

# MULTIFRACTAL FORMALISM FOR GENERALISED LOCAL DIMENSION SPECTRA OF GIBBS MEASURES ON THE REAL LINE

JOHANNES JAERISCH

*Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya, 464-8602 Japan*

HIROKI SUMI

*Course of Mathematical Science, Department of Human Coexistence, Graduate School of Human and Environmental Studies, Kyoto University Yoshida-nihonmatsu-cho, Sakyo-ku, Kyoto 606-8501, Japan*

ABSTRACT. We extend the multifractal formalism for the local dimension spectrum of a Gibbs measure  $\mu$  supported on the attractor  $\Lambda$  of a conformal iterated functions system on the real line. Namely, for  $\alpha \in \mathbb{R}$ , we establish the multifractal formalism for the Hausdorff dimension of the set of  $x \in \Lambda$  for which the  $\mu$ -measure of a ball of radius  $r_n$  centred at  $x$  obeys a power law  $r_n^\alpha$ , for a sequence  $r_n \rightarrow 0$ . This allows us to investigate the Hölder regularity of various fractal functions, such as distribution functions and conjugacy maps associated with conformal iterated function systems.

---

*E-mail addresses:* jaerisch@math.nagoya-u.ac.jp (corresponding author), sumi@math.h.kyoto-u.ac.jp.

*Key words and phrases.* Multifractal formalism, conformal iterated function systems, fractal functions, local dimension spectrum.  
15th October 2019.

## 1. INTRODUCTION

Multifractal analysis has its origin in statistical physics (see [Man74, FP85, HJK<sup>+</sup>86]). For the mathematical theory of multifractal formalism and its relation to thermodynamic formalism, we recommend [Pes97]. In this article we extend the multifractal formalism for local dimension spectra of Gibbs measures supported on the attractor  $\Lambda \subset \mathbb{R}$  of a conformal iterated function system on the real line. For a Borel measure  $\mu$  on  $\mathbb{R}$  and with  $B(x, r) := \{y \in \mathbb{R} \mid |y - x| < r\}$ , the local dimension spectrum of  $\mu$  is given by the level sets

$$\mathcal{F}(\alpha) := \left\{ x \in \Lambda \mid \lim_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right\}, \quad \alpha \in \mathbb{R}.$$

It is well known ([Pat97, PW97a, PW97b, Pes97]) for finitely generated contracting conformal iterated function systems satisfying the open set condition that, if  $\mu$  is the Gibbs measure of a Hölder continuous potential, then the dimension spectrum  $f$  given by

$$f(\alpha) := \dim_H \mathcal{F}(\alpha)$$

is equal to the Legendre transformation of a certain function  $t$  defined implicitly by solving a topological pressure equation. To denote this state of affairs, we say that the multifractal formalism holds. In our setting, the function  $t$  will be given by equation (2.1) below. For related results on the multifractal formalism for non-uniformly hyperbolic systems and graph-directed constructions, we refer to [JR11, RU09].

For self-conformal measures of finitely generated contracting conformal iterated function systems it is also known ([BS00]) that the set of divergence points

$$\left\{ x \in \Lambda \mid \liminf_{r \rightarrow \infty} \frac{\log \mu(B(x, r))}{\log r} < \limsup_{r \rightarrow \infty} \frac{\log \mu(B(x, r))}{\log r} \right\}$$

has full Hausdorff dimension, unless the the spectrum  $f$  is degenerate. In [LWX12] it is shown that, for  $x \in \Lambda$ , the set of accumulation points  $A(x)$  of  $(\log(r)^{-1} \log \mu(B(x, r)))_{r \geq 0}$  is either a singleton or a closed interval. Moreover, it is shown that

$$\dim_H \{x \in \Lambda \mid A(x) = [a, b]\} = \inf_{\alpha \in [a, b]} \dim_H \mathcal{F}(\alpha).$$

Recently, the following modified level sets have attracted a lot of attention in studying the regularity of various fractal functions ([JS15], [JS18], [Ota18], [BKK18], [All18]). For  $\alpha \in \mathbb{R}$ , we define

$$R_*(\alpha) := \left\{ x \in \Lambda \mid \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right\}, \quad R^*(\alpha) := \left\{ x \in \Lambda \mid \limsup_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right\}.$$

In [JS15] it is shown that the level sets  $R_*(\alpha)$  and  $R^*(\alpha)$  satisfy the multifractal formalism in the context of semigroups of rational maps on the Riemann sphere satisfying the separating condition. Allaart ([All18]) proved the multifractal formalism for the level sets  $R_*(\alpha)$  for self-similar measures satisfying the open set condition. In [JS18] we established the multifractal formalism for  $R_*(\alpha)$  for self-similar measures supported on attractors of conformal iterated function systems satisfying the open set condition. Allaart ([All18]) raised the question whether the multifractal formalism holds for  $R^*(\alpha)$ . Moreover, it is of interest whether the formalism for  $R_*$  and  $R^*$  extends to arbitrary Gibbs measures of Hölder continuous potentials.

In this article we establish the multifractal formalism for the generalised level sets

$$\mathcal{F}^*(\alpha) := \left\{ x \in \Lambda \mid \exists (r_k) \rightarrow \infty \text{ as } k \rightarrow \infty : \lim_{k \rightarrow \infty} \frac{\log \mu(B(x, r_k))}{\log r_k} = \alpha \right\}, \quad \alpha \in \mathbb{R},$$

for Gibbs measures of Hölder continuous potentials supported on attractors of conformal iterated function systems satisfying the open set condition. Combining with the previously known results for  $\mathcal{F}(\alpha)$ , we thus obtain that the multifractal formalism holds in particular for the level sets  $R_*(\alpha)$  and  $R^*(\alpha)$ .

## 2. STATEMENT OF MAIN RESULTS

Let us now introduce the necessary terminology to state our main theorem precisely. For an index set  $I$  with  $\#I < \infty$  let  $\Phi = (\phi_i)_{i \in I}$ ,  $\phi_i : X \rightarrow X$ , be a (contracting) orientation-preserving  $\mathcal{C}^{1+\varepsilon}$  conformal iterated function system on a compact interval  $X \subset \mathbb{R}$  with non-empty interior. Here, by orientation-preserving  $\mathcal{C}^{1+\varepsilon}$  conformal we mean that each  $\phi_i$  extends to a  $\mathcal{C}^{1+\varepsilon}$  differentiable map on a neighborhood of  $X$  which satisfies  $0 < \phi_i'(x) < 1$  for every  $x \in X$  and for each  $i \in I$ . We refer to [MU96] for further details and basic properties of conformal iterated function systems. Let  $\pi : I^{\mathbb{N}} \rightarrow \mathbb{R}$  denote the coding map of  $\Phi$  which is for  $\omega = (\omega_1, \omega_2, \dots) \in I^{\mathbb{N}}$  given by

$$\bigcap_{n \geq 1} \phi_{\omega_1} \circ \dots \circ \phi_{\omega_n}(X) = \{\pi(\omega)\}.$$

Let  $\Lambda := \pi(I^{\mathbb{N}})$ . We say that  $\Phi$  satisfies the open set condition if there exists a non-empty open interval  $U \subset \mathbb{R}$  such that  $\phi_i(U) \subset U$ , for all  $i \in I$ , and  $\phi_i(U) \cap \phi_j(U) = \emptyset$  for all  $i, j \in I$  with  $i \neq j$ . To utilize the symbolic thermodynamic formalism (see [Bow75]) we will also need the following definitions. We denote by  $\sigma : I^{\mathbb{N}} \rightarrow I^{\mathbb{N}}$  the left-shift on  $I^{\mathbb{N}}$  which becomes a compact metric space endowed with the shift metric. Let  $\varphi : I^{\mathbb{N}} \rightarrow \mathbb{R}$  be the geometric potential of  $\Phi$  given by

$$\varphi(\omega) := \log \phi'_{\omega_1}(\pi(\sigma(\omega))), \quad \omega = (\omega_1, \omega_2, \dots) \in I^{\mathbb{N}}.$$

Since each  $\phi_i$  is  $\mathcal{C}^{1+\varepsilon}$  differentiable, it is easy to verify that  $\varphi$  is Hölder continuous. Let  $\psi : I^{\mathbb{N}} \rightarrow \mathbb{R}$  be Hölder continuous and let  $\mu_\psi$  denote the unique Gibbs measure for  $\psi$  in the sense of Bowen ([Bow75]). For  $n \geq 1$  we denote by  $S_n \psi := \sum_{k=0}^{n-1} \psi \circ \sigma^k$  the ergodic sum. The range of the multifractal spectrum is defined by

$$\alpha_- := \inf_{\omega \in I^{\mathbb{N}}} \liminf_{n \rightarrow \infty} \frac{S_n \psi(\omega)}{S_n \varphi(\omega)}, \quad \alpha_+ := \sup_{\omega \in I^{\mathbb{N}}} \limsup_{n \rightarrow \infty} \frac{S_n \psi(\omega)}{S_n \varphi(\omega)}.$$

Recall that  $\alpha_- \leq \dim_H \Lambda$  with equality if and only if  $\alpha_- = \alpha_+$  (see [Pes97]). Since  $\varphi < 0$  we have that for each  $\beta \in \mathbb{R}$  there exists a unique  $t(\beta)$  such that

$$(2.1) \quad \mathcal{P}(t(\beta)\varphi + \beta\psi) = 0,$$

where  $\mathcal{P}(f)$  refers to the topological pressure of a continuous function  $f : I^{\mathbb{N}} \rightarrow \mathbb{R}$  with respect to  $\sigma$ . We denote by

$$t^*(\alpha) := \sup_{x \in \mathbb{R}} (\alpha x - t(x))$$

the Legendre transform of  $t$ .

**Theorem 2.1.** *Let  $\Phi = (\phi_i : X \rightarrow X)_{i \in I}$  be an orientation-preserving  $\mathcal{C}^{1+\varepsilon}$  conformal iterated function system on  $X \subset \mathbb{R}$  satisfying the open set condition. Let  $\psi : I^{\mathbb{N}} \rightarrow \mathbb{R}$  be Hölder continuous and let  $\mu := \mu_\psi \circ \pi^{-1}$ . Then we have for all  $\alpha \in [\alpha_-, \alpha_+]$ ,*

$$\dim_H R_*(\alpha) = \dim_H R^*(\alpha) = \dim_H \mathcal{F}(\alpha) = \dim_H \mathcal{F}^*(\alpha) = -t^*(-\alpha),$$

and for  $\alpha \notin [\alpha_-, \alpha_+]$  we have  $R_*(\alpha) = R^*(\alpha) = \mathcal{F}(\alpha) = \mathcal{F}^*(\alpha) = \emptyset$ .

We proceed with two applications of our main result to the regularity of fractal functions.

**2.1. Distribution functions of Gibbs measures.** For a function  $F : \mathbb{R} \rightarrow \mathbb{R}$  we define the pointwise Hölder exponent of  $F$  at  $x \in \mathbb{R}$  by

$$\text{Höl}(F, x) := \sup \left\{ \alpha > 0 \mid \limsup_{y \rightarrow x} \frac{|F(y) - F(x)|}{|y - x|^\alpha} < \infty \right\}.$$

It is shown in [JS15, Lemma 5.1] that for any bounded function  $F : \mathbb{R} \rightarrow \mathbb{R}$  we have

$$(2.2) \quad \text{Höl}(F, x) = \liminf_{r \rightarrow 0} \frac{\log \sup_{y \in B(x, r)} |F(y) - F(x)|}{\log r}.$$

Under the assumptions of Theorem 2.1, we consider the distribution function of  $\mu = \mu_\psi \circ \pi^{-1}$  given by

$$F_\mu : \mathbb{R} \rightarrow [0, 1], \quad F_\mu(x) := \mu((-\infty, x]).$$

By (2.2) we have

$$(2.3) \quad \text{Höl}(F_\mu, x) = \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r}.$$

**Corollary 2.2.** *Under the assumptions of Theorem 2.1, the distribution function  $F_\mu : \mathbb{R} \rightarrow [0, 1]$  of  $\mu = \mu_\psi \circ \pi^{-1}$  satisfies for every  $\alpha \in [\alpha_-, \alpha_+]$*

$$\dim_H \{x \in \Lambda \mid \text{Höl}(F_\mu, x) = \alpha\} = -t^*(-\alpha),$$

where  $t : \mathbb{R} \rightarrow \mathbb{R}$  is defined implicitly by  $\mathcal{P}(t(\beta)\varphi + \beta\psi) = 0$ .

For results on points of non-differentiability and related properties of  $F_\mu$  we refer to [Fal04, KS09, Tro14].

**2.2. Conjugacy maps between expanding piecewise  $\mathcal{C}^{1+\varepsilon}$  interval maps.** In this section we apply the multifractal formalism to conjugacy maps between expanding piecewise  $\mathcal{C}^{1+\varepsilon}$  interval maps as considered in [JKPS09]. In fact, we slightly extend the framework by allowing one of the repellers of the expanding interval maps to be a proper subset of  $[0, 1]$ , whereas in [JKPS09] both repellers are equal to  $[0, 1]$ .

Let us now briefly introduce the setting. Let  $f$  be an expanding piecewise  $\mathcal{C}^{1+\varepsilon}$  interval map with  $s \geq 2$  full branches, i.e., there exist closed intervals  $J_1, \dots, J_s \subset [0, 1]$  with non-empty, pairwise disjoint interiors such that  $f|_{J_i}$  has a  $\mathcal{C}^{1+\varepsilon}$  extension to a neighbourhood of  $J_i$  satisfying  $f'|_{J_i} > 1$  and  $f(J_i) = [0, 1]$ , for  $1 \leq i \leq s$ . We will always assume that the intervals  $J_1, \dots, J_s$  are given in increasing order (i.e.,  $\sup J_i \leq \inf J_j$  if  $i < j$ ). The repeller of  $f$  is denoted by  $\Lambda$  and the restriction  $f|_\Lambda : \Lambda \rightarrow \Lambda$  is semi conjugate to the shift  $\sigma : I^\mathbb{N} \rightarrow I^\mathbb{N}$  with  $I := \{1, \dots, s\}$ . The semi conjugacy is given by the coding map  $\pi_f : I^\mathbb{N} \rightarrow \Lambda$  of the conformal iterated function systems  $\Phi_f$ , which is given by the contracting inverse branches  $(f|_{J_i})^{-1} : [0, 1] \rightarrow [0, 1]$ ,  $1 \leq i \leq s$ . Similarly, let  $g : [0, 1] \rightarrow [0, 1]$  be an expanding  $\mathcal{C}^{1+\varepsilon}$  interval map with  $s$  full branches. We assume that the repeller of  $g$  is the interval  $[0, 1]$ . Again,  $g$  is semi conjugate to  $(I^\mathbb{N}, \sigma)$  with conjugacy map given by the coding map  $\pi_g : I^\mathbb{N} \rightarrow [0, 1]$  of the associated conformal iterated function system  $\Phi_g$ . Note that  $\pi_f^{-1}\{x\}$  has at most two elements, for every  $x \in \Lambda$ , and that  $\pi_g(\pi_f^{-1}\{x\})$  is always a singleton. We can thus define a conjugacy map  $\Theta : \Lambda \rightarrow [0, 1]$  satisfying  $\Theta \circ f|_\Lambda = g \circ \Theta$  which is given by  $\{\Theta(x)\} = \pi_g(\pi_f^{-1}\{x\})$  for  $x \in \Lambda$  (see [JKPS09] for the case  $\Lambda = [0, 1]$ ).  $\Theta$  is a non-decreasing function on the real line, which is strictly increasing function on  $\Lambda$ . We have for  $x \in \Lambda$

$$(2.4) \quad \pi_g^{-1}((-\infty, \Theta(x))) = \pi_f^{-1}((-\infty, x]).$$

Denote by  $\varphi_g : I^\mathbb{N} \rightarrow \mathbb{R}$  (resp.  $\varphi_f : I^\mathbb{N} \rightarrow \mathbb{R}$ ) the geometric potential of  $\Phi_g$  (resp.  $\Phi_f$ ) given by

$$\varphi_g := -\log g' \circ \pi_g, \quad \varphi_f := -\log f' \circ \pi_f.$$

Let  $\lambda$  denote the Lebesgue measure on  $[0, 1]$  and recall that  $\lambda = \mu_{\varphi_g} \circ \pi_g^{-1}$  where  $\mu_{\varphi_g}$  is the unique Gibbs measure for  $\varphi_g$  on  $I^\mathbb{N}$ . By (2.4) we then have for  $x \in \Lambda$ ,

$$\Theta(x) = \mu_{\varphi_g} \circ \pi_g^{-1}((-\infty, \Theta(x))) = \mu_{\varphi_g} \circ \pi_f^{-1}((-\infty, x]).$$

So, the conjugacy  $\Theta : \Lambda \rightarrow [0, 1]$  coincides with the distribution function of  $\mu_{\phi_g} \circ \pi_f^{-1}$  (cf. [JKPS09] for the case  $\Lambda = [0, 1]$ ). Therefore, we have

$$\{x \in \Lambda \mid \text{Höl}(\Theta, x) = \alpha\} = \{x \in \Lambda \mid \text{Höl}(F_\mu, x) = \alpha\},$$

where  $\mu = \mu_{\phi_g} \circ \pi_f^{-1}$ . Hence, with

$$\alpha_- = \inf_{\omega \in I^{\mathbb{N}}} \liminf_{n \rightarrow \infty} \frac{S_n \phi_g(\omega)}{S_n \phi_f(\omega)}, \quad \alpha_+ := \sup_{\omega \in I^{\mathbb{N}}} \limsup_{n \rightarrow \infty} \frac{S_n \phi_g(\omega)}{S_n \phi_f(\omega)},$$

we obtain the following corollary as a consequence of Corollary 2.2.

**Corollary 2.3.** *Let  $f$  and  $g$  be two expanding piecewise  $\mathcal{C}^{1+\varepsilon}$  interval maps with  $s \geq 2$  orientation-preserving full branches and coding maps  $\pi_f, \pi_g : I^{\mathbb{N}} \rightarrow [0, 1]$ . Let  $\Lambda := \pi_f(I^{\mathbb{N}})$  and suppose that  $\pi_g(I^{\mathbb{N}}) = [0, 1]$ . Then the conjugacy map  $\Theta : \Lambda \rightarrow [0, 1]$ , given by  $\{\Theta(x)\} = \pi_g(\pi_f^{-1}\{x\})$  for  $x \in \Lambda$ , satisfies for every  $\alpha \in [\alpha_-, \alpha_+]$*

$$\dim_H \{x \in \Lambda \mid \text{Höl}(\Theta, x) = \alpha\} = -t^*(-\alpha),$$

where  $t : \mathbb{R} \rightarrow \mathbb{R}$  is defined implicitly by  $\mathcal{P}(t(\beta)\phi_f + \beta\phi_g) = 0$ .

### 3. PROOF OF THEOREM 2.1

*Proof.* First observe that  $\mathcal{F}(\alpha)$  is a subset of each of the level sets considered in the theorem. Therefore, the lower bound for the Hausdorff dimension follows from the well-known multifractal formalism for  $\mathcal{F}(\alpha)$ . Therefore, to complete the proof of the theorem, it suffices to show the upper bound for the Hausdorff dimension of  $\mathcal{F}^*(\alpha)$ . Throughout, we may assume  $I = \{1, \dots, s\}$ , for  $s \geq 2$ , and  $\phi_i(x) \leq \phi_j(y)$  for all  $i, j \in I$  with  $i < j$  and for all  $x, y \in U$ , where  $U$  is the open set in the open set condition.

Denote by  $I^* := \bigcup_{k \geq 1} I^k$  the set of finite words in the alphabet  $I$ . It is well known that  $\Phi$  has the bounded distortion property, i.e.

$$\sup_{k \geq 1} \sup_{\omega \in I^k} \sup_{x, y \in X} \frac{(\phi_{\omega_1} \circ \dots \circ \phi_{\omega_k})'(x)}{(\phi_{\omega_1} \circ \dots \circ \phi_{\omega_k})'(y)} < \infty.$$

From this one derives the existence of  $C > 1$  such that for all  $\gamma \in I^*$  and  $i \in I$ ,

$$C \cdot \text{diam}(\pi[\gamma i]) \geq \text{diam}(\pi[\gamma]),$$

where  $\text{diam}(A) := \sup\{|x - y| \mid x, y \in A\}$  refers to the diameter of a set  $A \subset \mathbb{R}$ .

Let  $x \in \mathcal{F}^*(\alpha)$ ,  $x = \pi(\omega)$  for some  $\omega \in I^{\mathbb{N}}$ . There exists a sequence  $r_k \rightarrow 0$  such that

$$\lim_{k \rightarrow \infty} \frac{\log \mu(B(x, r_k))}{\log r_k} = \alpha.$$

We define for  $k \geq 1$

$$n_k := \min \{n \geq 1 \mid \text{diam}(\pi[\omega_1, \dots, \omega_n]) < C^2 r_k\}.$$

For  $a \in I$  and  $m \in \mathbb{N}$  we denote  $a^m := (a, \dots, a) \in I^m$ . We define two sequences of words  $(v_k), (v'_k) \in I^*$ ,  $k \geq 1$ , as follows. If  $(\omega_1, \dots, \omega_{n_k})$  takes the form

$$(\omega_1, \dots, \omega_{n_k}) = (\tau j s^{\ell_k}),$$

for some  $\tau \in I^*$ ,  $j \leq s - 1$  and  $\ell_k \geq 1$  then let

$$\ell'_k := \max \left\{ \ell \geq 1 \mid \text{diam} \left( \pi \left[ \tau (j+1) 1^\ell \right] \right) \geq r_k \right\}.$$

Note that  $\ell'_k$  is well defined, since our definition of  $n_k$  implies that  $\text{diam}(\pi[\tau]) \geq C^2 r_k$ , and thus,  $\text{diam}(\pi[\tau(j+1)1]) \geq r_k$ . We then define

$$v_k := (\tau j s^{\ell_k - 1}) = (\omega_1, \dots, \omega_{n_k - 1}), \quad v'_k := (\tau(j+1)1)^{\ell'_k}.$$

If  $(\omega_1, \dots, \omega_{n_k})$  takes the form  $(\omega_1, \dots, \omega_{n_k}) = (\tau(j+1)1)^{\ell_k}$ , then we define  $v_k := (\omega_1, \dots, \omega_{n_k - 1})$  and  $v'_k := (\tau j s^{\ell'_k})$  where

$$\ell'_k := \max \left\{ \ell \geq 1 \mid \text{diam}(\pi[\tau j s^\ell]) \geq r_k \right\}.$$

Finally, if  $(\omega_1, \dots, \omega_{n_k})$  satisfies  $\omega_{n_k} \notin \{1, s\}$  then we define  $v_k := v'_k := (\omega_1, \dots, \omega_{n_k - 1})$ . Let  $n'_k := |v'_k|$ .

It is important to note that, by the definition of  $v_k$  and  $v'_k$ , we have as  $k \rightarrow \infty$ ,

$$(3.1) \quad \text{diam}(\pi[v_k]) \asymp \text{diam}(\pi[v'_k]) \asymp r_k,$$

where, for sequences of positive numbers  $(a_k)$  and  $(b_k)$  the notation  $a_k \asymp b_k$  means that  $a_k/b_k$  is bounded away from zero and infinity. We will show that this implies the existence of a uniform constant  $D$  such that

$$(3.2) \quad \ell'_k \cdot \varphi(\bar{1}) - D \leq \ell_k \cdot \varphi(\bar{s}) \leq \ell'_k \cdot \varphi(\bar{1}) + D,$$

where we have set  $\bar{\gamma} := (\gamma\gamma\dots) \in I^{\mathbb{N}}$ , for  $\gamma \in I^*$ . To prove (3.2) suppose that  $(\omega_1, \dots, \omega_{n_k}) = (\tau j s^{\ell_k})$ . The other case  $(\omega_1, \dots, \omega_{n_k}) = (\tau(j+1)1)^{\ell_k}$ , can be handled analogously. By the bounded distortion property of the geometric potential we have, as  $k \rightarrow \infty$ ,

$$\begin{aligned} \text{diam}(\pi[v_k]) &= \text{diam}\left(\pi\left[(\tau j s^{\ell_k - 1})\right]\right) \asymp \text{diam}(\pi[\tau]) e^{S_{\ell_k} \varphi(\bar{s})} \\ \text{diam}(\pi[v'_k]) &= \text{diam}\left(\pi\left[\tau(j+1)1^{\ell'_k}\right]\right) \asymp \text{diam}(\pi[\tau]) e^{S_{\ell'_k} \varphi(\bar{1})}, \end{aligned}$$

which proves (3.2).

We will show that for  $x \in \Lambda = \pi(I^{\mathbb{N}})$  and  $k \geq 1$  we have

$$B(x, r_k) \cap \Lambda \subset \pi([v_k]) \cup \pi([v'_k]).$$

First suppose that  $(\omega_1, \dots, \omega_{n_k}) = (\tau j s^{\ell_k})$ . Then  $\pi[v_k] = \pi[(\tau j s^{\ell_k - 1})] \supset \pi[(\tau j s^{\ell_k - 1}1)]$ . Since  $x \in \pi[(\tau j s^{\ell_k})]$  we have  $x \geq \max \pi[(\tau j s^{\ell_k - 1}1)]$ . By the definition of  $C$  and  $n_k$  we have

$$\text{diam}\left(\pi\left[(\tau j s^{\ell_k - 1}1)\right]\right) \geq C^{-1} \text{diam}\left(\pi\left[(\tau j s^{\ell_k - 1})\right]\right) = C^{-1} \text{diam}(\pi[(\omega_1, \dots, \omega_{n_k - 1})]) \geq C r_k.$$

Hence,

$$\pi[v_k] \supset [x, x - r_k] \cap \Lambda.$$

Further, by the definition of  $\ell'_k$  we have

$$\text{diam}(\pi[v'_k]) = \text{diam}\left(\pi\left[\tau(j+1)1^{\ell'_k}\right]\right) \geq r_k,$$

so  $[x, x + r_k] \cap \Lambda \subset \pi[v_k] \cup \pi[v'_k]$ . This proves that  $B(x, r_k) \cap \Lambda \subset \pi([v_k]) \cup \pi([v'_k])$ .

Let  $\varepsilon > 0$ . We will derive from our assumption  $x \in \mathcal{F}^*(\alpha)$  that there exists  $N \geq 1$  such that for all  $k \geq N$ ,

$$(3.3) \quad \frac{S_{|v_k|} \psi(\bar{v}_k)}{S_{|v_k|} \varphi(\bar{v}_k)} \leq \alpha + \varepsilon \quad \text{or} \quad \frac{S_{|v'_k|} \psi(\bar{v}'_k)}{S_{|v'_k|} \varphi(\bar{v}'_k)} \leq \alpha + \varepsilon.$$

To prove (3.3), we first note that by the Gibbs property of  $\mu_\psi$  we have for every  $\gamma \in I^*$ ,

$$\mu_\psi([\gamma]) \leq C_\psi \exp(S_{|\gamma|} \psi(\bar{\gamma})),$$

where  $C_\psi \geq 0$  is a uniform constant depending on  $\psi$ . Suppose for a contradiction that (3.3) does not hold. Then, by passing to a subsequence of  $(n_k)$  we may assume that for all  $k$  and for all  $v \in \{v_k, v'_k\}$  we have

$S_{|v|}\psi(\bar{v})/S_{|v|}\varphi(\bar{v}) > \alpha + \varepsilon$ , and so, by the bounded distortion property of Hölder continuous potentials, taking  $C'$  large enough we have

$$\mu_\psi([v]) \leq C_\psi \exp(S_{|v|}\psi(\bar{v})) \leq C' r_k^{\alpha+\varepsilon}.$$

Since  $B(x, r_k) \subset \pi([v_k]) \cup \pi([v'_k])$ , we conclude that

$$\lim_{k \rightarrow \infty} \frac{\log \mu(B(x, r_k))}{\log r_k} \geq \alpha + \varepsilon.$$

This contradiction proves (3.3).

We will prove the theorem only in the case when  $\alpha \leq \alpha_0 := \int \psi d\mu_{t(0)\varphi} / \int \varphi d\mu_{t(0)\varphi}$ . The case  $\alpha \geq \alpha_0$  can be considered in a similar fashion (see also Remark 3.1 below). Let  $\beta > 0$ ,  $\eta > 0$  and let  $b = t(\beta) + \beta(\alpha + \varepsilon) + \eta$ . We define

$$\mathcal{C}_{\alpha+\varepsilon} := \left\{ \tau \in I^* \mid \frac{S_{|\tau|}\psi(\bar{\tau})}{S_{|\tau|}\varphi(\bar{\tau})} \leq \alpha + \varepsilon \right\}.$$

We obtain a covering of  $\mathcal{F}^*(\alpha)$  by cylinders  $\mathcal{C}$  of sufficiently small diameters as follows. For each  $x \in \mathcal{F}^*(\alpha)$  we define the sequences  $(n_k)$ ,  $(v_k)$  and  $(v'_k)$  as above. We then pick  $v(x) := v_k$  such that  $\pi([v_k])$  is sufficiently small,  $x \in \pi([v_k])$  and (3.3) holds. We define

$$\mathcal{C} := \{v(x) \mid x \in \mathcal{F}^*(\alpha)\}.$$

To verify that the corresponding sum of diameters  $\sum_{v \in \mathcal{C}} \text{diam}(\pi([v]))^b$  converges, we proceed as follows. If  $v \notin \mathcal{C}_{\alpha+\varepsilon}$  then, by (3.3), we can replace  $v$  by  $v' \in \mathcal{C}_{\alpha+\varepsilon}$ , because  $\text{diam}(\pi([v])) \asymp \text{diam}(\pi([v']))$  by (3.1). This defines a map  $v \mapsto v'$  from  $\mathcal{C} \setminus \mathcal{C}_{\alpha+\varepsilon}$  to  $\mathcal{C}_{\alpha+\varepsilon}$ . Since the involved numbers  $\ell_k$  and  $\ell'_k$  in the definition of  $v$  and  $v'$  satisfy (3.2), we have that the map  $v \mapsto v'$  is at most  $M$ -to-1 for some uniform constant  $M \in \mathbb{N}$ . Since

$$(3.4) \quad \sum_{\omega \in \mathcal{C}_{\alpha+\varepsilon}} \text{diam}(\pi([\omega]))^b \asymp \sum_{\omega \in \mathcal{C}_{\alpha+\varepsilon}} e^{(t(\beta)+\beta(\alpha+\varepsilon)+\eta)S_{|\omega|}\varphi(\bar{\omega})} \leq \sum_{\omega \in \mathcal{C}_{\alpha+\varepsilon}} e^{(t(\beta)+\eta)S_{|\omega|}\varphi(\bar{\omega})+\beta S_{|\omega|}\psi(\bar{\omega})} < \infty,$$

we conclude that the  $b$ -dimensional Hausdorff measure of  $\mathcal{F}^*(\alpha)$  is bounded by  $M \cdot \sum_{\omega \in \mathcal{C}_{\alpha+\varepsilon}} \text{diam}(\pi([\omega]))^b < \infty$ . Now, first assume that  $\alpha \in [\alpha_-, \alpha_0]$ . Since  $\varepsilon$  and  $\eta$  are arbitrary, it follows that

$$\dim_H \mathcal{F}^*(\alpha) \leq \inf_{\beta > 0} \{t(\beta) + \beta\alpha\} = -t^*(-\alpha),$$

where we have used  $\alpha \leq \alpha_0$  for the last equality. Finally, if  $\alpha < \alpha_-$  then  $\mathcal{C}_{\alpha+\varepsilon} = \emptyset$  if  $\alpha + \varepsilon < \alpha_-$ . By the above construction of the covering of  $\mathcal{F}^*(\alpha)$  it thus follows that  $\mathcal{F}^*(\alpha) = \emptyset$ . The proof is complete.  $\square$

*Remark 3.1.* For the dimension spectrum of the level sets  $R_*(\alpha)$  with  $\alpha \geq \alpha_0$ , the upper bound of the Hausdorff dimension can also be deduced from [LWX12, Theorem 1.1 (2)] (see also [All18, Proof of Proposition 7.6 ii]).

*Acknowledgement 3.2.* The research of Johannes Jaerisch was supported by the Japan Society for the Promotion of Science, Kakenhi 15H06416 and 17K14203. The research of Hiroki Sumi was supported by the Japan Society for the Promotion of Science, Kakenhi 15K04899 and 18H03671.

## REFERENCES

- [All18] P. C. Allaart, *Differentiability and Hölder spectra of a class of self-affine functions*, Adv. Math. **328** (2018), 1–39. MR 3771122
- [BKK18] B. Bárány, G. Kiss, and I. Kolossváry, *Pointwise regularity of parameterized affine zipper fractal curves*, Nonlinearity **31** (2018), no. 5, 1705–1733. MR 3816651

- [Bow75] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Springer-Verlag, Berlin, 1975, Lecture Notes in Mathematics, Vol. 470. MR MR0442989 (56 #1364)
- [BS00] L. Barreira and J. Schmeling, *Sets of “non-typical” points have full topological entropy and full Hausdorff dimension*, Israel J. Math. **116** (2000), 29–70. MR MR1759398 (2002d:37040)
- [Fal04] K. Falconer, *One-sided multifractal analysis and points of non-differentiability of devil’s staircases*, Math. Proc. Cambridge Philos. Soc. **136** (2004), no. 1, 167–174. MR 2034020 (2005b:28013)
- [FP85] U. Frisch and G. Parisi, *On the singularity structure of fully developed turbulence*, Turbulence and predictability in geophysical fluid dynamics and climate dynamics (North Holland Amsterdam), 1985, pp. 84–88.
- [HJK<sup>+</sup>86] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. J. Shraiman, *Fractal measures and their singularities: The characterization of strange sets*, Phys. Rev. A **85** (1986), no. 33, 1141–1151.
- [JKPS09] T. Jordan, M. Kesseböhmer, M. Pollicott, and B. O. Stratmann, *Sets of nondifferentiability for conjugacies between expanding interval maps*, Fund. Math. **206** (2009), 161–183. MR 2576266
- [JR11] T. Jordan and M. Rams, *Multifractal analysis of weak Gibbs measures for non-uniformly expanding  $C^1$  maps*, Ergodic Theory Dynam. Systems **31** (2011), no. 1, 143–164. MR 2755925
- [JS15] J. Jaerisch and H. Sumi, *Multifractal formalism for expanding rational semigroups and random complex dynamical systems*, Nonlinearity **28** (2015), 2913–2938.
- [JS18] ———, *Spectral gap property for random dynamics on the real line and multifractal analysis of generalised takagi functions*, Preprint available on arXiv:1808.08421 (2018).
- [KS09] M. Kesseböhmer and B. O. Stratmann, *Hölder -differentiability of Gibbs distribution functions*, Math. Proc. Cambridge Philos. Soc. **147** (2009), no. 2, 489–503. MR MR2525939
- [LWX12] J. Li, M. Wu, and Y. Xiong, *Hausdorff dimensions of the divergence points of self-similar measures with the open set condition*, Nonlinearity **25** (2012), no. 1, 93–105. MR 2864378
- [Man74] B. B. Mandelbrot, *Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier*, Journal of Fluid Mechanics Digital Archive **62** (1974), no. 02, 331–358.
- [MU96] R. D. Mauldin and M. Urbański, *Dimensions and measures in infinite iterated function systems*, Proc. London Math. Soc. (3) **73** (1996), no. 1, 105–154. MR 1387085 (97c:28020)
- [Ota18] A. Otani, *Fractal dimensions of graph of Weierstrass-type function and local Hölder exponent spectra*, Nonlinearity **31** (2018), no. 1, 263–292. MR 3746637
- [Pat97] N. Patzschke, *Self-conformal multifractal measures*, Adv. in Appl. Math. **19** (1997), no. 4, 486–513. MR 1479016
- [Pes97] Y. B. Pesin, *Dimension theory in dynamical systems*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1997, Contemporary views and applications. MR MR1489237 (99b:58003)
- [PW97a] Y. Pesin and H. Weiss, *A multifractal analysis of equilibrium measures for conformal expanding maps and Moran-like geometric constructions*, J. Statist. Phys. **86** (1997), no. 1-2, 233–275. MR 1435198 (97m:58118)
- [PW97b] ———, *The multifractal analysis of Gibbs measures: motivation, mathematical foundation, and examples*, Chaos **7** (1997), no. 1, 89–106. MR 1439809 (98e:58130)
- [RU09] M. Roy and M. Urbański, *Multifractal analysis for conformal graph directed Markov systems*, Discrete Contin. Dyn. Syst. **25** (2009), no. 2, 627–650. MR 2525196
- [Tro14] S. Troscheit, *Hölder differentiability of self-conformal devil’s staircases*, Math. Proc. Cambridge Philos. Soc. **156** (2014), no. 2, 295–311. MR 3177871

URL: <http://www.math.nagoya-u.ac.jp/~jaerisch/index.html>

URL: <http://www.math.h.kyoto-u.ac.jp/~sumi/index.html>