi'[1-(1/2)(Az''_Q)^2x^2+\cdots]+\sin\varphi'[Az''_Qx+\cdots]\\ &=\cos\varphi'-\sin\varphi''[(1/2)(Az''_Q)^2x^2+\cdots]+\cos\varphi''[Az''_Qx+\cdots] \end{aligned} \tag{A7}$$

The ratio of distances and of areas are given by

$$\frac{r'^2}{r^2}=\frac{x^2+z_P^2}{x^2+(z-z_P)^2}=1-\frac{z_P/x}{1+(z_P/x)^2}z''_Q x+\cdots$$
$$\frac{dS}{dS'}=\frac{1}{\cos\delta}=1+\frac{1}{2}(z''_Q)^2x^2+\cdots \tag{A8}$$

The integrand of Eq. (A1) can be transformed to

$$\frac{\cos\varphi}{r^2}dS=\frac{\cos\varphi'}{r'^2}dS'$$
$$+\frac{dS'}{r'^2}\left\{-\sin\varphi''\left[\frac{(Az''_Q)^2}{2}x^2+\cdots\right]+\cos\varphi''[Az''_Q x+\cdots]\right\}, \tag{A9}$$

where $\varphi''=\tan^{-1}(z_P/x)$. If the surface has a finite z''_Q, the integrand function approaches in any extent to the corresponding function to the projected element by taking the region closer to Q, resulting in

$$\frac{\cos\varphi}{r^2}dS \to \frac{\cos\varphi'}{r'^2}dS' \tag{A10}$$

The integration in Eq. (A1) can be performed for the corresponding integrand function in its projected plane tangential at Q.[18]

Appendix 2. Limit of Surface Integrations: *I*, *J* and *K*

In the present theory parts of surface element have different tangent at the trailing edge, but they are continuous in their own regions. With similar considerations to the case of ordinary surface point it is assumed that the usual process of limit calculations of surface integrals performed on the projected plane element can be applied to the present individual parts of surface element.

In order to calculate I, J and K of Eq. (40), integrations are first performed for a fixed point P and satisfying the higher order approach limit of $\overline{PT}\to 0$ is calculated and then limit of $\varDelta S\to 0$ is taken. Values for a left- and right-hand parts of surface elements are distinguished by subscripts L and R, which substitutes u, l' and $\underline{w}$, $\bar{w}'$, respectively.

Expanding the terms of surface integral for I_L given by

$$\int_{-b}^{b}dh_1\int_{-c}^{0}\frac{s-s_1}{r^3}ds_1$$
$$=-\ln\frac{h-b+\sqrt{s^2+(h-b)^2+n^2}}{h+b+\sqrt{s^2+(h+b)^2+n^2}}+\ln\frac{h-b+\sqrt{(s+c)^2+(h-b)^2+n^2}}{h+b+\sqrt{(s+c)^2+(h+b)^2+n^2}}$$

into small magnitude of s, h and n comparing to b and c, we have limiting values of the first and second terms by taking $\overline{PT}\to 0$, i. e. s, h, $n\to 0$, as follows:

$$\lim_{\substack{s,h,n\to 0\\(b\to 0)}} -\ln\frac{h-b+\sqrt{s^2+(h-b)^2+n^2}}{h+b+\sqrt{s^2+(h+b)^2+n^2}}=\lim_{\delta\to 0}\ln\frac{1}{\delta^2},$$

where $\delta=\varepsilon/2b=\sqrt{s^2+n^2}/2b$, and

$$\lim_{\substack{s,h,n\to 0\\(b,c\to 0)}} \ln\frac{h-b+\sqrt{(s+c)^2+(h-b)^2+n^2}}{h+b+\sqrt{(s+c)^2+(h+b)^2+n^2}}=\ln\frac{-b+\sqrt{c^2+b^2}}{b+\sqrt{c^2+b^2}}$$
$$=\ln[(1-\sin\omega)/(1+\sin\omega)],$$

where $\sin\omega=b/\sqrt{c^2+b^2}$, and therefore, this value depends on the ratio b/c.

Introducing a logarithmic singularity:

$$\varkappa=(1/4\pi)\lim_{\delta\to 0}\ln(1/\delta^2), \tag{A11}$$

and an indefinite function depend on the form of surface element which is defined by

$$\lambda=(1/4\pi)\ln[(1-\sin\omega)/(1+\sin\omega)], \tag{A12}$$

we have

$$I_L=\lim_{\substack{s,h,n\to 0\\(\Delta S_L\to 0)}}\iint_{\Delta S_L}\frac{s-s_1}{4\pi r^3}dS=\varkappa+\lambda. \tag{A13}$$

The similar limiting process on the integration:

$$\int_{-b}^{b}dh_1\int_0^c\frac{s-s_1}{r^3}ds_1$$

$$=-\ln\frac{h-b+\sqrt{(s-c)^2+(h-b)^2+n^2}}{h+b+\sqrt{(s-c)^2+(h+b)^2+n^2}}+\ln\frac{h-b+\sqrt{s^2+(h-b)^2+n^2}}{h+b+\sqrt{s^2+(h+b)^2+n^2}}$$

gives for a right-hand part of surface element

$$I_R=\lim_{\substack{s,h,n\to 0\\(\Delta S_R\to 0)}}\iint_{\Delta S_R}\frac{s-s_1}{4\pi r^3}dS=-(\varkappa+\lambda). \tag{A14}$$

When the same process is applied to an ordinary surface element of $2b\times 2c$, which is given by taking $\Delta S=\Delta S_L+\Delta S_R$ and $\theta_L=\theta_R$, the function I for the ordinary surface point can be calculated by I_L+I_R, resulting in

$$I=I_L+I_R=0. \tag{A15}$$

Integrations for J_L and J_R are calculated by

$$\int_{-c}^{0}ds_1\int_{-b}^{b}\frac{h-h_1}{r^3}dh_1$$

$$=-\ln\frac{s+\sqrt{s^2+(h-b)^2+n^2}}{s+c+\sqrt{(s+c)^2+(h-b)^2+n^2}}+\ln\frac{s+\sqrt{s^2+(h+b)^2+n^2}}{s+c+\sqrt{(s+c)^2+(h+b)^2+n^2}}$$

and by

$$\int_0^c ds_1 \int_{-b}^{b} \frac{h-h_1}{r^3} dh_1$$

$$= -\ln \frac{s-c+\sqrt{(s-c)^2+(h-b)^2+n^2}}{s+\sqrt{s^2+(h-b)^2+n^2}} + \ln \frac{s-c+\sqrt{(s-c)^2+(h+b)^2+n^2}}{s+\sqrt{s^2+(h+b)^2+n^2}},$$

respectively. With the limit of $\overline{PT}\to 0$, i. e. s, h, $n\to 0$, J_L and J_R are

$$J_L = \lim_{\substack{s,\,h,\,n\to 0\\ (\Delta S_L\to 0)}} \iint_{\Delta S_L} \frac{h-h_1}{4\pi r^3} dS = 0, \tag{A16}$$

and

$$J_R = \lim_{\substack{s,\,h,\,n\to 0\\ (\Delta S_R\to 0)}} \iint_{\Delta S_R} \frac{h-h_1}{4\pi r^3} dS = 0. \tag{A17}$$

Similar to the previous process J for an ordinary element can be calculated by

$$J = J_L + J_R = 0. \tag{A18}$$

Taking the limit process $\overline{PT}\to 0$, i. e. s, h, $n\to 0$, on

$$\int_{-b}^{b} dh_1 \int_{-c}^{0} \frac{n}{r^3} ds_1 = \tan^{-1} \frac{(h-b)s}{n\sqrt{s^2+(h-b)^2+n^2}} - \tan^{-1} \frac{(h+b)s}{n\sqrt{s^2+(h+b)^2+n^2}}$$

$$- \tan^{-1} \frac{(h-b)(s+c)}{n\sqrt{(s+c)^2+(h-b)^2+n^2}} + \tan^{-1} \frac{(h+b)(s+c)}{n\sqrt{(s+c)^2+(h+b)^2+n^2}},$$

we have

$$K_L = \lim_{\substack{s,\,h,\,n\to 0\\ (\Delta S_L\to 0)}} \iint_{\Delta S_L} \frac{n}{4\pi r^3} dS$$

$$= \lim_{\substack{s,\,h,\,n\to 0\\ (b,\,c\to 0)}} \frac{1}{4\pi} \left[-2\tan^{-1}\frac{s}{n} + 2\tan^{-1} \frac{bc}{n\sqrt{c^2+b^2}} \right] = \pm\frac{1}{4} + \frac{\varphi_\pm}{2\pi} \tag{A19}$$

where $\pm$ sign represents values for the inner and outer side of a surface element, respectively, and φ is the path angle approaching P_C to T with $-\pi \leqq \varphi_\pm \leqq \pi$, as defined in Fig. 13. The similar limit process on

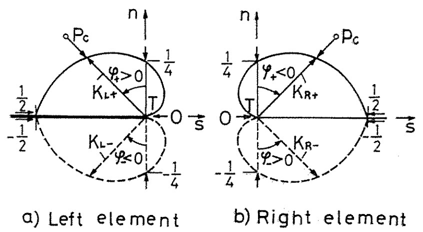

Fig. 13. Singular characteristics of limit of integrals: K_L and K_R

$$\int_{-b}^{b} dh_1 \int_0^c \frac{n}{r^3} ds_1 = \tan^{-1}\frac{(h-b)(s-c)}{n\sqrt{(s-c)^2+(h-b)^2+n^2}}$$

$$-\tan^{-1}\frac{(h+b)(s-c)}{n\sqrt{(s-c)^2+(h+b)^2+n^2}}-\tan^{-1}\frac{(h-b)s}{n\sqrt{s^2+(h-b)^2+n^2}}$$

$$+\tan^{-1}\frac{(h+b)s}{n\sqrt{s^2+(h+b)^2+n^2}}$$

leads to

$$K_R = \lim_{\substack{s,h,n\to 0\\(\Delta S_R\to 0)}} \iint_{\Delta S_R} \frac{n}{4\pi r^3} dS$$

$$= \lim_{\substack{s,h,n\to 0\\(b,c\to 0)}} \frac{1}{4\pi}\left[2\tan^{-1}\frac{bc}{n\sqrt{c^2+b^2}}+2\tan^{-1}\frac{s}{n}\right] = \pm\frac{1}{4}-\frac{\varphi_\pm}{2\pi} \tag{A20}$$

It is found that K_L and K_R contain a constant value and a term of path angle φ as shown in Fig. 13, referring to polar co-ordinates.

If the similar process is applied, K for an ordinary surface element is given by

$$K = K_L + K_R = \pm 1/2. \tag{A21}$$

This is the same result obtained by a direct calculation shown in Appendix 1.

Appendix 3. Limit of Line Integrations: L and M

The basic form of a line integral for a fixed space point P is given by

$$\int_{-b}^{b} \frac{dY_1}{r^3} = \frac{1}{X^2+Z^2}\left[-\frac{Y-b}{\sqrt{X^2+(Y-b)^2+Z^2}}+\frac{Y+b}{\sqrt{X^2+(Y+b)^2+Z^2}}\right].$$

Introducing a pole singularity:

$$\tau = (1/2\pi)\lim_{\varepsilon\to 0}(1/\varepsilon), \qquad \varepsilon = \sqrt{X^2+Z^2} \tag{A22}$$

the limit process of X, Y, $Z\to 0$, i. e. the approach of P to T, gives

$$\left.\begin{matrix} L_L \\ L_R \end{matrix}\right\} = \lim_{\substack{X,Y,Z\to 0\\(b\to 0)}} \int_{\mp b}^{\pm b} \frac{Z dY_1}{4\pi r^3} = \lim_{X,Y,Z\to 0} \frac{\pm 1}{2\pi}\frac{Z}{X^2+Z^2} = \pm\tau\cos\Psi \tag{A23}$$

and

$$\left.\begin{matrix} M_L \\ M_R \end{matrix}\right\} = \lim_{\substack{X,Y,Z\to 0\\(b\to 0)}} \int_{\mp b}^{\pm b} \frac{X dY_1}{4\pi r^3} = \lim_{X,Y,Z\to 0} \frac{\pm 1}{2\pi}\frac{X}{X^2+Z^2} = \mp\tau\sin\Psi \tag{A24}$$

It is found that L and M are pole singularities combined with functions of the path angle Ψ, which is defined in Fig. 10.