ON STEADY-STATE CHARACTERISTICS OF LINEAR MULTIVARIABLE SYSTEMS

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Chapter I. Introduction

The most typical realistic control systems are subjected to persistently acting, unknown, unmeasurable external disturbances. Further the systems are often required to maintain their outputs at nonzero constant set points, or make their outputs follow some reference inputs which are not known beforehand. Systems with these properties are called servomechanism systems, and the problem of designing such systems is called a servomechanism problem. In classical control theory, this problem is almost standard, and solutions to the problem are well established, although most applications of the results are limited to scalar systems. In modern treatments of multivariable systems, however, the problem has been often overlooked. As the results, considerably larger steady state errors have often been found in systems designed by modern linear control theory than in conventional control systems.

One of the main purposes of this article is to discuss a servomechanism problem by giving multivariable versions of ' type ℓ feedback system ' concept, on which many conventional servomechanism theory have been based. These versions will be shown to enable us to design servomechanism systems by a method which is conceptually similar to the classical one. To this subject, we shall devote two chapters; the first one for the case of reference inputs, and the second one for the case of disturbances. Further, in this article, a more general servomechanism problem will be discussed, in which reference inputs and external disturbances are completely arbitrary provided that they are describable by some linear autonomous differential equations. Necessary and sufficient conditions for the existence of state feedbacks that assure convergence of outputs to zero will be derived.

Chapter II. Type $[\ell_1,\dots,\ell_m]$ Feedback Systems ..... Reference Inputs Case

The "type ℓ feedback system" concept is often utilized in conventional control theory, and proved to be very useful for designing single-input, single-output servomechanism systems, and for identifying asymptotic behaviors of such systems. The purpose of this chapter is to extend some such results which are known for scalar systems to multivariable systems.

More specifically, as an extended concept of the type ℓ feedback system, a type $[\ell_1,\dots,\ell_m]$ feedback system (; in short, type $[\ell_1,\dots,\ell_m]$ f.s.) is defined. Here the integers ℓ_i (i=1, ..., m) are defined to indicate that the i-th output of an m-input, m-output unity feedback system (Fig. 2-1) will follow without steady state error the i-th reference input, which is a polynomial function of t , up to degree ℓ_i-1. Based on this definition the next two problems are considered and solved. The first is: (i) Given a system of the form of Fig. 2-1, where either the open-loop transfer function matrix or its state equations are known, then what is a convenient way to know the type of the feedback system (; that is the integers ℓ_i) ? The second is; (ii) Given an r-input, m-output controllable observable linear time-invariant plant, how can a type $[\ell_1,\dots,\ell_m]$ f.s. be synthesized ?

Another important concept which is introduced in this chapter is one of type $[k_1,\dots,k_m]$ transfer element (; in short, type $[k_1,\dots,k_m]$ t.e.). The concept will be seen in the subsequent developments of the study to play an essential role in solving both problems (i), (ii) indicated above.

Almost all the results in this paper will be stated in state space, whereas, in conventional control theory, they are described in frequency domain. This is only because doing so is in many cases more convenient for multivariable systems.

In the literature, Wiberg [1] and Sandell and Athans [2] have defined type L multivariable feedback systems, and have discussed similar problems. However, their definition is inadequate for multivariable systems, since their definition does not reflect the fact that asymptotic behaviors of the individual outputs (representing in general different physical quantities, such as velocity, position, or pressure) are not equal. Moreover in [1] and [2], to identify the type of a given feedback system, a fairly tedious numerical operations – trial and error computations, inversions of matrices containing the variable s, and computations of characteristic polynomials of transfer function matrices – have been required. These difficulties do not occure in the present method. As for the synthesis problem alone, a vast related litarature (e.g., [1], [3]-[10]) exists, although most of the works are intended not for extending classical design procedures which utilize the type ℓ feedback system concept, but for extending the state space method in regulator problems to include servomechanism problems. Sandell and Athans [1] have considered the problem on the basis of the type L multivariable feedback system concept. Davison's results in [3] can also be regarded to be based on the same concept, since m defferential equations which generates m reference inputs have been assumed to be the same. Youg and Willems [4] have considered the problem for the case L = 1.

2.1 Definitions

In this section, two definitions are introduced: a type $[k_1,\dots,k_m]$f.s., and a type $[\ell_1,\dots,\ell_m]$ t.e.. The relationship existing between the two definitions will be clarified in Section 2.2. Now let us first give the definition of a type $[\ell_1,\dots,\ell_m]$ f.s..

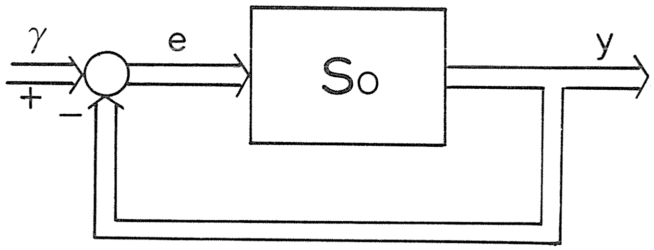

Fig. 2-1. m-input, m-output unity feedback systems.

Given the system of Fig. 2-1, which is assumed to be asymptotically stable when $\gamma(t) = 0$. So represents a linear time-invariant m-input, m-output system, and the output y, the reference input γ, and the error e are m-vectors. The error function corresponding to a reference input $\gamma(t)$ and the zero initial state of So will be denoted by $e[\gamma(t)]$. Then, type $[\ell_1,\dots,\ell_m]$ f.s. is defined as

Definition 2-1: Let the reference input of the form

$$\gamma_i^j(t) \triangleq t^{j-1}U(t)\varepsilon_i \tag{2-1}$$

be applied to the system of Fig. 2-1, where ϵ_i is the i-th unit vector, dim ϵ_i = dimγ_i^j = m, U(t) the unit step function, and $t^{-1} \equiv 0$.

Let

$$l_i \triangleq \max\left\{j : \lim_{t\to\infty} e[\gamma_i^j(t)] = 0,\ j = 0,1,2,\dots\right\} \qquad i = 1,\cdots, m. \tag{2-2}$$

Then the system of Fig. 2-1 is called a type $[\ell_1,\ldots,\ell_m]$ f.s..

If m=1, the above definition coincides clearly with the classical one for single-input, single-output systems. In [1] and [2], Wiberg, and Sandell and Athans have call the system in Fig. 2-1 a type L feedback system. The connection between the two definitions is expressed by

$$L = min\{l_1, \cdots, l_m\}. \tag{2-3}$$

It should be noted that a synthesis problem exists which have a solution if the asymptotic characteristics of the outputs are individually specified, but does not have a solution if the specification is given on the basis of type L feedback system concept. A trivial example is the following: The plant has transfer function matrix $G(s) = \text{diag}\ [\frac{1}{s+1}, \frac{s}{s+1}]$. It is desired to construct a feedback system such that $y_1(t) \mapsto r_{10}, y_2(t) \mapsto 0$, where r_{10} is an arbitrary constant. Clearly, this problem has a solution. However, we cannot construct a type one feedback system, since, to do this, the relation

$$y_2(t) \rightarrow r_{20}$$

also must be satisfied for an arbitrary constant r_{20}.
Notice also that if

$$\gamma(t) = \left\{ \sum_{i=0}^{l_1-1} \alpha_{1i} t^i, \cdots, \sum_{j=0}^{l_m-1} \alpha_{mi} t^j \right\} \tag{2-4}$$

in Fig. 2-1, then we have

$$\lim_{t\to\infty} e[\gamma(t)] = 0.$$

Therefore, a type $[\ell_1,\ldots,\ell_m]$ f.s. is asymptotically decoupled to the reference input (2-4).

Next, we shall give the definition of a type $[k_1,\ldots,k_m]$ t.e.. According to the definition to be proposed, it will be seen that every r-input, m-output linear time-invariant system is classified into type $[k_1,\ldots,k_m]$ t.e.. Two equivalent expressions of the definition will be given; the first correspondes to the case where a system is described by the state equations, and the second correspondes to the case where a system is described by the transfer function matrix.

In the first case, a system S is described by the equations

$$\dot{x} = Ax + Bu \tag{2-5a}$$

$$y = Cx + Du \tag{2-5b}$$

where the state x is an n-vector, the input u an r-vector, the output y an m-vector, A, B, C, and D are constant matrices of appropriate dimensions. It is assumed that (2-5) represents only the observable and controllable part of the system S. To state the definition in this case, it is necessary to bring (2-5) by an appropriate coordinates transformation into the following canonical form:

$$\begin{bmatrix} \dot{x}_a \\ \dot{x}_b \end{bmatrix} = \begin{bmatrix} A_a & O \\ O & A_b \end{bmatrix} \begin{bmatrix} x_a \\ x_b \end{bmatrix} + \begin{bmatrix} B_a \\ B_b \end{bmatrix} u \tag{2-6a}$$

$$y = [C_a \quad C_b] \begin{bmatrix} x_a \\ x_b \end{bmatrix} + Du \tag{2-6b}$$

where every eigenvalues of Aa are zero, $|A_b| \neq 0$, dim $x_a = n_a$, and dim $x_b = n_b$.

Throughout this paper, the following notations will be used:

(i) To any matrices X and Y such that products YX and XX are defined,

$$\Delta^{l}[Y, X] \triangleq \begin{bmatrix} Y \\ YX \\ \vdots \\ YX^{l} \end{bmatrix} \tag{2-7}$$

(ii) Given a matrix $X=[X_1', X_2', \ldots, X_s']'$, where X_i represents the i-th row or the i-th row-submatrix of X, ${}^{i}X$ denotes the matrix

$$ {}^{i}X = \begin{bmatrix} X_1 \\ \vdots \\ X_i \\ X_{i+1} \\ \vdots \\ X_s \end{bmatrix} \tag{2-8}$$

Definition 2-2a : Let the observable and controllable part of an r-input, m-output system S be described by (2-5). Let (2-5) be equivalent to (2-6). Then system S is call a type $[k_1, \ldots, k_m]$ t.e., where

$$\begin{aligned} k_i \triangleq & \operatorname{rank} \Delta^{n_a-1}[C_a, A_a] - \operatorname{rank} \Delta^{n_a-1}[{}^{i}C_a, A_a] \\ &= n_a - \operatorname{rank} \Delta^{n_a-1}[{}^{i}C_a, A_a]. \end{aligned} \tag{2-9}$$

Remark 2-1: Irrespectively of what coordinates transformation is chosen to obtain (2-6), the integers k_i are uniquely defined, since the two representations (Ca, Aa) and ($\tilde{C}a$, $\tilde{A}a$) which originate from two different coordinates transformations are related by some nonsingular constant matrix Ta as $Ca = \tilde{C}aTa$ and $Aa = Ta^{-1}AaTa$.

Often, multivariable systems are described by the transfer function matrices. In such cases, it is desirable to state the definition of a type $[k_1, \ldots, k_m]$ t.e. in the following form:

Definition 2-2b: Let G(s) be an mxr proper rational transfer function matrix of a linear time-invariant r-input, m-output system S. Define $G_a(s)$ and $G_b(s)$ uniquely as

$$G(s) = G_a(s) + G_b(s) \tag{2-10}$$

where the strictly proper rational matrix $G_a(s)$ has its every pole at s=0, and proper rational matrix $G_b(s)$ has no poles at s=0. Then, the system S is called a type $[k_1, \ldots, km]$ t.e., where

$$k_i \triangleq \delta[G_a(s)] - \delta[{}^{i}G_a(s)]. \tag{2-11}$$

$\delta[\cdot]$ denotes the McMillan degrees of proper rational matrices.

The equivalence of the two definitions 2-2a and 2-2b can be verified by constructing a minimal realization of G(s). The simple proof is omitted.

Remark 2-2: The following give one easy way to compute the right-hand side of (2-11). Expand Ga(s) into its partial fractions

$$G_a(s) = G_{a_1}/s + \cdots\cdots + G_{a_p}/s^{p}, \quad G_{a_p} \neq O.$$

Define matrix $\Gamma[G_a(s)]$ as (2-12)

$$\Gamma[G_a(s)] \triangleq \begin{bmatrix} G_{a_1} & \cdots\cdots\cdots & G_{a_p} \\ G_{a_2} & \cdots\cdots G_{a_p} & O \\ \vdots & & \\ G_{a_p} & & 0 \end{bmatrix} \tag{2-13}$$

Matrix $\Gamma[{}^{i}G_a(s)]$ is similarily defined. Then [11], the righthand sides of (2-11) equals to

$$\operatorname{rank} \Gamma[G_a(s)] - \operatorname{rank} \Gamma[{}^{i}G_a(s)].$$

2.2 Unity Feedback Systems Containing a Type[$k_1,\ldots,km$] t.e. in Cascade

In this section, we investigate asymptotic properties of an unity feedback system indicated in Fig. 2-2, in which S_C and S_L denote an m-input, r-output, and an r-input, m-output linear time-invariant systems respectively. For simplicity, it is assumed that the systems S_C and S_L are completely characterized by controllable and observable state equations; in other words, by transfer function matrices. The next theorem states the results:

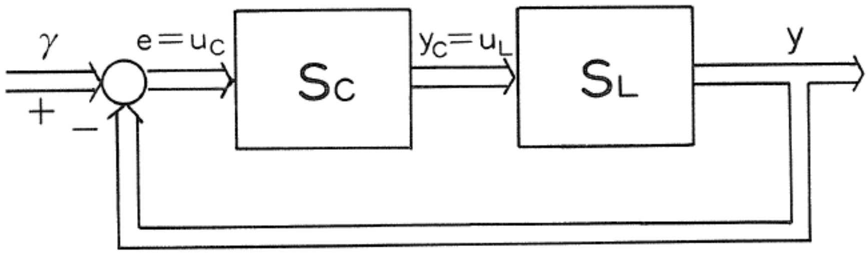

Fig. 2-2. Unity Feedback System containing a type [$\ell_1,\cdots,\ell_m$] t.e. in cascade.

Theorem 2-1: Let S_L in Fig. 2-2 be a type [$k_1,\ldots,km$] t.e.. Then, for any S_C, provided that the closed-loop system is asymptotically stable, the unity feedback system in Fig. 2-2 is of type [$\ell_1,\ldots,\ell m$], where ℓ_i are integers satisfying the inequalities

$$l_i \geqq k_i \qquad i = 1,\ldots, m \tag{2-14}$$

If, in addition, it is assumed that none of the eigenvalues of the representation of S_C is located at zero; in other words, the transfer function matrix of S_C has no poles at s=0, the inequalities (2-14) are replaced by the equalities.

This theorem is fundamental, in the sence that it serves as a key to the solution to both problems (i) and (ii) which are indicated at the introduction of this chapter. In fact, the solution to problem (i) follows immediately from it (; see below, Theorem 2-2). In section 2.3, we shall show that problem (ii) also can be solved by making use of this theorem.

Now, identifying S_O in Fig. 2-1 with S_L in Fig. 2-2 and setting $S_C=I_m$ in Fig. 2-2, we have Theorem 2-2: Let the closed-loop system of Fig. 2-1 be asymptotically stable when $\gamma\equiv 0$. The unity feedback system indicated in Fig. 2-1 is of type [$\ell_1,\ldots,\ell m$] if and only if S_O is a type [$\ell_1,\ldots,\ell m$] t.e.

This theorem enable us to know the type of the unity feedback system from computations on S_O. When S_O is described by the state equations, use Definition 2-2a, and when S_O is described by the transfer function matrix, use Definition 2-2b and Remark 2-2.

Proof of Theorem 2-1: We represent S_C and S_L by the following equations:

$$S_L: \dot{x}_L = A_L x_L + B_L u_L \tag{2-15a}$$
$$y = C_L x_L + D_L u_L \tag{2-15b}$$
$$S_C \quad \dot{x}_C = A_C x_C + B_C u_C \tag{2-16a}$$
$$y_C = C_C x_C + D_C u_C \tag{2-16b}$$

, which are observable and controllable by the assumption.

The dimensions of the vectors x_L, x_C, y_L, y_C, u_L, and u_C are n_L, n_C, m, r, r and m respectively. Among the variables $y_L,\ldots,y_C$, the following relations exist:

$$y_L = y, \quad u_L = y_C, \quad u_C = e = \gamma - y. \tag{2-17}$$

We describe the polynomial reference input $\gamma_i^j(t)$ also by the state equations

$$\dot{z} = Fz \tag{2-18a}$$
$$\gamma_i' = Hz \tag{2-18b}$$

where

$$F = \begin{bmatrix} 0 & 1 & & \\ & & & 1 \\ 0 & \cdots\cdots & & 0 \end{bmatrix} \begin{matrix} \uparrow \\ j \\ \downarrow \end{matrix} \tag{2-19a}$$

$$H = \varepsilon_i [1 \quad 0 \cdots\cdots 0 \;] \tag{2-19b}$$

where ϵ_i is the i-th unit m-vector.

Then, to prove the first assertion of the theorem, it is necessary to show, with the aid of (2-15)-(2-19), that, if $j \leq k_i$ in (2-18) and (2-19),

$$\lim_{t\to\infty} e(t) = 0 \tag{2-20}$$

for x_L (0)=0, x_C (0)=0, and any z(0). To do this, annex (2-15) to (2-18), and use (2-17). The result is

$$\dot{\hat{\xi}} = \hat{A}\,\hat{\xi} + Bu_L \tag{2-21a}$$
$$e = \hat{C}\hat{\xi} - D_L u_L \tag{2-21b}$$

where

$$\hat{\xi} = \begin{bmatrix} x_L \\ z \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} A_L & O \\ O & F \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_L \\ O \end{bmatrix}, \tag{2-22}$$
$$\hat{C} = [-C_L, \; H]$$

System (2-21) is not in general observable (; notice that unobservable states of (2-21) do not influence the output e(t)). In fact, we have the lemma (Proof is in appendix):

Lemma 1: Let S_L be a type $[k_1,\ldots,k_m]$ t.e., then the rank of the observability matrix for (2-21) is given by

$$\text{rank } \Delta^{n_L+j-1}[C, A] = \begin{cases} n_L & : if \;\; o \leqq j \leqq k_i \;\; in\ (2-19)\,, & (2-23a) \\ n_L+1 : if \;\; j = k_i+1 & in\ (2-19)\,. & (2-23b) \end{cases}$$

Using this lemma and the fact that (C_L, A_L) is observable (; that is, rank $\Delta^{n_L-1}[C_L, A_L] = n_L$), it is easy to see that, if $1 \leq j \leq k_i$, a $n_L \times (n_L+j)$ transformation matrix

$$T = [I_{n_L}, T_z]$$

exists such that

$$TA = A_L T, \quad C = -C_L T. \tag{2-25}$$

Therefore, by setting $\xi = T\hat{\xi}$, we have the following observable representation of (2-21):

$$\dot{\xi} = A_L\xi + B_L u_L \tag{2-26a}$$
$$e = -C_L\xi - D_L u_L \tag{2-26b}$$

Equations (2-26) together with (2-16) and (2-17) describes the dynamics of e(t), but, as is easily seen, the characteristic polynomial corresponding to these equations is the same as the closed-loop system's. So, we see that $\xi(t)$, $x_C(t)$, $e(t)$, $y_L(t) \to 0$ as $t \to \infty$, since from the assumption the closed-loop system is asymptotically stable. Therefore, (2-20) holds for any $x_L(0)$, and z(0). This proves the first assertion of the theorem.

Now, to prove the second assertion, let $j=k_i+1$ in (2-20). It is desired to show that

$$\lim_{t\to\infty} e(t) \neq 0 \qquad (2\text{-}27)$$

for some z(0). Since $j=k_i+1$ in this case, a $(n_L+1)\times(n_L+j)$ transformation matrix

$$T=\begin{pmatrix} I_{n_L} & T_{z_1} \\ o & t'_{z_2} \end{pmatrix} \qquad (2\text{-}28)$$

exists from Lemma 1 such that

$$T\hat{A}=\begin{pmatrix} A_L & \tilde{f}_L \\ o' & o \end{pmatrix}T,\quad \hat{C}=[-C_L,\ \tilde{h}]\,T \qquad (2\text{-}29)$$

where T_{Z_1} is an $n_L\times j$ matrix, t_{Z_2} a j-vector, f_L an n_L-vector, and $\tilde{h}$ an m-vector. Then, if we set $\xi=T\hat{\xi}$, one of the observable representation of (2-21) is obtained as

$$\dot{\xi}=\begin{pmatrix} A_L & f_L \\ o' & o \end{pmatrix}\xi+\begin{pmatrix} B_L \\ o \end{pmatrix}u_L \qquad (2\text{-}30a)$$

$$e=[-C_L,\ \tilde{h}]\,\xi-D_Lu_L \qquad (2\text{-}30b)$$

Thus, in this case, the dynamics of e(t) is seen to be described by (2-16), (2-17), and (2-30). Every mode of (2-30) is observable, whereas one mode which correspondes to the zero eigenvalue is clearly uncontrollable. This mode remaines uncontrollable even in the tandemly connected system of (2-30) and (2-16). Also, it remains observable, since from the assumption every mode of (2-16) is nonzero. Therefore, on noting further the fact that controllability and observability properties of linear systems are not affected by output feedbacks, we know that, even if the output feedback $u_C=e$ [(2-17)] is introduced to the tandemly connected system, the output e(t) of (2-30) (; that is, the output error) containes the observable but uncontrollable mode which correspondes to zero eigenvalue. This means $\lim_{t\to\infty} e(t)\neq 0$ for some $\xi(0)$ and z(0), that is for some $x_L(0)$, $x_C(0)$, and z(0). But, since the influence of $x_L(0)$ and $x_C(0)$ on e(t) decays to zero by the assumed closed-loop asymptotic stability, therefore $\lim_{t\to\infty} e(t)\neq 0$ for some z(0). This is the desired result.

2.3 Synthesis

Consider the r_p-input, m-output observable and controllable plant

$$S_P:\ \dot{x}_P=Ax_P+Bu_P \qquad (2\text{-}31a)$$

$$y=Cx_P+Du_P \qquad (2\text{-}31b)$$

where the state x_P is an n_P-vector, the input u_P an r-vector, and the output y an m-vector.

It is desired to synthesize an unity feedback system such that

(i) the i-th output y_i (i=1, ... , m) tracks, without steady state error, polynomial reference inputs whose degrees are less than or equal to ℓ_i^d-1, where ℓ_i^d are integers to be specified by the designers, and

(ii) the resulting closed-loop system has preassigned stable eigenvalues.

To the problem formulated, we propose the following synthesis procedure:

The procedure consists of the following three steps:

(a) Examine the type of the plant. Here, the plant (2-31) is assumed to be a type$[k_1,..,k_m]$t.e.

(b) As the result of (a), if $k_i\geq\ell_i^d$ for all i, this step is unnecessary. If $k_i<\ell_i^d$ for some i, then in this case, construct a cascade compensator S_Q so that the tandemly connected system $S_P.S_Q$, where S_P is followed by S_Q, is to be a type $[k_1^L,...,k_m^L]$t.e., and $k_i^L\geq\ell_i^d$ for all i.

(c) Construct a second controller S_R so that the specification (ii) is met.

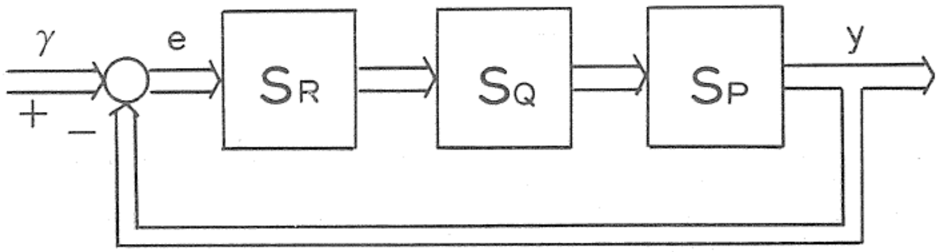

Fig. 2-3. Synthesis ot a type $[\ell_1,\cdots,\ell_m]$ feedback system.

Fig. 2-3 indicates a system which is constructed by the method just proposed. From Theorem 2-2 and the indicated procedure, it is seen that, if each step is performed successfully, the resulting system is a type $[k_1^L,\ldots,k_m^L]$ f.s., and satisfies specifications (i) and (ii).

Step (a) can be performed from Definition 2-2a. Soon after, it will be shown that a compensator S_Q is obtainable, under appropriate conditions, such that $S_P.S_Q$ is observable and controllable. Therefore, (c) reduces to the well known problem of assigning poles of closed-loop systems by output feedbacks, and, therefore, we can use the results, say in [12], [13]. Thus the only step which requires detailed consideration is (b).

In the following, we shall give a theorem which enable us, if it is used repeatedly, to construct a compensator S_Q. To present the theorem, let (2-31) be transformed by a coodinates transformation into its following equivalent equations:

$$\begin{pmatrix} \dot{x}_{P_a} \\ \dot{x}_{P_b} \end{pmatrix} = \begin{pmatrix} A_{P_a} & O \\ O & A_{P_a} \end{pmatrix} \begin{pmatrix} x_{P_a} \\ x_{P_b} \end{pmatrix} + \begin{pmatrix} B_{P_a} \\ B_{P_b} \end{pmatrix} u_P \tag{2-32a}$$

$$y = [C_{P_a},\ C_{P_b}] \begin{pmatrix} x_{P_a} \\ x_{P_b} \end{pmatrix} + Du_P \tag{2-32b}$$

where the subindices a, b are used to imply the same as in Section 2.1, dim $x_{Pa}=n_{Pa}$, and dim $x_{Pb}=n_{Pb}$. In the subsequent discussion, subscript P in $A_{Pa},\ldots,n_{Pb}$ will be dropped for national convenience. Notice that, from Definition 2-2a and the assumption on (2-31), the following hold.

$$k_i = \mathrm{rank}\ \Delta^{n_a-1}[C_a,\ A_a] - \mathrm{rank}\ \Delta^{n_a-1}[{}^iC_a,\ A_a] \tag{2-33}$$

for i=1, ... , m. To state the theorem, two matrices are defined as follows:

$$\hat{A}_a = \begin{pmatrix} A_a & B_a \\ O & O \end{pmatrix},\quad \hat{C}_a = [C_a,\ D-C_bA_b^{-1}B_b] \tag{2-34}$$

where the dimensions of $\hat{A}_a$ and $\hat{C}_a$ are $(n_a+r_P)\times(n_a+r_P)$ and $m\times(n_a+r_P)$ respectively.
Theorem 2-3: Assume that the r_P-input, m-output plant (2-31) is a type $[k_1,\ldots,k_m]$ t.e. and is equivalent to (2-32). Then, (i) if and only if the vector $\hat{c}_{ai}\hat{A}_a^{k_i}$ is independent of the rows of ${}^iW$, where

$$W = \begin{pmatrix} \Delta^{n_a-k_1}[[\hat{c}_{a_1}\hat{A}_a^{k_1},\ \hat{A}_a] \\ \Delta^{n_a-k_2}[[\hat{c}_{a_2}\hat{A}_a^{k_2},\ \hat{A}_a] \\ \vdots \\ \Delta^{n_a-k_m}[\hat{c}_{a_m}\hat{A}_a^{k_m},\ \hat{A}_a] \end{pmatrix}$$

where $\hat{c}_{aj}$ are the j-th rows of $\hat{C}_a$, a compensator S_{Qi} of the form

$$\dot{x}_q = v \tag{2-36a}$$

$$y_q = c_q x_q + w \tag{2-36b}$$

exists, where the state x_q and the input v are scalars, the output y_q the input w, and c_q are r_p-vectors, such that the following hold:

(1) The tandemly connected system $S_p.S_{Qi}$, where $u_p=y_q$, preserves the controllability and the observability of the plant.

(2) $S_p.S_{Qi}$ is a type $[k_1',\dots,k_m']$ t.e., where $k_j'=k_j+1$ for j=i, and $k_j \leqq k_j' \leqq k_j+1$ for $j\neq i$.

(ii) If the condition in (i) is satisfied, c_q in (2-36) is obtained to satisfy (1) and (2) by solving the following equations:

$$\begin{pmatrix} \hat{c}_{a_i}\hat{A}_a^{\kappa_i} \\ \cdots\cdots\cdots \\ {}^iW \end{pmatrix}\begin{pmatrix} \tilde{c}_q \\ \\ c_q \end{pmatrix} = \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix} \tag{2-37}$$

where $\tilde{c}_q$ is an n_a-vector, and 0 is the zero $(\sum_{j\neq i}(n_a-k_j+1))$-vector.

Proof: Consider the tandemly connected system $S_p.S_{Qi}$. From the assumption on S_p, $S_p.S_{Qi}$ is equivalent to the system described by

$$\frac{d}{dt}\begin{pmatrix} x_{p_a} \\ x_{p_b} \\ x_q \end{pmatrix} = \begin{pmatrix} A_a & O & B_a c_q \\ O & A_b & B_b c_q \\ O & O & O \end{pmatrix}\begin{pmatrix} x_{p_a} \\ x_{p_b} \\ x_q \end{pmatrix} + \begin{pmatrix} B_a & 0 \\ B_b & 0 \\ O & 1 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix} \tag{2-38a}$$

$$y = [C_a,\ C_b \quad Dc_q]\begin{pmatrix} x_{p_a} \\ x_{p_b} \\ x_q \end{pmatrix} + Dw \tag{2-39}$$

To put these equations in the form of (2-32), define

$$\eta = \begin{pmatrix} I & O & O \\ O & O & I \\ O & I & \alpha \end{pmatrix}\begin{pmatrix} x_{p_a} \\ x_{p_b} \\ x_q \end{pmatrix} \tag{2-40}$$

where $-A_b\alpha+B_bc_q=0$, i.e., $\alpha = A_b^{-1}B_bc_q$. It is easily shown that

$$\dot{\eta} = \begin{pmatrix} A_a & B_a c_q & O \\ O & O & O \\ O & O & A_b \end{pmatrix}\eta + \begin{pmatrix} B_a & O \\ O & 1 \\ B_b & \alpha \end{pmatrix}\begin{pmatrix} x_{p_a} \\ x_{p_b} \\ x_q \end{pmatrix} \tag{2-41a}$$

$$y = [C_a,\ (D-C_bA_b^{-1}B_b)c_q,\ C_b]\eta + Dw. \tag{2-41b}$$

Therefore, from the definition, S_pS_{Qi} is a type $[k_1',\dots,k_m']$t.e., where

$$k_j' = \operatorname{rank}\ \Delta^{na}[\hat{C}_a,\ \hat{A}_a]\cdot E - \operatorname{rank}\ \Delta^{na}[{}^j\hat{C}_a,\ \hat{A}_a]\cdot E \tag{2-42}$$

where

$$E = \begin{pmatrix} I_{na} & O \\ O & c_q \end{pmatrix} \tag{2-43}$$

Permutating the rows of the matrices in (2-42), it becomes

$$k_j' = \text{rank}\ \ PE - \text{rank}\ \ {}^jPE \tag{2-44}$$

where

$$P = \begin{pmatrix} \Delta^{n_a}[\hat{c}_{a_1},\ \hat{A}_a] \\ \Delta^{n_a}[\hat{c}_{a_2},\ \hat{A}_a] \\ \cdots\cdots\cdots\cdots \\ \Delta^{n_a}[\hat{c}_{a_m},\ \hat{A}_a] \end{pmatrix} \tag{2-45}$$

Now, we shall prove the theorem by making use of the following facts, whose proof will be found in Appendix of this chapter:

(a) Any one of the first k_j rows of the submatrix $\Delta^{n}a[\hat{c}_{aj}, \hat{A}_a]\cdot E$ of PE cannot be represented as a linear conbination of the other rows of PE.

(b) The (k_j+1)-, (k_j+3)-,..., (n_a+1)-th rows of $\Delta^{n}a[\hat{c}_{aj}, \hat{A}_a]\cdot E$ are linearly dependent on the rows of ${}^jWE$.

(c) $k_j \leqq k_j' \leqq k_j + 1$, $j = 1,\ldots,m$

To prove the necessity, assume that $S_P.S_{Qi}$ is a type $[\ldots, k_i+1,\ldots]$ t.e.. Then from (2-44)

$$k_i + 1 = \text{rank}\ \ PE - \text{rank}\ \ {}^iPE. \tag{2-46}$$

, each terms in the right-hand side of which are computed from (a) as

$$\text{rank}\ \ PE = \sum_{i=1}^{m} k_j + \text{rank}\ \ WE \tag{2-47}$$

$$\text{rank}\ \ {}^iPE = \sum_{j\neq i}^{m} k_j + \text{rank}\ \ {}^iWE \tag{2-48}$$

Thus, from (2-46) we have

$$\text{rank}\ \ WE - \text{rank}\ \ {}^iWE = 1. \tag{2-49}$$

But, from (b), every row of $\Delta^{n_a-k_i}[\hat{c}_{ai}\hat{A}_a^{k_i}, A_a]E$ except $\hat{c}_{ai}\hat{A}_a^{k_i}E$ is linearly dependent on the rows of ${}^iWE$. Therefore, from (2-49), $\hat{c}_{ai}\hat{A}_a^{k_i}E$ must be linearly independent of the rows of ${}^iWE$. Therefore $\hat{c}_{ai}\hat{A}_a^{k_i}$ is linearly independent of the rows of ${}^iW$.

Sufficiency will be proved by showing that (ii) is true. Chose c_q to satisfy (2-37). This is always possible by the assumption. It is now desired to show that $S_P.S_{Qi}$ is a type $[k_1',\ldots,k_m']$ t.e., where $k_j \leqq k_j' \leqq k_j+1$ for $j\neq i$ and $k_j'=k_j+1$ for $j=i$, and is observable and controllable. The first relations on k_j' were proved in (c). The second relation is obtained as follows:

$$k^i = \text{rank}\ \ PE - \text{rank}\ \ {}^iPE \ \ (;\ \text{from (2-44)})$$

$$= k_i + \text{rank}\ \ WE - \text{rank}\ \ {}^iWE \ \ (;\ \text{from (2-47), (2-48)}) \tag{2-50}$$

$$= k_i + \text{rank}\begin{pmatrix} \hat{c}_{a_i}\hat{A}_a^{k_i} \\ \\ {}^iW \end{pmatrix} E - \text{rank}\ \ {}^iWE \ \ (;\ \text{from (b)}) \tag{2-51}$$

$$= k_i + \text{rank}\begin{pmatrix} 0 & \cdots 0 & 1 \\ & \cdots\cdots\cdots & 0 \\ & X & \vdots \\ & & \vdots \end{pmatrix} - \text{rank}\ \ X \ \ (;\ \text{from (2-37) and (2-43)}) \tag{2-52}$$

$$= k_i + 1, \tag{2-53}$$

where $X = {}^iW[I_{na},0]'$. Notice that eqs. (2-37), (2-43), and elementary column operations were used to derive (2-52).

From (2-38) or (2-41), it is easy to see that $S_P S_{Qi}$ is controllable. Observe that this does not depend on the choice of c_q. To prove that $S_P S_{Qi}$ is also observable, it suffices to show that

$$\left\{ [C_a,\ (D+C_b A_b^{-1} B_b) c_q],\ \begin{bmatrix} A_a & B_a c_q \\ O' & O \end{bmatrix} \right\} \tag{2-54}$$

is observable, for A_b and $\begin{bmatrix} A_a & B_a c_q \\ 0' & 0 \end{bmatrix}$ in (2-39) do not have eigenvalues in common and (C_b, A_b) is observable. The observability matrix of (2-54) is given by PE, the rank of which can be computed as follows: From (a), and Lemma 2-2 in Appendix,

rank PE - rank P $[I_{na}, 0]'$

$$= \sum_{i=1}^{m} k_i + \text{rank}\ WE - \left(\sum_{i=1}^{m} k_i + \text{rank}\ W[I_{n_a},\ 0]' \right)$$

$$= \text{rank}\ WE - \text{rank}\ {}^iW[I_{n_a},\ 0]' \tag{2-55}$$

which is equal to one from (2-43) and from the assumption on c_q of the theorem. From (2-45), we also have rank $P[I_{na}, 0]' = \text{rank}\ \Delta^{n_a}[C_a, A_a]$, which is equal to n_a, since (c_a, A_a) is observable. Therefore

$$\text{rank}\ PE = n_a + 1, \tag{2-56}$$

which proves that (2-54) is the observable pair. This completes the proof of Theorem 2-3.

The next theorem presents a sufficient condition for one in Theorem 2-3-(i).

Theorem 2-4: If S_P in Theorem 2-3 satisfies

$$\text{rank} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = n + m \tag{2-57}$$

, then the condition in Theorem 2-3-(i) is satisfied for every i=1, ... , m.

Proof: Since (2-31) is equevalent to (2-32) and since $|A_b| \neq 0$, it is easily verified that

$$\text{(2-57)} = \text{rank} \begin{pmatrix} A_a & O & B_a \\ O & A_a & B_b \\ C_a & C_b & D \end{pmatrix} = \begin{pmatrix} A_a & O & B_a \\ O & A_b & O \\ C_a & O & D\text{-}C_b A_b^{-1} B_b \end{pmatrix}$$

$$= \text{rank} \begin{pmatrix} A_a & B_a \\ C_a & D\text{-}C_b A_b^{-1} B_b \end{pmatrix} + n_b$$

Therefore

$$\text{rank} \begin{pmatrix} A_a & B_a \\ C_a & D\text{-}C_b A_b^{-1} B_b \end{pmatrix} = n_a + m \tag{2-58}$$

, which implies that the matrix in the left-hand side has full row rank. So,

$$\text{rank}\ P = \text{rank}\ \Delta^{n_a}[\hat{C}_a,\ \hat{A}_a]$$

$$= \text{rank} \begin{pmatrix} O & I_m \\ \cdots\cdots & \cdots\cdots \\ \Delta^{n_a-1}[C_a,\ A_a] & O \end{pmatrix} \begin{pmatrix} A_a & B_b \\ C_a & D\text{-}C_b A_b^{-1} B_b \end{pmatrix}$$

$$= \text{rank}\ \Delta^{n_a-1}[C_a,\ A_a] + m$$

, and, similarly,

$$\text{rank}\ {}^iP = \text{rank}\ \Delta^{n_a-1}[{}^iC_a,\ A_a] + m - 1. \tag{2-59}$$

Thus, from (2-33),

$$\text{rank}\ P - \text{rank}\ {}^iP = k_i + 1$$

Using this, we can complete the proof almost in the same manner as was done in proving, from (2-46), the necessity in Theorem 2-3-(i), and, so, omit the rest of the proof.

2.4 Examples

In this section two examples are given to illustrate the theory of this chapter.

Example 2-1 : It is desired to find the type of a stable unity feedback system which has the following open-loop transfer function matrix:

$$G(s)=\begin{bmatrix}(-6+9s)/s^3 & (-1-s^2)/s^2 \\ (1+12s+3s^2)/s^2 & 0.1/s+1\end{bmatrix}$$

Theorem 2-2 and Definition 2-2b are applied. According to (2-10), expand G(s) into the form

$$G(s)=\begin{bmatrix}(-6+9s)/s^2 & -1/s^2 \\ (1+12s)/s^2 & 0\end{bmatrix}+\begin{bmatrix}0 & -1 \\ 3 & 0.1/s+1\end{bmatrix}.$$

Then, $\Gamma[Ga(s)]$ and $\Gamma[{}^1Ga(s)]$ in (2-14) become, respectively, as

$$\begin{bmatrix}0 & 0 & 9 & -1 & -6 & 0 \\ 12 & 0 & 1 & 0 & 0 & 0 \\ 9 & -1 & -6 & 0 & & \\ 1 & 0 & 0 & 0 & & \\ -6 & 0 & & & \bigcirc & \\ 0 & 0 & & & & \end{bmatrix}, \quad \begin{bmatrix}12 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

Therefore, from (2-14)

$$k_1=\text{rank } \Gamma[G_a(s)]-\text{rank } \Gamma[{}^1G_a(s)]=4-2=2$$

Similarly, we have $k_2=1$.

Thus, the system is a type [2,1] f.s.. It should be observed that the detailed asymptotic properties have been derived with less computational efforts than in [1] and [2].

Example 2-2 : Given the two-input, two-output controllable observable plant

$$\dot{x}_p=\left[\begin{array}{cc:c}0 & 1 & 0 \\ 0 & 0 & 0 \\ \hdashline 0 & 0 & 1\end{array}\right]x_p+\left[\begin{array}{cc}0 & 0 \\ 1 & 0 \\ \hdashline 1 & 1\end{array}\right]u_p, \quad y=\left[\begin{array}{cc:c}0 & 1 & 1 \\ 1 & 1 & 1\end{array}\right]x_p+\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}u_p, \tag{2-60}$$

it is desired to construct an unity feedback system such that $y_1\to r_{10}$, $y_2\to r_{20}$, where r_{10} and r_{20} are arbitrary constants.

On partitioning the matrices in (2-60) as indicated, and on applying Definition 2-2a, we see that the plant is a type [0 1] t.e.. So, a compensator S_Q in step (ii) is needed. To apply Theorem 2-3, the vector $\hat{c}_{a1}\hat{A}_a^0=\hat{c}_{a1}$ and the matrix ${}^1W$ must be computed. In this example, they are:

$$\hat{c}_{a_1}=[0\ \ 1\ \ 0\ -1]$$

$${}^1W=\Delta^{n_a-1}[\hat{c}_{a_1},\ \hat{A}_a]=\begin{bmatrix}0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0\end{bmatrix}.$$

Thus, $\hat{c}_{a1}$ is independent of the rows of ${}^1W$, so that the condition in Theorem 2-3-(i) is satisfied. From Theorem 2-3-(ii), and solving (2-37), we have the compensator

$$S_Q:\dot{x}_q=v,\quad y_q=\begin{pmatrix}0 \\ -1\end{pmatrix}x_q+w. \tag{2-61}$$

The tandemly connected system of (2-60) and (2-61) is observable and controllable, and is at least a type [1 1] t.e.. In fact, on applying again Definition 2-2a, we can verify that $S_P S_Q$ is a type [1 , 2] t.e.. Finally, by constructing a compensator S_R in step (iii) to $S_P S_Q$ so that the closed-loop specifications are met, we have a desired system.

Appendix

To prove Lemma 2-1, the next lemma is needed.

Lemma 2-2 : Let S, which is described by (2-5), be a type $[k_1,\ldots,k_m]$ t.e., and let (2-5) be state equivalent to (2-6). The following hold for every $i=1,\ldots,m$;

(i) If $s \in [0,\ldots,k_i-1]$, the row $c_{ai}A_a^s$ of $\Delta^{n_a-1}[C_a, A_a]$ is linearly independent of the other rows of $\Delta^{n_a-1}[C_a, A_a]$.

(ii) If $s \in [k_i,\ldots, n_a-1]$, the row vector $c_{ai}A_a^s$ is linearly dependent on the rows $c_{aj}A_a^{s_j}$: $j=1,\ldots, i-1, i+1,\ldots, m$ and $s_j=k_j,\ldots,n_a-1$.

Proof : It is enough to prove for the case i=1. From Definition 2-2a.

$$k_1 = \text{rank } \Delta^{n_a-1}[C_a,\ A_a] - \text{rank } \Delta^{n_a-1}[{}^1C_a,\ A_a] \tag{A-1}$$

Permutating the rows of the matrices, (A-1) becomes

$$k_1 = \text{rank}\begin{bmatrix} \Delta^{n_a-1}[c_{a_1},\ A_a] \\ \Delta^{n_a-1}[c_{a_2},\ A_a] \\ \vdots \\ \Delta^{n_a-1}[c_{a_m},\ A_a] \end{bmatrix} - \text{rank}\begin{bmatrix} \Delta^{n_a-1}[c_{a_2},\ A_a] \\ \vdots \\ \Delta^{n_a-1}[c_{a_m},\ A_a] \end{bmatrix}. \tag{A-2}$$

To prove (i), assume that υ is the smallest integer such that $c_{a1}A_a^{\upsilon-1}$ is obtained as a linear combination of the other rows of $\Delta^{n_a-1}[C_a, A_a]$.

Then, we can write $c_a A_a^{\upsilon-1}$ as

$$c_{a_1} A_a^{\nu-1} = \sum_{t=\nu}^{n_a-1} \beta_t\, c_{a1} A^t + \sum_{j=2}^{m} \sum_{t=0}^{n_a-1} \gamma_{jt}^{(0)}\ c_{a_j} A^t, \tag{A-3}$$

where β_t and $\gamma_{jt}^{(0)}$ are constants.

Notice that $A^{n_a} = A^{n_a+1} = \ldots = 0$, since every eigenvalue of A_a is zero. So, postmultiplying (A-3) by $A^{n_a-\upsilon}$ gives

$$c_{a_1} A_a^{n_a-1} = \sum_{j=2}^{m} \sum_{t=0}^{n_a-1} \gamma_{jt}^{(n_a-\nu)} c_{a_j} A^t \tag{A-4}$$

where $\gamma_{jt}^{(n_a-\upsilon)}$ are some other constants. By multiplying by successively lower powers of A, and substituting the results into the right-hand side of (A-3), we see that the rows $c_{a1}A^{\upsilon}$, $\ldots, c_{a1}A^{n_a-1}$ are linearly dependent on the rows of $\Delta^{n_a-1}[{}^1C_a, A_a]$. From this and (A-2), and on noting that if $c_{a1}A^j$ is dependent on the rows of $\Delta^{n_a-1}[{}^1C_a, A_a]$ for some j, $c_{a1}A^{j+1}$, $c_{a1}A^{j+2}$, ... are dependent on the same set of rows, υ must be equal to k_1. Thus, (i) has been proved.

Now if we use the result just provea, it is easy to see, from (A-2), that

$$\text{rank}\begin{bmatrix} \Delta^{n_a-k_1-1}[c_{a_1}A_a^{k_1},\ A_a] \\ \Delta^{n_a-k_2-1}[c_{a_2}A_a^{k_1},\ A_a] \\ \vdots \\ \Delta^{n_a-k_m-1}[c_{a_m}A_a^{k_m},\ A_a] \end{bmatrix} - \text{rank}\begin{bmatrix} \Delta^{n_a-k_2-1}[c_{a_2}A_a^{k_2},\ A^a] \\ \vdots \\ \Delta^{n_a-k_m-1}[c_{a_m}A_a^{k},\ A_a] \end{bmatrix} = 0$$

, which proves that every row of $\Delta^{n_a-k_1-1}[c_{a1}A_a^{k_1}, A_a]$ is linearly dependent on the rows of $\Delta^{n_a-k_2-1}[c_{a2}A_a^{k_2}, A_a],\ldots, \Delta^{n_a-k_m-1}[c_{am}A^{k_m}, A_a]$, and therefore proves (ii).

Proof of Lemma 1 : Since the left hand-side of (2-23) remains unchanged by a transformation of the form : $T\hat{A}T^{-1}$, $\hat{C}T^{-1}$, A_L and C_L in (2-22) can be assumed to be of the forms,

$$\begin{bmatrix} A_{La} & 0 \\ 0 & A_{La} \end{bmatrix}, \quad [C_{La}, \quad C_{Lb}] \tag{A-5}$$

where the subindices a, b are used to indicate the same as in (2-6), and the sizes of A_{La} and A_{Lb} are $n_{La}\times n_{La}$ and $n_{Lb}\times n_{Lb}$ respectively. Then, the left-hand side of (2-23) can be written as

$$\operatorname{rank}\ [\Delta^{n_L+j-1}[-C_{La}, A_{La}] \ \vdots\ \Delta^{n_L+j-1}[-C_{Lb}, A_{La}] \ \vdots\ \Delta^{n_L+j-1}[H, F]]$$

, which is equal to

$$\operatorname{rank}\ [\Delta^{n_L+j-1}[-C_{La}, A_{La}] \ \vdots \Delta^{n_L+j-1}[H, F]] + \operatorname{rank}\ [\Delta^{n_L+j-1}[-C_{Lb}, A_{Lb}]] \tag{A-6}$$

since A_{La} and F have no eigenvaluse in common with A_{Lb} (; see [14]). On noting that rank $\Delta^{n_L+j-1}[-C_{Lb}, A_{Lb}]=n_b$, since (C_{Lb}, A_{Lb}) is observable, and that

$$\Delta^{n_L+j-1}[h_i, F] = \begin{bmatrix} I_j \\ 0 \end{bmatrix}, \quad \Delta^{n_L+j-1}[{}^iH, F] = 0$$

where h_i the i-th row of H, we obtain,

$$(A\text{-}6) = \operatorname{rank}\begin{bmatrix} \Delta^{n_L+j-1}[-c_{Lai}, A_{La}] & \vdots & \Delta^{n_L+j-1}[h_i, F] \\ \cdots\cdots & \vdots & \cdots\cdots \\ \Delta^{n_L+j-1}[-{}^iC_{La}, A_{La}] & \vdots & \Delta^{n_L+j-1}[-{}^iH, F] \end{bmatrix} + n_{Lb}$$

$$= \operatorname{rank}\begin{bmatrix} \Delta^{n_j-1}[-c_{Lai}, A_a] & \vdots & I_j \\ \Delta^{n_L-1}[-c_{La}A_{La}^{n_j}, A_a] & \vdots & 0 \\ \cdots\cdots & \vdots & \cdots \\ \Delta^{n_L+j-1}[-{}^iC_{La}, A_a] & \vdots & 0 \end{bmatrix} + n_{Lb} \tag{A-7}$$

If $1\leq j\leq k_i$ in (A-7), the first term in the last equation of (A-7) remains unchanged even if the term I_j is removed from it, since every row of $\Delta^{j-1}[-c_{Lai}, A_a]$ is linearly independent of the other rows from Lemma 2-2. Therefore, in this case, the first term of (A-7) equals to n_{La}, and the right-hand side of (2-22) equals to $n_{La}+n_{Lb}=n_L$, which proves (2-23a).
Eq. (2-23b) is similarly proved from Lemma 2-2 and (A-7).

Proof of (a), (b), (c) in Section 2.3 :
Observe that the s-th row of $\Delta^{n_a}[\hat{c}_{ai}, \hat{A}_a]E$ has the form

$$\begin{aligned} &[c_{ai}A_a^{s-1}, \quad c_{ai}A_a^{s-2}B_a]E \ : \ j\geq 2, \\ &[c_{ai}, \quad d_i - c_{bi}A_b^{-1}B_a]E \ : \ j=1. \end{aligned} \tag{A-8}$$

Therefore, if we restrict our attention on the first n_a element of rows, (a) is clear from Lemma 2-2.

From the same lemma, constant row vectors β_t exist such that

$$c_{aj}A_a^{k_j} = \sum_{t\neq j}^{m} \beta_t\, \Delta^{n_a-k_t-1}[c_{at}A_a^{k_t}, \ A_a].$$

From this and (A-8), the (k_j+s)-th row of $\Delta^{n_a}[\hat{c}_{aj}, \hat{A}_a]E$ $(s\geq 2)$ can be written as

$$[c_{aj} A_a^{k_j+s-1}, \ c_{aj} A_a^{k_j+s-2} B_a] E$$

$$= \sum_{t \neq j}^{m} \beta_t \left\{ \Delta^{n_a - k_t - 1}[c_{at} A_a^{k_t+s-1}, \ A_a], \ \Delta^{n_a - k_t - 1}[c_{at} A_a^{k_t+s-2}, \ A_a] B_a \right\} E \tag{A-9}$$

Comparing this with the rows of ${}^jW$ and noting again $A^{n_a}=0$, we have (b). The inequalities in (c) follows immediately from (a), (b) and (2-44).

Chapter III Type[$\ell_1,\ldots,\ell_p$]Feedback Systems .... Disturbance inputs Case

Almost any realistic control systems will be in the environment where external disturbances exist. Among them, disturbances that drift over fairly long time interval, such as variations of atomospheric temperature in process control systems, variations of loads in servo systems, etc., have especially unfavorable influences upon steady state performances of control systems. To such disturbances, we shall invoke the argument that they are approximately representable as some polynomial functions of time, such as the step or ramp functions (; This is quite common in engineerings). Then, this enable us to formulate problems that are quite analogus to the ones discussed in Chapter II. By replacing reference inputs in Definition 2-1 with disturbances, we shall define type [$\ell_1,\ldots,\ell_p$] feedback systems, where the subscript p denotes the numbers of disturbances. Based on the definition, algorithms will be derived to compute the integers $\ell_1,\ldots,\ell_p$ in the case where either open-loop transfer function matrix or its state equations are known.

3.1 Definition of a type [$\ell_1,\ldots,\ell_p$] system to disturbance inputs

In this section, the definition of a type [$\ell_1,\ldots,\ell_p$] system to disturbance inputs is given. To introduce the concept, consider the unity feedback single-input, single-output system,

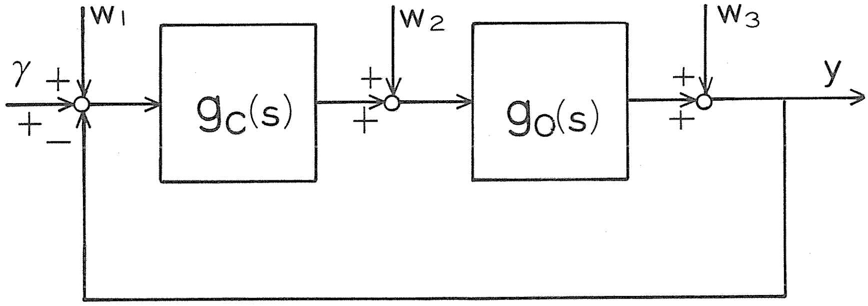

Fig. 3-1. Single-input, single-output feedback system with disturbances.

indicated in Fig. 3-1, whose closed-loop system is asymptotically stable and transfer functions $g_c(s)$, $g_o(s)$ can be written in the forms

$$g_c(s) = \frac{k \prod_{i=1}^{q} (s+z_i)}{s^l \prod_{i=1}^{n-l} (s+p_i)} \qquad g_o(s) = \frac{k' \prod_{i=1}^{q'} (s+z_i')}{s^{l'} \prod_{i=1}^{n'-l} (s+p_i')}$$

where $\ell, \ell' \geqq 0$ and $-z_i, -z_i'$ and $-p_i, -p_i'$ are the non-zero finite zeros and poles of gc(s), $g_0(s)$, respectively. In the classical control theory, this system is called a type 0 system, a type ℓ system or a type $(\ell+\ell')$ system to disturbance inputs w_1, w_2 or w_3, respectively. This statement can be expressed briefly by calling the system in Fig. 3-1 as a type $[0, \ell, \ell+\ell']$ system to the disturbance inputs w_1, w_2, w_3. Now in this way any m-inputs, m-outputs unity feedback system can be classified as follows:

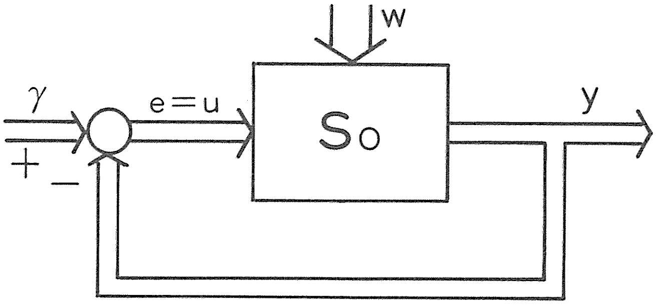

Fig. 3-2. m-input, m-output feedback system with p-dimensional disturbances.

Definition : Suppose that unity feedback m-inputs, m-outputs system, indicated in Fig. 3-2 is asymptotically stable and is subjected to disturbance inputs w $(=[w_1,\dots,w_p])$.
By assuming that

$$w(t) = w_i^j(t) \triangleq t^{j-1} U(t) \varepsilon_i, \tag{3-1}$$

where ϵ_i is the i-th unit vector, dim ϵ_i=dim w_i^j=dim w=p, U(t) the unit step functions, and $t^{-1} \equiv 0$, define the integers $\ell_1,\dots,\ell_p$ by the equations

$$l_i \triangleq \max \{ j : \lim_{t\to\infty} e[w_i^j(t)] = 0, \quad j = 0, 1, 2, \dots \}. \qquad i = 1, \dots, p \tag{3-2}$$

Then, the system is called a type $[\ell_1,\dots,\ell_p]$ feedback system to the vector disturbance w.

In the Definition, $e[w_i^j(t)]$ denotes the error that originates when the initially rest system of Fig. 3-2 is subjected to the disturbance $w_i^j(t)$.

Clearly from the definition, and from the superposition property of linear systems, we see that a type $[\ell_1,\dots,\ell_p]$ feedback system to the disturbance w can asymptotically reject disturbances of the form

$$w(t) = \left[\sum_{i=0}^{l_1-1} \alpha_{1j} t^j, \dots, \sum_{j=0}^{l_p-1} \alpha_{pj} t^j \right] \tag{3-3}$$

, where α_{ij} $(i = 1,\dots,p,\ j = 1,\dots,\ell_i-1)$ are arbitrary constants.

3.2 A type $[\ell_1,\dots,\ell_p]$ system to disturbances

In this section, methods for determining a type of a closed-loop system will be derived. The m-inputs, m-outputs unity feedback system, to be considered is indicated in Fig. 3-2, whose open-loop characteristic is represented by

$$\begin{cases} \dot{x} = Ax + Bu + Dw \\ y = Cx + Ew \end{cases} \tag{3-4}$$

where x is an n_I-vector describing the state of the system, u is an m-vector of control inputs, and w is a p-vector of unmeasurable disturbance inputs. A,B,C,D,E are matrices of appropriate dimensions. It is assumed that (A,B) are controllable, (A,C) are observable and that the closed-loop system is asymptotically stable. It is also assumed that the reference inputs γ are identically zero. The frequency domain structure of the system can be written in the form

$$y(s) = G_1(s)u(s) + [G_2(s) + E]$$

$$= C(sI-A)^{-1}Bu(s) + [C(sI-A)^{-1}D + E]w(s). \tag{3-5}$$

Now, let us consider the following question:
Given (3-4) or (3-5), how can the integers ℓ_i be computed?
To answer this, set

$$w(t) = \tilde{w}_i^j(t) = Gz(t) \tag{3-6a}$$

$$\dot{z}(t) = F z(t) \tag{3-6b}$$

where

$$G = \varepsilon_i \overbrace{[1\ 0\ \cdots\cdots\ 0]}^{j} \tag{3-7a}$$

$$F = \left.\begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ \vdots & & & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}\right\} j \tag{3-7b}$$

It is clearly seen that $\lim_{t\to\infty} e[\tilde{w}_i^j(t)] = 0$ if and only if $\lim_{t\to\infty} e\,[w_i^j(t)] = 0$. On noting this, we shall first derive the implicit condition that guarantees $\lim_{t\to\infty} e[w_i^j(t)] = 0$.
Substituting (3-6) into (3-4), we obtain

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A, & DG \\ 0, & F \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u \tag{3-8a}$$

$$y = [C,\ \ EG]\begin{pmatrix} x \\ z \end{pmatrix}. \tag{3-8b}$$

Define

$$\hat{A} = \begin{pmatrix} A, & DG \\ 0, & F \end{pmatrix}, \quad \hat{C} = [C,\ \ EG] \tag{3-9}$$

and

$$q = \operatorname{rank} \Delta^{n_1+j-1}[\hat{C},\ \ \hat{A}] \tag{3-10}$$

Then it is clearly seen that the integer q depends on the number j in (3-7), and that q satisfies

$$n_1 \leqq q \leqq n_1 + j \tag{3-11}$$

because by the assumption (A,C) are observable.
Lemma 3-1 : Necessary and sufficient condition that guarantees $\lim_{t\to\infty} e\,[w_i^j(t)] = 0$ is $q = n_1$.
Proof : Suppose $q > n_1$. Then the observable part of (3-8) can be written in the form

$$\dot{\eta} = \begin{pmatrix} A, & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix}\eta + \begin{pmatrix} B \\ 0 \end{pmatrix} u \tag{3-12a}$$

$$y = \bar{C}\eta \tag{3-12b}$$

where

$$\eta = \begin{bmatrix} I_{n_1}, & T_{12} \\ 0 \ , & T_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = T \begin{bmatrix} x \\ z \end{bmatrix},$$

$$\left\{ \begin{bmatrix} A, & \alpha_{12} \\ 0 \ , & \alpha_{22} \end{bmatrix}, \ \bar{C} \right\} \quad \text{are observable,}$$

$$\dim \ \eta = q.$$

and a matrix T satisfies the equation

$$T \begin{bmatrix} A, & DG \\ 0 \ , & F \end{bmatrix} = \begin{bmatrix} A, & \alpha_{12} \\ 0 & \alpha_{22} \end{bmatrix} T \tag{3-13}$$

From (3-13) and the form of T, α_{22} has unstable eigenvalues of F [15]. Therefore system (3-13) has unstable uncontrollable modes which are observable. These uncontrollable modes remain observable under the output feedback $u=-y$, so $e=-y\not\to 0$ as $t\to\infty$.

Now, suppose $q = n_1$. Then (3-12) becomes

$$\dot{\eta} = A\eta + Bu \tag{3-14a}$$

$$y = C\eta. \tag{3-14b}$$

By the control law $u=-y$ and from the assumed stability of the unity feedback system (3-4), it follows $e=-y\to 0$ as $t\to\infty$.

This completes the proof.

From this lemma, we can easily derive a computational algorithm of ℓ_i, for the case where open-loop system is described by (3-4).

Theorem 3-1 : Assume that the open-loop transfer characteristics of the system in Fig. 3-2 is described by (3-4). The system is of type $[\ell_1,\ldots,\ell_p]$ to the disturbance inputs w, in which ℓ_i are the integers determined by the equations

$$l_i = \max \{ j ; \ \text{rank} \ L_i^j = n_1 \} \tag{3-15a}$$

$$L_i^j = \begin{bmatrix} C & e_i & & & \\ CA & Cd_i & e_i & & \\ CA^2 & CAd_i & Cd_i & e_i \ddots & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & e_i \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots \\ CA^{n_1+j-1} & CA^{n_1+j-2}d_i & \cdots\cdots & & CA^{n_1-1}d_i \end{bmatrix} \tag{3-15b}$$

where

$$E = [e_1, \ldots, e_p], \quad D = [d_1, \ldots, d_p] \tag{3-16}$$

Note : In this chapter, e_i and d_i will be used to denote the columns of the corresponding matrices. In Chapter II, symbol d_i was used to denote the i-th row of D.

Proof : By substituting (3-7) and (3-9) into (3-10), and on noting

$$[1, 0 \cdots 0] F^k = \begin{cases} [0 \cdots 0 \ \overset{\overset{k+1}{\downarrow}}{1} \ 0 \cdots 0] & \text{if } 0 \leqq k \leqq j-1 \\ 0 & \text{if } j \leqq k \end{cases}$$

we obtain

$$
\begin{aligned}
q &= \text{rank}\begin{bmatrix} C & EG \\ CA & CDG+EGF \\ CA^2 & CADG+CDGF+EGF^2 \\ \cdots\cdots & \\ CA^{n_1+j-1} & \sum_{k=0}^{n_1+j-2} CA^{n_1+j-k-2}DGF^k+EGF^{n_1+j-1} \end{bmatrix} \\
&= \text{rank}\begin{bmatrix} C & e_i & 0 & 0 & \bigcirc \\ CA & Cd_i & e_i & 0 & \\ CA^2 & CAd_i & Cd_i & e_i & \\ \cdots\cdots & & & & \ddots \\ CA^{n_1+j-1} & CA^{n_1+j-2}d_i & \cdots & \cdots & e_i \end{bmatrix}\begin{bmatrix} I_n & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & I_j \\ \cdots & & \cdots \\ & \bigcirc & \end{bmatrix} \\
&= \text{rank } L_i^j
\end{aligned}
\tag{3-17}
$$

From this and Lemma 1, we obtain (3-15a).

Now, to consider the case where systems are described by transfer function matrices, we shall prove the following lemma:

Lemma 3-2 : The following three conditions are equivalent.

(i) $q = n_1$

(ii) There exists an $n_1 \times (n_1+j)$-matrix U which satisfies the equation

$$\hat{C}e^{\hat{A}t} = Ce^{At}U. \tag{3-18}$$

(iii) There exists an $n_1 \times j$-matrix $\tilde{U}$ which satisfies the equation

$$[C(sI-A)^{-1}D+E]G(sI-F)^{-1} = C(sI-A)^{-1}\tilde{U}. \tag{3-19}$$

Proof :

<(i) → (ii)> If (i) is true, there exists a matrix T such that

$$\hat{C} = CT, \quad T\hat{A} = AT,$$

(; see the proof of Lemma 1.)

Then the following equations are derived.:

$$
\begin{aligned}
\hat{C}e^{\hat{A}t} &= \hat{C}(I+\hat{A}t+\frac{1}{2!}\hat{A}^2t^2+\cdots) \\
&= C(I+At+\frac{1}{2!}A^2t^2+\cdots)\,T \\
&= Ce^{At}T
\end{aligned}
$$

<(ii) → (i)> Differentiate the both hand sides of (3-18) successively and put t=0, then

$$\hat{C}\hat{A}^i = CA^iU\,;\ i = 0, 1, 2, \cdots$$

$$
\text{Therefore, } n_1 \leqq q = \text{rank}\begin{bmatrix} \hat{C} \\ \hat{C}\ \hat{A} \\ \vdots \end{bmatrix}
= \text{rank}\begin{bmatrix} C \\ C\ A \\ \vdots \end{bmatrix} U
$$

$$\leqq \operatorname{rank} \begin{bmatrix} C \\ CA \\ \vdots \end{bmatrix}$$

$$= n_1$$

<(ii)→(iii)> Take the Laplace transform of (3-19) and rewrite $\hat{A}$, $\hat{C}$ using (3-9). Then the equivalence of (ii) and (iii) can easily be verified.

With the help of the two lemmas, the following theorem can be proved.

Theorem 3-2: Let $\ell_1,\ldots,\ell_p$ be the integers defined below ((3-20)-(3-24)). Then an m-inputs, m-outputs unity feedback system, shown in Fig. 3-2 is a type $[\ell_1,\ldots,\ell_p]$ system to disturbance inputs $w_1,\ldots,w_p$.

$$l_i = \max \{k_i;\ \operatorname{rank} \tilde{M} = \operatorname{rank} [\tilde{M} : \tilde{N}_{ik_i}] \tag{3-20}$$

where $\tilde{M}$, $\tilde{N}_{ik_i}$ are constant matrices defined from the open-loop transfer function matrices $G_1(s)$, $G_2(s)+E$ as follows:

$$G_1(s) = C(sI-A)^{-1}B$$

$$= \sum_{i=1}^{l} \frac{M_i}{s^i} + \tilde{G}_1(s) \tag{3-21}$$

$$G_2(s) = \left[\sum_{i=1}^{r_1} \frac{d_{ij}}{s^j} + \tilde{h}_1(s), \ldots, \sum_{j=1}^{r_p} \frac{d_{pj}}{s^j} + \tilde{h}_p(s) \right] \tag{3-21}$$

$$\tilde{M} = \begin{bmatrix} M_1 & M_2 & \cdots\cdots & M_l \\ M_2 & M_3 & \cdots & M_l \\ \vdots & & \cdot^{\cdot^{\cdot}} & \bigcirc \\ M_l & & & \end{bmatrix} \quad (m\, l \times m l) \tag{3-23}$$

$$\tilde{N}_{ik_i} = \begin{bmatrix} \tilde{h}_i(0) + e_i & \cdots & \tilde{h}_i^{(k_i-1)}(0)/(k_i-1)! \\ d_{i1} & \ddots & \vdots \\ \vdots & \ddots & \tilde{h}_i(0) + e_i \\ d_{ir_i} & \ddots & d_{i1} \\ 0 & \ddots & \vdots \\ \vdots & & d_{ir_i} \\ 0 & & 0 \end{bmatrix} \tag{3-24}$$

where $\tilde{G}_1(s)$, $\tilde{h}_j(s)$ have no poles at the origin.

Proof :

Using the definition of the type and lemma 3-2-(iii), it can be easily verified that ℓ_i is the maximal integer k_i such that there exists a vector v which satisfies the equation

$$[C(sI-A)^{-1}d_i + e_i]\frac{1}{s^{k_i}} = C(sI-A)^{-1}v. \tag{3-25}$$

In Appendix I it will be shown that the condition for the existence of a vector v in (3-25) is equivalent to the condition which guarantees the existence of a vector $\tilde{v}$ which satisfies the equation

$$\sum_{j=1}^{r_i} \frac{d_{ij}}{s^{j+k_i}} + \frac{e_i}{s^{k_i}} + \frac{\tilde{h}_i(0)}{s^{k_i}} + \cdots + \frac{\tilde{h}_i^{(k_i-1)}(0)}{(k_i-1)!\,s} \tag{3-26}$$

$$= \tilde{C}_1(sI-\tilde{A}_1)^{-1}\tilde{v}$$

where $\widetilde{A}_1$, $\widetilde{C}_1$ satisfy the following conditions,

(i) There exists a matrix $\widetilde{B}_1$ such that $(\widetilde{A}_1, \widetilde{B}_1)$ are controllable,

$$\text{(ii)} \quad \tilde{C}_1(sI-\tilde{A})^{-1}\tilde{B}_1 = \sum_{i=1}^{l} \frac{M_i}{s^i} \tag{3-27}$$

Now put

$$\tilde{B}_1 = \begin{bmatrix} I_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \tilde{A}_1 = \begin{bmatrix} 0 & & & \bigcirc \\ I_m & \ddots & & \\ & \ddots & \ddots & \\ \bigcirc & & I_m & 0 \end{bmatrix}$$

and

$$\tilde{C}_1 = [M_1, \; M_2, \; \ldots, \; M_l].$$

Then (3-26) becomes

$$\sum_{j=1}^{r_i} \frac{d_{ij}}{s^{j+k_i}} + \frac{e_1}{s^{k_i}} + \frac{\tilde{h}_1(0)}{s^{k_i}} + \cdots + \frac{\tilde{h}_1^{(k_i-1)}(0)}{(k_1-1)!\,s}$$

$$= [M_1, \ldots, M_l] \begin{bmatrix} I_m/s & & & \bigcirc \\ I_m/s^2 & \ddots & & \\ \vdots & & \ddots & \\ I_m/s^l & \cdots & & I_m/s \end{bmatrix} \tilde{v}. \tag{3-28}$$

On comparing both sides of this equation, and from the remark given at the beginning of the proof we have (3-20).

3.3 Relation to a type $[k_1,\ldots,k_p]$ transfer element

In this section the results obtained in Section 3.2 will be applied to a feedback system shown in Fig. 3-3, which is a special case of Fig. 3-2, and a much simpler criterion which resembles the one for a single-input, single-output system will be developed.

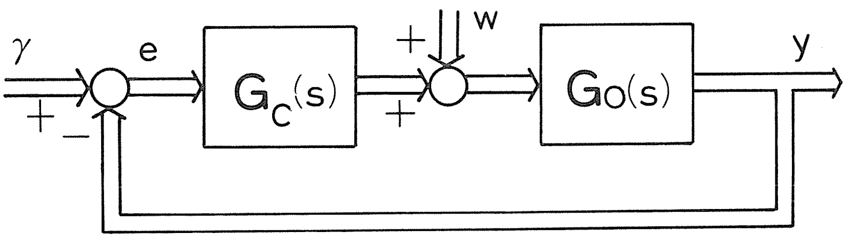

Fig. 3-3. m-input, m-output feedback system with simple structure with respect to disturbances.

Consider the m-inputs, m-outputs unity feedback system, indicated in Fig. 3-3, whose closed-loop system is asymptotically stable. It is supposed that tandem system of Go(s) followed by Gc(s) maintains controllability and observability.

The minimal realizations of Gc(s), Go(s) are written in the form

$$G_c(s): \dot{x}_c = A_c x_c + B_c u_c \tag{3-29a}$$

$$y_c = C_c x_c \tag{3-29b}$$

$$G_0(s): \dot{x}_0 = A_0 x_0 + B_0 u_0 \tag{3-30a}$$

$$y_0 = C_0 x_0 + D_0 u_0 \tag{3-30b}$$

Comparing these with (3-4), we get

$$A = \begin{pmatrix} A_c & 0 \\ B_0 C_c, & A_o \end{pmatrix}, \quad B = \begin{pmatrix} B_c \\ 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ B_o \end{pmatrix}$$

$$C = (D_0 C_c, \ C_0), \quad E = D_0$$

By these relations, the condition (iii) of Lemma 3-2 is rewritten as follows;

Condition (iv): There exist matrices $\widetilde{U}_1$, $\widetilde{U}_2$ such that

$$G_0(s) G(sI-F)^{-1}$$
$$= G_0(s) C_c(sI-A_c)^{-1} \tilde{U}_1 + C_0(sI-A_0)^{-1} \tilde{U}_1. \tag{3-32}$$

Then it is clear that the following condition (v) is a sufficient condition to condition (iv).

Condition (v) : There exists a matrix $\widetilde{U}_1$ such that

$$G(sI-F)^{-1} = C_c(sI-A_c)^{-1} \tilde{U}_1. \tag{3-33}$$

Then the following theorem is obtained.

Theorem 3-3: In the m-inputs, m-outputs unity feedback system indicated in Fig. 3-3, if Gc(s) is a type $[\ell_1,\ldots,\ell_p]$ transfer element, the system is at least a type $[\ell_1,\ldots,\ell_p]$ system to the disturbance input w.

Moreover if p=m, and if $G_0^{-1}(s)$ exists and $G_0(s)$ has no zeros at the origin, then the system is a type $[\ell_1,\ldots,\ell_m]$ system to the disturbance input w.

Proof :

Following the development given in the proof of Lemma 2-1, and using the definition of a type $[\ell_1,\ldots,\ell_p]$ t.e., it is easily shown that if Gc(s) is a type $[\ell_1,\ldots,\ell_p]$ transfer element, condition (v) holds for $k_i = \ell_i$ (i=1,...,p).

Then the first half statement of the theorem is obviously true.

If $G_0^{-1}(s)$ exists, (3-32) can be rewritten in the form

$$G(sI-F)^{-1}$$
$$= C_c(sI-A_c)^{-1}\tilde{U}_1 + G_0^{-1}(s) C_0(sI-A_0)^{-1} \tilde{U}_2.$$

From Appendix II, $G_0^{-1}(s)C_0(sI-A_0)^{-1}$ has no poles at the origin. Therefore condition (v) is a necessary and sufficient condition to condition (iv). This completes the proof.

Remark : We can apply Theorem 3-2 to the synthesis problem as follows:

(i) Given transfer function matrices $G_1(s)$, $G_2(s)$ indicated in Fig. 3-4, where $G_2(s)$ is a type $[\ell_1,\ldots,\ell_p]$ transfer element.

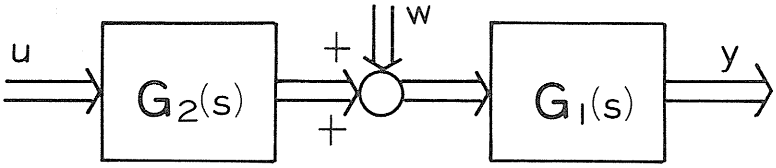

Fig. 3-4. Plant with disturbances.

(ii) It is required to design a tandem connection compensator $G_3(s)$, indicated in Fig. 3-5, such that the obtained closed-loop system becomes at least a type $[\ell_1{}^0,...,\ell_p{}^0]$ system to the disturbance input w.

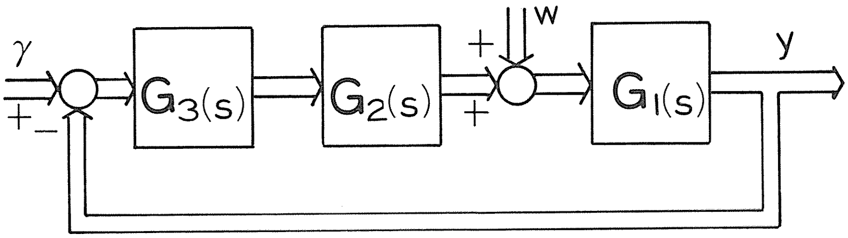

Fig. 3-5. Synthesis of a type $[\ell_1,\cdots,\ell_p]$ system to the distarbances W.

Using the method obtained by in Chapter II, we can easily design a transfer function matrix $G_3(s)$ such that $G_2(s)\cdot G_3(s)$ is a type $[\ell_1{}^c,...,\ell_p{}^c]$ transfer element, where $\ell_i{}^c \geq \ell_i{}^0$ for i=1,...,p, and that the closed-loop system indicated in Fig. 3-5 is asymptotically stable.

3.4 An Example

Let us consider the steady state characteristic of the two-inputs, two-outputs unity feedback stable system, indicated in Fig. 3-6, to the disturbance inputs w_1, w_2. The frequency domain structure of the system can be written in the form

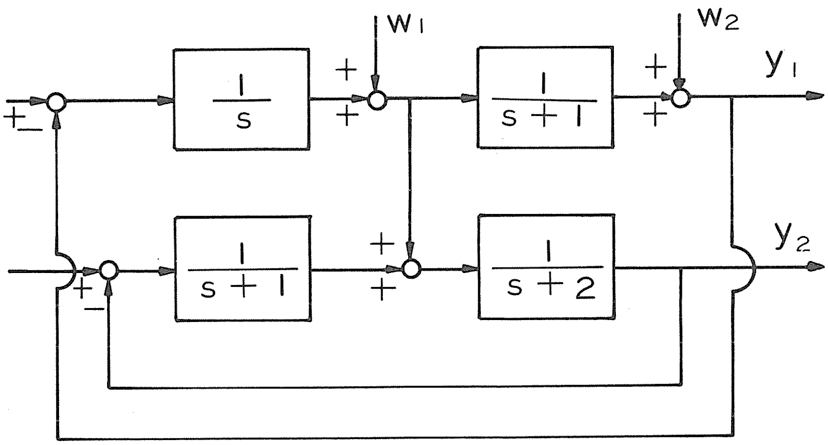

Fig. 3-6. Example.

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s(s+1)}, & 0 \\ \dfrac{2}{s(s+2)}, & \dfrac{1}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix} + \begin{bmatrix} \dfrac{1}{s+1}, & 1 \\ \dfrac{1}{s+2}, & 0 \end{bmatrix} \begin{bmatrix} w_1(s) \\ w_2(s) \end{bmatrix}$$

Matrices M, $\tilde{N}_{11}$, $\tilde{N}_{21}$ defined in theorem 3-2 can be written in the form

$$M = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}, \quad \tilde{N}_{11} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}, \quad \tilde{N}_{21} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore the system is a type [1, 0] system to the disturbance inputs w_1, w_2.

Appendix I

It can be assumed that A is in Jordan form. Only a special case where A has only two Jordan blocks is proved. General cases can be treated similarly. So, set

$$A = \begin{bmatrix} J_1, & 0 \\ 0, & J_2 \end{bmatrix}$$

where

$$J_1 = \underbrace{\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}}_{p_1} \qquad J_2 = \underbrace{\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix}}_{p_2}$$

Corresponding to the division of A, matrices and vectors C, B, di, v in (3-25) are divided as follow;

$$C = [C_1, \ C_2], \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$d_i = \begin{bmatrix} \tilde{d}_i \\ \bar{d}_i \end{bmatrix} \qquad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

From (3-5) and (3-22), we obtain

$$C_1(sI - J_1)^{-1}\tilde{d}_i = \sum_{j=1}^{r_1} \frac{d_{ij}}{s^j}$$

$$C_2(sI - J_2)^{-1}\tilde{d}_i = \tilde{h}_i(s)$$

$$C_1(sI - J_1)^{-1}B_1 = \sum_{j=1}^{l} \frac{M_i}{s^i} \tag{A-1}$$

Then by substituting these equations, (3-25) can be rewritten as follows:

$$\sum_{j=1}^{r_1} \frac{d_{ij}}{s^{j+k_i}} + \frac{e_i}{s^{k_i}} + \frac{\tilde{h}_i(0)}{s^{k_i}} + \cdots + \frac{\tilde{h}_i^{(k_i-1)}(0)}{(k_i-1)!\,s} + h_i(s) = C_1(sI-J_1)^{-1}v_1 + C_2(sI-J_2)^{-1}v_2. \tag{A-2}$$

where $h_i(s) = \dfrac{\tilde{h}_i(s)}{s^{k_i}} - \dfrac{\tilde{h}_i(0)}{s^{k_i}} - \cdots - \dfrac{\tilde{h}_1^{(k_i-1)}(0)}{(k_i-1)!\,s}$ has no poles at the origin.

On minding that the j-th column of $(1/s^{k_i})\,(sI-J_2)^{-1}$ can be written in the form

$$\frac{1}{s^{k_i}}\begin{pmatrix} \frac{1}{s-\lambda} & \cdots & \frac{1}{(s-\lambda)^{p_2}} \\ & \ddots & \\ \bigcirc & & \frac{1}{s-\lambda} \end{pmatrix}\begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{s-\lambda} & \cdots & \frac{1}{(s-\lambda)^{p_2}} \\ & \ddots & \vdots \\ \bigcirc & & \frac{1}{s-\lambda} \end{pmatrix}\begin{pmatrix} \frac{1}{(j-1)!}\left(\frac{1}{s^{k_i}}\right)^{(j-1)}_{s=\lambda} \\ \vdots \\ \left(\frac{1}{s^{k_i}}\right)^{(0)}_{s=\lambda} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

it is easily seen that a vector d exists such that

$$h_i(s) = C_2(sI-J_2)^{-1}d.$$

Put v_2=d in Eq. (A-2). Then Eq. (A-2) can be rewritten in the form

$$\sum_{j=1}^{r_i} \frac{d_{ij}}{s^{j+k_i}} + \frac{c_i}{s^{k_i}} + \frac{\tilde{h}_i(0)}{s^{k_i}} + \cdots + \frac{\tilde{h}_i^{(k_i-1)}(0)}{(k_i-1)!\,s} = C_1(sI-J_1)^{-1}v_1.$$

$$\sum_{j=1}^{r_i} \frac{d_{ij}}{s^{j+k_i}} + \frac{c_i}{s^{k_i}} + \frac{\tilde{h}_i(0)}{s^{k_i}} + \cdots + \frac{\tilde{h}_i^{(k_i-1)}(0)}{(k_i-1)!\,s} = c_1(sI-J_1)^{-1}v_1. \tag{A-3}$$

With the help of the following lemma, we can put

$J_1 = \widetilde{A}_1$, $C_1 = \widetilde{C}_1$ in (A-3). This completes the proof.

Lemma (Simple proof is omitted):

If (A_1, B_1, C_1), (A_2, B_2, C_2) are controllable realization of the same transfer function, that is

$C_1(sI - A_1)^{-1}B_1 = C_2(sI - A_2)^{-1}B_2$, then there always exist matrices V_1 , V_2 such that

$$C_1(sI-A_1)^{-1} = C_2(sI-A_2)^{-1}V_1$$
$$C_2(sI-A_2)^{-1} = C_2(sI-A_1)^{-1}V_2\,.$$

Appendix II

It will be shown here that if $G_0(s)$ has no zeros at the origin, $G_0^{-1}(s)C_0(sI-A_0)^{-1}$ has no poles at the origin. To prove this, the system matrix introduced by Rosenbrock [11] will be used.

Let $[\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}(s)]$ be a minimal realization of $G_0^{-1}(s)$ $(=\tilde{C}(sI-\tilde{A})^{-1}\tilde{B}+\tilde{D}(s))$,where $\tilde{D}(s)$ is a polynomial matrix. Then $\tilde{A}$ has no eigenvalues at the origin by the assumption that $G_0(s)$ has no zeros at the origin (see also Rosenbrock [11]). It is clear that the following system matrix

$$\left(\begin{array}{cc:c} sI-\tilde{A}, & -\tilde{B}C_0 & \tilde{B}D_0 \\ 0, & sI-A_0 & B_0 \\ \hdashline -\tilde{C}, & -\tilde{D}(s)C_0 & \tilde{D}(s)D_0 \end{array}\right) \qquad \text{(A-4)}$$

gives rise to a transfer function matrix $G_0^{-1}(s)\,G_0(s)$ $(=I_m)$. Multiply this system matrix by a unimodular polynomial matrix of the form

$$\left(\begin{array}{cc:c} I & & 0 \\ \hdashline 0 & M(s) & I_m \end{array}\right)$$

where M(s) is a polynomial matrix such that

$$M(s)(sI-A_0)-\tilde{D}(s)C_0=-\bar{C}_0$$

and $\bar{C}_0$ is an appropriate constant matrix.

The system matrix (A-4) now has the form

$$\left(\begin{array}{cc:c} sI-\tilde{A}, & -\tilde{B}C_0 & \tilde{B}D_0 \\ 0, & sI-A_0 & B_0 \\ \hdashline -\tilde{C}, & -\bar{C}_0 & M(s)B_0+\tilde{D}(s)D_0 \end{array}\right) \qquad \text{(A-6)}$$

and gives rise to the same transfer function matrix I_m.

Then it is apparent that the system matrix

$$\left(\begin{array}{cc:c} sI-\tilde{A}, & -\tilde{B}C_0 & \tilde{B}D_0 \\ 0, & sI-A_0 & B_0 \\ \hdashline -\tilde{C}, & -\bar{C}_0 & 0 \end{array}\right) \qquad \text{(A-7)}$$

gives rise to a transfer function matrix O_m.

Since the system matrix (A-7) is in state-space form, there exists a nonsingular matrix H_1 such that

$$\begin{pmatrix} H_1 & 0 \\ 0 & I_m \end{pmatrix}\left(\begin{array}{cc:c} sI-\tilde{A}, & -\tilde{B}C_0 & \tilde{B}D_0 \\ 0, & sI-A_0 & B_0 \\ \hdashline -\tilde{C}, & -\bar{C}_0 & 0 \end{array}\right)\begin{pmatrix} H_1^{-1} & 0 \\ 0 & I_m \end{pmatrix}$$

$$= \left[\begin{array}{cc:c} sI-\hat{A}, & 0 & 0 \\ *, & sI-\hat{A}_0 & \hat{B}_0 \\ \hdashline -\hat{C}, & 0 & 0 \end{array}\right] \tag{A-8}$$

where $(\hat{A}, \hat{C})$ are observable, $(\hat{A}_0, \hat{B}_0)$ are controllable and $\hat{A}$ has no eigenvalues at the origin.

On the other hand $G_0^{-1}(s)C_0(sI - A_0)^{-1}$ is obtained by a system matrix

$$\left(\begin{array}{cc:c} sI-\tilde{A}, & -\tilde{B}C_0 & 0 \\ 0 & , sI-A_0 & I \\ \hdashline -\tilde{C}, & -\tilde{D}(s)C_0 & 0 \end{array}\right)$$

The similar transformations that were applied to the system matrix (A-4) reduce the system matrix (A-8) to the form

$$\left(\begin{array}{cc:c} sI-\hat{A}, & 0 & * \\ *, & sI-\hat{A}_0 & * \\ \hdashline -\hat{C}, & 0 & M(s) \end{array}\right)$$

Since $\hat{A}$ has no eigenvalues at the origin, $G_0^{-1}(s)C_0(sI - A_0)^{-1}$ has no poles at the origin.

Chapter IV Zeroing the Output by State Feedback

In the previous two chapters, servomechanism problems have been discussed for the case that reference inputs and disturbances inputs are described as polynomial functions of t. The results obtained should be most practical in engineering applications, since, in engineerings, it is quite common that actual inputs are approximately represented as low-order polynomials in time over sufficiently short intervals of time. However, some practical needs [17] and also theoretical interest (in greater part) propose the following generalization of the problem:

Let ψ be a linear time-invariant system representation described by the equations

$$\psi: \quad \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

where x(t) is an n-vector, u(t) an r-vector, y(t) an m-vector and A, B, C, and D are respectively constant matrices of dimension $n \times n$, $n \times r$, $m \times n$ and $m \times r$.

It is required to find a state feedback such that $y(t) \to 0$ as $t \to \infty$. The result to be obtained has applications to the problem of tracking command signals and rejecting disturbances. An example is the following : It is desired to make the output of a controlled system S_1, which is given by the equations

$$S_1: \quad \begin{aligned} \dot{\xi}(t) &= R\xi(t) + S\eta(t) \\ \mu(t) &= T\xi(t) + U\eta(t), \end{aligned} \tag{4-2}$$

track the output of a command system S_2, which is given by the equations

$$S_2: \quad \begin{aligned} \dot{\zeta}(t) &= G\zeta(t) \\ \rho(t) &= H\zeta(t). \end{aligned}$$

This problem is equivalent to the problem of stabilizing the output of composite system S_3, which is represented by the equations

$$S_3:\quad \begin{bmatrix} \dot{\xi}(t) \\ \zeta(t) \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} \xi(t) \\ \zeta(t) \end{bmatrix} + \begin{bmatrix} S \\ 0 \end{bmatrix} \mu(t)$$

$$[\mu(t) - \rho(t)] = [\,T - H\,] \begin{bmatrix} \xi(t) \\ \zeta(t) \end{bmatrix} + U\eta(t)\ .$$

S. P. Battacharyya and J. B. Pearson [5] geometrically solved this problem when D=0. L. M. Silverman and H. J. Payne [16] discussed the conditions to maintain the output of a linear system at zero. In this chapter, first, the unobservable subspace of a linear system is considered. Using the result, the problem of stabilizing the output is solved. A condition for the system ψ to be stabilizable is algebraically derived, because of convenience for applications.

4.1 Unobservable Subspace

Applying a state feedback

$$u(t) = Fx(t) + v(t) \tag{4-3}$$

to system ψ, where v(t) is an r-vector, ψ becomes ψ' which is represented by the equations

$$\psi':\quad \begin{aligned} \dot{x}(t) &= (A+BF)x(t) + v(t) \\ y(t) &= (C+DF)\,x(t) + v(t)\,. \end{aligned}$$

Then the unobservable subspace of linear system ψ is defined as follows:

Definition:

The vector space ν_F defined by

$$\nu_F \triangleq \{x;\ (C+DF)(A+BF)^i x = 0 \quad i = 0, 1, 2, \ldots, (n-1)\} \tag{4-4}$$

is called the unobservable subspace of system ψ (under the state feedback u = Fx+v).

In general, the unobservable subspace ν_F changes by selections of F. Moreover, the matrix F which gives a fixed unobservable subspace is not uniquely determined. So, the class of matrices which give an unobservable subspace ν will be denoted by $F(\nu)$. Then the following theorem is established.

Theorem 4-1:

System ψ has always the maximal unobservable subspace. That is to say, for some F_0 and any F, unobservable subspaces satisfy following relations:

$$\nu_F \subseteq \nu_{F_0} \triangleq \nu_{max} \tag{4-5}$$

Moreover, $\nu_{max} = N(L_\beta)$ and $-\bar{D}_a^{+}\,\bar{C}_a$ belongs to $F(\nu_{max})$, where $N(L_\beta)$ is the null space of L_β, $\bar{D}_a^{+}$ is the generalized inverse of $\bar{D}_a$, L_β, $\bar{D}_a$ and $\bar{C}_a$ are the matrices that are defined in reference [16].

To prove this theorem the following lemmas are needed.

Lemma 4-1:

When a system is given by equations

$$\dot{x}(t) = Ax(t)\ , \quad x(0) = x_0\ ,$$
$$y(t) = Cx(t)\ ,$$

the output y(t) is identically zero on $[\,0, \infty)$ if and only if the initial state x_0 belongs to the unobservable subspace ν_0 of the system.

Proof is omitted for easiness.

Lemma 4-2: (Silverman and Payne)

System ψ has the following properties.:

(i) There exists an input u(t) such that y(t)=0 on $[0, \infty)$ if and only if x_0 is in $N(L_\beta)$.

(ii) Let x_0 be in $N(L_\beta)$. Then y(t)=0 on $[0, \infty)$ if and only if u(t) can be expressed as the output of the system

$$\dot{z}(t) = (A - B\bar{D}_a^+ \bar{C}_a) z(t) + BKv(t) \tag{4-6a}$$

$$u(t) = -\bar{D}_a^+ \bar{C}_a z(t) + Kv(t) \tag{4-6b}$$

for some v(t) and z(0)=x_0, where K is a matrix whose columns form a basis for the null space of $\bar{D}a$.

Proof of Theorem 4-1.

First, $\mathcal{V}_{-\bar{D}_a^+ \bar{C}_a} = N(L_\beta)$ will be proved. Suppose

$$x_0 \in \mathcal{V}_{-\bar{D}_a^+ \bar{C}_a}$$

Applying the state feedback

$$u(t) = -\bar{D}_a^+ \bar{C}_a x(t) \tag{4-7}$$

to system ψ, ψ becomes as

$$\dot{x}(t) = (A - B\bar{D}_a^+ \bar{C}_a) x(t), \quad x(0) = x_0, \tag{4-8a}$$

$$y(t) = (C - D\bar{D}_a^+ \bar{C}_a) x(t). \tag{4-8b}$$

So, by lemma 4-1,

$$y(t) = 0 \text{ on } [0, \infty]. \tag{4-9}$$

Therefore from lemma 4-2-(i)

$$x_0 \in N(L_\beta).$$

Conversely, suppose the relation (4-9) holds. Set v(t)=0 on $[0, \infty)$ in equations (4-6), it follows

$$\dot{z}(t) = (A - B\bar{D}_a^+ \bar{C}_a) z(t), \quad z(0) = x_0 \tag{4-10a}$$

$$u(t) = -\bar{D}_a \bar{C}_a^+ z(t). \tag{4-10b}$$

Using the output of system (4-10) as the input of system ψ, it follows by lemma 4-2 ii) that

$$y(t) = 0 \text{ on } [0, \infty)$$

Comparing (4-10) with (4-8), it is seen that both x(t) and z(t) obey to the same equation and that the initial states are the same, so

$$x(t) = z(t) \text{ on } [0, \infty)$$

This means that u(t) determined by (4-10b) is just the state feedback (4-7). Therefore, from lemma 4-1.

$$x_0 \in \mathcal{V}_{-\bar{D}_a^+ \bar{C}_a}$$

Next, $\mathcal{V}_{-\bar{D}_a^+ \bar{C}_a} = \mathcal{V}_{max}$ will be proved by contradiction.
Suppose $\mathcal{V}_{-\bar{D}_a^+ \bar{C}_a}$ is not the maximal unobservable subspace.
Then there exists an n-vector x_0 such that

$$x_0 \notin \mathcal{V}_{-\bar{D}_a^+ \bar{C}_a} \tag{4-11}$$

$$x_0 \in \mathcal{V}_F \tag{4-12}$$

for some F. Applying the state feedback

$$u(t) = Fx(t)$$

to system ψ, it follows from lemma 4-1 that

$$y(t) = 0 \text{ on } [0, \infty).$$

Therefore, from lemma 4-2-(i),

$$x_0 \in N(L_\beta) \tag{4-13}$$

Since $N(L_\beta)$ is equal to $\nu_{-\overline{D}_a^+\overline{C}_a}$ the relation (4-13) contradicts to (4-11). Concludingly, it has been proved that there exists the maximal unobservable subspace and it equals to $\nu_{-\overline{D}_a^+\overline{C}_a}$ It is clear from the above arguments that $-\overline{D}_a^+\overline{C}_a \epsilon F(\nu_{max})$.
The proof is complete.

4.2 Stabilizing the Output

Choose $F_0 \epsilon F(\nu_{max})$ and write $A_0 = A+BF_0$ and $C_0 = C+DF_0$.
Under the state feedback

$$u(t) = F_0 x(t) + v(t), \tag{4-14}$$

system ψ becomes system ψ^0 written by the equations

$$\psi^0 : \begin{array}{ll} \dot{x}(t) = A_0 x(t) + Bv(t), \quad x(0) = x_0 & \text{(4-15a)} \\ y(t) = C_0 x(t) + Dv(t) & \text{(4-15b)} \end{array}$$

where v(t) is an r-vector. Let system ψ^0 be transformed to the observable canonical form. By a transformation matrix T, the following system $\psi^{1,2}$ is obtained.

$$Tx(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\psi^{1,2} : \begin{array}{ll} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} v(t) & \text{(4-16a)} \\ y(t) = [C_1 \ \ 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + Dv(t) & \text{(4-16b)} \end{array}$$

Under this transformation, coefficient matrices satisfy the following relations

$$TA_0T^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad TB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{4-17}$$

$$C_0T^{-1} = [C_1 \ \ 0].$$

Corresponding to the representation (4-16), the state of the system $\psi^{1,2}$ can be expressed as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2(t) \end{bmatrix}. \tag{4-18}$$

The components $x_1(t)$ and $x_2(t)$, respectively, represent the observable and the unobservable component. Then it is clear from the definition of system $\psi^{1,2}$ that the vectors set

$$\left\{ \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \right\}$$

is equal to the maximal unobservable subspace of the system $\psi^{1,2}$, namely, $T\mathcal{V}_{max}$. Therefore if new state feedback $v(t)=G[x_1'(t)x_2'(t)]'$ is applied to the system $\psi^{1,2}$, where G is any $r\times n$ matrix, the observable component will be enlarged in general. From system $\psi^{1,2}$, a completely observable system ψ^1 is obtained as

$$\psi^1: \quad \begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + B_1v(t) && (4\text{-}19a) \\ y(t) &= C_1x_1(t) + Dv(t). && (4\text{-}19b) \end{aligned}$$

Then the following theorem can be concluded about the output stabilizing problem.

Theorem 4-2:

Whatever the initial state is, the output of the system ψ is stabilizable by a state feedback if and only if the state of the system ψ^1 is stabilizable by a state feedback, that is to say, there exists some F_{11} such that the real parts of all eigenvalues of $(A_{11}+B_1F_{11})$ are negative.

Remark:

Note that the stabilizability of the state of the system does not depend on the special selection of the transformation matrix T. (See (4-17))

To prove the Theorem 4-2 the following lemma is needed.

Lemma 4-3:

When a system is given by the equations

$$\dot{x}(t) = Ax(t)$$
$$y(t) = Cx(t),$$

the output tends to zero if and only if the observable component of the state tends to zero.

Proof is omitted for easiness.

Proof of Theorem 4-2

Let the state of the system ψ be stabilizable by a state feedback

$$v(t) = F_{11}x_1(t).$$

Then, whatever the initial state is, the output of the system $\psi^{1,2}$ is stabilizable by a state feedback

$$v(t) = [F_{11}\ 0]\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

This shows that the output of system ψ is stabilizable by a state feedback

$$u(t) = (F_0 + [F_{11}\ 0]T)x(t),$$

whatever the initial state is. So sufficiency was proved.

Now, suppose the pair (A_{11}, B_1) is not stabilizable. Then, for some initial state x_{10}, the state of the system ψ^1 never tends to zero by any state feedback and therefore by any input. So it follows that the observable component of the state of the system $\psi^{1,2}$ never tends to zero under any state feedback when the initial state is $[x_{10}'\ 0']$, since the state feedback to the system $\psi^{1,2}$ results in enlarging the observable component as previously stated. By lemma 4-3, the output of the system $\psi^{1,2}$ does not tend to zero, too. So, the output of the system ψ is not stabilizable by any state feedback when the initial state is

$$T^{-1}\begin{bmatrix} x_{10} \\ 0 \end{bmatrix}$$

The necessity was proved.

Theorem 4-2 gives the condition under which the output of the system is stabilizable by a state feedback. But, in many cases, it is not sufficient only to assure the stability of the output. It may be practically necessary that the output of the system is stabilizable at optional speed. Then the following theorem can be stated by similar way to Theorem 4-3.

Theorem 4-3:

Whatever the initial state is, the output of the system ψ is stabilizable by a state feedback at any speed if and only if the system ψ^1 is completely controllable.

Proof is omitted.

Chapter V Conclusion and Acknowledgement

The results obtained in this article constitute an extension of the conventional servomechanism theory that utilize type ℓ system concept to multivariable problems. Specifically, on the basis of the new concepts (; that is, the type $[\ell_1,...,\ell_m]$ feedback system concept, and the type $[\ell_1,...,\ell_m]$ transfer element concept), we have developed a procedure for synthesizing linear time-invariant multivariable servomechanism systems, and a method of finding the types of unity feedback systems in the case where either open-loop transfer function matrices or their state representations are known. In the both, type $[\ell_1,...,\ell_m]$ transfer element concept has been proved particularly useful. Further, in this article, a more general servomechanism problem has been discussed by formulating the problem as the one of zeroing the outputs of linear time-invariant systems. A necessary and sufficient condition for the existence of a state feedback that assures convergence of outputs has been derived.

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