HIERARCHICAL STUDIES OF DENDROLANGUAGES WITH AN APPLICATION TO CHARACTERIZING DERIVATION TREES OF PHRASE STRUCTURE GRAMMARS

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(Received June 6, 1973)

Abstract

Dendrolanguage generating systems form a fairly-broad class of tree-manipulating systems and have many applications in the fields of logic, linguistics and automata theory.

In this paper, various types of dendrolanguage generating systems are proposed and their properties are studied: a hierarchical studies on dendrolanguages, relations to dendro-automata which are acceptors of dendrolanguages, closure properties of dendrolanguages under several operations on trees, decidable problems concerning dendrolanguages, properties of two kind of dendrolanguage generating systems with control and characterization of derivation trees of phrase structure grammars.

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1. Introduction

Many tree-manipulating devices, *e.g.*, tree generating regular system[13] and tree automata[12], have been considered in logic, mathematical linguistics and automata theory. These devices are ordinarily defined sui generis, rather than as special cases of a broad category that includes them all. This paper studies "dendrolanguage generating systems", a fairly broad class of tree-manipulating systems. The systems will find their applications in many fields of logics, formal language manipulations, automata theory and so on.

Some common mathematical terminologies and concepts are reviewed in chapter 2. In chapters 3 we define several types of "dendrolanguage generating systems" which we call context sensitive DS (CSDS), scattered context DS (SCDS), context-free DS (CFDS), linear DS (LDS), right-linear DS (RLDS) and left-linear DS (LLDS). These systems are natural generalizations of tree generating regular systems and constitute a very similar hierachy to that of phrase structure grammars as studied in chapter 5. We denote the respective families of the dendrolanguages by $\mathscr{T}_{CS}$, $\mathscr{T}_{SC}$, $\mathscr{T}_{CF}$, $\mathscr{T}_{L}$, $\mathscr{T}_{RL}$ and $\mathscr{T}_{LL}$. One of main results is that $\mathscr{T}_{CS} \supsetneqq \mathscr{T}_{SC} \supsetneqq \mathscr{T}_{CF} \supsetneqq \mathscr{T}_{L} \supsetneqq \mathscr{T}_{RL}$, $\mathscr{T}_{LL}$. Dendrolanguage generating systems are considered as generators. As the corresponding accepters we introduce dendroautomata in chapter 6. They are related to each other just like grammars and automata. In chapter 7, many closure properties of dendrolanguages are studied and some decidable problems concerning dendrolanguages are also investigated in chapter 8. In chapter 9, a context-free dendrolanguage generating system, called a state dendrolanguage generating system, is defined. Using this system, we find an infinite hierarchy between $\mathscr{T}_{CF}$ and $\mathscr{T}_{CS}$. Another extension of context-free dendrolanguage generating system, which is called a string-dendrolanguage generating system, is studied in chapter 10, where we also find that there exists another infinite hierarchy of dendrolanguages. Finally, in chapter 11, a characterization of derivation trees of phrase structure grammars are investigated as an application of our dendrolanguage generating systems.

2. Trees and basic definitions

In this chapter, several basic concepts, definitions and notations concerning trees are introduced.

Definition 2.1. Let N be the set of positive integers. Let N^* be the free monoid generated from N under the operation $\cdot$ (concatenation). The identity of N^* is denoted by 0.

Let us define a relation $\preceq$ over N^* as follows: For any m, n in N^*, $n \preceq m$ if and only if there exists $l \in N^*$ such that $n \cdot l = m$.

Defintion† *2.2.* A subset D of N^* is called a *tree domain* if and only if the following conditions (i) and (ii) are satisfied:

(1) If $m \in D$ then $n \preceq m$ implies $n \in D$.

(2) If $m \cdot j \in D$ then for any $i \in N$ such that $i < j$, $m \cdot i$ is contained in D.

An element of D will be called a *node* and if n and $n \cdot i$ is in D then the pair $(n, n \cdot i)$ will be called a *branch*, in the sequel.

Example 2.1. $D = \{0, 1, 2, 1\cdot1, 1\cdot2, 1\cdot2\cdot1, 1\cdot2\cdot2\}$ is a tree domain, of which topological representation is given in Fig. 2.1.

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† This notation is well-known as "Dewey decimal notation[23]".

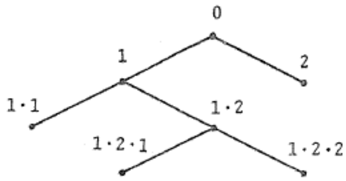


FIG. 2.1. A topological representation of a tree domain D given in Example 2.1.

Definition 2.3. The *depth of a node* n in D, denoted by $d(n)$, is defined as follows:

(1) $d(0)=0$,

(2) If $n=m\cdot i$ then $d(m\cdot i)=d(m)+1$ for all $i\in N$

The *depth of a tree domain* D, which is denoted by $d(D)$, is defined by

$$d(D)=\max\{d(n)\,|\,n\in D\} \tag{2.1}$$

Definition 2.4. For a tree domain D, define a subset

$$\overline{D}=\{n\,|\,n\in D,\ n\cdot 1\notin D\} \tag{2.2}$$

which will be called the *leaf set* of D.

Definition 2.5. The adjacency relation $\sim$ on $\overline{D}$ is defined as follows: for any m, n in $\overline{D}$, m is adjacent to n and denoted as $m\sim n$, if and only if (1) $m=n$ or (2) there exists x in D such that $m=x\cdot i_1\cdot i_2\cdot\cdots\cdot i_l$ $(l\geqq 1)$ and $n=x\cdot j_1\cdot i_2\cdot\cdots\cdot j_k$ $(k\geqq 1)$, where (i) $j_1=i_1+1$, (ii) $j_r=1$ or 0 $(2\leqq r\leqq k)$ and (iii) for each i_q $(2\leqq q\leqq l)$, $i_q=\max\{i\,|\,x\cdot i_1\cdot\cdots\cdot i_{q-1}\cdot i\in D,\ i\in N\}$.

Definition 2.6. A *ranked alphabet* (r.a.) is a pair (Ω, σ), where Ω is a finite set of symbols and σ is a mapping from Ω into $N\cup\{0\}$, which will be called a *ranking mapping*. For a symbol A in Ω, if $\sigma(A)=r\in(N\cup\{0\})$, it means that A has rank r.

Let us denote the set of symbols A with rank i by Ω_i, *i.e.*, $\Omega_i=\{A\,|\,A\in\Omega, \sigma(A)=i\}$. It should be noted that if $i\neq j$ then $\Omega_i\cap\Omega_j=\phi$ (empty). In the sequel, we will simply write Ω for r.a. (Ω, σ).

Definition 2.7. A *tree* over r.a. Ω (*i.e.*, (Ω, σ)) is a mapping from a tree domain D into Ω, *i.e.*,

$$t : D\rightarrow\Omega \tag{2.3}$$

such that for any $n\in D$, $\sigma(t(n))=\max\{i\,|\,n\cdot i\in D\}$, which means that the rank of the symbol assigned to a node n is equal to the degree of the node n, *i.e.*, the number of branches connected to the node n.

Definition 2.8. Depth of a tree is that of the tree domain.

Example 2.2. Let us consider the tree domain D of Example 2.1 and r.a. $\Omega=\{X, Y, a\}$, where $\sigma(X)=\sigma(Y)=2$ and $\sigma(a)=0$. A mapping $t : D\rightarrow\Omega$ is a tree, where $t(0)=X$, $t(1)=X$, $t(1\cdot 2)=Y$, $t(1\cdot 1)=a$, $t(1\cdot 2\cdot 1)=a$, $t(1\cdot 2\cdot 2)=a$ and $t(2)=a$. This

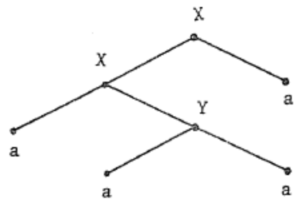


FIG. 2.2. A topological representation of a tree of Example 2.2.

tree t is topologically represented in Fig. 2.2. The depth of t is 3.

It is often convenient to introduce an alternative representation of a tree: consider a tree $t : D \to \Omega$. If $t(n) = X$ then we denote it by (n, X), which means that a symbol X is assigned to a node n by $t : D \to \Omega$. Then the tree t with a tree domain D, which is denoted by t_D, can be represented by

$$t_D = \{(n, X) \mid n \in D, \ t(n) = X\} \tag{2.4}$$

For an example, the tree of Example 2.2 is represented by

$$t_D = \{(0, X), (1, X), (1 \cdot 2, Y), (1 \cdot 1, a), (1 \cdot 2 \cdot 1, a), (1 \cdot 2 \cdot 2, a), (2, a)\}.$$

On the other hand, we will write D_t to represent the tree domain of a tree t in order to distinguish it from the others.

Furthermore, we introduce linear representations of a tree, prefix and suffix notation.

Notation 2.1. For a tree $t : D \to \Omega$, $\mu(t)$ stands for a *prefix notation* of t, and $\varphi(t)$ a *suffix notation* of t.

We can recursively obtain a prefix notation $\mu(t)$ and a suffix notation $\varphi(t)$ as follows: If a tree t is

$$t_{D_0} = \{(0, X)\} \cup \bigcup_{i=1}^{k} \{(i \cdot n, Y) \mid (n, Y) \in t_{D_i}\},$$

where t_{D_i} $(i = 1, \ldots, k)$ is a tree, then

$$\mu(t) = \mu(t_{D_0}) = X\mu(t_{D_1})\,\mu(t_{D_2}) \cdots \mu(t_{D_k}) \tag{2.5}$$

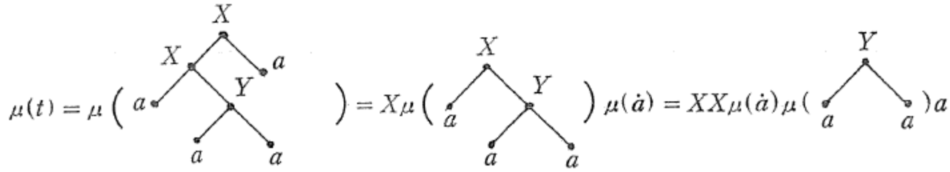
and

$$\varphi(t) = \varphi(t_{D_0}) = \varphi(t_{D_1})\,\varphi(t_{D_2}) \cdots \varphi(t_{D_k})\,X. \tag{2.6}$$

If t is $t_{D_0} = \{(0, X)\}$ then $\mu(t) = X$ and $\varphi(t) = X$.

Example 2.3. The prefix notation of the tree t of Example 2.2 can be obtained as follows†:

† $\dot{a}$ is the topological representation of a tree $\{(0, a)\}$.



$= XXaY\mu(\dot{a})\,\mu(\dot{a})\,a = XXaYaaa$

The suffix notation $\varphi(t) = aaaYXaX$ is obtained similarly.

Here, it should be noted that since $\Omega_i \cap \Omega_j = \phi\ (i \neq j)$, any tree over a ranked alphabet Ω can be uniquely determined from its prefix notation (or suffix notation). Thus, a tree and its prefix notation (or suffix notation) have one to one correspondence.

3. Dendrolanguage generating systems

Brainerd introduced a tree generating system which he called tree generating regular system [13], and characterized generation trees of the context-free languages. In this chapter, we introduce more general tree generating systems including the Brainerd's one as a special case of ours. This system will be called *dendrolanguage generating system*. In the later chapters, it will play a fundamental role in developing our discussions.

A tree $t: D \to \Omega$ is called a finite tree if its tree domain is finite. Let us denote *the set of all finite trees* over r.a. Ω by $\mathscr{T}_\Omega$ and *the set of all finite trees with depth less than or equal to k* by $\mathscr{T}_\Omega^k$. Now, let us define dendrolanguage generating system.

Definition 3.1. A *context-sensitive dendrolanguage generating system* (abbreviated CSDS) is a 5-tuple

$$S = (\Omega,\ V,\ \Sigma,\ P,\ \lambda_0), \tag{3.1}$$

where each of five entries are as follows:

(1) Ω: a finite ranked alphabet (Ω, σ).

(2) $V \subseteq \Omega$: a set of terminal node symbols.

In the followings, $\Omega - V$ is denoted by Λ, which will be called set of nonterminal node symbols. Here, we assume that

$$\sigma(\lambda) = 0 \quad \text{for all} \quad \lambda \in \Lambda. \qquad (3.2)^\dagger$$

(3) Σ is the subset of V such that $\Sigma = \{a \mid \sigma(a) = 0,\ a \in V\}$, which will be called a set of leaf node symbols.

(4)†† P is a finite subset of $\bigcup_{l=1}^{m} \underbrace{\mathscr{T}_\Lambda \times \cdots \times \mathscr{T}_\Lambda}_{l\ \text{times}} \times \underbrace{\mathscr{T}_\Omega \times \cdots \times \mathscr{T}_\Omega}_{l\ \text{times}}$ (m: a finite

† This assumpution means that nonterminal node symbols can appear only at leaf nodes and so generation rules can be applied only to leaf nodes which rewrite the nodes by appropriate trees.

†† $\underbrace{\mathscr{T}_\Lambda \times \cdots \times \mathscr{T}_\Lambda}_{l\ \text{times}}$ means a direct product of l sets of $\mathscr{T}_\Lambda$.

positive integer), called a set of dendrolaguage generating rules. An element $(s_1, s_2, \ldots, s_l, t_1, t_2, \ldots, t_l)$ in P is called dendrolanguage generating rule (or simply, rule) and usually written as $(s_1, s_2, \ldots, s_l) \to (t_1, t_2, \ldots, t_l)$. Since $s_i (1 \leqq i \leqq l) \in \mathscr{T}_\Lambda$ means from the assumption (3.2) that $s_i = \{(0, \lambda_i)\}$ for some λ_i in Λ, we will often write $(\dot{\lambda}_1, \dot{\lambda}_2, \ldots, \dot{\lambda}_l) \to (t_1, t_2, \ldots, t_l)$ instead of $(s_1, s_2, \ldots, s_l) \to (t_1, t_2, \ldots, t_l)$.

(5) $\lambda_0 \in \Lambda$: an initial nonterminal node symbol.

Definition 3.2. Let $S = (\Omega, V, \Sigma, P, \lambda_0)$ be a CSDS. For any α and β in $\mathscr{T}_\Omega$, we write $\alpha \underset{S}{\Longrightarrow} \beta$ (or $\alpha \Longrightarrow \beta$ when S is understood) if there exist $x_i \in \Omega^*$ $(0 \leqq i \leqq l)$, $\xi_i \in \Lambda$ $(1 \leqq i \leqq l)$ and t_i $(1 \leqq i \leqq l)$ such that the following three conditions (1), (2) and (3) are satisfied†:

(1) $\mu(\alpha) = x_0 \xi_1 x_1 \cdots x_{l-1} \xi_l x_l$,
$\mu(\beta) = x_0 \mu(t_1) x_1 \cdots x_{l-1} \mu(t_l) x_l$,

(2) $(\dot{\xi}_1, \ldots, \dot{\xi}_l) \to (t_1, \ldots, t_l)$ is in P, and

(3)†† $\Subset_\alpha(\xi_1) \sim \Subset_\alpha(\xi_2) \sim \cdots \Subset_\alpha(\xi_l)$,

where $\Subset_\alpha(\xi)$ denotes a node n to which the symbol ξ is assigned, *i.e.*, the node n such that (n, ξ) in α_D, the representation of α in the form of (2.4). (Here, it should be noted that $\Subset_\alpha(\xi_i)$ is in the leaf set $\overline{D}_\alpha$ of α.)

Next, for any α and β in $\mathscr{T}_\Omega$, we write $\alpha \overset{*}{\Longrightarrow} \beta$ if ejther $\alpha = \beta$ or there exist $\alpha_0, \ldots, \alpha_r$ such that $\alpha_0 = \alpha$, $\alpha_r = \beta$ and $\alpha_i \Longrightarrow \alpha_{i+1}$ for each i $(0 \leqq i \leqq r-1)$.

Definition 3.3. Let $S = (\Omega, V, \Sigma, P, \lambda_0)$ be a CSDS, then the subset of $\mathscr{T}_V$,

$$T(S) = \{t \mid \dot{\lambda}_0 \overset{*}{\Longrightarrow} t \in \mathscr{T}_V\}$$

is called a *context-sensitive dendrolanguage* (abbreviated CSDL). $T(S)$ is said to be generated by S. Each tree in $T(S)$ is said to be *generated* (from $\dot{\lambda}_0$).

An example of a CSDS:

Example 3.1. Consider $S = (\Omega, V, \Sigma, P, \lambda)$ specified as follows:

(1) $\Omega = \{\lambda, \xi, \xi', \bar{\xi}, \eta, \bar{\eta}, \eta', \zeta, A, a\}$, where $\Omega_0 = \Omega - \{A\}$, $\Omega_2 = \{A\}$.

(2) $V = \{A, a\}$, $\Sigma = \{a\}$.

(3)* $P = \{$(1) $(\lambda) \to (A\xi'\xi')$, (2) $(\lambda) \to (Aaa)$, (3) $(\zeta) \to (a)$,
(4) $(\xi', \xi') \to (A\eta'\eta, A\eta\eta')$,
(5) $(\eta', \eta) \to (\xi', \bar{\xi})$, (6) $(\bar{\xi}, \eta) \to (\xi, \bar{\xi})$,
(7) $(\bar{\xi}, \eta') \to (\xi, \xi')$, (8) $(\xi', \xi) \to (\zeta, \zeta)$,
(9) $(\zeta, \xi) \to (\zeta, \zeta)$, (10) $(\zeta, \xi') \to (\zeta, \zeta)$,

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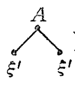
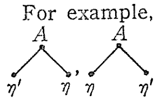
† Ω^* denotes the set of all strings of elements in the alphabet Ω, including the empty string ε.

†† $\Subset_\alpha(\xi_1) \sim \Subset_\alpha(\xi_2) \sim \cdots \sim \Subset_\alpha(\xi_l)$ means that $\Subset_\alpha(\xi_1) \sim \Subset_\alpha(\xi_2)$, $\Subset_\alpha(\xi_2) \sim \Subset_\alpha(\xi_3)$, $\ldots$, and $\Subset_\alpha(\xi_{l-1}) \sim \Subset_\alpha(\xi_l)$.

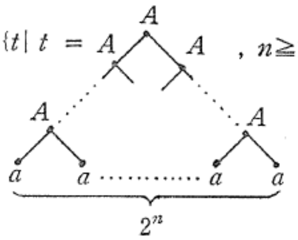
* Because of the limitation of space, the generation rules are given by using the prefix notations. For example, $(\lambda) \to (A\xi'\xi')$ means $(\lambda) \to ($ 

$)$ and $(\xi', \xi') \to (A\eta'\eta, A\eta\eta')$ means $(\dot{\xi}', \dot{\xi}') \to ($ 

$)$

(11) $(\xi', \xi) \to (A\eta'\eta,\ A\eta\bar{\eta})$,
(12) $(\bar{\eta}, \xi) \to (\eta,\ A\eta\bar{\eta})$,
(13) $(\bar{\eta}, \xi') \to (\eta,\ A\eta\eta')\}$.

This system S is a CSDS and generates a CSDL

$T(S) = \{t \mid t =$  $, n \geqq 1\}$

The generation process of a tree† t ($\mu(t) = A^3a^2Aa^2A^2a^2Aa^2$) in $T(S)$ is illustrated in Fig. 3.1.

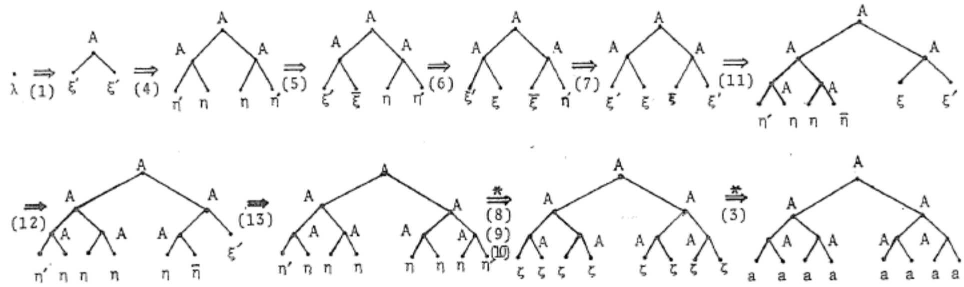


FIG. 3.1. The topological representation of the generation process of a tree t ($\mu(t) = A^3a^2Aa^2A^2a^2Aa^2$). The numbers attached to the arrows ⟹ means the rule numbers used there.

Let us define several special cases of CSDS in the following. The first is scattered-context dendrolanguage generating system.

Definition†† *3.4.* Let $S = (\Omega, V, \Sigma, P, \lambda_0)$ be a CSDS. If missing the adjacency condition (3) of definition 3.2, we call S a *scattered-context dendrolanguage generating system* (abbreviated SCDS). When clearer distinction is desired we write a SCDS, $\overset{*}{S}$. The subset $T(\overset{*}{S})$ of $\mathscr{T}_V$ generated by SCDS is called a scattered-context dendrolanguage (SCDL).

In the following three definitions, let $S = (\Omega, V, \Sigma, P, \lambda_0)$ be a CSDS.

Definition 3.5. If P is a finite subset of $\mathscr{T}_\Delta \times \mathscr{T}_\Omega$, *i.e.*, each rule of P is in the form $(\dot{\xi}) \to (t)$, $\xi \in \Lambda$, $t \in \mathscr{T}_\Omega$, then S is called a *context-free dendrolanguage generating system* (CFDS). A subset $T(S)$ of $\mathscr{T}_V$ generated by a CFDS, S is called a *context-free dendrolanguage* (CFDL).

† For a symbol string x, x^n means an n times concatenated string of x.

†† The idea of SCDS is originated from Greibach and Hopcroft[15].

Definition† *3.6.* If each rule of P is in the form $(\dot{\xi}) \to (t)$, $\xi \in \Lambda$, $\mu(t) \in (V^*\Lambda V^* \cup V^+)$, *i.e.*, a tree in the righthand side of each rule contains at most one nonterminal node symbol, then S is called a *linear dendrolanguage generating system*.(LDS). The subset $T(S)$ generated by a LDS, is called a *linear dendrolanguage* (LDL).

Definition 3.7. If each rule of P is in the form $(\dot{\xi}) \to (t)$, $\xi \in \Lambda$, $\mu(t) \in (V^*\Lambda \cup V^+)$ then S is called a *right-linear dendrolanguage generating system* (RLDS). The subset $T(S)$ generated by a RLDS, S is called a *right-linear dendrolanguage* (RLDL).

A *left-linear dendrolanguage generating system* (LLDS) and a *left-linear dendrolanguage* (LLDL) can be defined in a similar way.

Here, it should be noted that, in the cases of CFDS, LDS, RLDS and LLDS, we do not need the adjacency condition (3) of Definition 3.2, since each rules of these dendrolanguage generating system has only one nonterminal node symbol in its lefthand side. Thus, we need the adjacency condition only for a CSDS.

In the followings, these dendrolanguage generating systems and dendrolanguages will be generically named dendrolanguage generating system (DS) and dendrolanguage (DL), respectively. Two DS, S_1 and S_2 are said to be equivalent and written $S_1 = S_2$ if and only if $T(S_1) = T(S_2)$, *i.e.*, DL's generated by them are the same.

In closing of this chapter, some examples of DS and DL are given:

Example 3.2. An example of RLDS:

$$S = (\Omega, V, \Sigma, P, \lambda),$$

where $\Omega = \{\lambda, A, a\}$, $\Omega_0 = \{\lambda, a\}$, $\Omega_2 = \{A\}$, $V = \{A, a\}$, $\Sigma = \{a\}$ and $P = \{(\lambda) \to (Aa\lambda), (\lambda) \to (Aaa)\}$.

A generation process of a tree $t(\mu(t) = (Aa)^3 a)$ is shown in Fig. 3.2.

FIG. 3.2. A topological representation of the generation process of a tree $t(\mu(t) = (Aa)^3 a)$ in $T(S)$.

The RLDL, $T(S)$ generated by this S is given by

$$T(S) = \{t \mid \mu(t) = (Aa)^n a,\ n \geqq 1\}.$$

Example 3.3. An example of LDS:

$$S = (\Omega, V, \Sigma, P, \lambda),$$

where $\Omega = \{\lambda, A, a\}$, $\Omega_0 = \{\lambda, a\}$, $\Omega_3 = \{A\}$, $V = \{A, a\}$, $\Sigma = \{a\}$ and $P = \{(\lambda) \to (Aa\lambda a), (\lambda) \to (a)\}$.

† V^+ denotes VV^*, which is a set of all strings over an alphabet V not containing ε.

$T(S)=\{t \mid \mu(t)=(Aa)^n a^{n+1},\ n \geqq 0\}$ is the LDL generated by the LDS, S. An element $t(\mu(t)=(Aa)^3 a^4)$ in $T(S)$ is shown in Fig. 3.3.

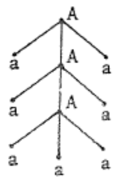


FIG. 3.3. A topological representation of $t(\mu(t)=(Aa)^3 a^4)$.

Example 3.4. An example of CFDS is given as follows:

(1) $\Omega=\{\lambda,\ A,\ a\}$, $\Omega_0=\{\lambda,\ a\}$, $\Omega_2=\{A\}$.

(2) $V=\{A,\ a\}$, $\Sigma=\{a\}$.

(3) $P=\{(\lambda)\rightarrow(a),\ (\lambda)\rightarrow(A\lambda\lambda)\}$.

This CFDS generates a CFDL

$$T(S)=\{t \mid D_t \subseteq \{1,2\}^*,\ {}^\forall m \in \overline{D}_t,\ t(m)=a,\ {}^\forall n \in (D_t-\overline{D}_t),\ t(n)=A\},$$

which is a set of all binary trees whose leaf nodes have a symbol a and the other nodes A as shown in Fig. 3.4.

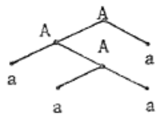


FIG. 3.4. A topological representation of a tree $t(\mu(t)=A^2aAa^3)$ in $T(S)$.

Example 3.5. The following DS is an example of SCDS:

$$\overset{+}{S}=(\Omega,\ V,\ \Sigma,\ P,\ \lambda),\ \text{where}$$

(1) $\Omega=\{\lambda,\ \xi,\ A,\ a,\ b\}$, $\Omega_0=\{\lambda,\ \xi,\ a,\ b\}$, $\Omega_2=\{A\}$.

(2) $V=\{A,\ a,\ b\}$, $\Sigma=\{a,\ b\}$.

(3) $P=\{(\lambda)\rightarrow(A\xi\xi),\ (\xi,\ \xi)\rightarrow(Aa\xi,\ Aa\xi),\ (\xi,\ \xi)\rightarrow(Ab\xi,\ Ab\xi),$
$(\xi,\ \xi)\rightarrow(Aaa,\ Aaa)\}$.

This SCDS, $\overset{+}{S}$ generates a SCDL

$$T(\overset{+}{S})=\{t \mid \mu(t)=A(xAaa)^2,\ x \in (\{A\}\cdot\Sigma)^*\}.$$

The generation process of a tree $t(\mu(t)=A(AaAbAaa)^2)$ in $T(\overset{+}{S})$ is given in Fig. 3.5.

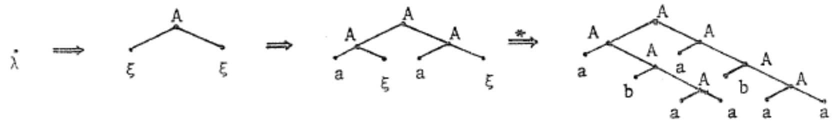


FIG. 3.5. A topological representation of a generation process of a tree $t(\mu(t)=A(AaAbAaa)^2)$ in $T(\overset{+}{S})$.

Here, it should be noted that since we need not consider the adjacency condition (3) of Definition 3.2, we can apply, say, a rule $(\xi, \xi) \to (Aa\xi, Aa\xi)$ to the third tree of Fig. 3.5. This is impossible if the DS of this example is considered as a CSDS, S and so the adjacency condition must be taken into consideration. In this case, we have $T(S) = \{t \mid \mu(t) = A(Aaa)^2\}$ as easily understood.

4. Canonical form of DS

In this chapter, we present a canonical form of DS. It will help us in the discussions in the following chapters.

First, we prepare several lemmas and corollaries.

Lemma 4.1. For arbitrary DS, $S=(\Omega, V, \Sigma, P, \lambda)$, there exists an equivalent DS, $S_1=(\Omega_1, V_1, \Sigma_1, P_1, \lambda_1)$ such that P_1 is a finite subset of $(\mathscr{T}_{\Lambda_1} \times \mathscr{T}_{\Omega_1}) \cup (\mathscr{T}_{\Lambda_1} \times \mathscr{T}_{\Lambda_1} \times \mathscr{T}_{\Omega_1} \times \mathscr{T}_{\Omega_1})$. That is, S_1 is a DS such that $T(S)=T(S_1)$ and any rule of P_1 is in either form of

$$(\dot{\xi}) \to (t) \tag{4.1}$$

or

$$(\dot{\xi}_1, \dot{\xi}_2) \to (t_1, t_2), \tag{4.2}$$

where $\dot{\xi}, \dot{\xi}_1, \dot{\xi}_2 \in \mathscr{T}_{\Lambda_1}$ and $t, t_1, t_2 \in \mathscr{T}_{\Omega_1}$.

Proof We construct DS, S_1 from S as follows:
First, let Λ' be the new nonterminal node symbol set defined by

$$\Lambda' = \{[i, j] \mid i : (\dot{\lambda}_1, \ldots, \dot{\lambda}_{l_i}) \to (t_1, \ldots, t_{l_i}) \text{ in } P,\ 1 \leqq i \leqq \# P,\ 2 \leqq j \leqq l_i - 1\}, \quad (4.3)^\dagger$$

where we assume that rules in P are appropriately numbered. Putting $\Omega_1 = \Omega \cup \Lambda'$, $V_1 = V$, $\Sigma_1 = \Sigma$, $\lambda_1 = \lambda$, we determine P_1 as follows: For each $i : (\dot{\lambda}_1, \ldots, \dot{\lambda}_{l_i}) \to (t_1, \ldots, t_{l_i})$ in P, $1 \leqq i \leqq \# P$,

(i) if $l_i \leqq 2$ then it is contained in P_1,

and

(ii) if $l_i \geqq 3$ then the following $(l_i - 1)$ rules are contained in P_1:

$$\left.\begin{array}{l} (\dot{\lambda}_1, \dot{\lambda}_2) \to (t_1, [i, \dot{2}]) \\ ([i, \dot{j}], \dot{\lambda}_{j+1}) \to (t_j, [i, \dot{j+1}]);\quad 2 \leqq j \leqq l_i - 2 \\ ([i, \dot{l}_{i-1}], \dot{\lambda}_{l_i}) \to (t_{l_i - 1}, t_{l_i}) \end{array}\right\} \tag{4.4}$$

From the above construction procedure, we can directly know that S_1 is a DS having the required property of this lemma.

Here, we should note that this lemma is trivial for CFDS, LDS, RLDS, and LLDS, since they have only the type of rules in the form of the first in (4.1).

Lemma 4.2. For any DS, $S=(\Omega, V, \Sigma, P, \lambda)$, there exists an equivalent DS, $S_2=(\Omega_2, V_2, \Sigma_2, P_2, \lambda_2)$ of which rules satisfy the following conditions:

(i) For any rule of type $(\dot{\xi}) \to (t)$ in P, the tree t is in $\mathscr{T}_{(\Omega_2 - \Sigma_2)}$ or in $\mathscr{T}_{\Sigma_2}$,

(ii) For any rule of type $(\dot{\xi}_1, \ldots, \dot{\xi}_l) \to (t_1, \ldots, t_l)$, $l \geqq 2$ in P, the trees t_i

---

† $\# P$ denotes the number of elements of P.

$(1 \leqq i \leqq l)$ are in $\mathscr{T}_{(\Omega_2 - \Sigma_2)}$.

Proof Let an arbitrary DS be $S=(\Omega, V, \Sigma, P, \lambda)$.
We define a mapping $\eta: \Omega \to (\Omega - \Sigma) \cup \{\eta_a | a \in \Sigma\}$ by

$$\begin{aligned} \eta(X) &= X \quad \text{for any } X \text{ in } (\Omega - \Sigma) \\ \eta(a) &= \eta_a \quad \text{for any } a \text{ in } \Sigma \end{aligned} \tag{4.5}$$

The mapping η can be extended to $\Omega^* \to ((\Omega - \Sigma) \cup \{\eta_a | a \in \Sigma\})^*$, through the conventional way, *i.e.*, for any $x = Xy \in \Omega^*$, $X \in \Omega$, $y \in \Omega^*$, $\eta(x) = \eta(X)\eta(y)$.

Using the help of this mapping, we construct a DS, $S_2 = (\Omega_2, V_2, \Sigma_2, P_2, \lambda_2)$ from S as follows: First, put $\Omega_2 = \Omega \cup \{\eta_a | a \in \Sigma\}$, $V_2 = V$, $\Sigma_2 = \Sigma$, $\lambda_2 = \lambda$. Next, P_2 is constructed by the following two rules:

(i) For all rule $(\dot{\xi}_1, \ldots, \dot{\xi}_l) \to (t_1, \ldots, t_l)$ in P, $(\dot{\xi}_1, \ldots, \dot{\xi}_l) \to (t'_1, \ldots, t'_l)$ is contained in P_2, where $\mu(t'_i) = \eta(\mu(t_i))$ for all i, $1 \leqq i \leqq l$.

(ii) For all η_a in $\{\eta_a | a \in \Sigma\}$, $(\dot{\eta}_a) \to (\dot{a})$ is contained in P_2.

As easily known from the construction rules (i) and (ii), S_2 satisfies the conditions (i) and (ii) of this lemma. It is also obvious that $T(S) = T(S_2)$.

Combining the Lemmas 4.1 and 4.2, we can obtain the following corollary:

Corollary 4.1. For any DS, $S=(\Omega, V, \Sigma, P, \lambda)$ there exists an equivalent DS, $S_3 = (\Omega_3, V_3, \Sigma_3, P_3, \lambda_3)$ of which rules are in either one form of the following three:

(i) $(\dot{\xi}) \to (\dot{a})$, (4.6)

(ii) $(\dot{\xi}) \to (t)$, (4.7)

(iii) $(\dot{\xi}_1, \dot{\xi}_2) \to (t_1, t_2)$, (4.8)

where ξ, ξ_1, ξ_2 in Λ_3, a in Σ_3 and t, t_1, t_2 in $\mathscr{T}_{(\Omega_3 - \Sigma_3)}$.

Next, let us introduce a concept of order of DS. Let $S=(\Omega, V, \Sigma, P, \lambda)$ be a DS. If, for any rule $(\dot{\xi}_1, \ldots, \dot{\xi}_l) \to (t_1, \ldots, t_l)$ in P, the depths $d(t_i)$ of t_i $(1 \leqq i \leqq l)$ are less then or equal to n, then we say that the DS, S has *order n.*

Lemma 4.3. Let S be a CSDS of order n. Then, if $n \geqq 2$, there exists an equivalent CSDS, S' of order $(n-1)$.

Proof Put $S=(\Omega, V, \Sigma, P, \lambda)$. From the S we construct a CSDS, $S'=(\Omega', V', \Sigma', P', \lambda')$ as follows:

(1) P' is constructed by the following rules: For each rule

$$(\dot{\xi}_1, \ldots, \dot{\xi}_l) \to (t_1, \ldots, t_l) \tag{4.9}$$

in P, (i) if $d(t_i) < n$ for all i $(1 \leqq i \leqq l)$, it should be contained in P', (ii) otherwise, *i.e.*, if there exist some t_i's such that $d(t_i) = n$, then

$$(\dot{\xi}_1, \ldots, \dot{\xi}_l) \to (t'_1, \ldots, t'_l) \tag{4.10}$$

should be contained in P', where if $d(t_i) < n$, then $t'_i = t_i$ and if $d(t_i) = n$, then t'_i is determined as follows:

(a) the tree domain $D_{t'_i} = \{m | d(m) \leqq 1, m \in D_{t_i}\}$,

(b) $t'_i(0) = t_i(0)$, (c) for $m(\neq 0)$ in $D_{t'_i}$, $t'_i(m)$ is $t_i(m)$ if $\sigma(t_i(m)) = 0$

and is ξ_{im} if $\sigma(t_i(m)) \neq 0$. Furthermore, coressponding to the last cases, *i.e.*, the cases of $\sigma(t_i(m)) \neq 0$, the rules

$$(\dot{\xi}_{im}) \rightarrow (s_{im}) \tag{4.11}$$

should be included in P', where $D_{s_{im}} = \{n \mid n \in N^*,\ m \cdot n \in D_{t_i}\}$ and $s_{im}(n) = t_i(m \cdot n)$.

(2) Ω' is the union of Ω and the set of variables ξ_{im}'s introduced in the above, $V' = V$, $\Sigma = \Sigma'$ and $\lambda' = \lambda$.

As easily known from the above construction procedure of S', it is a CSDS with order $(n-1)$ and S' is equivalent to S, *i.e.*, $T(S') = T(S)$.

The following two lemmas can be proved in the same way as Lemma 4.3.

Lemma 4.4. For any SCDS of order n $(n \geqq 2)$, there exists an equivalent SCDS of order $(n-1)$.

Lemma 4.5. For any CFDS of order n $(n \geqq 2)$, there exists an equivalent CFDS of order $(n-1)$.

Repetitive applications of Lemmas 4.3, 4.4 and 4.5 deduce the following corollary.

Corollary 4.2. For any DS which is CSDS, SCDS or CFDS, there exists an equivalent DS of order 1 which is of the respective type of DS.

Next, turning our attention to LDS, RLDS and LLDS, we find that such a fine result as above can not hold for them.

Lemma 4.6. There exists an RLDL which can be generated by an RLDS of order n but not by any RLDS with order less than n.

Proof Consider the following RLDS, S: For any $n (\geqq 1)$,

$$S^{(n)} = (\Omega,\ V,\ \Sigma,\ P,\ \lambda) \tag{4.12}$$

where $\Omega = \{\xi\} \cup V$, $V = \{X, Y_1, Y_2, \ldots, Y_{n-1}, a\}$, $\Sigma = \{a\}$, $\lambda = \xi$ and P is constituted from the following two rules:

$$(\dot{\xi}) \rightarrow (t)\ ;\ \ \mu(t) = XY_1Y_2, \ldots, Y_{n-1}a\xi \tag{4.13}$$

$$(\dot{\xi}) \rightarrow (\dot{a}) \tag{4.14}$$

Here, $\sigma(X) = 2$, $\sigma(Y_i) = 1$ $(1 \leqq i \leqq n-1)$, $\sigma(a) = \sigma(\xi) = 0$.

Obviously, the DL, $T(S^{(n)})$ generated by $S^{(n)}$ of (4.12) is an RLDL generated by an RLDS of order n. But, this DL can not be generated by any RLDS of

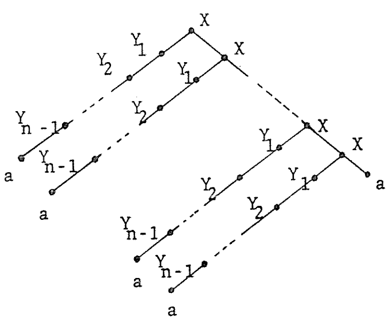


FIG. 4.1. A typical example of trees contained in $T(S^{(n)})$ of Lemma 4.6.

order $(n-1)$. This can easily be understood by considering the DL, $T(S^{(n)})$, which contains trees shown in Fig. 4.1. That is, if we want to generate $T(S^{(n)})$ by a DS, S' of order less than n, then some rules of S' should contain more than one nonterminal node symbols in their right-hand side. But such a DS can not be an RLDS.

Similar discussions as above yield the following lemmas:

Lemma 4.7. There exists an LLDL which can be generated by an LLDS of order n but not by any LLDS of order less than n.

Lemma 4.8. There exists an LDL which can be generated by an LLDS of order n but not by any one of order less than n.

Now, combining Corollaries 4.1 and 4.2, we directly prove the following theorem:

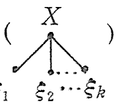
Theorem 4.1. For any CSDS (SCDS), there exists an equivalent CSDS (SCDS), $S=(\Omega, V, \Sigma, P, \lambda)$ of which rules are in forms of the following three:

(i) $(\dot{\xi})\to(\dot{a})$; ξ in $\Lambda=\Omega-V$, a in Σ, (4.15)

(ii) $(\dot{\xi})\to(t)$; ξ in Λ, t in $\mathscr{T}^1_{(\Omega-\Sigma)}$, (4.16)†

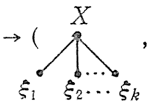
(iii) $(\dot{\xi}_1, \dot{\xi}_2)\to(t_1, t_2)$; ξ_1, ξ_2 in Λ, t_1, t_2 in $\mathscr{T}^1_{(\Omega-V)}$. (4.17)†

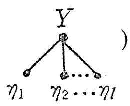
Note that an explicit representation of the types (ii) and (iii) of rules is as follows:

(ii-1) $(\dot{\xi})\to($ 

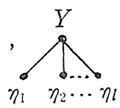
$)$; X in $V-\Sigma$, $\xi, \xi_1, \ldots, \xi_k$ in Λ, $k\geqq 1$, (4.18)

(ii-2) $(\dot{\xi})\to(\dot{\eta})$; ξ, η in Λ, (4.19)

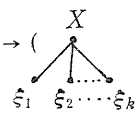
(iii-1) $(\dot{\xi}, \dot{\eta})\to($ 

, 

$)$; X, Y in $V-\Sigma$, (4.20)
ξ, ξ_i in Λ,
η, η_j in Λ,
$(1\leqq i\leqq k,\ 1\leqq j\leqq l)$,

(iii-2) $(\dot{\xi}, \dot{\eta})\to(\ \dot{\xi}_1,$ 

$)$; Y in $V-\Sigma$, (4.21)
ξ, ξ_1 in Λ,
η, η_j in Λ,
$(1\leqq j\leqq l)$,

(iii-3) $(\dot{\xi}, \dot{\eta})\to($ 

$, \dot{\eta}_1)$; X in $V-\Sigma$, (4.22)
ξ, ξ_i in Λ,
η, η_1 in Λ,
$(1\leqq i\leqq k)$,

(iii-4) $(\dot{\xi}, \dot{\eta})\to(\dot{\xi}_1, \dot{\eta}_1)$; ξ, ξ_1, η, η_1 in Λ. (4.23)

† $\mathscr{T}^1_{(\Omega-\Sigma)}$ means the set of all finite trees over r.a. $(\Omega-\Sigma)$ of which depth is less than or equal to 1.

A slight consideration with an appropriate introduction of distinct nonterminal node symbols asserts that by using only the rules of types (ii-1) and (iii-4) we can construct a set of rules equivalent to (iii-1), (iii-2) and (iii-3). Furthermore, we can take the set of rules $(\dot{\xi}, \dot{\zeta}) \to (\dot{\eta}, \dot{\zeta})$, $(\dot{\zeta}, \dot{\xi}) \to (\dot{\zeta}, \dot{\eta})$ for all ζ in Λ instead of (ii-2). Thus, we have the following theorem, which may be called a normal form theorem for CSDS and SCDS.

Theorem 4.2. For any CSDS (SCDS), there exists an equivalent CSDS (SCDS) of which rules are in either one form of (4.15), (4.18) or (4.23). That is,

$$\text{(i)}\ (\dot{\xi}) \to (\dot{a}),\quad \text{(ii)}\ (\dot{\xi}) \to \left(\begin{array}{c} X \\ \swarrow \downarrow \cdots \searrow \\ \xi_1\ \ \xi_2 \cdots \xi_k \end{array} \right)\quad \text{(iii)}\ (\dot{\xi}, \dot{\eta}) \to (\dot{\xi}_1, \dot{\eta}_1)$$

Similarly, taking account of the definition of CFDS and using Corollaries 4.1 and 4.2, we obtain the following theorem:

Theorem 4.3. For any CFDS, there exists an equivalent CFDS, $S=(\Omega, V, \Sigma, P, \lambda)$ of which rules are in either form of the following two:

$$\text{(i)}\ (\dot{\xi}) \to (\dot{a})\ ;\ \xi \text{ in } \Lambda = \Omega - V,\ a \text{ in } \Sigma, \tag{4.24}$$

$$\text{(ii)}\ (\dot{\xi}) \to \left(\begin{array}{c} X \\ \swarrow \downarrow \cdots \searrow \\ \xi_1\ \ \xi_2 \cdots \xi_k \end{array} \right)\ ;\ \xi,\ \xi_i \text{ in } \Lambda,\ (1 \leqq i \leqq k),\ X \text{ in } V - \Sigma,\ k \geqq 1. \tag{4.25}$$

Proof It is an immediate result of application of Corollaries 4.1 and 4.2 to a CFDS that we can construct an equivalent CFDS of which rules are in the forms of (4.15), (4.18) or (4.19). Deletion of the type of (4.19) from them can be verified by the same way of Theorem 4.4 in page 50 of Hopcroft and Ullman[19].

Concerning LDS, RLDS and LLDS, we can not have such normal form theorems, because of Lemmas 4.6, 4.7 and 4.8. Thus, we will take those of Definitions 3.6 and 3.7 as their normal forms.

5. Families of dendrolanguages

In the previous chapters, several types of dendrolanguage generating systems, *i.e.*, CSDS, SCDS, CFDS, LDS, RLDS and LLDS were introduced. The dendrolanguage generating power of these systems is a fundamental and important point to be revealed if we attempt an application of them to the formal language theory. It is, of course, very interesting in itself.

In this chapter, this problem is discussed and it is shown that there exists a hierarchy of dendrolanguages which is very closely related to that of the phrase structure languages. Let each of $\mathscr{T}_{RL}$, $\mathscr{T}_{LL}$, $\mathscr{T}_{L}$, $\mathscr{T}_{CF}$, $\mathscr{T}_{SC}$ and $\mathscr{T}_{CS}$ denote the family of RLDL, LLDL, LDL, CFDL, SCDL and CSDL, respectively. Then, it will be shown that the proper inclusion relations

$$\mathscr{T}_{RL},\ \mathscr{T}_{LL} \subsetneqq \mathscr{T}_{L} \subsetneqq \mathscr{T}_{CF} \subsetneqq \mathscr{T}_{SC} \subsetneqq \mathscr{T}_{CS}$$

hold. This is one of the main results of this chapter.

First, the following theorem is a direct result of the definitions.

Theorem 5.1. (1) There exists a DL, T in both $\mathcal{T}_{RL}$ and $\mathcal{T}_{LL}$.

(ii) There exists a DL, T in $\mathcal{T}_{RL}$ but not in $\mathcal{T}_{LL}$.

(iii) There exists a DL, T in $\mathcal{T}_{LL}$ but not in $\mathcal{T}_{RL}$.

Thus, we know that $\mathcal{T}_{RL} \cap \mathcal{T}_{LL} \fallingdotseq \phi$ and that $\mathcal{T}_{RL}$, or $\mathcal{T}_{LL}$, can not be contained in the other.

Proof (i) Let V be $\{A, a\}$, where $\sigma(a)=0$ and $\sigma(A)=1$. Consider a DL, T over a r.a. V defined by

$$T=\{t \mid \mu(t)=A^n a,\ n \geqq 1\} \tag{5.1}$$

This DL, T can obviously be generated by either an RLDS or an LLDS. Thus, T is in both $\mathcal{T}_{RL}$ and $\mathcal{T}_{LL}$.

(ii) and (iii) As easily understood, the RLDL, $T(S)$ of Example 3.2 cannot be generated by any LLDS. Conversely, we can easily give an example of an LLDL which can not be generated by any RLDS similarly to Example 3.2.

Here we should note that $\mathcal{T}_{RL}$ and $\mathcal{T}_{LL}$ are closely related. That is, if a DL, T is an RLDL, then $T'=\{t \mid \varphi(t)=\mu(t)^R,\ t \in T\}$ is an LLDL†. Conversely, if T is an LLDL, then $T'=\{t \mid \mu(t)=\varphi(t)^R,\ t \in T\}$ is an RLDL.

Now, we denote the families of regular[19], linear[19], context-free[19], scattered-context[15], and context-sensitive languages[19] by $\mathcal{L}_R$, $\mathcal{L}_L$, $\mathcal{L}_{CF}$, $\mathcal{L}_{SC}$ and $\mathcal{L}_{CS}$, respectively. Then, we can find useful relations between DL's and these languages, as stated in the following theorem. This theorem is a generalization of the Theorem 3.16 of Brainerd[13] which asserts that if T is a CFDL then the set of prefix notations of the trees in T is a context-free language. Here we use some helpful notations: For any DL, T, $\mu(T)$ denotes the set of prefix notations $\{\mu(t) \mid t \in T\}$ and for any family of DL, $\mathcal{T}$, $\mu(\mathcal{T})=\{\mu(T) \mid T \in \mathcal{T}\}$. As for the suffix notation φ, $\varphi(T)$ and $\varphi(\mathcal{T})$ are similarly defined.

Theorem 5.2. (i) $\mu(\mathcal{T}_{RL}) \subsetneqq \mathcal{L}_R,\ \varphi(\mathcal{T}_{LL}) \subsetneqq \mathcal{L}_R$ (5.2)

(ii) $\mu(\mathcal{T}_L) \subsetneqq \mathcal{L}_L,\ \varphi(\mathcal{T}_L) \subsetneqq \mathcal{L}_L$ (5.3)

(iii) $\mu(\mathcal{T}_{CF}) \subsetneqq \mathcal{L}_{CF},\ \varphi(\mathcal{T}_{CF}) \subsetneqq \mathcal{L}_{CF}$ (5.4)

(iv) $\mu(\mathcal{T}_{SC}) \subsetneqq \mathcal{L}_{SC},\ \varphi(\mathcal{T}_{SC}) \subsetneqq \mathcal{L}_{SC}$ (5.5)

(v) $\mu(\mathcal{T}_{CS}) \subsetneqq \mathcal{L}_{CS},\ \varphi(\mathcal{T}_{CS}) \subsetneqq \mathcal{L}_{CS}$ (5.6)

Proof First, we consider (i)~(iv). Let $S=(\Omega, V, \Sigma, P, \lambda)$ be an arbitrary SCDS. For this S, we can construct a scattered-context grammar $G=(\Omega, V, P', \lambda)$ as follows: For each rule $(\dot{\xi}_1, \dot{\xi}_2, \ldots, \dot{\xi}_l) \to (t_1, t_2, \ldots, t_l)$ in P let a rule $(\xi_1, \xi_2, \ldots, \xi_l) \to (\mu(t_1), \mu(t_2), \ldots, \mu(t_l))$ be contained in P' of G. Then, it is easily known that $L(G)=\{\mu(t) \mid t \in T(S)\}=\mu(T(S))$. Thus, for any SCDL, T, $\mu(T)$ is in $\mathcal{L}_{SC}$. Hence, $\mu(\mathcal{T}_{SC}) \subseteq \mathcal{L}_{SC}$.

The same techniques as the above can be applied to (i), (ii) and (iii) and it is proved that $\mu(\mathcal{T}_{CF}) \subseteq \mathcal{L}_{CF}$, $\mu(\mathcal{T}_L) \subseteq \mathcal{L}_L$, and $\mu(\mathcal{T}_{RL}) \subseteq \mathcal{L}_R$.

Next, we show the proper inclusion for each case.

First, DL, $T=\{t \mid \mu(t)=ww^R c,\ w \in \{A, B\}^*,\ \sigma(A)=\sigma(B)=1,\ \sigma(c)=0\}$ is not SCDL

† Here, for any string w of symbols, w^R means the reverse of w.

and its prefix notation $\mu(T)$ is CFL[19]. Hence, $\mu(\mathscr{T}_{SC}) \subsetneqq \mathscr{L}_{SC}$. Next, a DL, $T=\{t \mid \mu(t) = XA^n cB^n c,\ n \geqq 1,\ \sigma(X)=2,\ \sigma(A)=\sigma(B)=1,\ \sigma(c)=0\}$ is not CFDL but SCDL and its prefix notation $\mu(T)$ is CFL. Hence, $\mu(\mathscr{T}_{CF}) \subsetneqq \mathscr{L}_{CF}$.

A DL, $T=\{t \mid \mu(t) = XA^{n_1}cXA^{n_2}c \cdots XA^{n_k}cc,\ k \geqq 1,\ n_1, \ldots, n_k \geqq 1\}$ is not RLDL but CFDL and its prefix notation $\mu(T)$ is regular. Hence, $\mu(\mathscr{T}_{RL}) \subsetneqq \mathscr{L}_R$.

Similarly to RLDL, we obtain $\mu(\mathscr{T}_L) \subsetneqq \mathscr{L}_L$.

Concerning (v), it is sufficient to see that for each rule $(\dot{\xi}_1, \ldots, \dot{\xi}_l) \to (t_1, \ldots, t_l)$ in P of CSDS, $S=(\Omega, V, \Sigma, P, \lambda_0)$, the inequalities

$$|\mu(\dot{\xi}_i)| \leqq |\mu(t_i)|,\ (1 \leqq i \leqq l) \tag{5.7}$$

hold, where $|w|$ denotes the length of a string w. Thus, it can easily be understood that for any SCDS, S, we can construct a linear bounded automaton[18] A such that $\mu(T(S)) = M(A)$, where $M(A)$ is a set of strings accepted by A[18]. This means that $\mu(\mathscr{T}_{CS}) \subseteq \mathscr{L}_{CS}$. On the other hand, there exists a DL, $T=\{t \mid \mu(t) = ww^R c,\ w \in \{A, B\}^*,\ \sigma(A)=\sigma(B)=1,\ \sigma(c)=0\}$ which is not CSDL and whose prefix notation $\mu(T)$ is CFL. Hence, $\mu(\mathscr{T}_{CS}) \subsetneqq \mathscr{L}_{CS}$.

In the case of suffix notation φ, we can similarly prove that $\varphi(\mathscr{T}_{LL}) \subsetneqq \mathscr{L}_R$, $\varphi(\mathscr{T}_L) \subsetneqq \mathscr{L}_L$, $\varphi(\mathscr{T}_{CF}) \subsetneqq \mathscr{L}_{CF}$, $\varphi(\mathscr{T}_{SC}) \subsetneqq \mathscr{L}_{SC}$ and $\varphi(\mathscr{T}_{CS}) \subsetneqq \mathscr{L}_{CS}$.

Using Theorem 5.2, we can easily prove the following theorem:

Theorem 5.3.
$$\mathscr{T}_{RL},\ \mathscr{T}_{LL} \subsetneqq \mathscr{T}_L \subsetneqq \mathscr{T}_{CF} \subsetneqq \mathscr{T}_{SC} \tag{5.8}$$

Proof From definitions of DS's, it is obvious that $\mathscr{T}_{RL},\ \mathscr{T}_{LL} \subseteq \mathscr{T}_L \subseteq \mathscr{T}_{CF} \subseteq \mathscr{T}_{SC}$. First, consider a DL of Example 3.3. This DL, $T(S)$ is such one that $\mu(T(S))$ is in $\mathscr{L}_L$ but not in $\mathscr{L}_R$. Thus, the fact $\mathscr{L}_R \subsetneqq \mathscr{L}_L$ and (i) of Theorem 5.2 asserts that there exist a DL which is in $\mathscr{T}_L$ but neither in $\mathscr{T}_{RL}$ nor in $\mathscr{T}_{LL}$. With the fact that $\mathscr{L}_L \subsetneqq \mathscr{L}_{CF} \subsetneqq \mathscr{L}_{SC}$ and Theorem 5.2, similar discussions also prove that $\mathscr{T}_L \subsetneqq \mathscr{T}_{CF} \subsetneqq \mathscr{T}_{SC}$, where Examples 3.4 and 3.5 can work well.

This and the following theorem show not only that there exists a hiearchy very similar to that of phrase structure languages but also that there exists a good correspondence between the families of DL and those of phrase structure languages.

Theorem 5.4.
$$\mathscr{T}_{SC} \subseteq \mathscr{T}_{CS} \tag{5.9}$$

Proof This theorem is proved by showing that for any SCDS, $S=(\Omega, V, \Sigma, P, \lambda_0)$ we can construct a CSDS, S' such that $T(S)=T(S')$.

From Theorem 4.2, with no loss of generality we can assume that each rule of P is in one form of the three types (i), (ii) and (iii) of Theorem 4.2. The CSDS, $S'=(\Omega', V, \Sigma, P', \lambda_0)$ can be constructed as follows: Each rule in P, of type (i) and (ii) of Theorem 4.2, should be contained in P'. For each rule of type (iii), the following rules should be contained in P': That is, for a rule $(\dot{\lambda}_1, \dot{\lambda}_2) \to (\dot{\zeta}_1, \dot{\zeta}_2)$,

(1) $(\dot{\lambda}_1, \dot{\eta}) \to (\dot{\vec{\lambda}}_1, [\eta, \dot{\vec{\lambda}}_2])$ for all η in $\Lambda = \Omega - V$, (5.10)

(2) $([\eta, \dot{\vec{\lambda}}_2], \dot{\xi}) \to (\dot{\eta}, [\xi, \dot{\vec{\lambda}}_2])$ for all ξ, η in Λ, (5.11)

(3) $([\eta, \dot{\vec{\lambda}}_2], \lambda_2) \to ([\overleftarrow{\lambda}_1, \dot{\eta}], \dot{\zeta}_2)$ for all η in Λ, (5.12)

(4) $(\dot{\xi}, [\overleftarrow{\lambda}_1, \dot{\eta}]) \to ([\overleftarrow{\lambda}_1, \dot{\xi}], \dot{\eta})$ for all ξ, η in Λ, (5.13)

(5) $(\dot{\lambda}_1, [\overleftarrow{\lambda}_1, \dot{\eta}]) \to (\dot{\zeta}_1, \dot{\eta})$ for all η in Λ (5.14)

where $\vec{\lambda}_1$, $\overleftarrow{\lambda}_1$, $[\eta, \vec{\lambda}_2]$, $[\overleftarrow{\lambda}_1, \eta]$ are new nonterminal node symbols.

The CSDS, S' constructed in the above manner generates just the DL, $T(S)$.

Note that this theorem corresponds to the theorem of Greibach and Hopcroft[15], which states that $\mathcal{L}_{SC} \subseteq \mathcal{L}_{CS}$, and then the proofs for them are very similar. Thus, the reader may complete the details of the proof outlined above, if he wants.

Here, the problem whether $\mathcal{T}_{SC}$ is a proper subfamily of $\mathcal{T}_{CS}$ or not, *i.e.*, whether $\mathcal{T}_{SC} \subsetneqq \mathcal{T}_{CS}$ or not, is left as an open problem. But, note that if it be true that $\mathcal{T}_{SC} \subsetneqq \mathcal{T}_{CS}$ then we can prove that $\mathcal{L}_{SC} \subsetneqq \mathcal{L}_{CS}$, which is an open problem of Greibach and Hopcroft[15].

Thus, we have the main result of this chapter that $\mathcal{T}_{RL}$, $\mathcal{T}_{LL} \subsetneqq \mathcal{T}_{L} \subsetneqq \mathcal{T}_{CF} \subsetneqq \mathcal{T}_{SC} \subseteq \mathcal{T}_{CS}$. Finally, we present some results about order n DL, which is defined as a DL generated by a DS with order n. We denote the families of RLDL, LLDL, LDL, CFDL, SCDL, and CSDL which are generated by the respective DS of order n by $\mathcal{T}^{(n)}_{RL}$, $\mathcal{T}^{(n)}_{LL}$, $\mathcal{T}^{(n)}_{L}$, $\mathcal{T}^{(n)}_{CF}$, $\mathcal{T}^{(n)}_{SC}$ and $\mathcal{T}^{(n)}_{CS}$, respectively. Then, Corollary 4.2 asserts that $\mathcal{T}^{(1)}_{CF} = \mathcal{T}^{(n)}_{CF} = \mathcal{T}_{CF}$, $\mathcal{T}^{(1)}_{SC} = \mathcal{T}^{(n)}_{SC} = \mathcal{T}_{SC}$ and $\mathcal{T}^{(1)}_{CS} = \mathcal{T}^{(n)}_{CS} = \mathcal{T}_{CS}$ for any integer $n \geqq 1$. But, from Lemmas 4.6, 4.7 and 4.8, we must conclude that for any $n \geqq 1$, $\mathcal{T}^{(n-1)}_{RL} \subsetneqq \mathcal{T}^{(n)}_{RL}$, $\mathcal{T}^{(n-1)}_{LL} \subsetneqq \mathcal{T}^{(n)}_{LL}$, and $\mathcal{T}^{(n-1)}_{L} \subsetneqq \mathcal{T}^{(n)}_{L}$. Thus, we know that there exists an infinite hiearchical structure in $\mathcal{T}_{RL}$, $\mathcal{T}_{LL}$ and $\mathcal{T}_{L}$, respectively. On the other hand, it can easily be shown that $\mathcal{T}^{(n)}_{RL}$, $\mathcal{T}^{(n)}_{LL} \subsetneqq \mathcal{T}^{(n)}_{L}$ for any $n \geqq 1$. An infinite hiearchical structure between $\mathcal{T}_{CF}$ and $\mathcal{T}_{CS}$ will be discussed in later chapters.

6. Dendro-automata

In the previous chapters, we have discussed the DS's as the dendrolanguage generators. For these generators, it is meaningful to consider the corresponding acceptors of DL's. In this chapter, we introduce dendro-automata as the dendrolanguage acceptors and reveal the relations between DS's and the dendro-automata. Nondeterministic and deterministic behaiviour of dendro-automata are also discussed.

First, we define several types of dendro-automata:

Definition 6.1. A *nondeterministic context-sensitive dendro-automaton* (abbreviated N-CSDA) is a 5-tuple

$$A = (K,\ V,\ \Sigma,\ \delta,\ F) \tag{6.1}$$

where each of five entries are as follows:

(1) K: a nonempty and finite set of symbols, which is called a set of states. We also define rank of symbols p in K, which is assumed to be 0, *i.e.*, $\sigma(p) = 0$ for all p in K.

(2) V: a ranked alphabet.

(3) Σ $(= V \cap K \subseteq K)$: a set of leaf symbols. From the definition of K, for any symbol a in Σ, $\sigma(a) = 0$.

(4) $$\delta: \underbrace{\mathcal{T}_{(K\cup V)} \times \cdots \times \mathcal{T}_{(K\cup V)}}_{l\ \text{times}} \to 2^{\underbrace{\mathcal{T}_{(K-\Sigma)} \times \cdots \times \mathcal{T}_{(K-\Sigma)}}_{l\ \text{times}}},\quad l \geqq 1, \tag{6.2}$$

a mapping from $\mathcal{T}_{(K\cup V)} \times \cdots \times \mathcal{T}_{(K\cup V)}$ into the set of all subsets of

$\mathscr{T}_{(K-\Sigma)}\times\cdots\times\mathscr{T}_{(K-\Sigma)}$. Usually we represent δ by writing the list of the forms

$$\delta(t_1,\ t_2,\ \ldots,\ t_l)=\{(\dot{p}_1,\ \dot{p}_2,\ \ldots,\ \dot{p}_l)\} \tag{6.3}$$

which means that the ordered l-tuple $(t_1, t_2, \ldots, t_l)$ of trees in $\mathscr{T}_{K\cup V}$ can be reduced to any l-tuple $(\dot{p}_1, \dot{p}_2, \ldots, \dot{p}_l)$ in $\{(\dot{p}_1, \dot{p}_2, \ldots, \dot{p}_l)\}$, where p_i is not in Σ but in $(K-\Sigma)$ and $\dot{p}_i$ is a tree $\{(0, p_i)\}$.

(5) F: a distinct subset of $(K-\Sigma)$, which is called the set of final states.

In the above definition, if δ is a mapping from $\mathscr{T}_{(K-V)}\times\cdots\times\mathscr{T}_{(K\cup V)}$ into $2^{\mathscr{T}(K-V)}\times\cdots\times 2^{\mathscr{T}(K-V)}$ then an N-CSDA, A is called a *quasi-deterministic context-sensitive dendro-automaton* (Q-CSDA). If δ is a mapping from $\mathscr{T}_{(K\cup V)}\times\cdots\times\mathscr{T}_{(K\cup V)}$ into $\mathscr{T}_{(K-\Sigma)}\times\cdots\times\mathscr{T}_{(K-\Sigma)}$, then an N-CSDA, A is called a *deterministic context-sensitive dendro-automaton* (D-CSDA).

Now, define the accepting of N-CSDA:

Definition 6.2. Let $A=(K,\ V,\ \Sigma,\ \delta,\ F)$ be an N-CSDA. For any trees α and β in $\mathscr{T}_{(K\cup V)}$, we write $\alpha\vdash_{\overline{A}}\beta$ (or only $\alpha\vdash\beta$ when A is understood) if there exist $x_i\in(K\cup V)^*$ $(0\leqq i\leqq l)$, $p_i\in(K-\Sigma)$ $(1\leqq i\leqq l)$, and $t_i\in\mathscr{T}_{K\cup V}$ $(1\leqq t\leqq l)$ such that the following three conditions (1), (2) and (3) are satisfied:

(1) $\mu(\alpha)=x_0\mu(t_1)x_1\cdots x_{l-1}\mu(t_l)x_l$,

$\mu(\beta)=x_0p_1x_1\cdots x_{l-1}p_lx_l$

(2) $(\dot{p}_1,\ \dot{p}_2,\ \ldots,\ \dot{p}_l)\in\delta(t_1,\ t_2,\ \ldots,\ t_l)$

(3)† $\Subset_\beta(p_1)\sim\Subset_\beta(p_2)\sim\cdots\sim\Subset_\beta(p_l)$

Next, for any α and β in $\mathscr{T}_{(K\cup V)}$, we write $\alpha\overset{*}{\vdash}\beta$ if either $\alpha=\beta$ or there exist $\alpha_0, \ldots, \alpha_r$ such that $\alpha_0=\alpha$, $\alpha_r=\beta$ and $\alpha_i\vdash\alpha_{i+1}$ for each i $(0\leqq i\leqq r-1)$.

Definition 6.3. Let $A=(K,\ V,\ \Sigma,\ \delta,\ F)$ be an N-CSDA. The subset of $\mathscr{T}_V$,

$$M(A)=\{t\in\mathscr{T}_V\,|\,t\overset{*}{\vdash}\dot{p},\ p\in F\} \tag{6.4}$$

is called a *DL accepted by N-CSDA*, or a *nondeterministic context-sensitive DA dendrolanguage* (N-CSADL).

If A is a Q-CSDA, then $M(A)$ is called a *quasi-deterministic context-sensitive DA dendrolanguage* (Q-CSADL). The DL, $M(A)$ accepted by a D-CSDA is called a *deterministic context-sensitive DA dendrolanguage* (D-CSADL).

Now, let us introduce several types of DA as special cases of N-CSDA.

Definition 6.4. Let $A=(K,\ V,\ \Sigma,\ \delta,\ F)$ be an N-CSDA. When missing the adjacency condition (3) of Definition 6.2, we call A a *non-deterministic scattered-context dendro-automata* (N-SCDA). Similarly, A is called a *quasi-deterministic scattered-context dendro-automata* (Q-SCDA) or a *deterministic scattered-context dendro-automata* if it is a Q-CSDA or a D-CSDA, respectively.

In the following three definitions, let $A=(K,\ V,\ \Sigma,\ \delta,\ F)$ be a N-CSDA.

Definition 6.5. If δ is a mapping $\mathscr{T}_{(K\cup V)}\to 2^{\mathscr{T}(K-V)}$, then A is called a *non-*

† Refer to Definition 3.2.

deterministic context-free dendro-automaton (N-CFDA). If δ is a mapping $\mathscr{T}_{(K\cup V)} \rightarrow \mathscr{T}_{(K-\Sigma)}$ then A is called a *deterministic context-free dendro-automaton* (D-CFDA).

Definition 6.6. If the domain of δ is a subset $\mathscr{T}^{\text{dom } L}$ of $\{t \in \mathscr{T}_{(K\cup V)} \mid \mu(t) \in (V^*(K-\Sigma)V^* \cup V^+\}$ and if $\delta: \mathscr{T}^{\text{dom } L} \rightarrow 2^{\mathscr{T}(K-V)}$, then A is called a *nondeterministic linear dendro-automaton* (N-LDA). If $\delta: \mathscr{T}^{\text{dom } L} \rightarrow \mathscr{T}_{(K-\Sigma)}$ then A is called a *deterministic linear dendro-automaton* (D-LDA).

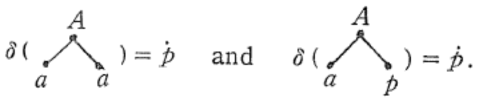
Definition 6.7. If the domain of δ is a subset $\mathscr{T}^{\text{dom } RL}$ of $\{t \in \mathscr{T}_{(K\cup V)} \mid \mu(t) \in V^*(K-\Sigma) \cup V^*\}$ and if $\delta: \mathscr{T}^{\text{dom } RL} \rightarrow 2^{\mathscr{T}(K-\Sigma)}$, then A is called a *nondeterministic right linear dendro-automaton* (N-RLDA). If $\delta: \mathscr{T}^{\text{dom } RL} \rightarrow \mathscr{T}_{(K-V)}$ then A is called a *deterministic right linear dendro-automaton* (D-RLDA).

A *nondeterministic left linear dendro-automaton* (N-LLDA) and a *deterministic left linear dendro-automaton* (D-LLDA) can be defined similarly.

Each DL accepted by N-CFDA, D-CFDA, N-LDA, D-LDA, N-RLDA, D-RLDA, N-LLDA and D-LLDA is called *nondeterministic context-free DA dendrolanguage* (N-CFADL), *deterministic context-free DA dendrolanguage* (D-CFADL), *nondeterministic linear DA dendrolanguage* (N-LADL), *deterministic linear DA dendrolanguage* (D-LADL), *nondeterministic right linear DA dendrolanguage* (N-RLADL), *deterministic right linear DA dendrolanguage* (D-RLADL), *nondeterministic left linear DA dendrolanguage* (N-LLADL), *deterministic left linear DA dendrolanguage* (D-LLADL), respectively.

In the followings, these dendro-automata, introduced by Definitions 6.1, 6.4~6.7, will be generically called dendro-automata (DA).

Example 6.1. Let $A=(K, V, \Sigma, \delta, F)$ be a DA, where $K=\{p\} \cup \Sigma$, $V=\{A, a\}$, $\Sigma=\{a\}$, $\sigma(A)=2$, $\sigma(a)=\sigma(p)=0$, $F=\{p\}$ and δ is defined by



This is a D-RLDA and we can easily know that

$$M(A)=\{t \mid \mu(t)=(Aa)^n a,\ n \geqq 1\}.$$

The accepting process of $t(\mu(t)=(Aa)^3 a)$ by A is illustrated in Fig. 6.1, and the readers easily conceive that it is a reverse process of the generation of t by the RLDS, S of Example 3.2, which is shown in Fig. 3.2.

Here we note that the D-RLADL, $M(A)$ is equal to the DL of Example 3.2.

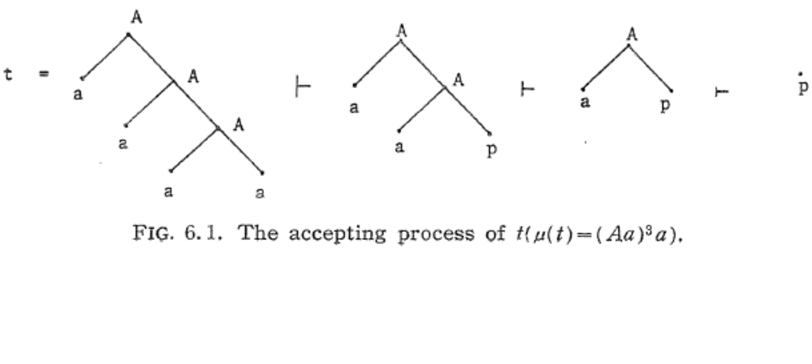


FIG. 6.1. The accepting process of $t(\mu(t)=(Aa)^3 a)$.

Similarly, we can construct D-LDA, D-CFDA, D-SCDA and N-CSDA which accepts precisely the DL's generated by the DS of Example 3.3, the CFDS of Example 3.4, the SCDS of Example 3.5 and the CSDS of Example 3.6. Generally each family of DL's accepted by D-RLDA's, D-LDA's, D-CFDA's, N-SCDA's and N-CSDA's coincides with each one of RLDL's, LDL's, CFDL's, SCDL's and CSDL's, respectively. The proofs for these are given in the followings. The families of DL's accepted by N-CSDA, Q-CSDA, D-CSDA, N-SCDA, Q-SCDA, D-SCDA, N-CFDA, D-CFDA, N-LDA, D-LDA, N-RLDA, D-RLDA, N-LLDA and D-LLDA will be denoted by $\mathcal{M}^N_{CS}$, $\mathcal{M}^Q_{CS}$, $\mathcal{M}^Q_{CS}$, $\mathcal{M}^N_{CS}$, $\mathcal{M}^Q_{SC}$, $\mathcal{M}^D_{SC}$, $\mathcal{M}^N_{CF}$, $\mathcal{M}^D_{CF}$, $\mathcal{M}^N_{L}$, $\mathcal{M}^D_{L}$, $\mathcal{M}^N_{RL}$, $\mathcal{M}^D_{RL}$, $\mathcal{M}^N_{LL}$ and $\mathcal{M}^D_{LL}$, respectively.

The relations among these which are directly followed from the definitions are summarized in Fig. 6.2.

$$
\begin{array}{ccccccccccc}
\mathcal{M}^D_{LL}, & \mathcal{M}^D_{RL} & \subseteq & \mathcal{M}^D_{L} & \subseteq & \mathcal{M}^D_{CF} & \subseteq & \mathcal{M}^D_{SC} & \subseteq & \mathcal{M}^D_{CS} \\
 & & & & & & & \cap\!\!| & & \cap\!\!| \\
\cap\!\!| & \cap\!\!| & & \cap\!\!| & & \cap\!\!| & & \mathcal{M}^Q_{SC} & \subseteq & \mathcal{M}^Q_{CS} \\
 & & & & & & & \cap\!\!| & & \cap\!\!| \\
\mathcal{M}^N_{LL}, & \mathcal{M}^N_{RL} & \subseteq & \mathcal{M}^N_{L} & \subseteq & \mathcal{M}^N_{CF} & \subseteq & \mathcal{M}^N_{SC} & \subseteq & \mathcal{M}^N_{CS}
\end{array}
$$

FIG. 6.2. Some relations among the families of DL's accepted by DA's.

Lemma 6.1. For any N-CSDA, $A=(K, V, \Sigma, \delta, F)$, there exists a CSDS, $S=(\Omega, V, \Sigma, P, \lambda)$ *such that* $M(A)=T(S)$.

Proof Consider a CSDS, $S=(\Omega, V, \Sigma, P, \lambda_0)$ which is effectively constructed from a given N-CSDA, $A=(K, V, \Sigma, \delta, F)$ as follows: Let V and Σ for S and A be the same, and put $\Omega=K\cup V\cup\{\lambda_0\}$ and $\Lambda(=\Omega-V)=(K-\Sigma)\cup\{\lambda_0\}$. P is the set of rules determined as follows:

(i) If $(\dot{p}_1, \ldots, \dot{p}_l)$ is contained in $\delta(t_1, \ldots, t_l)$ then the rule $(\dot{p}_1, \ldots, \dot{p}_l)\rightarrow(t_1, \ldots, t_l)$ should be an element of P.

(ii) For all elements f of F, $(\dot{\lambda}_0)\rightarrow(\dot{f})$ should be in P.

From the construction procedure of S described above, it is obvious that for any α in $\mathscr{T}_V$, $\lambda_0 \overset{*}{\underset{S}{\Rightarrow}} \alpha$ if and only if there exist f in F such that $\alpha \overset{*}{\underset{A}{\vdash}} \dot{f}$. Thus, $M(A)=T(S)$.

Lemma 6.2. Let $S=(\Omega, V, \Sigma, P, \lambda_0)$ be an arbitrary CSDS. Then, we can construct an N-CSDA, $A=(K, V, \Sigma, \delta, F)$ such that $T(S)=M(A)$.

Proof The N-CSDA, A can be effectively constructed as follows: First, put $K=(\Omega-V)\cup\Sigma$ and let V and Σ for both S and A be the same. And put $F=\{\lambda_0\}$. The function δ is determined as follows: Let $X_{(t_1,\ldots,t_l)}$ be the set $\{(\dot{\xi}_1, \ldots, \dot{\xi}_l) \mid (\dot{\xi}_1, \ldots, \dot{\xi}_l)\rightarrow(t_1, \ldots, t_l) \text{ in } P\}$, which is the set of all left hand sides of the rules in P of which right hand sides are $(t_1, \ldots, t_l)$. Then the function δ is defined by

$$\delta(t_1, \ldots, t_l)=X_{(t_1,\ldots,t_l)} \tag{6.5}$$

where if $X_{(t_1,\ldots,t_l)}=\phi$ then $\delta(t_1, \ldots, t_l)=\phi$, which means that δ is undefined for the $(t_1, \ldots, t_l)$.

The construction procedure of A asserts itself that for any α in $\mathscr{T}_V$, $\dot{\lambda}_0 \overset{*}{\underset{S}{\Rightarrow}} \alpha$

if and only if $\alpha \overset{*}{\underset{A}{\vdash}} \lambda_0$. Thus, $T(S)=M(A)$.

These two Lemmas 6.1 and 6.2 directly yield the following theorem:

Theorem 6.1. $\mathscr{T}_{CS}=\mathscr{M}^N_{CS}$.

Similar discussions to the above give us the following theorems:

Theorem 6.2. $\mathscr{T}_{SC}=\mathscr{M}^N_{SC}$.

Theorem 6.3. $\mathscr{T}_{CF}=\mathscr{M}^N_{CF}$.

Theorem 6.4. $\mathscr{T}_{L}=\mathscr{M}^N_{L}$.

Theorem 6.5. $\mathscr{T}_{RL}=\mathscr{M}^N_{RL}$, $\mathscr{T}_{LL}=\mathscr{M}^N_{LL}$.

From Theories 5.3, 5.6 and Theories 6.1～6.5, the following corollary results.

Corollary 6.1. $\mathscr{M}^N_{LL}$, $\mathscr{M}^N_{RL} \subsetneqq \mathscr{M}^N_{L} \subsetneqq \mathscr{M}^N_{CF} \subsetneqq \mathscr{M}^N_{SC} \subseteq \mathscr{M}^N_{CS}$.

Next, we discuss about quasi-deterministic and deterministic behaiviours of DA. For two DA's A_1 and A_2, if $M(A_1)=M(A_2)$ then A_1 and A_2 are said to be equivalent.

Theorem 6.6. For any Q-CSDA, $A_Q=(K_Q, V, \Sigma, \delta_Q, F_Q)$, there xists an equivalent D-CSDA, $A_D=(K_D, V, \Sigma, \delta_D, F_D)$.

Proof The D-CSDA, A_D equivalent to A_Q can be constructed as below:

Put $K_D=2^{K_Q}$ and $F_D=\{Q_f \in K_D \mid Q_f \cap F \neq \phi\}$. The function δ_D is defined as follows:

If for $(t_1, \ldots, t_l) \in \mathscr{T}_{(K_Q \cup V)} \times \cdots \times \mathscr{T}_{(K_Q \cup V)}$ such that

$$\mu(t_i)=x^i_0 p^i_1 \cdots p^i_{h(i)} x^i_{h(i)} \quad (1 \leqq i \leqq l) \tag{6.6}$$

where $x^i_j \in (V-\Sigma)^*$ $(1 \leqq i \leqq l,\ 0 \leqq j \leqq h(i))$ and $p^i_j \in K_Q$ $(1 \leqq i \leqq l,\ 1 \leqq j \leqq h(i))$, we have

$$\delta_Q(t_1, \ldots, t_l)=\{(p_1, \ldots, p_l) \mid (p_1, \ldots, p_l) \in Q_1 \times \cdots \times Q_l,\ Q_i \subseteq K_Q\} \tag{6.7}$$

then we determine

$$\delta_D(t'_1, \ldots, t'_l)=(Q_1, \ldots, Q_l) \in K_D \times \cdots \times K_D \tag{6.8}$$

for all $(t'_1, \ldots, t'_l) \in \mathscr{T}_{(K_D \cup V)} \times \cdots \times \mathscr{T}_{(K_D \cup V)}$ such that

$$\mu(t'_i)=x^i_0 Q^i_1 x^i_1 \cdots Q^i_{h(i)} x^i_{h(i)} \quad (1 \leqq i \leqq l) \tag{6.9}$$

where $Q^i_j \cap \{p^i_j\} \neq \phi$ $(1 \leqq i \leqq l,\ 1 \leqq j \leqq h(i))$.

Such a construction of A_D is a so-called subset construction. Thus, a similar technique to the case of finite automata[20] can prove that $M(A_Q)=M(A_D)$ and the detailed proof is left to the reader.

Similarly we can also prove the following theorems:

Theorem 6.7. For an arbitrary Q-SCDA, A_Q, there exists an equivalent D-SCDA, A_D.

Theorem 6.8. For an arbitrary N-CFDA, A_N, there exists an equivalent D-CFDA, A_D.

Theorem 6.9. For an arbitrary N-LDA, N-RLDA, and N-LLDA, there exists an equivalent D-LDA, D-RLDA, and D-LLDA, respectively.

These theorems can be proved by the same way as Theorem 6.6, but easier than it. Theorem 6.8 corresponds to Theorem 1 of Thatcher[12]. Tree automaton of Thatcher[12] is differently defined from our CFDA. But they are essentially the same. That is, by using techniques similar to that used to derive our Theorem 4.3, we can know that the family of DL's accepted by CFDA's precisely coincides with that of DL's accepted by tree automata. Moreover, we can know that for any CFDS in normal form, which is given in Theorem 4.3, the construction procedure of equivalent CFDA, which is similar to that of N-CSDA, will give a CFDA which becomes a tree automaton through a slightly modified interpretation. Similar statements can also stand for CSDA and SCDA. That is, because of Theorem 4.1 or 4.2, we can construct CSDA and SCDA in the form similar to tree automata of Thatcher. But, these discussions can not be cases of LDA, RLDA and LLDA, because of Lemma 4.6, 4.7 and 4.8.

From Theorems 6.6~6.9, we can obtain the following corollaries:

Corollary 6.2. $\mathscr{M}^Q_{CS}=\mathscr{M}^D_{CS}$, $\mathscr{M}^Q_{SC}=\mathscr{M}^D_{SC}$.

Corollary 6.3. $\mathscr{M}^N_{CF}=\mathscr{M}^D_{CF}$, $\mathscr{M}^N_{L}=\mathscr{M}^D_{L}$, $\mathscr{M}^N_{RL}=\mathscr{M}^D_{RL}$ and $\mathscr{M}^N_{LL}=\mathscr{M}^D_{LL}$.

Finally, we summarize in Fig. 6.3 the relations among many families of DL's obtained in this chapter.

$$
\begin{array}{cccccc}
\mathscr{M}^D_{LL} & \mathscr{M}^D_{RL} & \mathscr{M}^D_{L} & \mathscr{M}^D_{CF} & \mathscr{M}^D_{SC} & \mathscr{M}^D_{CS} \\
 & & & & \| & \| \\
\| & \| & \| & \| & \mathscr{M}^Q_{SC} & \mathscr{M}^Q_{CS} \\
 & & & & \cap| & \cap| \\
\mathscr{M}^N_{LL} & \mathscr{M}^N_{RL} & \mathscr{M}^N_{L} & \mathscr{M}_{CF} & \mathscr{M}^N_{SC} & \mathscr{M}^N_{CS} \\
\| & \| & \| & \| & \| & \| \\
\mathscr{T}_{LL}, & \mathscr{T}_{RL} \subsetneq & \mathscr{T}_{L} \subsetneq & \mathscr{T}_{CF} \subsetneq & \mathscr{T}_{SC} \subseteq & \mathscr{T}_{CS}
\end{array}
$$

FIG. 6.3. Relations among the families of DL's.

The problems whether $\mathscr{M}^D_{SC}=\mathscr{M}^N_{SC}$ or not and whether $\mathscr{M}^D_{CS}=\mathscr{M}^N_{CS}$ or not are left open. They may be equivalent to that of equivalence of deterministic and nonderterministic linear bounded automata[18]. The solution for the former would also be a solution for the latter.†

7. Closure properties of dendrolanguages

In this and the next chapters, we discuss about the further properties of dendrolanguages. That is, we define several operations on dendrolanguages and investigate the closure properties under them. In the next chapter, we discuss some decision problems on dendrolanguages.

Definition 7.1. For any two subsets T_1 and T_2 of $\mathscr{T}_V$, the *union* of them is

† Refer to Chapter 12, where $\mathscr{T}_{CS}$ (and $\mathscr{M}^N_{CS}$) can characterize the family of the set of derivation trees of the context-senstive grammars.

defined by

$$T_1 \cup T_2 = \{t \mid t \in T_1 \text{ or } t \in T_2\} \tag{7.1}$$

Theorem 7.1. All families of $\mathscr{T}_{RL}$, $\mathscr{T}_{LL}$, $\mathscr{T}_{L}$, $\mathscr{T}_{CF}$, $\mathscr{T}_{SC}$ and $\mathscr{T}_{CS}$ are closed under union.

Proof Let $S_1 = (\Omega_1, V_1, \Sigma_1, P_1, \lambda_1)$ and $S_2 = (\Omega_2, V_2, \Sigma_2, P_2, \lambda_2)$ be arbitrary DS's of the identical type. Then, we can construct a DS, S_3 from them such that $T(S_3) = T(S_1) \cup T(S_2)$ by the similar way to the case of the conventional theory of phrase structure languages.

There is no loss of generality in assuming that $(\Omega_1 - V_1) \cap (\Omega_2 - V) = \phi$. Then, the DS, $S_3 = (\Omega_3, V_3, \Sigma_3, P_3, \lambda_3)$ constructed as follows is certainly the one such that $T(S_3) = T(S_1) \cup T(S_2)$: Let λ_3 be a symbol not in $\Omega_1 \cup \Omega_2$, and put $\Omega_3 = \{\lambda_3\} \cup \Omega_1 \cup \Omega_2$ and $P_3 = \{\dot{\lambda}_3 \to \dot{\lambda}_1,\ \dot{\lambda}_3 \to \dot{\lambda}_2\} \cup P_1 \cup P_2$.

That the DS, S_3 is of the same type as S_1 and S_2 is obvious from the determination of P_3.

Definition 7.2. The *reverse* of a tree t, denoted by t^R, is defined by

$$\varphi(t^R) = (\mu(t))^R \tag{7.2}$$ †

For any subset T of $\mathscr{T}_V$, T^R is defined as

$$T^R = \{t^R \mid t \in T\} \tag{7.3}$$

and is called the reverse of T.

Theorem 7.2. $\mathscr{T}_{RL}$ and $\mathscr{T}_{LL}$ is not closed under reverse.

Proof Suppose $\mathscr{T}_{RL}$ is closed under reverse, then $\mathscr{T}^R_{RL} = \{T^R \mid T \in \mathscr{T}_{RL}\} = \mathscr{T}_{RL}$. On the other hand, from definition 3.7, we know that $\mathscr{T}^R_{RL} = \mathscr{T}_{LL}$. Thus, $\mathscr{T}_{RL} = \mathscr{T}_{LL}$, which is a contradiction to Theorem 5.1.

Theorem 7.3. Each family of $\mathscr{T}_{L}$, $\mathscr{T}_{CF}$, $\mathscr{T}_{SC}$ and $\mathscr{T}_{CS}$ is closed under reverse.

Proof Let $S = (\Omega, V, \Sigma, P, \lambda)$ be an arbitrary SCDS. Consider a CSDS, $S' = (\Omega, V, \Sigma, P', \lambda)$ of which set of rules P' is determined by: For each rule $(\dot{\lambda}_1, \ldots, \dot{\lambda}_l) \to (t_1, \ldots, t_l)$ in P, let the rule $(\dot{\lambda}_1, \ldots, \dot{\lambda}_l) \to (t_1^R, \ldots, t_l^R)$ be in P'. Then, it is obvious that $T(S) = T^R(S')$.

The proofs for $\mathscr{T}_{SC}$, $\mathscr{T}_{CF}$ and $\mathscr{T}_{L}$ are similar to the above.

Next, we define some mappings on the set of trees and discuss the closure properties under these mappings.

Definition 7.3. A *relabeling mapping* I from the set of trees over an alphabet V_1 into that over V_2, that is,

$$I : \mathscr{T}_{V_1} \to \mathscr{T}_{V_2} \tag{7.4}$$

is defined by: for any $t \in \mathscr{T}_{V_1}$,

† For a string w, w^R is the reverse of w. That is, for any x in $\Omega \cup \{\varepsilon\}$, $X^R = X$ and for $w = Xx$ in Ω^*, $w^R = x^R X$.

$$I(t) = t' \in \mathscr{T}_{V_2}, \tag{7.5}$$

where if $\mu(t) = X\mu(t_1)\cdots\mu(t_n)$ then $\mu(t') = I(X)\mu(I(t_1))\cdots\mu(I(t_n))$. Here, $I(X) \in V_2$ and $\sigma(X) = \sigma(I(X))$.

Definition 7.4. A mapping $H_{(n)} : \mathscr{T}_{V_1} \to \mathscr{T}_{V_2}$ which is called *a finite tree substitution*, is defined as follows: For any t in $\mathscr{T}_{V_1}$,

$$H_{(n)}(t) = t' \in \mathscr{T}_{V_2}, \tag{7.6}$$

where if $\mu(t) = X\mu(t_1)\cdots\mu(t_m)$ then $\mu(t') = H_{(n)}(X)\,\mu(H_{(n)}(t_1))\cdots\mu(H_{(n)}(t_m))$. Here,

$$\left.\begin{array}{r} H_{(n)}(X) \in \mathscr{T}^n_{V_2}, \text{ if } \sigma(X) = 0 \\ H_{(n)}(X) = V_2 \text{ and } \sigma(X) = \sigma(H_{(n)}(X)), \text{ if } \sigma(X) \neq 0 \end{array}\right\} \tag{7.7}$$

Definition 7.5. A mapping $F : \mathscr{T}_{V_1} \to \mathscr{T}_{V_2}$, called a *tree substitution*, is defined by: For any t in $\mathscr{T}_{V_1}$,

$$F(t) = T_t \subseteq \mathscr{T}_{V_2}, \tag{7.8}$$

where if $\mu(t) = X\mu(t_1)\cdots\mu(t_m)$ then $\mu(T_t) = F(X)\,\mu(F(t_1))\cdots\mu(F(t_m))$. Here,

$$\left.\begin{array}{l} F(X) = T_X \subseteq \mathscr{T}_{V_2}, \text{ if } \sigma(X) = 0 \\ F(X) \in V_2, \text{ if } \sigma(X) \neq 0 \end{array}\right\} \tag{7.9}$$

Example 7.1. Consider $T_1 = \{t \mid \mu(t) = (Aa)^n cb^n,\ n \geqq 1\}$, where $\sigma(A) = 3$, and $\sigma(a) = \sigma(b) = \sigma(c) = 0$.

(a) If we define a mapping I by

$$I(A) = B,\ I(a) = d,\ I(b) = e \text{ and } I(c) = f,$$

where $\sigma(B) = 3$, $\sigma(d) = \sigma(e) = \sigma(f) = 0$, then for a tree t_1 such that $\mu(t_1) = (Aa)^2cb^2$, we have $I(t_1) = t_1'$, where $\mu(t_1') = (Bd)^2fe^2$ and $I(T_1) = \bigcup_{t \text{ in } T_1} I(t) = \{t \mid \mu(t) = (Bd)^n fe^n,\ n \geqq 1\}$.

(b) If we define $H_{(2)}$ by

$$\left.\begin{array}{l} H_{(2)}(A) = B;\ \sigma(B) = 3, \\ H_{(2)}(a) = t_a,\ \mu(t_a) = CBdefg;\ \sigma(C) = 2,\ \sigma(d) = \sigma(e) = \sigma(f) = \sigma(g) = 0, \\ H_{(2)}(b) = e,\ H_{(2)}(c) = f \end{array}\right\}$$

then for a tree t_1, we have $H_{(2)}(t_1) = t_2'$, where $\mu(t_2') = (BCBdefg)^2fe^2$ and $H_{(2)}(T_1) = \bigcup_{t \text{ in } T_1} H_{(2)}(t) = \{t \mid \mu(t) = (BCBdefg)^n fe^n,\ n \geqq 1\}$.

(c) Define a mapping F by

$$\left.\begin{array}{l} F(a) = T_a = \{t \mid \mu(t) = (Bd)^n e,\ n \geqq 1\}, \\ F(b) = T_b = \{t \mid \mu(t) = C^n gf^n,\ n \geqq 1\}, \\ F(c) = T_c = \{t \mid \mu(t) = E^n h,\ n \geqq 1\};\ \sigma(E) = 1, \\ F(A) = A \end{array}\right\}$$

and we have $F(t_1)=\{t\,|\,\mu(t)=(A(Bd)^n e)^2 E^m h(C^l gf^l)^2,\ n,\ m,\ l\geqq 1\}$ and $F(T_1)=\{t\,|\,\mu(t)=(A(Bd)^n e)^k E^m h(C^l gf^l)^k,\ n,\ m,\ k,\ l\geqq 1\}$.

As known from Definitions 7.3, 7.4, 7.5 and Example 7.1, a relabeling mapping is a special case of finite tree substitution, which is also a special case of tree substitution. The finite tree substitution and tree substitution correspond to homomorphism and substitution in the conventional mathematical theory of languages[16][19], respectively.

Theorem 7.4. All families of $\mathscr{T}_{RL}$, $\mathscr{T}_{LL}$, $\mathscr{T}_{L}$, $\mathscr{T}_{CF}$, $\mathscr{T}_{SC}$ and $\mathscr{T}_{CS}$ are closed under relabeling mappings I.

Proof For a given CSDS, S_1 and a relabeling mapping I, consider a CSDS, S_2 of which rules are ones produced by applying I to both sides of rules in the given ones. The CSDS, S_2 certainly generates just the set $I(T(S_1))$. For the other families of DL's, the proof can be done in the same way as above.

Theorem 7.5. Each family of $\mathscr{T}_{CS}$, $\mathscr{T}_{SC}$, $\mathscr{T}_{CF}$, $\mathscr{T}_{L}$, $\mathscr{T}_{RL}$ and $\mathscr{T}_{LL}$ is closed under finite tree substitution $H_{(n)}$ for any $n\geqq 1$.

Proof For a given CSDS, $S_1=(\Omega_1,\ V_1,\ \Sigma_1,\ P_1,\ \lambda_1)$ and a given finite tree substitution $H_{(n)}:\mathscr{T}_{V_1}\to\mathscr{T}_{V_2}$, consider a CSDS, $S_2=(\Omega_2,\ V_2,\ \Sigma_2,\ P_2,\ \lambda_2)$ determined as follows:

(1) $\Omega_2=\Lambda_1\cup V_2$, where $\Lambda_1=\Omega_1-V_1$,

(2) V_2 is the alphabet used to define $\mathscr{T}_{V_2}$,

(3) $\Sigma_2=\{a\,|\,a\in V_2,\ \sigma(a)=0\}$,

(4) For each rule $(\dot{\xi}_1,\ \ldots,\ \dot{\xi}_l)\to(t_1,\ \ldots,\ t_l)$ in P_1,

let the rule $(\dot{\xi}_1,\ \ldots,\ \dot{\xi}_l)\to(t_1',\ \ldots,\ t_l')$ be contained in P_2, where $t_i'=H_{(n)}(t_i)$ for each i. Here, we extend $H_{(n)}$ from $\mathscr{T}_{V_1}\to\mathscr{T}_{V_2}$ to $\mathscr{T}_{\Omega_1}\to\mathscr{T}_{\Omega_2}$ by adding the rule

$$H_{(n)}(\xi)=\xi \text{ for any } \xi \text{ in } \Lambda_1 \tag{7.10}$$

to (7.7).

(5) $\lambda_2=\lambda_1$.

Then, it is obvious that for any tree generating process $\dot{\lambda}_1\overset{*}{\underset{S_1}{\Rightarrow}}t$, there exists one such that $\dot{\lambda}_1\overset{*}{\underset{S_2}{\Rightarrow}}H_{(n)}(t)$. Conversely, for any $\dot{\lambda}_2\overset{*}{\underset{S_2}{\Rightarrow}}t$, there exists $\lambda_1\overset{*}{\underset{S_1}{\Rightarrow}}t'$ such that $H_{(n)}(t')=t$. Thus, $T(S_2)=H_{(n)}(T(S_1))$.

The similar considerations proves the assertions for $\mathscr{T}_{SC}$ and $\mathscr{T}_{CF}$.

For $\mathscr{T}_{L}$, $\mathscr{T}_{RL}$ and $\mathscr{T}_{LL}$, we have known that they are closed under the mapping $H_{(n)}$, but we can prove the following theorem by noting how the DS, S_2 was constructed in the above proof and by refering to Lemmas 4.6, 4.7 and 4.8.

Theorem 7.6. There exists an LDL, T generated by an LDS with order m and a mapping $H_{(n)}$ such that $H_{(n)}(T)$ can be generated by an LDS with order $(n+m)$ but not by any one with order less than $(n+m)$. The same statement for RLDL and LLDL also hold.

Theorem 7.7. Each family of $\mathscr{T}_{CS}$, $\mathscr{T}_{SC}$ and $\mathscr{T}_{CF}$ is closed under tree substitution F.

Proof Let $S=(\Omega_1,\ V_1,\ \Sigma_1,\ P_1,\ \lambda_1)$ and F be an arbitrarily given CSDS and tree substitution, respectively. For each symbol a in Σ_1, let $S_a=(\Omega_a, V_a, \Sigma_a, P_a, \lambda_a)$ be a CSDS such that $F(a)=T_a=T(S_a)$. With no loss of generality, we can assume that $(\Omega_a-V_a)\cap(\Omega_b-V_b)=\phi$ if $a\neq b$, and that $(\Omega_a-V_a)\cap(\Omega_1-V_1)=\phi$ for all a in Σ_1.

We define a mapping $h: \Omega_1\rightarrow(\Omega_1\cup\bigcup_{a\in\Sigma_1}\{\lambda_a\})$ as follows:

$$\left.\begin{array}{l} h(\xi)=\xi \text{ for each } \xi \text{ in } \Lambda_1(=\Omega_1-V_1) \\ h(a)=\lambda_a \text{ for each } a \text{ in } \Sigma_1 \\ h(X)=X \text{ for each } X \text{ in } (V_1-\Sigma_1) \end{array}\right\} \quad (7.11)$$

And we extend h to $\Omega^*\rightarrow(\Omega_1\cup\bigcup_{a\in\Sigma_1}\{\lambda_a\})^*$ by $h(\varepsilon)=\varepsilon$ and $h(Xx)=h(X)h(x)$ for X in Ω_1 and x in Ω_1^*.

Now, we determine a CSDS, $S_2=(\Omega_2, V_2, \Sigma_2, P_2, \lambda_2)$ as follows:

(1) $\Omega_2=\Omega_1\cup\bigcup_{a\in\Sigma_1}\Omega_a,\ V_2=V_1\cup\bigcup_{a\in\Sigma_1}V_a,\ \Sigma_2=\bigcup_{a\in\Sigma_1}\Sigma_a,\ \lambda_2=\lambda_1$

(2) $P_2=\{(\dot{\xi}_1, \ldots, \dot{\xi}_l)\rightarrow(t_1', \ldots, t_l')\mid(\dot{\xi}_1, \ldots, \dot{\xi}_l)\rightarrow(t_1, \ldots, t_l)$ in P_1 and $\mu(t_i')=h(\mu(F(t_i))),\ 1\leqq i\leqq l\}\cup\bigcup_{a\in\Sigma_1}P_a$.

That the CSDS, S_2 generates just the DL, $F(T(S_1))$ is easily known from the above determination of S_2.

The proofs for $\mathscr{T}_{SC}$ and $\mathscr{T}_{CF}$ are similar to the above.

Theorem 7.8. Any of $\mathscr{T}_L$, $\mathscr{T}_{RL}$ and $\mathscr{T}_{LL}$ are not closed under tree substitution F.

Proof Refer to (c) of Example 7.1, where T_1, T_a, T_b and T_c are LDL's but $F(T_1)=\{t\mid\mu(t)=(A(Bd)^ne)^kE^mh(C^lgf^l)^k,\ n, m, k, l\geqq 1\}$ is not LDL. Thus, $\mathscr{T}_L$ is not closed under tree substitution F. For $\mathscr{T}_{RL}$ and $\mathscr{T}_{LL}$, we can easily find such examples, too.

Lemma 7.1. For any tree substitution F such that $F(X)=T_X$ of (7.9) is an RLDL (LLDL), and for any RLDL (LLDL) T, $\mu(F(T))$ $(\varphi(F(T)))$ is contained in $\mathscr{L}_R$.

Proof From Theorem 5.2, we know that for any RLDL, T, $\mu(T)$ is in $\mathscr{L}_R$. On the other hand, it is obvious from definitions of F and substitution[19] that we can find a substitution τ such that

$$\mu(F(T))=\tau(\mu(T)).$$

Thus, since $\mathscr{L}_R$ is closed under substitution, $\mu(F(T))$ is in $\mathscr{L}_R$.

Similar proof also stands for LLDL.

Theorem 7.9. For any tree substitution F of Lemma 7.1, and for any DL, T in $\mathscr{T}_{RL}$ or $\mathscr{T}_{LL}$, $F(T)$ is in $\mathscr{T}_{CF}$. But, there exists a CFDL, T for which we can not find any RLDL (or LLDL), T' and any F such that $F(T')=T$.

Proof Similarly to the proof of Theorem 7.7, for any RLDL or LLDL T_1 and any F, we can construct a CFDS, S such that $T(S)=F(T_1)$. This proves the first half of the theorem.

On the other hand, consider the CFDL, T_2 of Example 3.4 of which prefix notation $\mu(T_2)$ can not be in $\mathscr{L}_R$. Thus, Lemma 7.1 verifies the remainder.

8. Some decision problems concerning dendrolanguages

In this chapter, we discuss some decision problem concerning dendrolanguages. The first is so-called membership problem for N-CSDA.

Theorem 8.1. Let $A=(K, V, \Sigma, \delta, F)$ be an N-CSDA. Then, for any tree $t \in \mathscr{T}_V$, it is decidable whether t is in $M(A)$ or not.

Proof For arbitrarily given $t \in \mathscr{T}_V$, t is in $M(A)$ if and only if there exists at least a saquence of $t_i \in \mathscr{T}_{(K \cup V)}$ $(i=1, \ldots, n)$ such that $t \vdash t_1 \vdash \cdots \vdash t_n \vdash \dot{f}$ and $f \in F$. Thus, in the followings, for any $t \in \mathscr{T}_{(K \cup V)}$ we will show by induction on the depth of t that we can determine in a finite number of steps whether there exists such a sequence of t_i begining with t, or not.

First, if the depth of t, $d(t)=0$, then $t=\dot{a}$ for some a in Σ. Thus, we can decide that whether f is in F or not such that $t \overset{*}{\vdash} \dot{f}$.

Next, let t be a given tree in $\mathscr{T}_{(K \cup V)}$ of which depth $d(t)=d$. An application of δ to t yields t' such that $t \vdash t'$. Then, two cases occurs:

(i) Case $d(t')<d(t)$: In this case, it is necessary only to assure that there exists at most a finite number of different t' and that we can enumerate them in a finite number of steps. But these are obvious from the finiteness property of δ defined in Definition 6.1.

(ii) Case where $d(t')=d(t)$: In this case, the tree domains D_t and $D_{t'}$ are same and thus t' is different from t only with respect to symbols assigned to their leaf nodes. The number of such the distinct trees is at most k^{r^d}, where $k=\# K$ and $r=\max\{\sigma(X) \mid X \in V\}$. Thus, by a finite number of steps, we can determine if the tree t can be transformed to t'' such that $d(t'')<d(t)$ and $t \overset{*}{\vdash} t''$. If such a tree t'' exists, the case is reduced to case (i). If otherwise, the tree can not be accepted by A.

Thus, the induction that for any tree t in $\mathscr{T}_{(K \cup V)}$ of which depth is less than d, we can determine if $t \overset{*}{\vdash} \dot{f}$ and $f \in F$, or not, by a finite number of steps proves the theorem.

The following corollary is a direct result of Theorem 8.1.

Corollary 8.1. The family of N-CSADL, $\mathscr{M}^N_{CS}$ is included in the family of the recursive subsets of $\mathscr{T}_V$.

The following two lemmas can be proved in a similar way to the case of usual theory of languages and automata. The readers who wants to prove them may refer to the text of Davis[24], for example.

Lemma 8.1. All of the N-CSDA's can be effectively enumerable.

Lemma 8.2. All of the trees of $\mathscr{T}_V$ can be effectively enumerable without overlapping.

Using these two lemmas, we can prove the following theorem.

Theorem 8.2. There exists a recursive subset of $\mathscr{T}_V$ which can not be accepted by any N-CSDA.

Proof Let $A_1, A_2, \ldots$ be an enumeration of all N-CSDA's and $t_1, t_2, \ldots$ be an enumeration of all trees in $\mathscr{T}_V$ without any overlappings. Consider a DL, T which is defined by

$$T = \{t_i \mid t_i \notin M(A_i)\}$$

which is certainly a recursive set as known from Corollary 8.1. But, this T can not be accepted by any N-CSDA, because if there exist an N-CSDA which accepts T, it contradicts to the fact that $A_1, A_2, \ldots$ is an enumeration of all N-CSDA.

Corollary 8.3. $\mathscr{M}^N_{CS}$ is a proper subfamily of the family of all the recursive subset of $\mathscr{T}_V$.

This corollary is a direct result of Theorem 8.2 with Corollary 8.1.

Theorem 8.3. If A is a CFDA, then the following problems are decidable:
(i) Is $M(A)$ empty? (ii) Is $M(A)$ finite? (iii) Is $M(A)$ infinite?

Proof The CFDA is essentially equivalent to the tree automaton defined by Thatcher and Wright[11], of which theorem 7 corresponds to this theorem. Thus, the proof is omitted.

Corollary 8.4. If A is LDA, RLDA or LLDA, then the problem whether (i) $M(A)=\phi$ (ii) $M(A)$ is finite or $M(A)$ is infinite is decidable.

9. Dendrolanguage generating systems with sets of states

In this chapter, we show an existence of the infinite subfamilies $\mathscr{T}^{scf}_1$, $\mathscr{T}^{scf}_2$, $\ldots$, $\mathscr{T}^{scf}_\infty$ and $\mathscr{T}^{scf}_\omega$ between $\mathscr{T}_{CF}$ and $\mathscr{T}_{CS}$ defined in chapter 3, that is,

$$\mathscr{T}_{CF} = \mathscr{T}^{scf}_1 \subsetneqq \mathscr{T}^{scf}_2 \subsetneqq \cdots \subsetneqq \mathscr{T}^{scf}_\infty \subsetneqq \mathscr{T}^{scf}_\omega = \mathscr{T}_{CS}.$$

Each dendrolanguage of $\mathscr{T}^{scf}_n$ is defined by a system, called a state dendrolanguage generating system, which may be thought of as a context-free dendrolanguage generating system with states.

9.1. State Context-free dendrolanguages

In this section, we will define a *state context-free dendrolanguage generating system* (scf DS), which is an extention of a CFDS.

Definition 9.1. A scf DS is a 7-tuple
$D=(\Omega, V, \Sigma, K, P, p_0, \lambda_0)$, where
(1) Ω, V, Σ and λ_0 are same as in Definition 3.1.
(2) K is a nonempty finite set of states.
(3) p_0 is a distinguished state in K, called an initial state.
(4) P is a finite subset of $K \times \mathscr{T}_\Lambda \times K \times \mathscr{T}_\Omega$.

An element $(p, \dot{\xi}, q, t)$† of P is called a *state dendrolanguage generating rule* (abbreviated *rule*) and is usually written $(p, \dot{\xi}) \to (q, t)$. An element of $\Omega - V$ is called a nonterminal node symbol (nns), and an nns ξ is said to be *applicable* under a state p if $(p, \dot{\xi}) \to (q, t)$ is in P for some q in K and some t in $\mathscr{T}_\Omega$.

Definition 9.2. Given a scf DS, $D=(\Omega, V, \Sigma, K, P, p_0, \lambda_0)$, let $\Longrightarrow$ be a relation $K \times \mathscr{T}_\Omega$ defined as follows: Let p_1 be in K and t_1 in $\mathscr{T}_\Omega$, where $\mu(t_1) = x\xi y$. If

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† $\dot{\xi}$ in a rule $(p, \dot{\xi}, q, t)$ is a tree $\{(0, \xi)\} \in \mathscr{T}_\Lambda$.

this ξ is the leftmost occurrence of nns in $\mu(t_1)$ being applicable under p_1 and $(p_1, \dot{\xi}) \to (p_2, s)$ in P, then we write $(p_1, t_1) \Longrightarrow (p_2, t_2)$, where $\mu(t_2) = x\mu(s)y$. If this ξ is the j-th nns in $\mu(t_1)$, then we sometimes write $\overset{j}{\Longrightarrow}$ instead of $\Longrightarrow$.

For α, $\beta \in K \times \mathscr{T}_\Omega$, we write $\alpha \overset{*}{\Longrightarrow} \beta$ if either $\alpha = \beta$ or there exist $\alpha_0, \dots, \alpha_r$ such that $\alpha_0 = \alpha$, $\alpha_r = \beta$ and $\alpha_i \Longrightarrow \alpha_{i+1}$ for each i $(0 \leqq i \leqq r-1)$. The sequence $\alpha_0, \dots, \alpha_r$ is called a *derivation* and is denoted by $\alpha_0 \Longrightarrow \alpha_1 \Longrightarrow \cdots \Longrightarrow \alpha_r$. The subset of $\mathscr{T}_V$,

$$T(D) = \{t \in \mathscr{T}_V \mid (p_0, \dot{\lambda}_0) \overset{*}{\Longrightarrow} (q, t), \ ^{\exists}q \in K\} \tag{9.1}$$

is called a *scf DL* and is said to *be generated by* scf DS, D.

Definition 9.3. Let $D = (\Omega, V, \Sigma, K, P, p_0, \lambda_0)$ be a scf DS and let n be a positive integer. An *n-limited derivation* is defined as a derivation $\alpha_0 \overset{j(1)}{\Longrightarrow} \alpha_1 \overset{j(2)}{\Longrightarrow} \cdots \overset{j(r)}{\Longrightarrow} \alpha_r$ such that $j(i) \leqq n$ for each i $(1 \leqq i \leqq r)$. In this case we sometimes write $\alpha_0 \overset{n*}{\Longrightarrow} \alpha_r$ instead of $\alpha_0 \overset{*}{\Longrightarrow} \alpha_r$. A subset $T(D;\ n)$ of $\mathscr{T}_V$ is defined as follows:

$$T(D;\ n) = \{t \in \mathscr{T}_V \mid (p_0, \dot{\lambda}_0) \overset{n*}{\Longrightarrow} (q, t) \text{ for some } q \text{ in } K\} \tag{9.2}$$

Obviously we have

$$T(D;\ n) \subseteq T(D;\ n+1) \tag{9.3}$$

for each positive integer n.

Definition 9.4. A scf DS, D is said to be *of degree n* if and only if $T(D) = T(D;\ n)$ for a positive integer n. A scf DL, T is said to be of degree n if there exists a scf DS, D of degree n with $T = T(D)$. If $T(D;\ n) \neq T(D)$ for all positive integers n, then D is said to be *of infinite degree* ω and the scf DL, $T(D)$ is of infinite degree ω.

For each positive integer n, let $\mathscr{T}_n^{\text{scf}}$ be the family of all scf DL's of degree n, and $\mathscr{T}_\omega^{\text{scf}}$ be that of degree ω. Put $\mathscr{T}_\infty^{\text{scf}} = \bigcup_{n=1}^{\infty} \mathscr{T}_n^{\text{scf}}$.†

By the definitions, it is obvious that

$$\mathscr{T}_1^{\text{scf}} \subseteq \mathscr{T}_2^{\text{scf}} \subseteq \cdots \subseteq \mathscr{T}_\infty^{\text{scf}} \subseteq \mathscr{T}_\omega^{\text{scf}} \tag{9.4}$$

9.2. Relations of scf DS to DS

In this section, we will show that

$$\mathscr{T}_1^{\text{scf}} = \mathscr{T}_{CF} \text{ and } \mathscr{T}_\omega^{\text{scf}} = \mathscr{T}_{CS}$$

First, by Definitions 3.5 and 9.1, it is obvious that $\mathscr{T}_1^{\text{scf}} \supseteq \mathscr{T}_{CF}$. Conversely, for any scf DS, D we can easily construct a CFDS, S_{CF} such that $T(D;\ 1) = T(S_{CF})$. Hence, we obtain the following theorem.

Theorem 9.1. $\mathscr{T}_1^{\text{scf}} = \mathscr{T}_{CF}$.

Theorem 9.2. For any scf DS, D and for any integer $n \geqq 1$, $T(D;\ n)$ is a scf

† $\mathscr{T}_\infty^{\text{scf}}$ means the family of all scf DL's of finite degree and $\mathscr{T}_\omega^{\text{scf}}$ that of all scf DL's, which contains a scf DL for which we can not define any finite degree.

DL of degree n, *i.e.*, there exists a scf DS, D' such that $T(D; n)=T(D'; n)=T(D')$.

Proof Let $D=(\Omega, V, \Sigma, K, P, p_0, \lambda_0)$. Then, consider a scf DS, $D'=(\Omega', V, \Sigma, K', P', (p_0, 1), \lambda_0)$ such that $\Omega'=\Omega\cup\{\bar{\lambda}\mid\lambda\in\Omega-V\}$, $K'=K\times\{1, 2, \ldots, n\}\cup\{\bar{p}\mid p\in K\}\times\{1, 2, \ldots, n\}\cup\{f\}$, $f\notin K$ and P' is defined as follows:

(1) If $(p, \dot{\lambda})\to(q, t)\in P$ then $((p, i), \dot{\xi})\to((\bar{q}, i), t)$ is in P' for each i $(1\leqq i\leqq n)$.

(2) If ξ is not applicable under p, then $((p, i), \dot{\xi})\to((p, i+1), \dot{\bar{\xi}})$ is in P' for $1\leqq i\leqq n-1$ and $((p, n), \dot{\xi})\to(f, \dot{\bar{\xi}})$ is in P'.

(3) For each ξ in Λ and p in K, $((\bar{p}, i), \dot{\bar{\xi}})\to((\bar{p}, i-1), \dot{\xi})$, $(2\leqq i\leqq n)$ and $(\bar{p}, 1), \dot{\xi})\to((p, 1), \dot{\xi})$ are in P'.

From the above construction of scf DS, D', we can easily prove that $T(D; n)=T(D'; n)=T(D')$.

Corollary 9.1. A scf DL, T is of degree n if and only if $T=T(D; n)$ for some scf DS, D.

Next, we show that $\mathscr{T}_\omega^{\mathrm{scf}}$ is identical to $\mathscr{T}_{CS}$.

Theorem 9.3. $\mathscr{T}_\omega^{\mathrm{scf}}=\mathscr{T}_{CS}$.

Proof First it is obvious that any scf DL is a CSDL. Now, let $S=(\Omega, V, \Sigma, P, \lambda_0)$ be a CSDS. Without loss of generality, by Lemma 4.7 we may assume that P consists of the following forms: (a) $(\dot{\lambda}_1, \dot{\lambda}_2)\to(t_1, t_2)$, (b) $(\dot{\lambda}_1)\to(t_1)$, (c) $(\dot{\lambda}_1)\to(t_3)$, where $\mu(t_1)$, $\mu(t_2)\in V\cdot\Lambda^*$, $\mu(t_3)\in\Sigma$.

Let $\Omega'=\Omega\cup\{\lambda'\mid\lambda\in\Lambda\}\cup\{\lambda''\mid\lambda\in\Lambda\}$, where $\lambda', \lambda''\notin\Lambda$. Let $K=\{f\}\cup\Lambda\times\Omega^m\cup\{p_i\mid 0\leqq i\leqq 2\}$, where Ω^m is a set of sequences on Ω of which length is m and m is a finite integer. Let $D=(\Omega', V, \Sigma, K, P', p_0, \lambda_0)$, where P' consists of the following rules:

(1) $(p_0, \dot{\xi})\to(p_0, \dot{\xi}')$ for all ξ in Λ.

(2) $(p_0, \dot{\xi})\to(p_1, t')$ for all $\xi\to t$ in P, where $\mu(t')=X\xi_1'\cdot\cdots\cdot\xi_{l-1}'\xi_l''$ and $\mu(t)=X\xi_1\cdot\cdots\cdot\xi_l$.

(3) $(p_0, \dot{\xi}_1)\to((\xi_2, \mu(t_2)), t_1')$ for all $(\dot{\xi}_1, \dot{\xi}_2)\to(t_1, t_2)$ in P, where $\mu(t_1)=X\tau_1\cdot\cdots\cdot\tau_m$ and $\mu(t_1')=X\tau_1'\cdot\cdots\cdot\tau_{m-1}'\tau_m''$.

(4) $((\xi_2, \mu(t_2)), \dot{\xi}_2)\to(p_1, t_2')$ for all $(\dot{\xi}_1, \dot{\xi}_2)\to(t_1, t_2)$ in P, where $\mu(t_2)=Y\delta_1\cdots\delta_n$ and $\mu(t_2')=Y\delta_1'\cdots\delta_{n-1}'\delta_n''$.

(5) $((\xi_2, \mu(t_2)), \dot{\zeta})\to(f, \dot{\zeta})$, where $\zeta\neq\xi_2$.

(6) $(p_1, \dot{\xi}')\to(p_1, \dot{\xi})$ for all ξ in Λ.

(7) $(p_1, \dot{\xi}'')\to(p_0, \dot{\xi})$ for all ξ in Λ.

(8) $(p_0, \dot{\xi})\to(p_2, t)$ for each $\xi\to t$ in P, where $\mu(t)=a\in\Sigma$.

(9) $(p_2, \dot{\xi})\to(p_2, t)$ for each $\xi\to t$ in P, where $\mu(t)=a\in\Sigma$.

From the above construction of scf DS, D, it can be proved easily that $T(S)=T(D)$.

9.3. $\mathscr{T}_\infty^{\mathrm{scf}}$ is a proper subfamily of $\mathscr{T}_\omega^{\mathrm{scf}}$

In a scf DS, $D=(\Omega, V, \Sigma, K, P, p_0, \lambda_0)$, denote each rule by an abstract symbol π_i $(i=1, \ldots, k$; k is the number of rules). Given a derivation

$$(p_0, \dot{\lambda}_0)=(p_0, t_0)\Longrightarrow(p_1, t_1)\Longrightarrow\cdots\Longrightarrow(p_r, t_r),$$

we associate it a sequence of rules $\pi_0\pi_1\cdots\pi_{r-1}$ such that each π_i is used in $(p_i, t_i)\Longrightarrow(p_{i+1}, t_{i+1})$. We call it a control word of the derivation. If the derivation is of length 0, then the control word is considered to be ε.

For each $n\geqq 1$, let $C(D;\ n)$ be the set of those control words which realize n-limited derivations from $(p_0, \dot{\lambda}_0)$, *i.e.*,

$$C(D;\ n)=\{c\,|\,c=\pi_0\cdots\pi_{r-1},\ (p_0, \dot{\lambda}_0)\underset{\pi_0}{\overset{n}{\Longrightarrow}}\cdots\underset{\pi_{r-1}}{\overset{n}{\Longrightarrow}}(p_r, t_r),\ t_r\in T(D;\ n)\} \quad (9.5)$$

Notation 9.1. Given a scf DS, $D=(\Omega, V, \Sigma, K, P, p_0, \lambda_0)$, let $Q^n_\alpha : K\times\Omega^*\rightarrow K\times\Omega^*$ be a function defined as follows: For any $(p, x_1\xi x_2)\in K\times\Omega^*$ where $\xi\in\Lambda$, if $(p, \dot{\xi})\overset{*}{\Longrightarrow}(q, t)$ is realized by a control word $\alpha\in P^*$, then we determine $Q^n_\alpha(p, x_1\xi x_2)=(q, x_1\mu(t)x_2)$, where $(q, t)\in K\times\mathscr{T}_\Omega$. If no such derivation exists $Q^n_\alpha(p, x_1\xi x_2)$ is undefined.

Note that $Q^n_\varepsilon(p, x_1\xi x_2)=(p, x_1\xi x_2)$ for all $(p, x_1\xi x_2)$ in $K\times\Omega^*$.

The following lemma can be obtained similarly to Lemma 1 of the reference[21].

Lemma 9.1. $C(D;\ n)$ is a CFL for any scf DS, D and any positive integer n.

Definition 9.5. Let N denote the set of nonnegative integers. For each integer $m\geqq 1$, let $N^m=N\times\cdots\times N$ (m times).

Then N^m is a commutative associative semigroup with identity under componentwise addition. A subset θ of N^m is said to be *linear* if there exist members $c, d_1, \ldots, d_s$ of N^m such that

$$\theta=\{x\,|\,x=c+k_1d_1+\cdots+k_sd_s,\ k_i \text{ in } N\}. \quad (9.6)$$

A subset θ of N^m is said to be *semilinear* if it is a finite union of linear sets.

Notation 9.2. Let $V=\{X_1, \ldots, X_m\}$. Let Φ be the function from V^* to N^m defined by $\Phi(w)=(|w|_{X_1}, \ldots, |w|_{X_m})$, where $|w|_{X_i}$ is the number of occurrences of the symbol X_i in the word w.

Thus, $\Phi(\varepsilon)=(0, \ldots, 0)=0^m$ and

It is well known that if $L\subseteq V^*$ is a CFL then $\Phi(L)$ is a semilinear set[10].

Theorem 9.4. If T is a scf DL of finite degree, then $\Phi(\mu(T))$ is a semilinear set.

Proof Let $D=(\Omega, V, \Sigma, K, P, p_0, \lambda_0)$ be a scf DS such that $T=T(D;\ n)=T(D)$ for some $n\geqq 1$. Let h be the homomorphism of P^* into Ω^* defined by $h((p, \dot{\xi})\rightarrow(q, t))=\mu(t)$ for each $(p, \dot{\xi})\rightarrow(q, t)$ in P. Next we extend the function Φ of Notation 9.2 from V^* to Ω^* by defining $\Phi(\lambda)=(0, \ldots, 0)$ for each $\lambda\in\Lambda$ and $\Phi(x_1\cdots x_s)=\sum_{i=1}^{s}\Phi(x_i)$ for each $x_i\in\Omega$. First we show that if $Q^n_\alpha(p_0, \lambda_0)=(q, \mu(t))$ then $\Phi(\mu(t))=\Phi(h(\alpha))$. We prove this by induction on the length of α.

(1) For $\alpha=\varepsilon$, it is true.

(2) Suppose that $\Phi(\mu(t))=\Phi(h(\alpha))$ for some α, and consider $Q^n_{\alpha\pi}(p_0, \lambda_0)=Q^n_\pi(p, \mu(t_1))=(q, \mu(t_2))$, where $\mu(t_1)=x\xi y$, $\mu(t_2)=x\mu(s)y$, $\pi:\ (p, \dot{\xi})\rightarrow(q, s)$ in P.

Then,
$$\begin{aligned}\Phi(\mu(t_2))&=\Phi(xy)+\Phi(\mu(s))=\Phi(x\xi y)+\Phi(\mu(s))\\&=\Phi(\mu(t_1))+\Phi(\mu(s))=\Phi(h(\alpha))+\Phi(h(\pi))\\&=\Phi(h(\alpha\pi)).\end{aligned} \quad (9.7)$$

Thus, it holds for all α that $\Phi(\mu(t))=\Phi(h(\alpha))$.

Hence,
$$\Phi(\mu(T))=\Phi(\mu(T(D;\ n)))=\{\Phi(\mu(t))\mid Q^n_\alpha(p_0,\ \lambda_0)=(q,\ \mu(t)),\ \alpha\in C(D;\ n)\}$$
$$=\{\Phi(h(\alpha))\mid \alpha\in C(D;\ n)\}=\Phi(h(C(D;\ n))). \tag{9.8}$$

Now by Lemma 9.1, $C(D;\ n)$ is a CFL and thus $h(C(D;\ n))$ is a CFL, since the homomorphic image of a CFL is also a CFL[16]. Thus, $\Phi(\mu(T))=\Phi(h(C(D;\ n)))$ is semilinear.

On the other hand, consider a DL, $T(S)$ of Example 3.1, which is a CSDL. For this $T(S)$, we know that $\Phi(\mu(T(S)))=\{(2^n-1,\ 2^n)\mid n\geqq 1\}$, where $\Phi(A)=(1,0)$ and $\Phi(a)=(0,1)$. Since $\{(2^n-1,\ 2^n)\mid n\geqq 1\}$ is not semilinear, $T(S)$ is not any scf DL of finite degree. Hence, noting Theorem 9.3, we have the following theorem.

Theorem 9.5. $\mathscr{T}^{\mathrm{scf}}_{\infty}\subsetneqq\mathscr{T}^{\mathrm{scf}}_{\omega}$.

9.4. An infinite hierarchy between $\mathscr{T}_{CF}$ and $\mathscr{T}_{CS}$

In this section, it is shown that $\mathscr{T}^{\mathrm{scf}}_{n}$ properly contains $\mathscr{T}^{\mathrm{scf}}_{n-1}$, *i.e.*, $\mathscr{T}^{\mathrm{scf}}_{n-1}\subsetneqq\mathscr{T}^{\mathrm{scf}}_{n}$, which means that there exists an infinite hierarchy such that

$$\mathscr{T}_{CF}=\mathscr{T}^{\mathrm{scf}}_{1}\subsetneqq\mathscr{T}^{\mathrm{scf}}_{2}\subsetneqq\cdots\subsetneqq\mathscr{T}^{\mathrm{scf}}_{n}\subsetneqq\cdots\subsetneqq\mathscr{T}^{\mathrm{scf}}_{\omega}=\mathscr{T}_{CS}.$$

Notation 9.3. Let $V=\{X,\ c\}\cup\{a_i\mid 1\leqq i\leqq 4n-2\}$, where

$$\sigma(X)=n,\quad \sigma(c)=0,\quad \sigma(a_{4i-3})=2;\ 1\leqq i\leqq n,\quad \sigma(a_{4i-2})=2;\ 1\leqq i\leqq n-1,$$
$$\sigma(a_{4i})=\sigma(a_{4i-1})=0;\ 1\leqq i\leqq n-1,\quad \sigma(a_{4n-2})=0.$$

Let T_n be a subset of $\mathscr{T}_V$ defined by

$$T_n=\{t\mid \mu(t)=Xa_1^k a_2^k ca_3^k a_4^k\cdots a_{4n-7}^k a_{4n-6}^k ca_{4n-5}^k a_{4n-4}^k a_{4n-3}^k ca_{4n-2}^k,\ k\geqq 1\}.$$

An element t of T_n is topologically represented in Fig. 9.1. In the followings, we show that $T_n\in\mathscr{T}^{\mathrm{scf}}_{n}$ and $T_n\notin\mathscr{T}^{\mathrm{scf}}_{n-1}$.

Theorem 9.6. $T_n\in\mathscr{T}^{\mathrm{scf}}_{n}$.

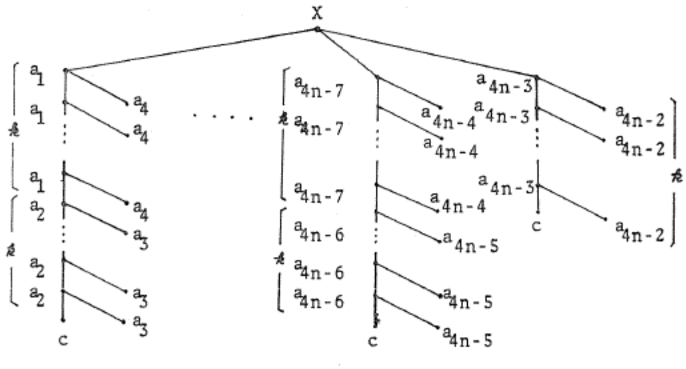

FIG. 9.1. Topological representation of t in T_n.

Proof Let $K_n=\{p_i|0\leqq i\leqq n\}\cup\{q_i|0\leqq i\leqq n\}\cup\{p'_i|1\leqq i\leqq n\}\cup\{f\}$, $\Sigma_n=\{c\}\cup\{a_{4i-1}|1\leqq i\leqq n-1\}\cup\{a_{4i}|1\leqq i\leqq n-1\}\cup\{a_{4n-2}\}$, $V_n=\{a_{4i-3}|1\leqq i\leqq n\}\cup\{a_{4i-2}|1\leqq i\leqq n-1\}\cup\{X\}\cup\Sigma_n$ and $\Omega_n=\{\lambda,\ \nu,\ \zeta\}\cup\{\xi_i|1\leqq i\leqq n\}\cup V_n$, where

$$\sigma(\lambda)=\sigma(\nu)=\sigma(\zeta)=\sigma(\xi_i)=0\,;\ \ 1\leqq i\leqq n.$$

Let $D_n=(\Omega_n,\ V_n,\ \Sigma_n,\ K_n,\ P_n,\ p_0,\ \lambda_0)$, where P_n is defined as follows:

(1) $(p_0,\ \dot{\lambda}_0)\rightarrow(p_1,\ t)$, where $\mu(t)=X\xi_1\cdots\xi_n$;

(2) $(p_i,\ \dot{\xi}_i)\rightarrow(p_{i+1},\ t)$, where $\mu(t)=a_{4i-3}\xi_i a_{4i}$,

$(p'_i,\ \dot{\xi}_i)\rightarrow(p'_{i+1},\ t')$, where $\mu(t')=a_{4i-3}\xi_i a_{4i}$,

$(q_i,\ \dot{\xi}_i)\rightarrow(q_{i+1},\ t'')$, where $\mu(t'')=a_{4i-2}\xi_i a_{4i-1}$,

$(q_i,\ \dot{\xi}_i)\rightarrow(q_{i+1},\ t''')$, where $\mu(t''')=a_{4i-2}ca_{4i-1}$; $1\leqq i\leqq n-1$;

(3) $(p_n,\ \dot{\xi}_n)\rightarrow(p'_1,\ t)$, where $\mu(t)=a_{4n-3}\xi_n\zeta$,

$(p'_n,\ \dot{\xi}_n)\rightarrow(p'_1,\ t')$, where $\mu(t')=a_{4n-3}\xi_n\nu$,

$(p'_n,\ \dot{\xi}_n)\rightarrow(q_1,\ t'')$, where $\mu(t'')=a_{4n-3}c\nu$,

$(q_n,\ \dot{\nu})\rightarrow(q_1,\ \dot{a}_{4n-2})$,

$(q_n,\ \dot{\zeta})\rightarrow(f,\ \dot{a}_{4n-2})$.

From the above construction of scf DS, D_n, it is clear that we have only the type of derivations: $(p_0,\ \dot{\lambda}_0)\overset{n*}{\Longrightarrow}(f,\ t)$, $\mu(t)=Xa_1^k a_2^k ca_3^k a_4^k\cdots a_{4n-7}^k a_{4n-6}^k ca_{4n-5}^k a_{4n-4}^k a_{4n-3}^k ca_{4n-2}^k$ for any $k\geqq 1$. Thus,

$$T(D_n;\ n)=T(D_n)=T_n.$$

Next we prepare a definition and some lemmas to show that $\mathscr{T}_{n-1}^{\rm scf}\subsetneqq\mathscr{T}_n^{\rm scf}$.

Definition 9.6. Let $D=(\Omega,\ V,\ \Sigma,\ K,\ P,\ p_0,\ \lambda_0)$ be a scf DS. For a tree t in $\mathscr{T}_\Omega$, $\tau(\mu(t))$ denotes the number of occurrences of the elements of Λ in $\mu(t)$. The function $\rho: P^*\rightarrow N$ is defined by

$$\rho(\pi)=\tau(\mu(t))-1 \text{ for each } \pi:(p,\ \dot{\xi})\rightarrow(q,\ t) \text{ in } P,$$
$$\rho(\varepsilon)=0, \text{ and}$$
$$\rho(\pi_1\cdots\pi_r)=\rho(\pi_1)+\cdots+\rho(\pi_r);\ \text{each } \pi_i \text{ is in } P.$$

The following lemma is immediately obtained from the definitions.

Lemma 9.2. If $Q_\alpha^n(p_1,\ \mu(t_1))=(p_2,\ \mu(t_2))$, then

$$\tau(\mu(t_2))=\tau(\mu(t_1))+\rho(\alpha) \tag{9.9}$$

Lemma 9.3. Let $D=(\Omega,\ V,\ \Sigma,\ K,\ P,\ p_0,\ \lambda_0)$ be a scf DS. If $C(D;\ n)$ is infinite, then $C(D;\ n)$ contains a control word $\gamma_1\beta_1\gamma_2\beta_2\gamma_3$ which satisfies the following conditions:

(1) $\beta_1\beta_2\neq\varepsilon$ and $\gamma_1\beta_1^k\gamma_2\beta_2^k\gamma_3$ is in $C(D;\ n)$ for all $k\geqq 0$;

(2) for all β'_1 in *init* (β_1), $\rho(\beta'_1)\geqq 0$;

(3) for all β'_2 in *init* (β_2), $\rho(\beta'_2)\geqq\rho(\beta_2)$,

where *init* (β_i) is the set of initial subwords[16] of β_i.

This lemma can be obtained by the same way as Lemma 3 in the reference[21].
Next we prove a key lemma:

Lemma 9.4. Let

$$T \subseteq \{t \mid \mu(t) \in X a_1^* a_2^* c a_3^* a_4^* \cdots a_{4m-7}^* a_{4m-6}^* c a_{4m-5}^* a_{4m-4}^* a_{4m-3}^* c a_{4m-2}^* \tag{9.10}$$

be a scf DL of degree n. Then if T is infinite then $\Phi(\mu(T))$ must contain a linear set $\{x \mid x = c + kd, k \geqq 1\}$ such that $d \neq 0^{4m}$ and d has less than $(4n+2)$ nonzero coordinates.

Proof Let Φ, h and τ be the same as in Theorem 9.4 and Lemma 9.2, respectively.

Let $D = (\Omega, V, \Sigma, K, P, p_0, \lambda_0)$ be a scf DS such that $T = T(D) = T(D; n)$. If T is infinite, then $C(D; n)$ is also infinite. Thus there exists a control word $\gamma_1 \beta_1 \gamma_2 \beta_2 \gamma_3$ which satisfies the conditions (1), (2) and (3) in Lemma 9.3. Therefore,

$$\{t_k \mid Q^n_{\gamma_1 \beta_1{}^k \gamma_2 \beta_2{}^k \gamma_3}(p_0, \lambda_0) = (q, \mu(t_k)), k \geqq 0\} \subseteq T \tag{9.11}$$

since the condition (1) assures that $\gamma_1 \beta_1^k \gamma_2 \beta_2^k \gamma_3$ is in $C(D; n)$ for all $k \geqq 0$.

From the first part of the proof for Theorem 9.4 it is known that if $Q^n_{\gamma_1 \beta_1{}^k \gamma_2 \beta_2{}^k \gamma_3}(p_0, \lambda_0) = (q, \mu(t_k))$ then

$$\begin{aligned} \Phi(\mu(t_k)) &= \Phi(h(\gamma_1 \beta_1^k \gamma_2 \beta_2^k \gamma_3)) \\ &= \Phi(h(\gamma_1 \gamma_2 \gamma_3)) + k\Phi(h(\beta_1 \beta_2)) \end{aligned} \tag{9.12}$$

Combining (9.11) and (9.12), we have

$$\{x \mid x = \Phi(h(\gamma_1 \gamma_2 \gamma_3)) + k\Phi(h(\beta_1 \beta_2))\} \subseteq \Phi(\mu(T)) \tag{9.13}$$

Thus, it suffices for completion of the proof to show that $\Phi(h(\beta_1 \beta_2))$ has at most $(4n+1)$ nonzero coodinates. To prove this, we have two cases, *i.e.*, $\rho(\beta_1) \geqq 1$ and $\rho(\beta_1) = 0$, where ρ is the mapping of Definition 9.5.

Case 1. Suppose that $\rho(\beta_1) \geqq 1$. Then, $\rho(\beta_2) = -\rho(\beta_1) \leqq -1$. Because if not then $\mu(t_k)$ would contain positive number of occurrences of elements in Λ, which contradicts the fact that t_k is in T.

That $\rho(\beta_1) \geqq 1$ assures that

$$\rho(\gamma \beta_1^n) = \rho(\gamma_1) + n\rho(\beta_1) \geqq n \tag{9.14}$$

Thus, $Q^n_{\gamma_1 \beta_1{}^n}(p_0, \lambda_0) = (p_1, \mu(t_1))$ contains more than n occurrences of nonterminal node symbols. Denote the n nonterminal node symbols from the left of $\mu(t_1)$ by $\xi_1, \xi_2, \ldots, \xi_n$, then

$$Q^n_{\gamma_1 \beta_1{}^n}(p_0, \lambda_0) = (p_1, x_1 \xi_1 x_2 \xi_2 \cdots x_n \xi_n x_{n+1}), \tag{9.15}$$

where $p_1 \in K$, $x_1, x_2, \ldots, x_n$ are in V^* and x_{n+1} is in Ω^*. On the other hand, the condition (2) of Lemma 9.3 assures that for all β_1' in $init(\beta_1^*)$, $Q^n_{\gamma_1 \beta_1{}^n \beta_1{}'}(p_0, \lambda_0)$ contains more than $(n + \tau(x_{n+1}))$ occurrences of nonterminal node symbols. Thus, $Q^n_{\beta_1{}^k}(p_1, \xi_1 \xi_2 \cdots \xi_n)$ is defined for all $k \geqq 0$. Then there exist p_2, p_3 in K, $y_1, y_2, \ldots, y_n$ in V^*, $y_{n+1}, z_1, z_2, \ldots, z_n$ in Ω^* and $\nu_1, \nu_2, \ldots, \nu_n$ in Λ such that

$$Q^{n}_{\beta_1\beta_2}(p_1,\ \xi_1\xi_2\cdots\xi_n) = Q^{n}_{\beta_2}(p_2,\ y_1\nu_1y_2\cdots y_n\nu_ny_{n+1})$$
$$= (p_3,\ y_1z_1y_2\cdots y_nz_ny_{n+1}) \tag{9.16}$$

Here, we define a function $g: \Omega\to V\cup\{\varepsilon\}$ by

$$\left.\begin{array}{ll} g(\xi)=\varepsilon\ ; & \text{for all } \xi \text{ in } \Lambda, \\ g(X)=X\ ; & \text{for all } X \text{ in } V \end{array}\right\} \tag{9.17}$$

and extend g to $\Omega^*\to V^*$ by $g(\varepsilon)=\varepsilon$ and $g(x_1x_2)=g(x_1)\,g(x_2)$ for any $x_1, x_2\in\Omega^*$.

Then, from (9.16) it follows that there exists a sequence of integers $1\leqq j_1\leqq\cdots\leqq j_{2i-1}\leqq j_{2i}\leqq\cdots\leqq j_{2n}\leqq 4m-2$ such that $g(z_i)$ is in $a^*_{j_{2i-1}}g^*_{j_{2i}}$ for each i. This comes from the following reasons: Assume that $g(z_i)$ contains a subword $a_{l_1}a^k_{l_2}a_{l_3}$ such that $l_1<l_2<l_3$ and $k>0$. Since

$$\Phi(h(\beta_1)) = \Phi(y_1\cdots y_{n+1}) = \Phi(z_1\cdots z_n) \tag{9.18}$$

(which is obtained by applying the same discussions as the first part of the proof for Theorem 9.4 to (9.16)), a_{l_2} occurs in some y_i. This means that in $y_1z_1y_2z_2\cdots y_nz_ny_{n+1}$, some a_{l_2} occurs either in the left to some a_{l_1} or in the right to some a_{l_3}. This contradicts the fact that $Q^{n}_{\gamma_1\beta_1{}^k\gamma_2\beta_2{}^k\gamma_3}(p_0,\ \lambda_0)=(q,\ \mu(t_k))$; $\mu(t_k) = Xa^k_1a^k_2ca^k_3a^k_4\cdots a^k_{4n-7}a^k_{4n-6}ca^k_{4n-5}a^k_{4n-4}a^k_{4n-3}ca^k_{4n-2}$ and thus the fact that the symbols a_i occur in the increasing order of suffixes in $y_1z_1y_2z_2\cdots y_nz_ny_{n+1}$. Thus $g(z_i)$ is in $a^*_{j_{2i-1}}a^*_{j_{2i}}$ and $g(z_1z_2\cdots z_n)$ is in $a^*_{j_1}a^*_{j_2}\cdots a^*_{j_{2n-1}}a^*_{j_{2n}}$. Thus, $\Phi(h(\beta_1))$ has at most $2n$ nonzero coordinates.

Next, let $s=-\rho(\beta_2)$ $(=\rho(\beta_1))$ and then $\rho(\gamma_1\beta_1^{n+2}\gamma_2)\geqq n+2s$. Thus, $Q^{n}_{\gamma_1\beta_1{}^{n+2}\gamma_2}(p_0,\ \lambda_0)=(\overline{p}_1,\ \mu(\overline{t}_1))$ contains more than $(n+2s)$ nonterminal node symbols. Denoting the $(n+2s)$ nonterminal node symbols from the left of $\mu(\overline{t}_1)$ by $\overline{\xi}_1, \overline{\xi}_2, \ldots, \overline{\xi}_{n+2s}$, we have

$$Q^{n}_{\gamma_1\beta_1{}^{n+2}\gamma_2}(p_0,\ \lambda_0) = (\overline{p}_1,\ \overline{x}_1\xi_1\overline{x}_2\cdots\overline{x}_{n+2s}\overline{\xi}_{n+2s}\overline{x}_{n+2s})$$

where $\overline{p}_1$ is in K, $\overline{x}_1, \ldots, \overline{x}_{n+2s}$ are in V^* and $\overline{x}_{n+2s+1}$ is in Ω^*. By the condition (1) and (3) in Lemma 9.3, $Q^{n}_{\beta_2{}^2}(\overline{p}_1,\ \overline{\xi}_1\overline{\xi}_2\cdots\overline{\xi}_{n+2s})$ is defined. Thus, there exist $\overline{p}_2$, $\overline{p}_3$ in K, $\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_{n+1}$ in V^*, $\overline{\nu}_1, \overline{\nu}_2, \ldots, \overline{\nu}_n$ in Λ and $\overline{z}_1, \overline{z}_2, \ldots, \overline{z}_n$ in Ω^* such that

$$Q^{n}_{\beta_2{}^2}(\overline{p}_1,\ \overline{\xi}_1\cdots\overline{\xi}_{n+2s}) = Q^{n}_{\beta_2}(\overline{p}_2,\ \overline{y}_1\overline{\nu}_1\cdots\overline{y}_n\overline{\nu}_n\overline{y}_{n+1}\overline{\xi}_{n+s+1}\cdots\overline{\xi}_{n+2s})$$
$$= (\overline{p}_3,\ \overline{y}_1\overline{z}_1\overline{y}_2\cdots\overline{y}_n\overline{z}_n\overline{y}_{n+1}\overline{z}_{n+1}) \tag{9.19}$$

Thus we have

$$\Phi(h(\beta_2)) = \Phi(\overline{y}_1\overline{y}_2\cdots\overline{y}_{n+1}) = \Phi(\overline{z}_1\overline{z}_2\cdots\overline{z}_{n+1})$$

And similarly to the above discussions, we can prove that there exists a sequence of integers $1\leqq j_1\leqq j_2\leqq\cdots\leqq j_{2n+1}\leqq 4m-2$ such that $g(z_i)$ is in $a^*_{j_{2i-1}}a^*_{j_{2i}}$ for $1\leqq i\leqq n$ and $g(\overline{z}_{n+1})$ is in $a^*_{j_{2n+1}}$. Suppose that $g(\overline{z}_i)$ contains a subword $a_{l_1}a_{l_2}$ with $l_1<l_2$. Since a_{l_2} occurs in some $\overline{y}_i$, $g(\overline{y}_1\overline{z}_1\cdots\overline{y}_{n+1}\overline{z}_{n+1})$ contains a subword of the form $a_{l_2}ua_{l_1}a_{l_2}$. This is a contradiction. Therefore $g(\overline{z}_1\overline{z}_2\cdots\overline{z}_{n+1})$ is in $a^*_{j_1}a^*_{j_2}\cdots a^*_{j_{2n+1}}$. Thus $\Phi(h(\beta_2))$ has at most $(2n+1)$ nonzero coordinates. Thus $\Phi(h(\beta_1\beta_2)) = \Phi(h(\beta_1))+\Phi(h(\beta_2))$ has at most $(4n+1)$ nonzero coodinates.

Case 2. Suppose that $\rho(\beta_1)=\rho(\beta_2)=0$. Let

$$Q^n_{\gamma_1}(p_0,\ \lambda_0)=(p_1,\ x_1\xi_1 x_2\cdots x_s\xi_s x_{s+1}), \tag{9.20}$$

where $s=\min(n,\ \rho(\gamma_1)+1)$, $x_1,\ldots,x_s$ in V^*, x_{s+1} in Ω^* and $\xi_1,\ldots,\xi_s$ in Λ. By the condition (2) in Lemma 9.3, $Q^n_{\beta_1^k}(p_1,\xi_1\cdots\xi_s)$ is defined for all $k\geqq 0$. Thus there exist p_2, p_3 in K, $y_1,\ldots,y_s$ in V^*, y_{s+1}, $z_1,\ldots,z_s$ in Ω^* and $\nu_1,\ldots,\nu_s$ in Λ such that

$$\begin{aligned}Q^n_{\beta_1^2}(p_1,\ \xi_1\cdots\xi_s)&=Q^n_{\beta_1}(p_2,\ y_1\nu_1 y_2\cdots y_s\nu_s y_{s+1})\\&=(p_3,\ y_1 z_1 y_2\cdots y_s z_s y_{s+1})\end{aligned} \tag{9.21}$$

Then there exist $1\leqq j_1\leqq j_2\leqq\cdots\leqq j_{2s}\leqq 4m-2$ such that $g(z_i)$ is in $a^*_{j_{2i-1}}a^*_{j_{2i}}$ for $1\leqq i\leqq s$. Therefore $\Phi(h(\beta_1))=\Phi(z_1\cdots z_s)$ has at most $2n$ nonzero coordinates. An anologous argument proves that $\Phi(h(\beta_2))$ has at most $2n$ nonzero coordinates. Thus $\Phi(h(\beta_1\beta_2))$ has at most $4n$ nonzero coordinates.

Both cases show that $\Phi(h(\beta_1\beta_2))$ has at most $(4n+1)$ coodinates and we have proved the theorem.

We are now ready for the main result of this section.

Theorem 9.7. T_{n+1} is not in $\mathscr{T}^{\mathrm{scf}}_n$.

Proof Suppose that $T_{n+1}=\{t\,|\,\mu(t)=Xa_1^k a_2^k c a_3^k a_4^k\cdots a_{4n-3}^k a_{4n-2}^k c a_{4n-1}^k a_{4n}^k a_{4n+1}^k c a_{4n+2}^k,\ k\geqq 1\}$ is of degree n. By Lemma 9.3, $\Phi(\mu(T_{n+1}))$ contains a linear set

$$U=\{x\,|\,x=c+kd,\ k\geqq 0\}$$

such that at most $(4n+1)$ coodinates of d are nonzero. But

$$\begin{aligned}\Phi(\mu(T_{n+1}))&=\{(|w|_X,\ |w|_c,\ |w|_{a_1},\ \ldots,\ |w|_{a_{4n+2}})\}\\&=\{x\,|\,x=(1,\ n+1,\ \overbrace{0\cdots 0}^{4n+2})+k((0,\ 0,\ \overbrace{1\,1\cdots 1}^{4n+2}),\ k\geqq 1\}.\end{aligned}$$

This is a contradiction. Thus T_{n+1} is not of degree n.

Theorem 9.8. $\mathscr{T}^{\mathrm{scf}}_n\subsetneqq\mathscr{T}^{\mathrm{scf}}_{n+1}$ for all $n\geqq 1$.

Proof By (9.4), Theorems 9.6 and 9.7.

10. Dendrolanguage generating systems on control sets of strings

In the previous chapter, we have discussed state dendrolanguage generating systems, which is an extension of CFDS and corresponds to state grammar[21]. In this chapter, we propose another extension of CFDS which corresponds to string grammar[22]. We will call it *string context-free dendrolanguage generating system* (gcf DS). We discuss its DL generating power and show an existence of an infinite hierarchy of DL, which is a similar result to Theorem 9.8 for scf DS.

10.1. String context-free dendrolanguage generating system

In this section, we will define a gcf DS and a gcf DL which is generated by a gcf DS.

Definition 10.1. A *gcf DS* is a 7-tuple $E=(\Omega, V, \Sigma, \Gamma, P, z_0, \lambda_0)$, where
(1) Ω, V, Σ and λ_0 are same as in Definition 3.1,
(2) Γ is a nonempty finite set of auxiliary symbols,
(3) Z_0 is a distinguished symbol in Γ, called an initial auxilary symbol,
(4) P is a finite subset of $\Gamma\times\mathscr{T}_\Lambda\times\Gamma^*\times\mathscr{T}_\Omega$.

An element $(A, \dot{\lambda}, \alpha, t)$ of P is called a *string context-free dendrolanguage generating rule* (or simply a *rule*) and usually writen $(A, \dot{\lambda})\rightarrow(\alpha, t)$.

Definition 10.2. Given a gcf DS, $E=(\Omega, V, \Sigma, \Gamma, P, Z_0, \lambda_0)$, let$\Longrightarrow$be a relation on $\Gamma^*\times\mathscr{T}_\Omega$ defined as follows: For any (β_1, t_1), (β_2, t_2) in $\Gamma^*\times\mathscr{T}_\Omega$, $(\beta_1, t_1)\Longrightarrow(\beta_2, t_2)$ if and only if there exist A in Γ, α in Γ^*, x, y in Ω^* such that the following conditions (1) and (2) hold:

(1) $(A, \dot{\lambda}, \alpha, s)\in P$,
(2) $\beta_1=A\beta'$, $\beta_2=\alpha\beta_1'$, $\mu(t_1)=x\lambda y$, $\mu(t_2)=x\mu(s)y$.

If at most j nns's are appeared in $\mu(t_1)$ then we write $\overset{j}{\Longrightarrow}$ instead of $\Longrightarrow$.

For α, $\beta\in\Gamma^*\times\mathscr{T}_\Omega$, we write $\alpha\overset{*}{\Longrightarrow}\beta$ if either $\alpha=\beta$ or there exist $\alpha_0, \dots, \alpha_r$ such that $\alpha_0=\alpha$, $\alpha_r=\beta$ and $\alpha_i\Longrightarrow\alpha_{i+1}$ for each i $(0\leqq i\leqq r-1)$.

The sequence $\alpha_0, \dots, \alpha_r$ is called a *derivation* and is denoted by $\alpha_0\Longrightarrow\alpha_1\Longrightarrow\cdots\Longrightarrow\alpha_r$.

The subset $T(E)$ of $\mathscr{T}_V$ is called *a gcf DL* generated by gcf DS, E and defined as follows:

$$T(E)=\{t\in\mathscr{T}_V|(Z_0,\ \dot{\lambda}_0)\overset{*}{\Longrightarrow}(\varepsilon,\ t)\}$$

Definition 10.3. Let $E=(\Omega, V, \Sigma, \Gamma, P, Z_0, \lambda_0)$ be a gcf DS and n be a positive integer. An *n-limited derivation* is a derivation $\alpha_0\overset{j(1)}{\Longrightarrow}\alpha_1\overset{j(2)}{\Longrightarrow}\cdots\overset{j(r)}{\Longrightarrow}\alpha_r$ such that $j(i)\leqq n$ for each $i(1\leqq i\leqq r)$. In this case we write $\alpha_0\overset{n*}{\Longrightarrow}\alpha_r$ instead of $\alpha_0\overset{*}{\Longrightarrow}\alpha_r$.

A subset $T(E;\ n)$ of $\mathscr{T}_V$ is defined as follows:

$$T(E;\ n)=\{t\in\mathscr{T}_V|(Z_0,\ \lambda_0)\overset{n*}{\Longrightarrow}(\varepsilon,\ t)\}$$

For each positive integer n, it is clear that

$$T(E;\ n)\subseteq T(E;\ n+1) \tag{10.1}$$

Definition 10.4. A gcf DS, E is said to be *of degree n* if and only if $T(E)=T(E;\ n)$ for some positive integer n. A gcf DL, T is said to be of degree n if there exists a gcf DS, E of degree n such that $T=T(E;\ n)$. If $T(D;\ n)\neq T(D)$ for all positive integers n, then E is said to be of infinite degree ω and the gcf DL, $T(E)$ is of infinite degree ω.

For each positive integer n, let $\mathscr{T}_n^{\mathrm{gcf}}$ be the family of gcf DL's of degree n, and $\mathscr{T}_\omega^{\mathrm{gcf}}$ be the family of all gcf DL's of degree ω. Define $\mathscr{T}_\infty^{\mathrm{gcf}}=\bigcup_{n=1}^{\infty}\mathscr{T}_n^{\mathrm{gcf}}$. By the definitions, it is clear that

$$\mathscr{T}_1^{\mathrm{gcf}}\subseteq\mathscr{T}_2^{\mathrm{gcf}}\subseteq\cdots\subseteq\mathscr{T}_\infty^{\mathrm{gcf}}\subseteq\mathscr{T}_\omega^{\mathrm{gcf}} \tag{10.2}$$

Example 10.1. Let $E=(\Omega, V, \Sigma, \Gamma, P, Z_0, \lambda_0)$ be a gcf DS, where

$$(1)\ \Sigma=\{c\},\ V=\{A,\ B\}\cup\Sigma,\ \Omega=\{\lambda_0,\ \eta\}\cup V,$$
$$\Gamma=\{Z_0,\ A,\ B\},$$
$$(2)\ P=\{(z_0,\ \dot{\lambda}_0)\rightarrow(z_0A,\ \begin{smallmatrix}A\\ |\\ \lambda_0\end{smallmatrix}),\ (Z_0,\ \dot{\lambda}_0)\rightarrow(Z_0B,\ \begin{smallmatrix}B\\ |\\ \lambda_0\end{smallmatrix}),$$
$$(z_0,\ \dot{\lambda}_0)\rightarrow(\varepsilon,\ \dot{\eta}),\ (A,\ \dot{\eta})\rightarrow(\varepsilon,\ \begin{smallmatrix}A\\ |\\ \eta\end{smallmatrix}),$$
$$(B,\ \dot{\eta})\rightarrow(\varepsilon,\ \begin{smallmatrix}B\\ |\\ \eta\end{smallmatrix}),\ (A,\ \dot{\eta})\rightarrow(\varepsilon,\ \begin{smallmatrix}A\\ |\\ c\end{smallmatrix}),$$
$$(B,\ \dot{\eta})\rightarrow(\varepsilon,\ \begin{smallmatrix}B\\ |\\ c\end{smallmatrix})$$

This gcf DS, E generates a gcf DL, $T(E)=T(E;\ 1)=\{t\mid\mu(t)=ww^Rc,\ w\in\{A,\ B\}^*\}$ and it is of degree 1.

10.2. Inclusion relations among $\mathscr{T}_{CS}$, $\mathscr{T}_n^{\text{gcf}}$ and $\mathscr{T}_\omega^{\text{gcf}}$

In this section we show that $\mathscr{T}_n^{\text{gcf}}\subsetneqq\mathscr{T}_{CS}\subsetneqq\mathscr{T}_\omega^{\text{gcf}}$ for any positive integer n.

Theorem 10.1. $\mathscr{T}_{CS}\subsetneqq\mathscr{T}_\omega^{\text{gcf}}$

Proof In a similar way to Theorem 9.3, we can prove that $\mathscr{T}_{CS}\subseteq\mathscr{T}_\omega^{\text{gcf}}$. On the other hand, we know that the DL of Example 10.1 is in $\mathscr{T}_\omega^{\text{gcf}}$ but not in $\mathscr{T}_{CS}$.

Definition 10.5. For a gcf DS, $E=(\Omega,\ V,\ \Sigma,\ \Gamma,\ P,\ Z_0,\ \lambda_0)$, we define a set $\overline{C}(t)$ for the derivation of t in $T(E)$ from $(Z_0,\ \dot{\lambda}_0)$ as follows:

$$\overline{C}(t)=\{A_0A_1\cdots A_m\mid(Z_0,\ \dot{\lambda}_0)=(A_0\beta_0,\ t_0)\Longrightarrow(A_1\beta_1,\ t_1)\Longrightarrow\cdots\Longrightarrow$$
$$(A_m\beta_m,\ t_m)\Longrightarrow(\varepsilon,\ t),\ A_i\in\Gamma,\ \beta_i\in\Gamma^*,\ 0\leqq i\leqq m\}\quad(10.3)$$

Next, we define a finite substitution $h:\Gamma^*\rightarrow 2^{P^*}$ by

$$\left.\begin{aligned}&h(A)=\{\pi_i\mid\pi_i:\ (A,\ \dot{\lambda})\rightarrow(\alpha,\ t)\in P\}\\ &h(x_1x_2)=h(x_1)\,h(x_2);\ \ x_1,\ x_2\in\Gamma^*\end{aligned}\right\}\quad(10.4)$$

Using the set $\overline{C}(t)$ and the finite substitution h, we define a set $C(E;\ n)$, of which element is called a control word, as follows:

$$C(E;\ n)=h(\overline{C}(T(E;\ n))),\quad(10.5)$$

where h and $\overline{C}$ are extended in a usual way such that $h(\overline{C}(T(E;\ n)))=\{h(x)\mid x\in\overline{C}(T(E;\ n))\}$ and $\overline{C}(T(E;\ n))=\{\overline{C}(t)\mid t\in T(E;\ n)\}$.

Similarly to Lemma 11 in the reference[22], under the above definition, we obtain the following lemma.

Lemma 10.1. For any gcf DL, $T(E)$ and any finite positive integer n, if $T(E;\ n)=T(E)$ then a set of control words, $C(T(E;\ n))$ is a CFL.

Definition 10.6. For a gcf DS, $E=(\Omega,\ V,\ \Sigma,\ \Gamma,\ P,\ Z_0,\ \lambda_0)$ we define a function $Q_\alpha^n:\Gamma^*\times\Omega^*\rightarrow\Gamma^*\times\Omega^*$ as follows:

For $\beta_1,\ \beta_2\in\Gamma^*$, $x_1,\ x_2\in\Omega^*$, $t_1,\ t_2\in\mathscr{T}_\Omega$ and any positive integer n, if $(\beta_1,\ t_1)\overset{n*}{\Longrightarrow}(\beta_2,\ t_2)$ is realized by a control word α in P^*, then $Q_\alpha^n(\beta_1,\ x_1\mu(t_1)x_2)=(\beta_2,\ x_1\mu(t_2)x_2)$ and if not then undefined. Morevoer, for the empty control word ε we define $Q_\varepsilon^n(\beta_1,\ x_1\mu(t_1)x_2)=(\beta_1,\ x_1\mu(t_1)x_2)$.

From the above definition, the following theorem can be proved similarly to Theorem 9.4.

Theorem 10.2. For any gcf DL, T of degree n, $\Phi(\mu(T))$ is a semilinear set, where Φ is same as in Theorem 9.4.

This theorem and Example 3.1 prove the following theorem which is similar to Theorem 9.5.

Theorem 10.3. $\mathscr{T}_{\infty}^{\text{gcf}} \subsetneqq \mathscr{T}_{\omega}^{\text{gcf}}$.

10.3. An infinite hierarchy of gcf DL

In this section, we show that $\mathscr{T}_{n}^{\text{gcf}} \subsetneqq \mathscr{T}_{n+1}^{\text{gcf}}$ for any integer $n \geqq 1$.

Notation 10.1. For any positive integer $n \geqq 1$, define a ranked alphabet $V_n = \{X, c\} \cup \{a_{4i-3} | 1 \leqq i \leqq n\} \cup \{a_{4i-2} | 1 \leqq i \leqq n\} \cup \{a_{4i-1} | 1 \leqq i \leqq n\} \cup \{a_{4i} | 1 \leqq i \leqq n\}$, where $\sigma(X) = n$, $\sigma(c) = 0$, $\sigma(a_{4i-3}) = \sigma(a_{4i-2}) = 2$, $\sigma(a_{4i-1}) = \sigma(a_{4i}) = 0$; $1 \leqq i \leqq n$. By T_n we denote a subset T_n of $\mathscr{T}_{V_n}$ such that

$$T_n = \{t | \mu(t) = Xa_1^k a_2^k c a_3^k a_4^k \cdots a_{4n-3}^k a_{4n-2}^k c a_{4n-1}^k a_{4n}^k,\ k \geqq 1\} \tag{10.6}$$

An element t of T_n is topologically represented in Fig. 10.1.

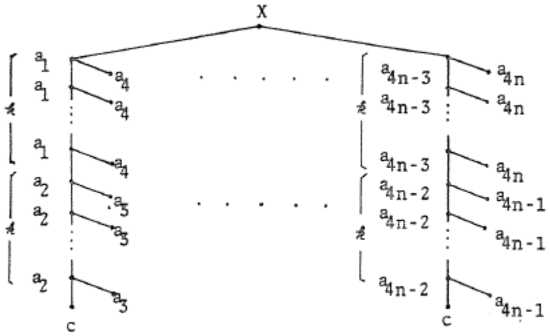

FIG. 10.1. Topological representation of an element of T_n.

Theorem 10.4. $T_n \in \mathscr{T}_{n}^{\text{gcf}}$.

Proof It is sufficient for the proof to show that there exists a gcf DS, E of degree n such that $T(E) = T_n$.

Consider a gcf DS, $E = (\Omega, V, \Sigma, \Gamma, P, Z_1, \lambda_1)$ effectively constructed as follows:

(1) $\Sigma = \{c\} \cup \{a_{4i-1},\ a_{4i} | 1 \leqq i \leqq n\}$

$V = \Sigma \cup \{a_{4i-1},\ a_{4i-3} | 1 \leqq i \leqq n\}$

$\Omega = \{\lambda_1, \xi_1, \xi_5, \ldots, \xi_{4n-3}, \eta_2, \eta_6, \ldots, \eta_{4n-2}\} \cup V$,

$\Gamma = \{Z_0, Z_1, Z_2, Z_4, \ldots, Z_{4n-2}, A, A_1, A_2, A_5, A_6, \ldots, A_{4n-3}, A_{4n-2}\}$

(2)† $P = \{(Z_1, \lambda_1) \to (Z_1 Z_1,\ X\xi_1 \xi_5 \cdots \xi_{4n-3})$,

$(Z_1, \xi) \to (A_1 A_5 \cdots A_{4n-3} Z_1 A,\ \xi_1)$,

† Here, a rule $(A, \dot{\lambda}) \to (\alpha, t)$ is represented in the form of prefix notation $(A, \lambda) \to (\alpha, \mu(t))$.

$$(A_{4i-3},\ \xi_{4i-3}) \to (\varepsilon,\ a_{4i-3}\xi_{4i-3}a_{4i});\ \ 1 \leqq i \leqq n,$$
$$(Z_1,\ \xi_1) \to (Z_2 Z_6 \cdots Z_{4n-2},\ \eta_2),$$
$$(Z_{4i-2},\ \xi_{4i-3}) \to (\varepsilon,\ \eta_{4i-2});\ \ 2 \leqq i \leqq n,$$
$$(A,\ \eta_2) \to (A_2 A_6 \cdots A_{4n-2},\ a_2\eta_2 a_3),$$
$$(A_{4i-2},\ \eta_{4i-2}) \to (\varepsilon,\ a_{4i-2}\eta_{4i-2}a_{4i-1});\ \ 2 \leqq i \leqq n,$$
$$(Z_1,\ \eta_2) \to (Z_0,\ c),$$
$$(Z_0,\ \eta_{4i-2}) \to (Z_0,\ c);\ \ 1 \leqq i \leqq n-1,$$
$$(Z_0,\ \eta_{4n-2}) \to (\varepsilon,\ c)\}.$$

The above construction of gcf DS, E assures that $T_n = T(E) = T(E;\ n)$.

In the followings we use the same functions τ, ρ as defined in Definition 9.6.

Then, the following lemma is immediately obtained by a similar way to Lemma 9.2.

Lemma 10.2. If $Q^n_\alpha(\beta_1,\ \mu(t_1)) = (\beta_2,\ \mu(t_2))$, then $\tau(\mu(t_2)) = \tau(\mu(t_1)) + \rho(\alpha)$.

Lemma 10.3. Let $E = (\Omega,\ V,\ \Sigma,\ \Gamma,\ P,\ Z_0,\ \lambda_0)$ be a gcf DS of degree n and $T = T(E)$, *i.e.*, $T(E;\ n) = T(E)$.

If the set $C(E;\ n)$ of control words of T is infinite then $C(E;\ n)$ contains a control word $\gamma_1\beta_1\gamma_2\beta_2\gamma_3$ which satisfies the following four conditions:

(1) $\beta_1\beta_2 \neq \varepsilon$ and $\gamma_1\beta_1^k\gamma_2\beta_2^\tau\gamma_3$ in $C(E;\ n)$ for all $k \geqq 0$,

(2) $\rho(\beta_1) = \rho(\beta_2) = 0$,

(3) For all β_1' in *init* (β_1), $n - \rho(\gamma_1) \geqq \rho(\beta_1') \geqq 0$,

(4) For all β_2' in *init* (β_2), $n - \rho(\gamma_1) - \rho(\gamma_2) \geqq \rho(\beta_2') \geqq 0$.

Proof (1) This is obtained by the same way as Lemma 3 in the reference[21].

(2) First if $\rho(\beta_1) \geqq 1$, then for an integer $k > n - \rho(\gamma_1) - 1$ (note $\rho(\gamma_1) > 0$) and for $Q^n_{\gamma_1\beta_1{}^k}(Z_0,\ \lambda_0) = (\alpha,\ \mu(t))$, we have $\tau(\mu(t)) > n$, which contradicts the fact that $T(E;\ n) = T(E)$. Second if $\rho(\beta_1) \leqq -1$, then for the integer k and for $Q^n_{\gamma_1\beta_1{}^k}(Z_0,\ \lambda_0) = (\alpha,\ \mu(t))$, we have $\tau(\mu(t)) < -n$, which means that there exists some integer $k > 0$ such that $Q^n_{\gamma_1\beta_1{}^k\gamma_2\beta_2{}^k\gamma_3}(Z_0,\ \lambda_0)$ is not defined. This contradicts the fact of (1). Thus, we have $\rho(\beta_1) = 0$. Similar arguments prove that $\rho(\beta_2) = 0$.

(3) and (4) can be proved by a similar way to Lemma 3 in the reference[21].

Lemma 10.4. Let $T \subseteq \{t \mid \mu(t) \in Xa_1^*a_2^*ca_3^*a_4^* \cdots a_{4m-3}^*a_{4m-2}^*ca_{4m-1}^*a_{4m}^*\}$ be a gcf DL of degree n. If T is infinite, then $\Phi(\mu(T))$ contains a linear set $\{x \mid x = c + kd,\ k \geqq 0\}$ such that $d \neq 0^{4m+2}$ and d has less than $4n$ nonzero coodinates.

This lemma can be proved in a same way as Lemma 9.4 with Lemmas 10.2 and 10.3 instead of Lemmas 9.2 and 9.3.

Theorem 10.5. T_{n+1} is not in $\mathscr{T}^{\mathrm{gcf}}_n$.

Proof Suppose that $T_{n+1} = \{t \mid \mu(t) = Xa_1^k a_2^k ca_3^k a_4^k \cdots a_{4n+1}^k a_{4n+2}^k ca_{4n+3}^k a_{4n+4}^k,\ k \geqq 1\}$ is of degree n. Then Lemma 10.4 requires that $\Phi(\mu(T_{n+1})) \supseteq \{x \mid x = c + kd,\ k \geqq 0\}$, where d had at most $4n$ nonzero coordinates. But

$$\Phi(\mu(T_{n+1})) = \{(|w|_X,\ |w|_c,\ \overbrace{|w|_{a_1},\ \ldots,\ |w|_{a_{4n+4}}}^{4n+4}) \mid w \text{ in } \mu(T_{n+1})\}$$
$$= \{x \mid x = (1,\ n+1,\ \overbrace{0,\ \ldots,\ 0}^{4n+4}) + k(0,\ 0,\ \overbrace{1,\ \ldots,\ 1}^{4n+4}),\ k \geqq 1\}.$$

This is a contradiction. Thus, T_{n+1} is not of degree n.

By (10.2), Theorems 10.4 and 10.5, we have the following results.

Theorem 10.6. $\mathscr{T}_n^{\mathrm{gcf}} \subsetneq \mathscr{T}_{n+1}^{\mathrm{gcf}}$ for all $n \geqq 1$.

11. A characterization of derivation trees of phrase structure grammars

We have studied various types of dendrolanguage generating systems and their properties from chapter 1 to the last chapter 10. In this chapter, we investigate a characterization of derivation trees of phrase structure grammars as an application of these systems. The results obtained here include that of Thatcher[12], which is a characterization of derivation trees of context-free grammars through tree automata. Our result not only generalizes Thatcher's result but shows that there exists a fine correspondence between various types of DS and those of phrase structure grammars.

First, various types of phrase structure grammars and their derivation trees are defined.

Definition 11.1. A *context-sensitive grammar* (CSG) is a 4-tuple $G=(V, \Sigma, P, S)$, where

(1) V: a finite set of symbols,

(2) $\Sigma \subseteq V$: a set of *terminal symbols*,

(3) P: a finite set of *rewriting rules* $uAv \to uyv$, with u, v in V^*, A in $(V-\Sigma)$ and y in V^+,

(4) S: a distinguished element of $(V-\Sigma)$ called an *initial symbol*.

For w and z in V^*, we write $w \underset{G}{\Longrightarrow} z$ if there exist x_0, x_1, uAv, uyv in V^* such that $w = x_0uAvx_1$, $z = x_0uyvx_1$ and $uAv \to uyv$ in P. For w and z in V^* we write $w \overset{*}{\underset{G}{\Longrightarrow}} z$ either if $w=z$ or if there exist $w_0, \ldots, w_r$ such that $w_0=w$, $w_r=z$, and $w_i \Longrightarrow w_{i+1}$ for each i.

The subset of Σ^*

$$L(G) = \{w \text{ in } \Sigma^* \mid S \overset{*}{\underset{G}{\Longrightarrow}} w\} \tag{11.1}$$

is called a *context-sensitive language* (SCL). $L(G)$ is said to be *generated* by G.

Definition 11.2[15]. A *scattered context-sensitive grammar* (CSG) is a 4-tuple $G=(V, \Sigma, P, S)$, where

(1) V, Σ and S are same as in Definition 11.1,

(2) P is a finite set of rewriting rules in the form of

$$(A_1, \ldots, A_l) \to (y_1, \ldots, y_l), \tag{11.2}$$

where A_i is in $(V-\Sigma)$ and y_i is in V^+ for $1 \leqq i \leqq l$ (l: finite integer).

Depending on the forms of rules in P, G is distinguished as follows:

(1) If $l=1$ for all rules in P, *i.e.*, P is consisted of the rules in the form of

$$A \to y;\ A \text{ in } (V-\Sigma),\ y \text{ in } V^+ \tag{11.3}$$

then G is called a *context-free grammar* (CFG).

(2) If all rules in P are in the form of

$$A \to uBv\,;\;\; A \text{ in } (V-\Sigma),\; B \text{ in } V,\; u,\, v \text{ in } \Sigma^* \tag{11.4}$$

then G is called a *linear grammar* (LG).

(3) If all rules in P are in the form of

$$A \to uB\,;\;\; A \text{ in } (V-\Sigma),\; B \text{ in } V,\; u \text{ in } \Sigma^* \tag{11.5}$$

then G is called a *right-linear grammar* (RLG). A *left-linear grammar* (LLG) is also defined.

For a SCG, G, we define a relation $\underset{G}{\Longrightarrow}$ on V^* as follows: For w and z in V^*, we write $w \underset{G}{\Longrightarrow} z$ if there exist $x_0, x_1, \ldots, x_l$, $y_0, y_1, \ldots, y_l$ in V^*, $A_0, \ldots, A_l$ in V such that $w = x_0 A_1 x_1 \cdots x_{l-1} A_l x_l$, $z = x_0 y_1 x_1 \cdots x_{l-1} y_l x_l$ and $(A_1, \ldots, A_l) \to (y_1, \ldots, y_l)$ is in P. The reflective and transitive closure of $\underset{G}{\Longrightarrow}$ is denoted by $\overset{*}{\underset{G}{\Longrightarrow}}$.

For CFG, LG, RLG and LLG, the relations $\underset{G}{\Longrightarrow}$ and $\overset{*}{\underset{G}{\Longrightarrow}}$ is defined in the same way, with only understanding $l=1$.

A subset defined by using $\overset{*}{\underset{G}{\Longrightarrow}}$

$$L(G) = \{w \in \Sigma^* \mid S \overset{*}{\underset{G}{\Longrightarrow}} w\}$$

is called *scattered-context langnage* (SCL), *context-free language* (CFL), *linear language* (LL), *right linear language* (RLL) and *left-linear language* (LLL), if G is SCG, CFG, LG, RLG and LLG, respectively.

Definition 11.3. Let $G=(V, \Sigma, P, S)$ be a CSG. For a w in $L(G)$ we have a derivation

$$S \Longrightarrow w_0 \underset{G}{\Longrightarrow} w_1 \underset{G}{\Longrightarrow} \cdots \underset{G}{\Longrightarrow} w_m = w. \tag{11.6}$$

For this derivation we iteratively define a *derivation tree* as follows:

(1) For $w_0(=S)$, let $\alpha^{w_0} = \{(0, S)\}$.

(2) For some $i(1 \leqq i \leqq m)$, assume that $\alpha^{w_{i-1}}$ is obtained.

If $w_{i-1} \underset{G}{\Longrightarrow} w_i$ is realized by $w_{i-1} = x_0 u A v x_1 \Longrightarrow x_0 u Y_1 \cdots Y_k v x_1 = w_i$ then

$$\alpha^{w_i} = \alpha^{w_{i-1}} \cup \{(m\cdot i,\; Y_i) \mid 1 \leqq i \leqq k,\; (m, A) \in \alpha^{x_0 u \dot{A} v x_1},\; m \in \overline{D}_{\alpha^{w_{i-1}}}\} \tag{11.7†}$$

The rules (1) and (2) define a tree $\alpha^w(=\alpha^{w_m})$, which is uniquely determined and characterized by the derivation (11.6). Thus α^w is called the derivation tree corresponding to the derivation (11.6).

The subset of $\mathscr{T}_V$ defined by

$$D_L(G) = \{\alpha^w \mid \text{there exists a derivation } S \overset{*}{\underset{G}{\Longrightarrow}} w \in L(G)\} \tag{11.8}$$

is the set of all derivation trees for $L(G)$.

Similarly we define derivation trees for SCG.

Definition 11.4. Let G be a SCG. For a derivation from S to w in $L(G)$,

$$S = w_0 \underset{G}{\Longrightarrow} w_1 \underset{G}{\Longrightarrow} \cdots \underset{G}{\Longrightarrow} w_m = w \tag{11.9}$$

† $(m, A) \in \alpha^{x_0 u \dot{A} v x_1}$ means that the symbol A is the one replaced by $Y_1 Y_2 \cdots Y_k$ and that m is the element of the tree domain with which the symbol A is associated.

the following rules (1) and (2) define the derivation tree corresponding to it:

(1) For $w_0(=S)$, let $\alpha^{w_0}=\{(0, S)\}$.

(2) For some $i(1\leqq i\leqq m)$, assume that $\alpha^{w_{i-1}}$ is obtained.

If $w_{i-1}\underset{G}{\Rightarrow}w_i$ is realized by $w_{i-1}=x_0A_1x_1\cdots x_{l-1}A_lz_l\underset{G}{\Rightarrow}x_0y_1x_1\cdots x_{l-1}y_lx_l=w_i$, where $y_i=Y_i^1\cdots Y_i^{k_i}$, $(1\leqq i\leqq l)$, then determine

$$\alpha^{w_i}=\alpha^{w_{i-1}}\cup\bigcup_{i=1}^{l}\{(m_i\cdot j,\ Y_i^j)\mid 1\leqq j\leqq k_i,$$
$$(m_i,\ A_i)\in\alpha^{x_0A_1x_1\cdots x_{i-1}\acute{A}_ix_i\cdots x_{l-1}A_lx_l},\ m_i\in\overline{D}_{\alpha w_{i-1}}\} \qquad (11.10)$$

The rule (1) and m iterative applications of rule (2) yield the derivation tree α^w. The set of derivation tree for a SCG, G is also defined by

$$D_L(G)=\{\alpha^w\in\mathscr{T}_V\mid \text{there exists a derivation } S\overset{*}{\underset{G}{\Rightarrow}}w\in L(G)\} \qquad (11.11)$$

Similarly, the sets of derivation trees for CFG, LG, RLG and can be defined. Although the details of their definitions are omitted, they becomes the same ones as conventional ones[19].

Definition 11.5. Let $G_r=(V_r,\ \Sigma_r,\ P_r,\ S_r)$ be a CSG. For any two rules in P_r, $uAv\rightarrow uxv$ and $u'B'v'\rightarrow u'y'v'$, if $A=B$ implies that $|x|=|y|$, then G_r is called a *ranked CSG* (r-CSG), and define $\sigma(A)=\sigma(B)=|x|\ (=|y|)$.

Let $G_r=(V_r,\ \Sigma_r,\ P_r,\ S_r)$ be a SCG. For any two rules† in P_r, $(A_1,\ldots,A_j,\ldots,A_l)\rightarrow(x_1,\ldots,x_j,\ldots,x_l)$ and $(B_1,\ldots,B_k,\ldots,B_m)\rightarrow(y_1,\ldots,y_k,\ldots,y_m)$, and for any j, k $(1\leqq j\leqq l,\ 1\leqq k\leqq m)$, if $A_j=B_k$ implies that $|x_j|=|y_k|$, then G_r is called a *ranked SCG* (r-SCG), and let $\sigma(A_j)=\sigma(B_k)=|x_j|\ (=|y_k|)$.

Similarly, ranked CFG (r-CFG), LG (r-LG), RLG (r-RLG) and LLG (r-LLG) can be defined.

Lemma 11.1. For any CSG, G, there exists an r-CSG, G_r such that (1) $L(G)=L(G_r)$ and (2) for all $w\in L(G)$, $D_{\alpha w|G_r}=\{0\}\cup\{1\cdot m\mid m\in D_{\alpha w|G}\}$, where $D_{\alpha w|G}$ denote the tree domain of α^w which is the derivation tree for w under a grammar G.

The condition (2) means that if we delete the root node $\{0\}$ from $D_{\alpha w|G_r}$, then it is identical to $D_{\alpha w|G}$. Namely, this lemma asserts that for any CSG, G we can construct an essentially equivalent r-CSG, G_r up to derivation trees.

Proof For a given CSG, $G=(V,\ \Sigma,\ P,\ S)$, define a ranked alphabet V_r by

$$V_r=\{A^n\mid uAv\rightarrow uxv \text{ in } P,\ n=|x|\}\cup\{S^0\}\cup\Sigma.$$

Clearly, $(V_r-\Sigma)\cap(V-\Sigma)=\phi$.

For a symbol A in V, denote the set $\{A^k\mid A^k\in V_r,\ k\text{: integer}\}$ by $[A]$.

Construct a r-CSG, $G=(V_r,\ \Sigma,\ P_r,\ S^0)$ as follows:

(1) For each S^i in $[S]$, let $S^0\rightarrow S^i$ be in P_r.

(2) For each rule $U_1\cdots U_iAW_1\cdots W_k\rightarrow U_1\cdots U_iX_1\cdots X_jW_1\cdots W_k$ in P, let rules $U_1'\cdots U_i'A^jW_1'\cdots W_k'\rightarrow U_1'\cdots U_i'X_1'\cdots X_j'W_1'\cdots W_k'$ be in P_r, where A^j in $[A]$, $U_1'\cdots U_i'$ in $[U_1]\cdots[U_i]$††, $X_1'\cdots X_j'$ in $[X_1]\cdots[X_j]$ and $W_1'\cdots W_k'$ in $[W_1]$

† The case where two rules are same should be taken into consideration.

†† $[U_1]\cdots[U_i]$ denotes the set concatenation of $[U_1],\ldots,[U_i]$.

$\cdots[W_k]$.

The above construction of G_r asserts the conditions (1) and (2).

A similar discussion gives the following lemma, again.

Lemma 11.2. Let G be any SCG, CFG, LG, RLG, or LLG. Then there exists a G_r, which is r-SCG, r-CFG, r-LG, r-RLG or r-LLG, such that (1) $L(G)=L(G_r)$ and (2) for all $w\in L(G)$, $D_{\alpha^w|G_r}=\{0\}\cup\{1\cdot m\mid m\in D_{\alpha^w|G}\}$.

Lemmas 11.1 and 11.2 allow us to consider only the grammars on a ranked alphabet. Thus their derivation trees are ones over a ranked alphabet and then a tree α^w uniquely determines its prefix notation $\mu(\alpha^w)$, and vice versa.

Now we are ready to describe the main results of this chapter.

Theorem 11.1. For any r-CSG, G, there exist a CSDS, S such that $D_L(G)=T(S)$.

Proof Let $G=(V, \Sigma, P, S)$ be a r-CSG. Define a set of symbols Λ by $\Lambda=\{\lambda_A\mid A \text{ in } V\}$. Define a mapping $\lambda: V\to\Lambda$ by $\lambda(X)=\lambda_X$ for X in V, and extend it to $V^*\to\Lambda^*$ by $\lambda(\varepsilon)=\varepsilon$ and $\lambda(x_1x_2)=\lambda(x_1)\lambda(x_2)$ for x_1, x_2 in V^*.

Using these notations, we construct a CSDS, $S=(\Omega, V, \Sigma, P_S, \lambda_S)$ as follows:

(1) $\Omega=V\cup\Lambda$, where $\sigma(\lambda)=0$ for any λ in Λ.

(2) For each rule $U_1\cdots U_iAW_1\cdots W_k\to U_1\cdots U_iX_1\cdots X_jW_1\cdots W_k$ in P, let $(\dot\lambda_{U_1},\ldots,\dot\lambda_{U_i},\dot\lambda_A,\dot\lambda_{W_1},\ldots,\dot\lambda_{W_k})\to(\dot\lambda_{U_1},\ldots,\dot\lambda_{U_i},t,\dot\lambda_{W_1},\ldots,\dot\lambda_{W_k})$ be in P_S, where $\mu(t)=A\lambda(X_1\cdots X_j)$. And for each $\lambda_X\in\{\lambda_{U_1},\ldots,\lambda_{U_i},\lambda_{W_1},\ldots,\lambda_{W_k}\}$ such that X is in Σ, let $(\dot\lambda_X)\to(\dot X)$ be in P_S.

(3) $\lambda_S=\lambda(S)$ where S is the initial symbol S of G.

From the above construction, we can easily prove that $D_L(G)=T(S)$.

Similar constructive discussions prove the following theorems.

Theorem 11.2. For any r-SCG, G, there exists a SCDS, S such that $D_L(G)=T(S)$.

Theorem 11.3. Let G be any r-CFG, r-LG, r-RLG or r-LLG. Then there exists a DS, S, which is CFDS, LDS, RLDS or LLDS, such that $T(S)=D_L(G)$.

This theorem is the one describing the result of Thatcher[12] in terms of DS.

From Theorems 11.1, 11.2, 11.3 and 5.2, the following corollary results.

Corollary 11.1. Let G_{CS}, G_{SC}, G_{CF}, G_L, G_{RL} and G_{LL} be CSG, SCG, CFG, LG, RLG and LLG, respectively. Then (i) $\mu(D_L(G_{CS}))\in\mathscr{L}_{CS}$, (ii) $\mu(D_L(G_{SC}))\in\mathscr{L}_{SC}$, (iii) $\mu(D_L(G_{CF}))\in\mathscr{L}_{CF}$, (iv) $\mu(D_L(G_L))\in\mathscr{L}_L$, (v) $\mu(D_L(G_{RL}))\in\mathscr{L}_R$ and (vi) $\mu(D_L(G_{LL}))\in\mathscr{L}_R$.

Define a mapping $\tau: V^*\to\Sigma^*$ defined by $\tau(\varepsilon)=\varepsilon$; $\tau(a)=a$ for a in Σ; $\tau(X)=\varepsilon$ for X in $(V-\Sigma)$. We have the following corollary.

Corollary 11.2.

(i) $\tau(\mu(D_L(G_{CS})))=L(G_{CS})$, (ii) $\tau(\mu(D_L(G_{SC})))=L(G_{SC})$,

(iii) $\tau(\mu(D_L(G_{CF})))=L(G_{CF})$, (iv) $\tau(\mu(D_L(G_L)))=L(G_L)$,

(v) $\tau(\mu(D_L(G_{RL})))=L(G_{RL})$, (vi) $\tau(\mu(D_L(G_{LL})))=L(G_{LL})$.

We have characterized the set of derivation trees of phrase structure grammars by the corresponding dendrolanguage generating systems. Closing this chapter, we note that the sets of derivation trees of state grammars[21] and string grammars[22] can also be characterized by scf DS and gcf DS, respectively.

12. Conclusions

In this paper, we have introduced various types of dendrolanguage generating systems, which can be considered as a fairly broad class of tree manipulating systems. A hierarchical studies on them have been done and many properties concerning dendrolanguages have been revealed.

Dendrolanguage generating systems discussed in this paper will be used in various fields. A good application field is that of syntax directed translation. Another example of the application is the using of the systems as a tool for assigning meanings to phrase structure languages. We will be able to device a more general tree manipulating system, *e.g.*, a subtree replacement system. But these problems are left for the future studies.

Acknowledgement

The authors would like to express their thanks to Professor Namio Honda of Tohoku University for his valuable discussions concerning this works. They also thank their colleagues for their encouragement to conduct this research.

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