A FEW CONSIDERATIONS ON THE TRAILING VORTEX APPEARING IN THE AXISYMMETRIC THEORY AND THE SECONDARY FLOW THEORY *

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(Received May 2, 1974)

Summary

It is shown that a same equation on the vorticity in flow is derived from both axisymmetric theory and secondary flow theory, but the value of trailing vortex or streamwise vorticity cannot be obtained from this equation. The condition which decides the strength of trailing vortex in axisymmetric theory is expressed by the flow equation downstream of blade row. In the case of secondary flow theory, the strength of trailing vortex is decided from the boundary condition of the Trefftz plane at the exit of blade row. The latter coincide with the former at the limit of pitch $\rightarrow 0$.

In the case of axisymmetric flow, provided the exit angle of blade row being of free vortex type, we get the result that the streamwise vorticity is zero in the downstream of blade row. This means that there exists no secondary flow in the downstream. The same thing is applied on the straight cascade. Because this result seems to be curious, verifications were tried from other angles.

1. Introduction

Axisymmetric theory (when the axial length of blade row is infinitesimally small, actuator disc theory) and secondary flow theory are two representative theories to approach the three-dimensional flow in axial-flow turbomachinery. The fomer is understood to be a treatment of the case in which the tangential variation of three-dimensional flow is ignored or the number of blades is infinite, and the latter is thought to be a method to deal with the flow in blade passage and the change of outlet flow caused by it. To clarify the connection of the both was the main object of this report, but it became clear in the study that the particular examination of the trailing vortex was the most important. Because results were so unsuspected and considerations were concentrated on this point, the report was entitled as shown.

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* The 26th Report of the Study on Axial-flow Turbomachines

In axisymmetric theory or actuator disc theory it is a problem how much trailing vortices are contained in the outlet flow of blade row, and a few answers were presented,[3],[9],[10] but we could not find any one which was theoretically satisfactory, and we must notice that we cannot get the trailing vortex from an ordinary handling of the axisymmetric equation.

On the other hand the character of trailing vortex is known pretty well in the secondary flow theory.[4],[5] Therefore, it is generally supposed that if we bring this theory into infinitesimally small pitch form we can get the answer to the above problem. Taking an opinion that to have infinitesimally small pitch is same as to take the averaged value along the tangential direction and making such trials, we find, to our surprise, the trailing vortex disappeared. In fine we cannot get the trailing vortex.

Special considerations are needed to get the trailing vortex, and circumstances not to be able to get the trailing vortex and ways to get it are discussed in this report, together with considerations on the relation between axisymmetric theory and secondary flow theory.

Only the flow in the stationary blade row is treated in the report. Although treatments about the moving blade row may be possible by uses of the idea of relative energy[2],[6] (in axisymmetric theory) and the consideration of secondary flow in moving blade, they are problems to be solved in the future. Assumptions of inviscid and incompressible flows are also employed.

Notation and Co-ordinate

Co-ordinates and components of velocity and vorticity are defined as shown in Fig. 1.

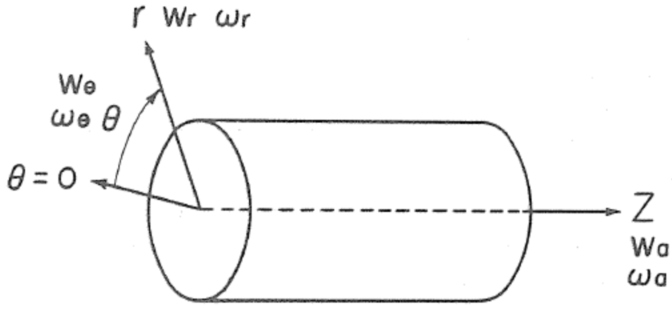

Fig. 1 Co-ordinate

a	pitch of stagnation stream lines
a'	distance of stagnation stream lines or wakes (see Fig. 3)
B	a half of blade span
g	gravity acceleration
H	total enthalpy
n	normal co-ordinate to meridional stream line
O[ ]	order

r	radial co-ordinate
V	absolute velocity
w	velocity component
x	co-ordinate in Trefftz plane
y	co-ordinate in Trefftz plane
Z	axial co-ordinate
Γ	circulation
γ	angle of stream line in axisymmetric stream surface (see Fig. 3)
δ	boundary layer (shear flow layer) thickness (see Fig. 4)
η	co-ordinate in Trefftz plane
θ	tangential co-ordinate
κ	inclination angle of vortex line (see Fig. 3)
ξ	co-ordinate in Trefftz plane
ρ	density
τ	normal co-ordinate to the stream line in axisymmetric stream surface
ψ	stream function in Trefftz plane at the blade row exit
ω	vorticity

Subscripts

1	before blade row
2	behind blade row
a	axial
e	at exit of blade row
m	meridional
p	passage vortex
s	along stream line
T	trailing vortex
t	trailing edge
T(F)	trailing filament vortex
T(S)	trailing shed vortex
θ	tangential
τ	normal to stream line in axisymmetric stream surface

2. Axisymmetric Theory

Wislicenus published a theory easy to understand on the axisymmetric theory,[2][6] and we follow it.

Let us consider two closely adjacent meridional streamlines as shown in Fig. 2, and let the total enthalpies along these two streamlines be H and H + dH. Then we have

$$dH=\left(\frac{\partial H}{\partial n}\right)_1 dn_1=\left(\frac{\partial H}{\partial n}\right)_2 dn_2=\text{constant} \qquad (2-1)$$

Therefore

$$\left(\frac{\partial H}{\partial n}\right)_2=\left(\frac{\partial H}{\partial n}\right)_1\frac{dn_1}{dn_2} \qquad (2-2)$$

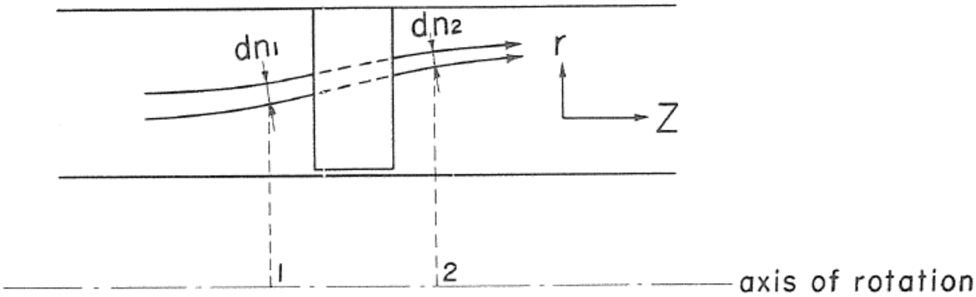

Fig. 2. The blade system and meridional stream lines

From the condition of continuity, we get

$$\frac{r_1 W_{m1}}{r_2 W_{m2}} = \frac{\rho_2 dn_2}{\rho_1 dn_1} \tag{2-3}$$

Next relationship was also got by Wislicenus.[2],[6]

$$g\frac{\partial H}{\partial n} = |\overleftarrow{V} \times \overleftarrow{\omega}| \tag{2-4}$$

This is valid at both stations 1 and 2.

From equations (2–2), (2–3) and (2–4)

$$\frac{|\overleftarrow{V}_1 \times \overleftarrow{\omega}_1|}{\rho_1 r_1 W_{m1}} = \frac{|\overleftarrow{V}_2 \times \overleftarrow{\omega}_2|}{\rho_2 r_2 W_{m2}} \tag{2-5}$$

or

$$\frac{W_{\theta 1}\omega_{m1} - W_{m1}\omega_{\theta 1}}{\rho_1 r_1 W_{m1}} = \frac{W_{\theta 2}\omega_{m2} - W_{m2}\omega_{\theta 2}}{\rho_2 r_2 W_{m2}} \tag{2-5'}$$

This is the equation correlating conditions before and behind the blade row in the axisymmetric theory.

But, we must be careful about the fact that we cannot get the component of ω_2 parallel to V_2 from equation (2–5) or (2–5'). This is because the vector product of two vectors which are parallel to each other is zero. Therefore, the vortex shed from a system such as the trailing vortex cannot be obtained from this axisymmetric equation.

For the convenience of comparison with the secondary flow theory, we take the incompressible flow assumption and a case in which the meridional streamline can be regarded to be parallel to the axis. From (2–5') we have

$$\frac{W_{\theta 1}\omega_{a1} - W_{a1}\omega_{\theta 1}}{r_1 W_{a1}} = \frac{W_{\theta 2}\omega_{a2} - W_{a2}\omega_{\theta 2}}{r_2 W_{a2}}$$

and using next relations

$$\frac{W_{\theta 1}}{W_{a1}} = -\tan\gamma_1, \quad \frac{W_{\theta 2}}{W_{a2}} = -\tan\gamma_2 \tag{2-6}$$

we get the following equation after a readjustment.

$$\frac{\omega_{\theta 1}}{r_1}+\frac{\omega_{a1}}{r_1}\tan\gamma_1=\frac{\omega_{\theta 2}}{r_2}+\frac{\omega_{a2}}{r_2}\tan\gamma_2 \tag{2-7}$$

3. Secondary Flow Theory

The flow pattern of blade system is illustrated in Fig. 3. This is the development of the flow shown in Fig. 2 along the stream line. Following the ordinary secondary flow theory, we start with assumptions that the vorticity in flow being small and the streamline distortion being also small.

Vortices in the flow are known to be consisted of [3],[4],[5]

(1) passage vortex (vorticity) which is the result of the vorticity ω_1 in the upstream being changed to ω_{2p} after passing through the blade passage,

(2) trailing filament vortex which is produced by ω_1 being cut by the blade profile, and

(3) trailing shed vortex which is produced by the spanwise (r–wise) variation of circulation of blade.

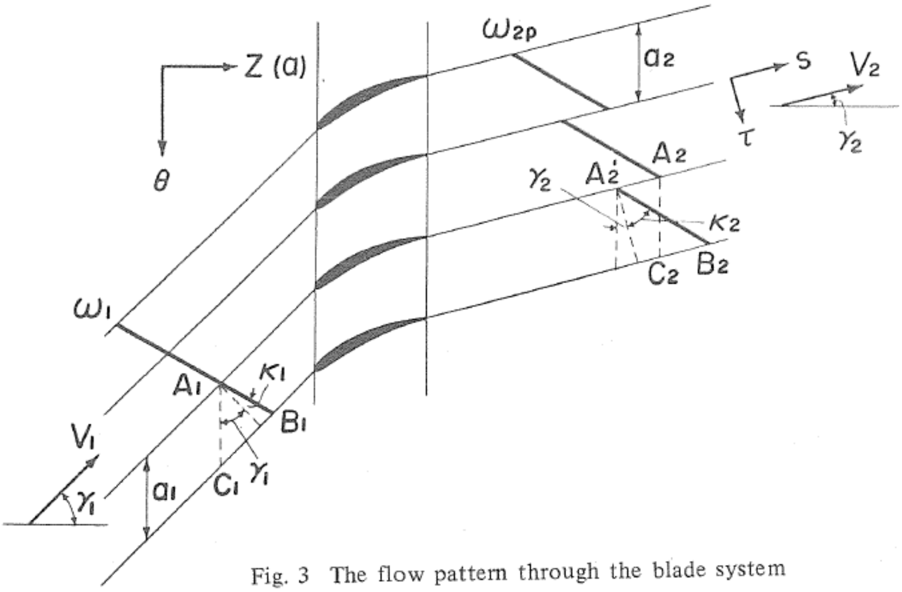

Fig. 3 The flow pattern through the blade system

3.1 Passage Vortex

Expressing components of ω_1 and ω_{2p} parallel and normal to the stream by suffixes s and τ, we have from reference (7) or (8),

$$\frac{\omega_{2p\tau}}{\omega_{1\tau}}=\frac{\cos\gamma_2}{\cos\gamma_1}\cdot\frac{r_2}{r_1} \tag{3-1}$$

This can be easily obtained from the Helmholtz's vortex law expressing that the vortex drifts with the flow. And we have

$$\frac{\omega_{2PS}}{\omega_{1\tau}}=\frac{\cos\gamma_2}{\cos\gamma_1}\cdot\frac{r_2}{r_1}\cdot\tan\kappa_2 \tag{3-2}$$

κ_2 can be obtained if we know the flow in blade passage or the time difference needed by a particle on a stagnation stream line to pass through blade upper or lower surfaces. But since this is not an urgent matter to be known, we leave it intact.

3.2 Trailing Filament Vortex

Denoting the strength of trailing filament vortex by $\Gamma_{T(F)}$ (the strength of trailing vortex contained in unit span),

$$\Gamma_{T(F)}=-\omega_{2P\tau}\cdot\overline{A_2'A_2} \tag{3-3}$$

where $A_2'A_2$ is the distance related to time difference needed by a particle on a stagnation stream line to pass through blade upper and lower surfaces and indicated in Fig. 3. Because times needed by particles to pass C_1B_1 and C_2B_2 are same, we get

$$\frac{\overline{C_1B_1}}{V_1}=\frac{\overline{C_2B_2}}{V_2} \tag{3-4}$$

And

$$\overline{C_1B_1}=a_1\sin\gamma_1+a_1\cos\gamma_1\cdot\tan\kappa_1 \tag{3-5}$$

Condition of continuity is

$$V_1\cos\gamma_1\cdot a_1\cdot dr_1=V_2\cos\gamma_2\cdot a_2\cdot dr_2 \tag{3-6}$$

or

$$\frac{w_{a1}}{w_{a2}}=\frac{r_2dr_2}{r_1dr_1} \tag{3-6'}$$

And

$$\overline{A_2'A_2}=a_2\sin\gamma_2+a_2\cos\gamma_2\cdot\tan\kappa_2-\overline{C_2B_2} \tag{3-7}$$

Using equations (3–4), (3–5) and (3–6) we have

$$A_2'A_2=a_1\left[\frac{r_2}{r_1}\sin\gamma_2+\frac{r_2}{r_1}\cos\gamma_2\cdot\tan\kappa_2-\frac{\sin\gamma_1\cdot\cos\gamma_1}{\cos\gamma_2}\frac{r_1dr_1}{r_2dr_2}-\frac{\cos^2\gamma_1}{\cos\gamma_2}\tan\kappa_1\frac{r_1dr_1}{r_2dr_2}\right] \tag{3-8}$$

From this equation and (3–1), (3–2) and (3–3) $\Gamma_{T(F)}$ becomes

$$\Gamma_{T(F)}=-\omega_{1\tau}a_2\cos\gamma_2\left[\frac{\sin\gamma_2}{\cos\gamma_1}\frac{r_2}{r_1}-\frac{\sin\gamma_1}{\cos\gamma_2}\frac{r_1dr_1}{r_2dr_2}+\frac{\omega_{2PS}}{\omega_{1\tau}}-\frac{\omega_{1S}}{\omega_{1\tau}}\frac{\cos\gamma_1}{\cos\gamma_2}\frac{r_1dr_1}{r_2dr_2}\right] \tag{3-9}$$

The passage vortex (vorticity) is a distributed vortex in the flow and the trailing vortex is a vortex existing in the wake of blade. But, if we consider from axisymmetrical view points, the trailing vortex must be considered to be distributed in θ (or τ) direction. Let $\omega_{2T(F)}$ be the vorticity of distributed $\Gamma_{T(F)}$, then

$$\omega_{2T(F)}=\frac{\Gamma_{T(F)}}{a_2\cos\gamma_2} \tag{3-10}$$

Using (3–9) we get

$$\omega_{2T(F)}=-\omega_{1\tau}\left[\frac{\sin\gamma_2}{\cos\gamma_1}\frac{r_2}{r_1}-\frac{\sin\gamma_1}{\cos\gamma_2}\frac{r_1dr_1}{r_2dr_2}\right]+\omega_{1s}\frac{\cos\gamma_1}{\cos\gamma_2}\frac{r_1dr_1}{r_2dr_2}-\omega_{2Ps} \tag{3-11}$$

3.3 Trailing Shed Vortex

Denoting the strength of trailing shed vortex by $\Gamma_{T(S)}$ and the blade circulation by Γ, we have

$$\Gamma_{T(S)}=\frac{d\Gamma}{dn} \tag{3-12}$$

Let us consider this vortex to be distributed like the trailing filament vortex, then its vorticity $\omega_{2T(S)}$ becomes

$$\omega_{2T(S)}=\frac{\Gamma_{T(S)}}{a_2\cos\gamma_2} \tag{3-13}$$

3.4 Vorticity in the Downstream

We express the vorticity in the downstream of blade system by ω_2, then we have

$$\omega_{2s}=\omega_{2Ps}+\omega_{2T(F)}+\omega_{2T(S)} \tag{3-14}$$

and using (3–11)

$$\omega_{2s}=\omega_{1\tau}\left[\frac{\sin\gamma_1}{\cos\gamma_2}\frac{r_1dr_1}{r_2dr_2}-\frac{\sin\gamma_2}{\cos\gamma_1}\frac{r_2}{r_1}\right]+\omega_{1s}\frac{\cos\gamma_1}{\cos\gamma_2}\frac{r_1dr_1}{r_2dr_2}+\omega_{2T(S)} \tag{3-15}$$

It is worth noticing that κ_2 is not contained in this expression.

Because the trailing vortex has no component normal to the stream, $\omega_{2\tau}$ becomes same as τ component of the passage vortex. We get from (3–1)

$$\omega_{2\tau}=\omega_{2P\tau}=\omega_{1\tau}\frac{\cos\gamma_2}{\cos\gamma_1}\frac{r_2}{r_1} \tag{3-16}$$

In the case of axial-flow machine, expressions using ω_a and ω_θ are more convenient than ω_s and ω_τ. Relations which are

$$\left.\begin{aligned} \omega_{1s} &= \omega_{a1}\cos\gamma_1 - \omega_{\theta 1}\sin\gamma_1 \\ \omega_{1\tau} &= \omega_{a1}\sin\gamma_1 + \omega_{\theta 1}\cos\gamma_1 \end{aligned}\right\} \tag{3–17}$$

and

$$\left.\begin{aligned} \omega_{a2} &= \omega_{2s}\cos\gamma_2 + \omega_{2\tau}\sin\gamma_2 \\ \omega_{\theta 2} &= -\omega_{2s}\sin\gamma_2 + \omega_{2\tau}\cos\gamma_2 \end{aligned}\right\} \tag{3–19}$$

are used together with equation (3–6') to rewrite equations (3–15) and (3–16), then we get

$$\omega_{a2} = \omega_{a1}\frac{W_{a2}}{W_{21}} + \omega_{2T(S)}\cos\gamma_2 \tag{3–19}$$

$$\omega_{\theta 2} = \omega_{a1}\left[\frac{r_2}{r_1}\tan\gamma_1 - \frac{W_{a2}}{W_{a1}}\tan\gamma_2\right] + \omega_{\theta 1}\frac{r_2}{r_1} - \omega_{2T(S)}\sin\gamma_2 \tag{3–20}$$

Eliminating w_{a2}/w_{a1} from (3–19) and (3–20), we get

$$\frac{\omega_{\theta 1}}{r_1} + \frac{\omega_{a1}}{r_1}\tan\gamma_1 = \frac{\omega_{\theta 2}}{r_2} + \frac{\omega_{a2}}{r_2}\tan\gamma_2 \tag{3–21}$$

The trailing shed vortex has disappeared in the process of introduction of this equation. We can understand from this situation that the trailing shed vortex cannot be obtained from the secondary flow solution alone.

4. Interim Considerations

We can find a perfect coincidence of equations (2–7) and (3–21). And it was shown in the course of derivations that the strength of trailing shed vortex could not be determined.

Although the coincidence of results from axisymmetric theory and secondary flow theory is natural because they start from the same physical law, it may be worth noticing that thinking about the assumptions of distributing trailing vortex in θ–direction or taking the averaged value in θ–direction in the secondary flow theory without assuming the infinitesimally small blade pitch we can get the coincidence of results of both theories.

In the next place it is an important matter that we could not get the strength of trailing vortex in the above analyses. The trailing vortex are decided in the real turbo-machine provided the geometry of blade row and the inlet flow are given. It was hoped, therefore, to get it by the analysis we did, but it was not so. Let us examine its reason in following paragraphs.

The comparison in the above was limited to the case in which meridional stream lines could be considered being parallel to the axis, but the extension of results to the case being not parallel is hoped to be possible.

5. Determination of the Trailing Vortex

It is an idea of secondary flow theory that the strength of total trailing vortex is decided by spanwise velocities induced by the passage vortex at walls of passage (upper and lower surfaces of blades). In the case of axisymmetric solution, the author will show that this is determined from the condition at blade row exit (exit flow direction).

Vortices in the flow just behind the blade row are consisted of passage vorticity ω_{2p} and trailing vortex Γ_T ($\Gamma_T = \Gamma_{T(F)} + \Gamma_{T(S)}$) (if we consider the trailing vortex being distributed, it is trailing vorticity ω_{2T}), and the direction of flow is equal to the exit direction of blade row and given by γ_{2e}. Vorticities parallel to the flow (s-direction) and normal to the flow and the span (τ-direction) are respectively.

$$\omega_{2s} = \omega_{2Ps} + \omega_{2T} \tag{5-1}$$

$$\omega_{2\tau} = \omega_{2P\tau} \tag{5-2}$$

For the sake simplifying explanation, let us consider a case in which radial component of velocity can be neglected, then we have

$$\left.\begin{aligned} \omega_s &= \omega_a \cos\gamma - \omega_\theta \sin\gamma \\ \omega_\tau &= \omega_a \sin\gamma + \omega_\theta \cos\gamma \end{aligned}\right\} \tag{5-3}$$

$$\left.\begin{aligned} \omega_a &= \frac{\partial w_\theta}{\partial r} + \frac{w_\theta}{r} \\ \omega_\theta &= -\frac{\partial w_a}{\partial r} \end{aligned}\right\} \tag{5-4}$$

$$\left.\begin{aligned} w_a &= V\cos\gamma \\ w_\theta &= -V\sin\gamma \end{aligned}\right\} \tag{5-5}$$

From these we get following equations just behind the exit of blade row,

$$\omega_{2s} = -V_{2e}\frac{\partial \gamma_{2e}}{\partial r} - \frac{V_{2e}}{r_2}\cos\gamma_{2e}\cdot\sin\gamma_{2e} \tag{5-6}$$

$$\omega_{2\tau} = -\frac{\partial V_{2e}}{\partial r_2} - \frac{V_{2e}}{r_2}\sin^2\gamma_{2e} \tag{5-7}$$

(where subscript e which must be attached to ω was omitted.) From equations (5–1), (5–2), (5–6) and (5–7) we get

$$-V_{2e}\frac{\partial \gamma_{2e}}{\partial r_2} - \frac{V_{2e}}{r_2}\cos\gamma_{2e}\cdot\sin\gamma_{2e} = \omega_{2Ps} + \omega_{2T} \tag{5-8}$$

$$-\frac{\partial V_{2e}}{\partial r_2} - \frac{V_{2e}}{r_2}\sin^2\gamma_{2e} = \omega_{2P\tau} \tag{5-9}$$

Since according to the assumption mentioned above we can consider $r_2 \fallingdotseq r_1$, we can get $\omega_{2p\tau}$ from equation (3–1) and V_{2e} *from equation* (5–9). We can have, therefore, $\omega_{2pS} + \omega_{2T}$ from equation (5–8). If we want to get ω_{2pS} and ω_{2T} separately, we get at first ω_{2pS} with the aid of the knowledge of flow in blade passage and then get ω_{2T}, but provided that we know $\omega_{2pS} + \omega_{2T}$ we can get the vorticity components just behind blade row from equations

$$\left.\begin{aligned} \omega_{a2} &= (\omega_{2PS} + \omega_{2T})\cos\gamma_{2e} + \omega_{2PT}\sin\gamma_{2e} \\ \omega_{\theta 2} &= -(\omega_{2PS} + \omega_{2T})\sin\gamma_{2e} + \omega_{2PT}\cos\gamma_{2e} \end{aligned}\right\} \quad (5-10)$$

If there exists an assumption that the distortion of flow is small, these can be regarded immediately as the vorticity components in the downstream.

Thus we have been able to get the vorticity (or trailing vortex) in the downstream by the consideration of conditions just behind the exit of blade row.

5.1 Special Cases

As a special case, let us consider a case in which γ_{2e} is of free vortex type i.e.

$$\tan\gamma_{2e} = \frac{K}{r_2} \quad (5-11)$$

where K is a constant.
Substituting this into (5−8) we have

$$O = \omega_{2PS} + \omega_{2T} \quad (5-12)$$

Of course same result is also obtained in the case of straight cascade. Accordingly, we recognize that in axisymmetric condition the free vortex type of blading or the straight cascade has no streamwise vorticity or so-called secondary flow in the downstream of blade row. This result coincide with the result obtained by Preston.[3] The author wants to suggest to say this as the vortex rectification of cascade.

5.2 The case of Cascade of Infinitesimal Blade Pitch

We have found that there exists no secondary vorticity in the downstream of blade row, provided the exit flow angle is of free vortex type equation (5−11) . Is it possible to say the same thing when we consider the secondary flow theory? Let us examine it here about the straight cascade having infinitesimally small blade pitch.

When the pitch of blade becomes infinitesimally small, the flow component normal to blade surfaces in blade passage and at the exit of blade row (trailing edge) is negligiblly small (this component is finite when the pitch is finite). Namely, we can consider the secondary velocity normal to the main flow and the span being zero. We may be able to think, therefore, that the flow angle γ_{2e} just behind the blade row exit is same as the exit angle of blade row (and γ_{2e} may be taken as same as the exit flow angle of two-dimensional cascade of blade row....... but we need further sufficient considerations to say this, and this is not the problem of the present report).

As aforesaid, we take the idea that the strength of total trailing vortex is decided by spanwise velocities induced by the passage vortex at upper and lower surfaces of blades. Let us take the straight cascade to simplify the consideration, and note the Trefftz plane at the exit which is illustrated as shown in Fig. 4(a).

ω_{2pS} can be generally regarded being constant in y-direction and the function of x, but let us consider it having constant value ω in the range of δ as illustrated

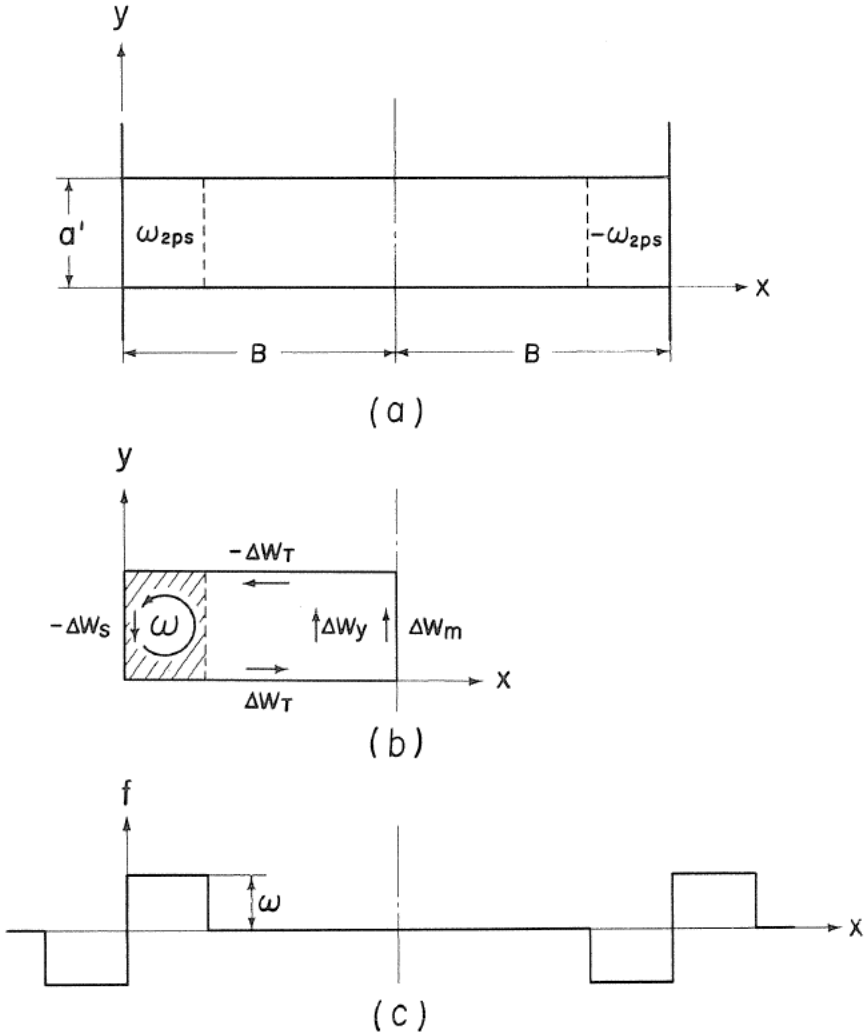

Fig. 4 The vorticity in rectangle

in Fig. 4 (b) and (c). When ω_{2pS} is a function of x, we can think it being combinations of cases in which ω and δ take varieties of values, and, therefore, the idea of constant ω has no fear that it lose generality.

x and y components of velocity in this rectangle can be expressed, using the stream function ψ, as follows,[1]

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\omega \tag{5-13}$$

$$\triangle w_x=-\frac{\partial\psi}{\partial y} \tag{5-14}$$

$$\triangle W_y = \frac{\partial \psi}{\partial x} \tag{5-15}$$

When the flow in straight cascade is bilaterally symmetrical along the span, we can treat the problem with a half of rectangle as illustrated in Fig. 4 (b). The problem is to solve the equation (5−13) under the boundary condition that ψ=0 at four sides of the rectangle. The solution has already obtained. [see the Appendix] Δw_T which corresponds to the trailing vortex is,

$$\triangle w_T = -\left(\frac{\partial \psi}{\partial y}\right)_{y=0}$$

$$= \frac{2B}{\pi^2}\omega \sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right]\cdot\tanh\left(\frac{n\pi a'}{2\,B}\right) \tag{5-16}$$

In the case of infinitesimal pitch *i.e.* a' → o we have,

$$\triangle w_T \fallingdotseq a'\omega\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right] \tag{5-17}$$

On the other hand, $\omega \sim x$ relation shown in Fig. 4(c) can be expressed by the Fourier's series,[12] as

$$f = \omega\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right] \tag{5-18}$$

We have finally

$$\left.\begin{aligned}\triangle w_T &= \frac{a'}{2}f = \frac{a'\omega}{2} \quad (0<x<\delta)\\ &= 0 \qquad\qquad (\delta<x\leqq B)\end{aligned}\right\} \tag{5-19}$$

The strength of trailing vortex in unit span is

$$\Gamma_T = -2\Delta w_T$$

and if we regard this being distributed along pitch direction, we get

$$\left.\begin{aligned}\omega_{2T} &= \frac{\Gamma_T}{a'} = -\omega_{2PS} \quad (0<x<\delta)\\ &= o \qquad\qquad (\delta<x\leqq B)\end{aligned}\right\} \tag{5-20}$$

where

$$\omega_{2PS} = \omega$$

We have, therefore, the following relation

$$\omega_{2PS} + \omega_{2T} = 0 \tag{5-21}$$

which is quite same as equation (5−12).

Another method of verification to attain to the same result is explained in the Appendix.

The treatment mentioned in the above was on the straight cascade of infinitesimal pitch, but it is unquestionable that same results are expected to be obtained on the stationary blade row of axial-flow machine.

6. Conclusions

We have found that the same equation on flow vorticity is obtained by both the axisymmetric theory and the secondary flow theory. But we can not get the streamwise vorticity from this equation.

It is the equation of flow in the downstream of blade row which determine the strength of trailing vortex in the axisymmetric theory, and the strength is calculated from the flow condition just behind the exit of blade row.

In the secondary flow theory, it is the boundary condition of Trefftz plane at the blade row exit which determine the strength of trailing vortex, and it coincides with the result of axisymmetric theory when we consider the limiting case of pitch $\rightarrow$ 0 on the straight cascade.

In the axisymmetric theory, if the exit angle of blade row is of free vortex type, we get the result that the streamwise vorticity in the downstream of blade row is zero, that is, there exists no secondary flow. The situation is same on the straight cascade. A same result is obtained from the secondary flow theory in the limiting case of pitch $\rightarrow$ 0 on straight cascade. Because of the curiosity of the result, the author has tried another proof from different angle in the Appendix.

7. References

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[There exists a misunderstanding in this report in expressing that the secondary vorticity is the difference of actual vorticity and axisymmetric vorticity. The difference must be the trailing vorticity as can be clarified from the present papers. The axisymmetric vorticity expressed in this report differs from the vorticity appeared in the axisymmetric theory of the present papers. This corresponds to the quasi vortex explained in the reference (7). Same pointing out can be seen in the reference (9).]

(6) Wislicenus, G.F., Fluid Mechanics of Turbomachinery. Dover Publications, Inc., New York, 1965

(7) Otsuka, S., A Theory about the Secondary Flow in Cascades. Proc. 6th Japan National Congress for Applied Mechanics, P. 327 1956
[A conclusion stated in this report that the blade circulation does not change by the secondary flow is applicable only to the case without trailing shed

circulation.]
(8) Otsuka, S., A Theory about the Secondary Flow in Axial-Flow Turbo-Machinery. Proc. 9th Japan National Congress for Applied Mechanics, P. 241 1959
[Same defect exists as the reference (7)]
(9) Hawthorne, W.R. and Novak, R.A., The Aerodynamics of Turbo-Machinery. Annual Review of Fluid Mechanics, Vol. 1 P. 341 1969
(10) Horlock, J.H. and Marsh, H., The Use of Averaged Flow Equations of Motion in Turbomachinery Aerodynamics. The Second International JSME Symposium Fluid Machinery and Fluidics, Tokyo, Sept. 1972
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8. Appendix

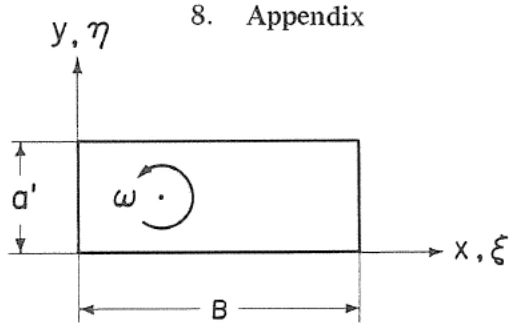

Fig. A-1. The vorticity in rectangle

The flow in a rectangle in which the vortex exists as illustrated in Fig. A-1 can be expressed by the stream function as follows,[11]

$$\psi = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(n\pi\frac{x}{B}\right)}{n\sinh\left(n\pi\frac{a'}{B}\right)}\int_0^B \sin\left(n\pi\frac{\xi}{B}\right)d\xi$$
$$\times\left\{\int_0^y \sinh\left(n\pi\frac{\eta}{B}\right)\cdot\sinh\left(n\pi\frac{a'-y}{B}\right)\cdot\omega(\xi,\eta)d\eta\right.$$
$$\left.+\int_y^{a'}\sinh\left(n\pi\frac{y}{B}\right)\cdot\sinh\left(n\pi\frac{a'-\eta}{B}\right)\cdot\omega(\xi,\eta)d\eta\right\}$$

When ω is constant in y-direction and the function of x alone, we have

$$\psi = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(n\pi\frac{x}{B}\right)}{n\sinh\left(n\pi\frac{a'}{B}\right)}\int_0^B \omega(\xi)\sin\left(n\pi\frac{\xi}{B}\right)d\xi$$
$$\times\left\{\sinh\left(n\pi\frac{a'-y}{B}\right)\int_0^y \sinh\left(n\pi\frac{\xi}{B}\right)d\eta\right.$$

$$+ \sinh\left(n\pi\frac{y}{B}\right)\int_{y}^{a'} \sinh\left(n\pi\frac{a'-\eta}{B}\right)d\eta \quad \Big\}$$

Integrations in $\Big\{ \quad \Big\}$ can be done easily, and in the condition shown in Fig. 4, namely,

$$\omega(\xi) = \omega \qquad 0<\xi<\delta$$

$$\omega(\xi) = 0 \qquad \delta<\xi\leqq B$$

we have the integration on ξ as

$$\int_0^B \omega(\xi)\sin\left(n\pi\frac{\xi}{B}\right)d\xi = \omega\int_0^{\sigma} \sin\left(n\pi\frac{\xi}{B}\right)d\xi$$

$$= \omega\frac{B}{n\pi}\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right]$$

Therefore, we have after calculations

$$\psi = -\frac{2B^2}{\pi^3}\omega\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right]$$

$$\times\left[1-\frac{\sinh\left(n\pi\frac{y}{B}\right)+\sinh\left(n\pi\frac{a'-y}{B}\right)}{\sinh\left(n\pi\frac{a'}{B}\right)}\right] \qquad \text{(A-1)}$$

The spanwise velocity Δw_T at the blade trailing edge, which corresponds to the trailing vortex, is

$$\Delta w_T = -\left(\frac{\partial\psi}{\partial y}\right)_{y=0} = \frac{2B}{\pi^2}\omega\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(n\pi\frac{x}{B}\right)$$

$$\times\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right]\cdot\tanh\left(\frac{n\pi}{2}\frac{a'}{B}\right) \qquad \text{(A-2)}$$

And y-component of induced velocity Δw_y at any point is

$$\Delta w_y = \frac{\partial\psi}{\partial x}$$

$$= -\frac{2B}{\pi^2}\omega\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right]$$

$$\times\left[1-\frac{\sinh\left(n\pi\frac{y}{B}\right)+\sinh\left(n\pi\frac{a'-y}{B}\right)}{\sinh\left(n\pi\frac{a'}{B}\right)}\right] \qquad \text{(A-3)}$$

Let us now examine the circulation along the path ABCDA in Fig. A–2,

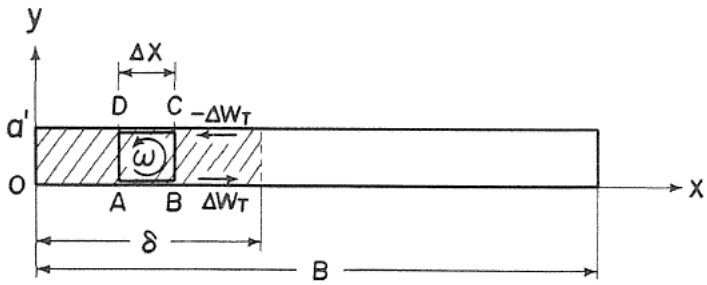

Fig. A-2. The vorticity in rectangle

$$2\Delta w_T \cdot \Delta x + \int_0^{a'} \Delta w_y(x+\Delta x)dy + \int_{a'}^{0} \Delta w_y(x)dy$$

$$= \omega \cdot a' \cdot \Delta x \qquad \text{(A–4)}$$

and consider the limiting case of a' → o. The first term becomes from equation (A–2)

$$\lim_{a'\to 0} \Delta w_T \doteqdot \frac{2B}{\pi^2}\omega \sum_{n=1}^{\infty} \frac{1}{n^2}\sin\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right]$$

$$\times \frac{n\pi}{2}\frac{a'}{B} = 0\left[\frac{a'}{B}\right] \qquad \text{(A–5)}$$

Concerning second and third terms, the inside of [ ] of equation (A–3) becomes

$$\left[1-\frac{\sin h\left(n\pi\frac{y}{B}\right)+\sin h\left(n\pi\frac{a'-y}{B}\right)}{\sin h\left(n\pi\frac{a'}{B}\right)}\right] = \left[1-\frac{\cos h\left\{\frac{n\pi}{B}\left(y-\frac{a'}{2}\right)\right\}}{\cos h\left(n\pi\frac{a'}{2B}\right)}\right]$$

$$= \left[\frac{2\sin h\left(\frac{n\pi}{B}\frac{y}{2}\right)\cdot \sinh\left(\frac{n\pi}{B}\frac{a'-y}{2}\right)}{\cos h\left(n\pi\frac{a'}{2B}\right)}\right] \doteqdot \left[2\sin h\left(\frac{n\pi}{B}\frac{y}{2}\right)\cdot\sin h\left(\frac{n\pi}{B}\frac{a'-y}{2}\right)\right]$$

$$\doteqdot 2\frac{n\pi}{B}\frac{y}{2}\cdot\frac{n\pi}{B}\frac{a'-y}{2} < \frac{1}{2}\frac{n^2\pi^2}{B^2}a'\cdot a' = \frac{1}{2}\frac{n^2\pi^2}{B^2}a'^2$$

where we note that $0<y<a'$
Therefore, we have

$$\lim_{a'\to o} |\Delta w_y| < \left(\frac{a'}{B}\right)^2 \omega B \sum_{n=1}^{\infty} \cos\left(n\pi\frac{x}{B}\right)\cdot\left[1-\cos\left(n\pi\frac{\delta}{B}\right)\right]$$

$$= 0\left[\left(\frac{a'}{B}\right)^2\right] \qquad \text{(A–6)}$$

Now (A–4) becomes

$$2\Delta w_T \cdot \Delta x = \omega \cdot a' \cdot \Delta x$$

or

$$\Delta w_T = \frac{a'\omega}{2} \tag{A–7}$$

This is quite same as equation (5–19)