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PERFECTLY DIAMAGNETIC PINCHED PLASMA

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Abstract

A solution of a steady pinched plasma is given by using the London equation with the condition of magnetostatic equilibrium. It is clarified that the London equation for the collisionless plasma is given from the total energy minimum condition with respect to the magnetic field.

1. Introduction

As it was pointed out by A. Simon[1], a set of equations governing the so-called magnetohydrostatic equilibrium do not yield a unique solution. For this problem, it was shown in the earlier note by one of authors[2], S. Miyajima, that if the current density is determined by London's equation then a unique solution is obtained. The reason that London's equation is applied to the collisionless plasma is that the collisionless plasma whose electric resistance is zero is phenomenologically similar to the state of superconductivity of a low tempetature metal. But a question arises in the process of leading London's equation from the two fluids model of plasma[3] namely, according to the generalized vortex theory, if $\nabla \times m_s V_s + e_s \boldsymbol{B} = 0$ (where s stands for e for electrons and i for ions, m_s is the mass, $\boldsymbol{V}_s$ the velocity, e_s the charge of a particle and $\boldsymbol{B}$ the magnetic field) at $t=0$, then $\nabla \times m_s \boldsymbol{V}_s + e_s \boldsymbol{B} = 0$ at an arbitrary time. Therefore we obtain the following London's equation,

$$\nabla \times \frac{\boldsymbol{J}}{n_e} = -\frac{e^2}{m_e}\boldsymbol{B}$$

Here $\boldsymbol{J}$ is the current density and n_e the number density. Now, a question is why $\nabla \times m_s V_s + e_s \boldsymbol{B}$ is equal to zero at $t=0$. About this question, it will be shown in the next paragraph on the basis of the variational principle that the total macroscopic energy is minimum with respect to the magnetic field when London's equation is valid.

While, concerning an equilibrium of collisionless plasma there are many reports which are based on the Vlasov equation[4)~10)], but we must note on these reports that choice of the steady state distribution function or the unperturbed distribution function is different by the research workers. Taking the Z pinch, for example, the distribution function given by E. G. Harris[4] gives the profile of number density obtained by W. H. Bennett[12], whereas the distribution function given by E. S. Weibel[5] is not so and it leads to London's equation as an equation of the current density. And also, in the paper of Von D. Pfirsch. London's equation is reduced only for the plane Θ pinch as an equation of the current density, and for the plane Z pinch, the cylindrical Z pinch and the cylindrical Θ pinch, the unperturbed distribution functions chosen by Pfirsch do not give London's equation. In any way there are many possible solutions of the Vlasov equation so that the above investigators have choesn a plausible solution with parameters. These parameters can be conneted with the macroscopic quantities of electrons and ions; for instance, the fluid velocities or the angular velocities. But they have given only a special solution on the basis of hydrodynamics.

Apart from the calculations based on the Vlasov equation, P. C. Thonemann and W. T. Cowhig[15] got the same profiles of number density as that given by W. H. Bennett using the hydrodynamic model of plasma.

In the next section, we shall show that London's equation is obtained by taking the variation of the total energy with respect to the magnetic field and then give a new set of equations governing the magnetohydrostatic equilibrium. In the third section, the equilibrium of the collisionless plasma is calculated for the cylindrical Z pinch, the cylindrical Θ pinch and these mixed pinch using the new set of equations.

2. The Basic Equations

At first, we shall show that London's eqation is equivalent to the condition that the total energy is minimum with respect to the magnetic field on the basis of variational principle. We shall follow an instructive work done by Ginzburg and Landau[16] in which semi-quantum theory on the super conductor of a low temperature metal is being developed. They gave, at the first place, the free energy in thermodynamics, took the variation of the free energy with respect to the effective wave function and the vector potential and then got a non-linear Schrödinger equation and the London equation.

Now, as the macroscopic energy of the system under consideration, we take, for instance, the total energe, L, given by Van Kampen *et al.*[13], which is expressed as follows,

$$L=\int\left\{\frac{1}{2}\rho V^{2}+\rho_i\psi_i+\rho_e\psi_e+\frac{m_i m_e}{2e^2}\frac{J^2}{\rho}+\frac{B^2}{2\mu_0}+\frac{\varepsilon_0 E^2}{2}\right\}d^3r \tag{1}$$

where $\rho=\sum\rho_s=\sum n_s m_s, \quad V=\sum\rho_s V_s/\rho$

Here the subscript s stanbs for e for electrons and i for ions, ρ the mass density, V the average velosity and ψ_s the internal energy of gas per unit mass density. Van Kampen *et al.* showed that the time derivative of (1) is equal to the sum of Joule's heat and the radiation-energy by making use of an equation of motion and the generalized Ohm's law. Therefore, when the plasma is radiation-free and the resistance is zero, the quantity L will be conserved. It should be noted that the sum of the first term and fourth term of L does not coincide with the sum of the kinetic energy of unit volume for ions and electrons, namely $\sum \rho_s V_s^2/2$. So we shall take into account the cross term, $(m_e \tau / n_e e^2)\, \boldsymbol{V} \cdot \boldsymbol{J}$, which is neglected in (1), where τ is the space change density and is equal to $\sum e_s n_s$.

Now, by making use of $\mu_0 \boldsymbol{J} = \nabla \times \boldsymbol{B}$ we determine the condition that L is minimum with respect to the magnetic field $\boldsymbol{B}$ taking the variation. The result is written as

$$\mu_0 \delta L = \int \left(-\frac{m_e}{e^2} \nabla \times \frac{\tau}{n_e} \boldsymbol{V} + \frac{m_i m_e}{e^2} \nabla \times \frac{\boldsymbol{J}}{\rho} + \boldsymbol{B} \right) \delta \boldsymbol{B} d^3 \boldsymbol{r}$$

Therefore,

$$\frac{m_i m_e}{e^2} \nabla \times \frac{\boldsymbol{J}}{\rho} + \boldsymbol{B} - \frac{m_e}{e^2} \nabla \times \frac{\tau}{n_e} \boldsymbol{V} = 0 \tag{2}$$

which represents the generalized London's equation for the gaseous collisionless plasma and shows that the current density is determined by London's equation for an equilibrium by setting $\boldsymbol{V}=0$. Since $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, where $\boldsymbol{A}$ is the vector potential so that (2) becomes

$$\boldsymbol{J} = -\frac{e^2 n_e \boldsymbol{A}}{m_e} + \tau \boldsymbol{V} + \nabla f$$

where f is a scalar function and can be neglected by an appropriate gauge transformation. Thus, a set of equations governing a steady state of a collisionless plasma is written as the following equations.

$$\nabla \cdot \rho \boldsymbol{V} = 0, \tag{3}$$

$$\rho (\boldsymbol{V} \cdot \nabla) \boldsymbol{V} = \tau \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} - \nabla p - \frac{m_i m_e}{e^2} \nabla \frac{\boldsymbol{J}\boldsymbol{J}}{\rho}, \tag{4}$$

$$\frac{m_i m_e}{e^2 \rho} \nabla (\boldsymbol{V}\boldsymbol{J} + \boldsymbol{J}\boldsymbol{V}) = \boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} + \frac{1}{e n_e} \left(\nabla p_e - \boldsymbol{J} \times \boldsymbol{B} + \frac{m_e}{e^2} \nabla \frac{\boldsymbol{J}\boldsymbol{J}}{\rho} \right), \tag{5}$$

$$\boldsymbol{J} = -\frac{e^2 n_e \boldsymbol{A}}{m_e} + \tau \boldsymbol{V}, \tag{6}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}, \qquad \nabla \cdot \boldsymbol{B} = 0, \tag{7 a, b}$$

$$\nabla \times \boldsymbol{E} = 0, \qquad \varepsilon_0 \nabla \cdot \boldsymbol{E} = \tau \tag{8 a, b}$$

From the above equations, if $\tau=0$ then (4) becomes

$$\nabla \left(\frac{1}{2} V^2 + \int \frac{dp_i}{\rho} + \frac{1}{m_i} \int \frac{dp_e}{n_e} + \frac{m_i m_e}{2 \rho^2 e} J^2 \right) = 0 \tag{9}$$

provided that

$$\nabla \times \boldsymbol{V} = 0$$

The Eq. (9) implies the generlized Bernoulli's law for the gaseous plasma.

By putting $\boldsymbol{V}=0$, we get the equations of the magnetostatic equilibrium as follows

$$0 = J \times B - \nabla\left(p_e + p_i + \frac{m_e}{e^2 n_e} JJ\right), \tag{10}$$

$$\nabla \times \frac{J}{n_e} = -\frac{e^2}{m_e} B, \tag{11}$$

$$\nabla \times B = \mu_0 J, \qquad \nabla \cdot B = 0 \tag{12 a, b}$$

In (10), we see, as previously reported, that the total pressure of the plasma is not equal to the sum of the partial pressure, and the additional term $\nabla \frac{m_e}{e^2 n_e} JJ$ implies the centrifugal force arising from the curvature of the current J. Thus we can obtain the profiles of macroscopic quantities, just as the electron density, the magnetic field and so on, being in an equilibrium state.

3. Application to the Pinched Plasma

In this section we shall calculate the spatial distribution of the macroscopic quantities for the cylindrical Z pinch, the cylindrical Θ pinch and these mixed pinch by assuming the temperature T_s to be constant.

3.1. The cylindrical Z pinch

In the cylindrical coordinates (r, θ, z), it follows $B = (0, B_\theta, 0)$, $J = (0, 0, J_z)$, $A = (0, 0, A_z)$ for the configuration of the Z pinch. For such a system, the equation of magnetohydrostatic equilibrium are written as follows.

$$-J_z \times B_\theta = k(T_e + T_i)\frac{dn_e}{dr} \tag{13}$$

$$J_z = -\frac{e^2 n_e A_z}{m_e} \tag{14}$$

$$\frac{1}{r}\frac{d}{dr}(rB_\theta) = \mu_0 J_z \tag{15}$$

$$B_\theta = -\frac{dA_z}{dr} \tag{16}$$

where the boundary conditions are $n_e = n_0$, $A_z = -A_{z0}$, $J_z = J_0$ and $B_\theta = 0$ at $r = 0$. By making use of (13) ~ (16),

$$N = n_e/n_0 = \exp\{\alpha^2(1 - a^2)\}$$

is reduced where $a = A_z/A_{z0}$ and $\alpha^2 = J_0^2/2e^2 n_0^2 k(T_e + T_i)/m_e$. Furthermore an equation which determines the value of a is

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi \frac{da}{d\xi}\right) = a \exp\{\alpha^2(1 - a^2)\}$$

where $$(\mu_0 e^2 n_0/m_e) r^2 = r^2/r_0^2 = \xi^2.$$

From $a(\xi)$, we can get $b = B_\theta/B_{\theta 0}$,

where $$B_{\theta 0} = (\mu_0 e^2 n_0/m_e)^{1/2} A_{z0}.$$

In the Z pinch the total current which flows inside the radius r is an important quantity and it is written as follows.

$$i = 2\,\pi r B_\theta / \mu_0$$

Now let i_0 be $i_0 = 2\,\pi\,(m_e/\mu_0 e^2 n_0)\left\{2\,e^2 n_0^2 \dfrac{k(T_e+T_i)}{m_e}\right\}^{1/2}$, then i is given from the relation $J = i/i_0 = \alpha\xi b$. It is noted that J implies the scale of the current which flows inside the radius r. Thus we can calculate the quantities N, b and J as a function of ξ; *i.e.*, Fig. 1 shows the relation N vs. ξ, Fig. 2 the relation b vs. ξ and Fig. 3 the relation J vs. ξ where α is a parameter. It is interesting to compare the profile of number density $N(\xi)$ given by the present theory with that obtained by Bennett; *i.e.* according to Bennett's theory $N(\xi)$ is, in our notations, expressed as

$$N(\xi) = n_e/n_0 = \left(1 + \frac{1}{4}\alpha^2\xi^2\right)^{-2} \tag{18}$$

The dotted line in Fig. 1 shows the calculated result by Eq. (18) for two values

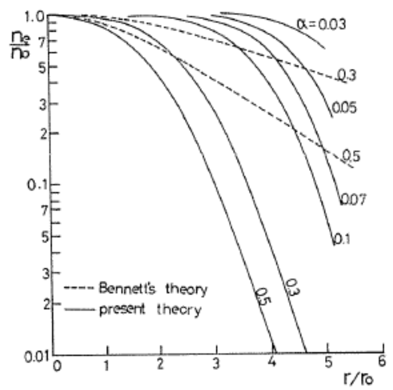

FIG. 1. The ratio $N = n_e/n_0$ versus the ratio $\xi = r/r_0$ in the cylindrical Z pinch where $r_0^2 = m_e/\mu_0 e^2 n_0$ and $\alpha^2 = j_0^2/2\,e^2 n_0 k\,(T_e + T_i)/m_e$.

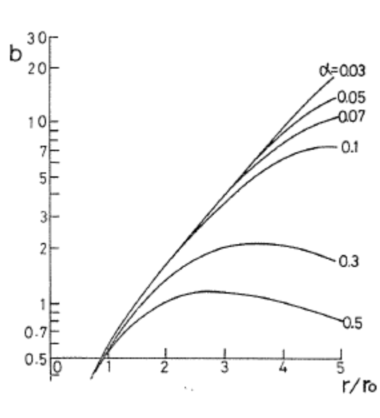

FIG. 2. The ratio $b = B_\theta/B_{\theta 0}$ versus the ratio $\xi = r/r_0$ in the cylindrucal Z pinch where $B_{\theta 0}^2 = \mu_0 e^2 n_0 / m_e A_{z0}^2$.

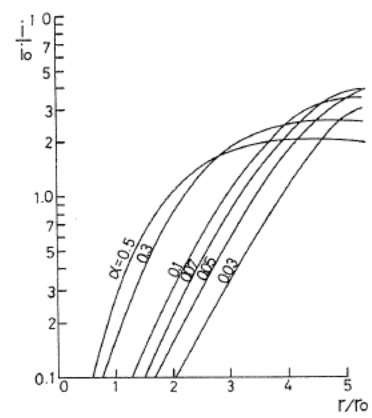

FIG. 3. The ratio $J = i/i_0$ versus the ratio $\xi = r/r_0$ in the cylindrical Z pinch.

of α. Furthermore, we see from Fig. 3 that J saturates at the smaller values of ξ for the larger value of α.

3.2. The cylindrical Θ pinch

For the configuration of cylindrical Θ pinch, $\boldsymbol{B}=(0,\ 0,\ B_z)$, $\boldsymbol{J}=(0,\ J_\theta,\ 0)$, $\boldsymbol{A}=(0,\ A_\theta,\ 0)$, so that basic equations are written as follows.

$$J_\theta B_z = k(T_e+T_i)\frac{dn_e}{dr} - \frac{m_e J_\theta^2}{e^2 r n_e} \tag{19}$$

$$J_\theta = -\frac{e^2 n_e A_\theta}{m_e} \tag{20}$$

$$B_z = \frac{1}{r}\frac{d}{dr}(rA_\theta) \tag{21}$$

$$-\frac{dB_z}{dr} = \mu_0 J_\theta \tag{22}$$

where the boundary conditions are $n_e=n_0$, $A_\theta=0$ and $B_z=B_{z0}$ at $r=0$. From Eq. (19) ~ Eq. (22) and the boundary conditions, the following equation is derived.

$$N = n_e/n_0 = \exp(-a^2) \tag{23}$$

where

$$a = eA_\theta/\{2m_e k(T_i+T_e)\}^{1/2}$$

The equation which determines a is

$$\frac{d}{d\xi}\left(\frac{a}{\xi}+\frac{da}{d\xi}\right) = a\exp(-a^2)$$

and B_z can be calculated by making use of the following equation,

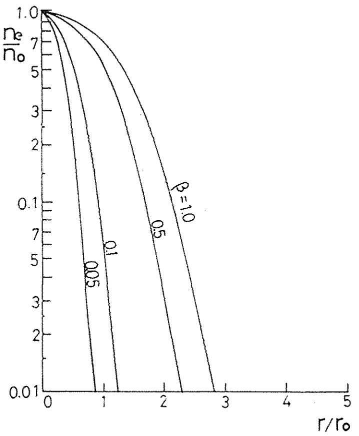

FIG. 4. The ratio $N=n_e/n_0$ versus the ratio $\xi=r/r_0$ in the cylindrical Θ pinch where $\beta=2\mu_0 n_0 k(T_e+T_i)/B_{z0}^2$.

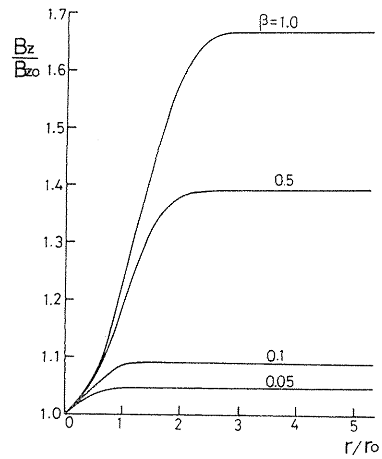

FIG. 5. The ratio $b=B_z/B_{z0}$ versus the ratio $\xi=r/r_0$ in the cylindrical Θ pinch.

$$b = \beta^{1/2} \frac{1}{\xi} \frac{d}{d\xi} (\xi a)$$

where $\beta = 2 \mu_0 n_0 k(T_e + T_i)/B_{z0}^2$ and $B_z/B_{z0} = b$. The relation between N and ξ has been shown in Fig. 4 and the relation between b and ξ in Fig. 5.

3.3. The cylindrical mixed pinch

For the configuration of cylindrical mixed pinch, it is chosen $\boldsymbol{B} = (0, B_\theta, B_z)$, $\boldsymbol{J} = (0, J_\theta, J_z)$ and $\boldsymbol{A} = (0, A_\theta, A_z)$. The basic equations in this case are written as follows.

$$J_\theta B_z - J_z B_\theta = k(T_e + T_i) \frac{dn_e}{dr} - \frac{m_e J_\theta^2}{e^2 n_e r} \tag{24}$$

$$J_\theta = - \frac{e^2 n_e A_\theta}{m_e} \tag{25}$$

$$J_z = - \frac{e^2 n_e A_z}{m_e} \tag{26}$$

$$\frac{1}{r} \frac{d}{dr} (r B_\theta) = \mu_0 J_z \tag{27}$$

$$-\frac{d}{dr} B_z = \mu_0 J_\theta \tag{28}$$

$$B_\theta = - \frac{dA_z}{dr} \tag{29}$$

$$B_z = \frac{1}{r} \frac{d}{dr} (r A_\theta) \tag{30}$$

where the conditions are $n_e = n_0$, $J_z = J_0$, $B_z = B_{z0}$, $J_\theta = 0$, $B_\theta = 0$, $A_\theta = 0$ and $A_z = -A_{z0}$. By making use of Eq. (24) ~ Eq. (30), we have

$$N = n_e/n_0 = \exp\{\alpha^2(1 - a_1^2 - a_2^2)\} \tag{31}$$

where α^2 equals to the quantity given in the paragraph (3.1) and where $a_1 = A_z/A_{z0}$ and $a_2 = A_\theta/A_{z0}$. The quantities a_1 and a_2 are determined by following equations,

$$\frac{1}{\xi} \frac{d}{d\xi} \left(\xi \frac{da_1}{d\xi} \right) = a_1 N$$

and

$$\frac{d}{d\xi} \left(\frac{1}{\xi} \frac{d}{d\xi} \xi a_2 \right) = a_2 N$$

from which $N(\xi)$, $B_\theta/B_{\theta 0} = b_1(\xi)$ and $B_z/B_{z0} = b_2(\xi)$ are obtained, where $B_{\theta 0} = (\mu_0 e^2 n_0/m_e)^{1/2} A_{z0}$. In Fig. 6, 7 and 8, we have shown the calculated results n_e/n_0, B_z/B_{z0} and $B_\theta/B_{\theta 0}$ as a function of $\xi = r/r_0$ respectively, where $\beta = 1$. Furthermore, the scale of the total current flowing inside of the radius r in the direction, $J = i/i_0$ has been shown in Fig. 9 as a function $\xi = r/r_0$ for $\beta = 1$. From these figures, it is seen that the relation between n_e/n_0 and r/r_0 is almost independent of α but the curves $B_\theta/B_{\theta 0}$, B_z/B_{z0} and i/i_0 are remarkably changed.

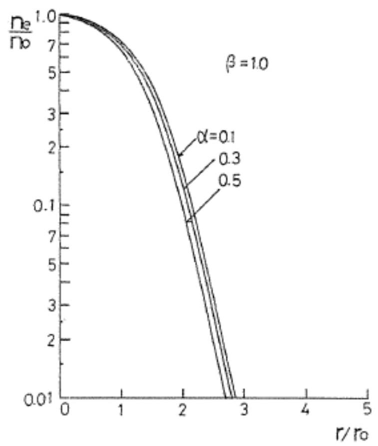

FIG. 6. The ratio $N=n_e/n_0$ versus the ratio $\xi=r/r_0$ in the cylindrical mixed pinch.

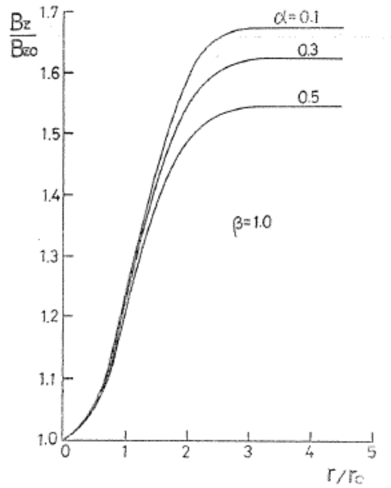

FIG. 7. The ratio $b_z=B_z/B_{z0}$ versus the ratio $\xi=r/r_0$ in the cylindrical mixed pinch.

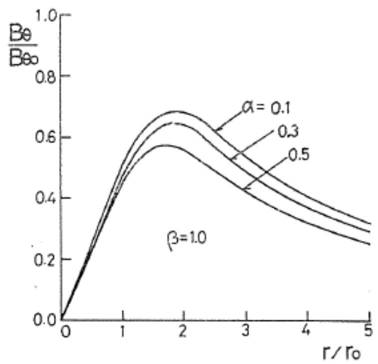

FIG. 8. The ratio $b_1=B_\theta/B_{\theta 0}$ versus the ratio $\xi=r/r_0$ in the cylindrical mixed pinch.

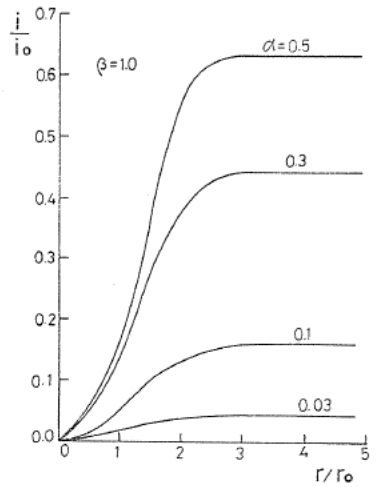

FIG. 9. The ratio $J=i/i_0=\alpha\xi b$ versus the ratio $\xi=r/r_0$ in the cylindrical mixed pinch where $\beta=1$.

4. Conclusion

We have here discussed the magnetohydrostatic equilibrium of a collisionless pinched plasma on the basis of modified magnetohydrostatic equations. The modified magnetohydrostatic equations consist of the usual hydrostatic equations and London's equation where the latter equation determines the current density of a collisionless plasma and gives the total energy minimum with respect to the magnetic field. Applying the modified magnetohydrostatic equation to the pinched plasma, we have obtained a unique solution of equilibrium for the cylindrical Z pinch, the cylindrical Θ pinch, and these mixed pinch. Finally it is noted the results given by the present theory do not always coincide with the results obtained by solving the Vlasov equation. This arises from the fact that the stationary distribution function of the Vlasov equation is not uniquely determined.

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