ZERO ASSIGNMENT IN LINEAR MULTIVARIABLE DECOUPLED CONTROL SYSTEM

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1. Introduction

The purpose of this paper is to discuss the zero assignment problem in linear multivariable decoupled control system. The zero locations in complex plane affect the input-output relationships of the linear systems, for instance, the step responses and the steady state errors vary with them. Therefore to derive the conditions under which desired zero configuration can be obtained is important in the synthesis of the linear systems. In the older literature, however, not so much attention is directed to the problems of zero assignment in contrast to the pole's[1][2].

In [3], zero assignment problem is considered for a single input-single output system and the conditions are derived where any desired pole-zero configuration can be obtained independently. This paper is an extention of that to the multivariable decoupled control system and gives necessary and sufficient conditions which assure the existence of an input matrix with which the multivariable system can be decoupled by state feedback and can have desired location of zeros.

The paper is organized as follows. In § 2 the characteristics of zeros in multivariable systems are reviewed and the precise statement of the problem considered in this paper is given. In § 3 the survey of the decoupling control is done and the problem given in § 2 is partially solved. § 4 gives the principal result of the paper and states it in theorem 3. In § 5 a numerical example is given to illustrate the computation of the input matrix. § 6 contains the concluding remarks.

2. Statement of the problem

Consider the linear time-invariant dynamical system

$$S_1 \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

where x is an n-vector, u and y are m-vectors, and A, B, C are real constant coefficient matrices of appropriate dimensions.

It is well known that the linear state feedback $u = Fx + Gw$ can be used to obtain any desired pole configuration if and only if S_1 is controllable. On the other hand it is also known that the zeros of the single input-single output system do not change their location by state feedback. So, in single input-single

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output system, if desired zero configuration can be obtained by some means, the poles of the system can be compensated without destroying the above obtained zero pattern, and therefore any pole-zero configuration is obtainable.

The zeros of S_1 can be altered by giving a different matrix to B or C. In applications, the output matrix C will usually be fixed by the physical structure of the systems, and more freedom is available in the selection of the input matrix B. In reference 3 it is shown that the input matrix B of the single inpt-single output system S_1, where A and B are given fixed matrices, can be found to obtain any desired zero configuration if and only if (C, A) is observable pair.

Unfortunately, the above results can not trivially be extended to the multivariable systems, the reasons are as follows. First, even if S_1 is observable all the zeros of the transfer matrix can not be assigned arbitrarily. This can be found simply by comparing the numbers of zeros with the numbers of the free parameter B. Secondly, in the case of multivariable systems, not only the poles but also the zeros are dependent on the feedback matrix F, and so even if the desired zero configuration is obtained by some means, it will generally be changed by application of state feedback for pole assignment.

The main difficulty in the pole-zero synthesis in multivariable systems lies in the fact that both poles and zeros are function of feedback matrix F. So, the great simplicity will be obtained if the above interactions of poles and zeros are eliminated. J. D. Simon and S. K. Mitter have considered this problem and showed[4] that, if the matrix A devides the whole space E^n into m cyclic invariant subspaces and if the input matrix B is constructed such that every column vector of it coincides with the generating vector of the m invariant subspaces, then the zeros of the system are invariant under state feedback. However, to require that the matrix A has exactly m invariant polynomials would be very unrealistic and furthermore even if this condition holds, their results only assure the invariancy of zeros, and does not give any assurance for the obtainability of desired zero configuration.

As stated above, the problem of pole-zero assignment in multivariable systems is very complicated and no so much work has been done yet. In this paper the zero assignment problem in multivariable decoupled control system is considered. This is somewhat restriction of the general problem where the zeros of any element of the transfer matrix is to be specified. It must, however, be noted that the restriction of this kind does come not only from the requirement for the simplification of the problem, but also from the actual importance of decoupling control in multivariable systems. The characteristics of the poles and zeros in decoupled system are fully examined in references[5], and the fundamentals of them which are required in this paper are briefly summarized in the next section.

At this point, the problem considered in this paper should be stated in more concrete fashion, and is as follows.

Given the matrices A and C, find the conditions for the existence of the input matrix B such that the system

$$S_1 \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

can be decoupled by state feedback $u = Fx + Gw$, and furthermore the decoupled system have a desired pole-zero configuration.

3. Decoupling control and the difference order

Consider again the m input-m output system

$$S_1 \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

The control law

$$u = Fx + Gw \tag{3-1}$$

is said to decouple S_1 if the closed loop transfer function

$$H(s,\ F,\ G) = C(sI - A - BF)^{-1}BG \tag{3-2}$$

becomes a nonsingular diagonal $m \times m$ matrix. Throughout the paper such a control and the system obtained by decoupling S_1 will be called the decoupling control law and the decoupled system of S_1 respectively. The $i-i$ th element of the transfer function matrix (3–2) represents the transfer function of the i-th subsystem of the decoupled system.

The necessary and sufficient condition for decoupling has been derived by P. L. Falb, W. A. Wolovich, E. G. Gilbert and others, and is given by the following theorem.

[Theorem 1] (P. L. Falb, W. A. Wolovich[5], E. G. Gilbert[6] and others)

The system S_1 can be decoupled by state feedback if and only if the matrix

$$D = \begin{pmatrix} c_1 A^{d_1} B \\ \cdot \\ \cdot \\ \cdot \\ c_m A^{d_m} B \end{pmatrix}$$

is nonsingular, where c_i is the i-th row of the matrix C and

$$d_i = \min\{j;\ c_i A^j B \neq 0,\ j \geq 0\}, \quad i = 1, \ldots, m$$

The following remark is the fundamentals for present paper and their proofs can be found in [6].

(*Remark 1*)

Given the system S_1 whose coefficient matrices A, B and C satisfy the required conditions in theorem 1, the transfer function of the decoupled system of S_1 is uniquely determined except for the location of poles, *i.e.*, the orders of the denominator and the numerator of every subsystem, and the locations of zeros do not depend on the particular choice of the decoupling control law of S_1. In addition, the location of the poles can be arbitrarily assigned by suitably choosing the decoupling control law. The difference of the orders in denominator and numerator of the i-th subsystem, which is seen from the above to be independent of the choice of decoupling control, is equal to d_i+1. The number d_i $(i=1, \dots, m)$ will be called the difference order of the i-th subsystem.

Now, in the followings, it is assumed that A and C are given matrices and B is the matrix of dimension $n \times m$ which can be arbitrarily chosen. Then the problem, as stated in the previous section, is to find B such that the system S_1 can be decoupled and the decoupled system has the required zero configuration.

For the former problem, following theorem holds.

[**Theorem 2**]

(i) There exists a matrix B which makes the system S_1 be decoupled by state feedback and makes the decoupled i-th subsystem have the difference order d_i if and only if.

(ii) The vectors

$$\{c_i A^j\} \qquad i=1, \dots, m, \quad j=0, \dots, d_i$$

are mutually independent.

(*Proof*)

Assume (i) holds, then from the definition of the difference order d_i and from theorem 1,

$$c_i A^j B = 0 \qquad i=1, \dots, m, \quad j=0, \dots, d_i-1, \quad \text{and}$$
$$c_i A^{d_i} B \qquad (i=1, \dots, m) \quad \text{are mutually independent.}$$

Suppose there are constants k_{ij} such that

$$\sum_{i=1}^{m}\sum_{j=0}^{d_i} k_{ij} c_i A^j = 0 \tag{3-3}$$

Postmultiplying both sides of this equation with B produces

$$k_{i d_i} = 0 \qquad i=1, \dots, m. \tag{3-4}$$

Now, introducing the above results into (3-3), and postmultiplying with AB gives

$$k_{i\, d_i-1} = 0 \qquad i=1, \dots, m.$$

Continuing in this manner proves (ii).

Next, (ii) is assumed. By the definition of the difference order d_i, it is enough to show that the linear squations

$$\begin{aligned} c_i A^j B &= 0 \\ c_i A^{d_i} B &= e_i \end{aligned} \qquad i=1, \dots, m, \quad j=0, \dots, d_i-1 \tag{3-5}$$

can be solved for B, where e_i is an unit vector whose elements are all zero except for i-th element which is one. From the assumption (ii)

$$\text{rank}\begin{pmatrix} c_1 \\ c_1 A \\ \cdot \\ c_1 A^{d_1} \\ c_2 \\ c_2 A \\ \cdot \\ c_m A^{d_m} \end{pmatrix} = \sum_{i=1}^{m} d_i + m,$$

and hence there always exists a matrix B which satisfies (3–5).

(*Rmark 2*)

If $\sum_{1}^{m}(d_i+1)=n$ in theorem 2, (3–5) have unique solution B, and in this case the decoupled system of S_1 does not have any zeros.

4. Integrator decoupled system and its zeros

In the followings, $\{d_i;\ i=1,\ldots,m\}$ are assumed to be the given set of nonnegative numbers which satisfy the condition (ii) of theorem 2, and $\sum_{i=1}^{m}(d_i+1)<n$. Then from theorem 2, there exists a input matrix B, not uniquely, with which S_1 becomes possible to be decoupled by state feedback and has difference orders d_i $(i=1,\ldots,m)$. Such a matrix B is given by solving equation (3–5).

Then the next problem to be considered is to find a matrix B, among the general solutions of (3–5), which makes the decoupled system of S_1 have the desired zero configuration.

Now, a system which has a general solution of (3–5) as its input matrix will be denoted as S_2 and is given by

$$S_2\begin{cases} \dot{x}=Ax+\overline{B}u \\ y=Cx \end{cases}$$

where

$$\overline{B}\in\mathfrak{B}\triangleq\{B;\ c_iA^{d_i}B=e_i,\ c_iA^jB=0;\ j=0,\ldots,d_i-1,\ i=1,\ldots,m\}.$$

If the control law

$$u=-A^{*}x+\overline{w} \tag{4-1}$$

where

$$A^{*}=\begin{pmatrix} c_1A^{d_1+1} \\ \cdot \\ \cdot \\ \cdot \\ c_mA^{d_m+1} \end{pmatrix}$$

is introduced to S_2, the closed loop transfer characteristics of S_2 with the feedback control (4–1) can be represented by

$$S_3 \begin{cases} \dot{x} = \overline{A}x + \overline{B}\overline{w} \\ y = Cx \end{cases}$$

where

$$\overline{A} = A - \overline{B}A^*, \qquad \overline{B} \in \mathfrak{B}.$$

Notice that the coefficients matrices in S_3 satisfy the relations,

$$c_i\overline{A}^j\overline{B} = 0, \qquad c_i\overline{A}^{d_i}\overline{B} = e_i, \qquad c_i\overline{A}^{d_i+1} = 0, \tag{4-2}$$

and it's transfer function matrix becomes

$$C(sI - \overline{A})^{-1}\overline{B} = \begin{bmatrix} 1/s^{d_1+1} \cdot & \cdot & \cdot & 0 \\ & & \cdot & \\ 0 & \cdot & \cdot & \cdot\, 1/s^{d_m+1} \end{bmatrix}.$$

E. G. Gilbert called such a system Integrator Decoupled System.

The control law given by eq. (4-1) is, of course, decoupling control law for S_2, but is a special one, which makes the i-th subsystem of S_3 have $1/s^{d_i+1}$ as its transfer function. On the other hand, according to the remark 1 the zeros of every subsystems of S_3 are known to be independent of the choice of decoupling control, and so the integrator decoupled system should also have zeros corresponding to the matrix $\overline{B}(\in \mathfrak{B})$.

Therefore, the fact that the transfer function of the i-th subsystem is represented by $1/s^{d_i+1}$ means the pole-zero cancellations have ocurred. In order to know which zeros have been cancelled out by poles, deeper structural consideration on S_3 is required.

[Lemma 1]

Let $\{\lambda_{ij};\ i=1, \ldots, m,\ j=1, \ldots, p_i\}$ be the set of mutually distinct nonzero numbers, where $p_i \geq 0$ and, $\sum_1^m (d_i + p_i + 1) \leq n$. When $p_i = 0$, the corresponding numders λ_{ij} are not included.

(i) If the system S_3 has its cancelled zeros at $s = \lambda_{ij}$ $(j=1, \ldots, p_i)$ in its i-th subsystem $(i=1, \ldots, m)$ for some $\overline{B} \in \mathfrak{B}$, S_3 can be transformed by $z = Tx$ into the following canonical form.

$$\dot{z} = \tilde{A}z + \tilde{B}\overline{\omega} = \left(\begin{array}{ccccc|c} A_1 & 0 & \cdot & 0 & & 0 \\ 0 & \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & 0 & & \cdot \\ 0 & \cdot & \cdot & A_2 & & 0 \\ \hline & & \gamma & & & \Phi \end{array}\right) z + \left(\begin{array}{cccc|c} b_1 & & & & \\ \tilde{b}_1 & & & & \\ & \cdot & & 0 & \\ 0 & & & \cdot & \\ & & & b_m & \\ & & & \tilde{b}_m & \\ \hline & & \alpha & & \end{array}\right) \overline{w},$$

$$y = \left(\begin{array}{ccccc|c} \theta_1 & 0 & \cdot & \cdot & 0 & \\ & & \cdot & & & 0 \\ 0 & \cdot & \cdot & 0 & \theta_m & \end{array}\right) z. \tag{4-3}$$

where
$$\theta_i = (\underbrace{1, 0, \ldots\ldots, 0}_{d_i + p_i + 1}),$$

$$A_i = \left[\begin{array}{ccccc|cccc} 0 & 1 & 0 & \cdot & 0 & & & & \\ 0 & 0 & 1 & \cdot & 0 & & & & \\ & & \cdot & & & & 0 & & \\ 0 & 0 & 0 & \cdot & 1 & & & & \\ 0 & \cdot & \cdot & \cdot & 0 & & & & \\ \hline & & & & & \lambda_{i1} & 0 & \cdot & 0 \\ & & 0 & & & 0 & \lambda_{i2} & \cdot & 0 \\ & & & & & & \cdot & \cdot & \\ & & & & & 0 & 0 & \cdot & \lambda_{ip_i} \end{array}\right] \begin{array}{l} \\ \\ d_i+1 \\ \\ \\ \\ p_i \\ \\ \\ \end{array} \qquad b_i = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix} d_i+1 \qquad \tilde{b}_i = \begin{pmatrix} \tilde{b}_{i1} \\ \tilde{b}_{i2} \\ \cdot \\ \tilde{b}_{ip_i} \end{pmatrix} p_i$$

and γ, Φ, α are matrices with appropriate dimensions. The transformation matrix T is enough to be of the form,

$$T = \begin{bmatrix} T_1 \\ \cdot \\ \cdot \\ \cdot \\ T_m \\ T_{m+1} \end{bmatrix} \tag{4-4}$$

where

$$T_i = \begin{bmatrix} c_i \\ c_i A \\ \cdot \\ c_i A^{d_i} \\ t_{i1} \\ \cdot \\ t_{ip_i} \end{bmatrix}, \qquad T_{m+1} = \begin{bmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ q_{n-\Sigma(d_i+p_i+1)} \end{bmatrix}$$

(ii) If there exists a matrix $\overline{B} \in \mathfrak{B}$ such that system S_1 can be transformed by the transformation of the form (4-4) into (4-3), and if all the element of $\tilde{b}_i$ $(i=1, \ldots, m)$ are nonzero numbers, then S_3 has its zeros at $s=\lambda_{ij}$.

(Proof)

(i) Let $\overline{B}=[\bar{b}_1, \ldots, \bar{b}_m]$ be the matrix stated in the lemma, and define $Q_i=\{\eta\,;\ \eta \bar{A}^j \bar{b}_k=0,\ j=0, 1, \ldots, n-1,\ k \neq i\}$. Q_i has following properties[6]. ① Q_i is row-invariant subspace with respect to $\bar{A}$, ② $Q_i \cap Q_j=\{0\}$ if $i \neq j$, ③ $c_i, c_i A, \ldots, c_i \bar{A}^{d_i}$ are linearly independent vectors of Q_i. Furthermore, eigenvalues of $\bar{A}$ on the invariant subspace Q_i include all the zeros of the i-th subsystem of S_3[6]. And hence, there always exists a eigenvector e_{ij} corresponding to the eigenvalue λ_{ij} such that $e_{ij} \in Q_i$.

Now, it will be proved $\{e_{ij},\ c_i \bar{A}^k,\ j=1, \ldots, p_i,\ k=0, \ldots, d_i\}$ is the set of mutually independent vectors in Q_i.

Suppose there exist scalars α_j $(j=1, \ldots, p_i)$ and β_k $(k=0, \ldots, d_i)$ such that

$$\sum_{j=1}^{p_i} \alpha_j e_{ij} + \sum_{k=0}^{d_i} \beta_k c_i \bar{A}^k = 0.$$

Postmultiplying both sides of this equation with $\bar{A}^l$, where l is sufficiently large number, yields

$$\sum_{j=1}^{p_i} \alpha_j (\lambda_{ij})^l e_{ij} = 0.$$

This implies $\alpha_j = 0$ $(j=1, \ldots, p_i)$, for λ_{ij} are, by assumption, mutually distinct nonzero numbers, and hence together with this the linear independence of $c_i\bar{A}^k$ $(k=0, \ldots, d_i)$ means $\alpha_j = \beta_k = 0$.

Now the matrix stated in the lemma can be defined as follows,

$$T = \begin{pmatrix} T_1 \\ T_2 \\ \cdot \\ \cdot \\ \cdot \\ T_m \\ T_{m+1} \end{pmatrix}, \qquad T_i = \begin{pmatrix} c_i \\ c_i\bar{A} \\ \cdot \\ c_i\bar{A}^{d_i} \\ e_{i1} \\ \cdot \\ e_{ip_i} \end{pmatrix}, \qquad T_{m+1} = \begin{pmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f_{n-\Sigma(p_i+d_i+1)} \end{pmatrix}$$

$$(i = 1, \ldots, m)$$

where $\{f_k\}$ is the set of independent row vectors with which the rows of T_i $(i=1, \ldots, m)$ span the whole space E^n. The fact TAT^{-1}, $T\bar{B}$ and CT^{-1} have the form in the lemma respectively can be easily verified by using the definition of T and the fact $\bar{B} \in \mathfrak{B}$.

(ii) Assume that the transformation $z = Tx$ brings S_3 into the form (4–3), then the control law

$$\bar{\omega} = \left(\begin{array}{ccc|ccc} \eta_1 & \cdots & 0 & 0 & \cdot & 0 \\ & \cdot & & \cdot & \cdot & \cdot \\ 0 & \cdots & \eta_m & 0 & \cdot & 0 \end{array}\right) Tx + \tilde{\omega} \qquad (4\text{-}5)$$

$$\eta_i = (\eta_{i1}, \ldots, \eta_{i\,p_i+d_i+1})$$

becomes a decoupling control law for S_3. It can easily be verified that the closed loop transfer function of S_3 with (4–5) has zeros in its i-th subsystem.

Now, using this lemma, the main results of the paper can be obtained. For the proof of the theorem next lemma must be prepared. In the next two theorems it is assumed that all the numbers λ_{ij} are real, mutually distinct and nonzero as in lemma 1, and furthermore it is also assumed they are not equal to the eigenvalues of A for simplicity.

[**Lemma 2**]

Linear first order equations,

$$(t_{ij}, \tilde{b}_{ij}) \begin{pmatrix} A - \lambda_{ij} I \\ -c_i A^{d_i+1} \end{pmatrix} = 0, \qquad i = 1, \ldots, m, \quad j = 1, \ldots, p_i \qquad (4\text{-}6)$$

where t_{ij} are n-vectors and $\tilde{b}_{ij}$ are scalars, can be solved for $(t_{ij}, \tilde{b}_{ij})$ uniquely up to the scalar multiplications. Further, $t_{ij} \neq 0$ if and only if $c_i A^{d_i+1} \neq 0$.

[**Theorem 3**]

The i-th subsystem of S_3 has zeros at $s = \lambda_{ij}$ $(i=1, \ldots, m,\ j=1, \ldots, p_i)$ if and only if

(a) $c_i A^{d_i+1}$ $(i=1, \ldots, m)$ are nonzero vectors, and

(b) $c_i A^j,\ t_{ik}\ (i=1, \ldots, m,\ j=0, 1, \ldots, d_i,\ k=1, \ldots, p_i)$

are mutually independent vectors, where $(t_{ik}, \tilde{b}_{ij})$ are nonzero solutions of eq. (4–6).

(*Proof*) Necessity. By lemma 1–(i), there exist $\overline{B} \in \mathfrak{B}$ and the transformation matrix T of the form (4–4) such that

$$T\bar{A}T^{-1} = \tilde{A} \tag{4-7}$$

$$T\overline{B} = \tilde{B} \tag{4-8}$$

$$\overline{B} \in \mathfrak{B} \tag{4-9}$$

where $\tilde{A}$ and $\tilde{B}$ are matrices defined in lemma 1.

Now, substituting the defining equation of $\bar{A}$ into (4–7) yields

$$TA - T\overline{B}A^* = \tilde{A}T. \tag{4-10}$$

From eqs. (4–8) and (4–10)

$$TA - \tilde{B}A^* = \tilde{A}T \tag{4-11}$$

follows. Substituting the defining equations of $\tilde{A}$ and $\tilde{B}$ into the above and equating both sidedes,

$$(t_{ij},\ \tilde{b}_{ij})\begin{pmatrix} A - \lambda_{ij} I \\ -c_i A^{d_i+1} \end{pmatrix} = 0 \qquad i=1, \ldots, m, \quad j=1, \ldots, p_i \tag{4-12}$$

are obtained. Note that $c_i A^{d_i+1}=0$ means $t_{ij}=0$ by lemma 2, and therefore it contradicts the nonsingularity of T. Consequently the necessary condition (a) is proved.

Next, assume $c_i A^{d_i+1} \neq 0$. In this case the equation (4–12) have nonzero unique solutions and $t_{ij} \neq 0$ by lemma 2. The necessary condition (b) can be verified by the nonsingularity of T.

Sufficiency. If (a) and (b) hold, then equation (4–12) have nonzero solutions $(t_{ij}, \tilde{b}_{ij})$ $(i=1, \ldots, m,\ j=1, \ldots, p_i)$ where both t_{ij} and $\tilde{b}_{ij}$ are not zero. Then, using these solution vectors t_{ij}, the transformation matrix T can be defined exactly in the same manner as is done in the proof of the lemma 1 and hence the derivation of it is omitted here.

It can be asily verified that the eqs. (4–8) and (4–9) are solvable for $\overline{B}$, by rewriting these equations into

$$\begin{pmatrix} t_{11} \\ t_{12} \\ \cdot \\ t_{1p_1} \\ t_{21} \\ \cdot \\ t_{mp_m} \\ c_1 \\ \cdot \\ c_1 A^{d_1} \\ \cdot \\ c_m A^{d_m} \end{pmatrix} \overline{B} = \begin{pmatrix} \tilde{b}_1 & \cdot & \cdot & \cdot & 0 \\ 0 & \tilde{b}_2 & \cdot & \cdot & 0 \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ 0 & \cdot & \cdot & \cdot & \tilde{b}_m \\ b_1 & 0 & \cdot & \cdot & 0 \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ 0 & 0 & \cdot & \cdot & b_m \end{pmatrix}, \tag{4-13}$$

where b_i are vectors defined in lemma 1.

It is easy to see that the equation (4–9) follows from equations (4–8) and (4–12).

Using these matrices T and $\bar{B}$, it finally follows from lemma 1–(ii) that the i-th subsystem of S_3 has zeros at $s=\lambda_{ij}$ and the theorem is proved.

5. Example

An infinitesimal transition matrix A and the output matrix C are given by

$$A=\begin{bmatrix}1&0&1&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{bmatrix},\qquad C=\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix}$$

respectively.

In order to examine the possibility for decoupling control, $c_i A^j$ must be calculated.

$$c_1=(1\ 0\ 0\ 0),\quad c_1A=(1\ 0\ 1\ 0),\quad c_1A^2=(1\ 1\ 1\ 0),\quad c_1A^3=(1\ 1\ 1\ 1)$$
$$c_2=(0\ 1\ 0\ 0),\quad c_2A=(0\ 0\ 0\ 1),\quad c_3A^2=(0\ 0\ 1\ 0),\quad c_2A^3=(0\ 1\ 0\ 0)$$

Threfore, by theorem 2, the decoupling control is possible for the case where the couples of difference orders for the first and second subsystem are $\{0, 0\}$, $\{0, 1\}$, $\{0, 2\}$, $\{1, 0\}$ and $\{1, 1\}$. For the case $\{0, 2\}$ and $\{1, 1\}$ there are no zeros in both decoupled systems (See Remark 2).

In the follwings the case $\{0, 0\}$ will be considered, *i.e.*, the transfer function of both decoupled subsystems has difference one in their orders of denominators and numerators. And it is further assumed that the first subsystem has two zeros at $s=\lambda_{11}$ and λ_{12} and the second has no zeros ($p_1=2$, $p_2=0$).

Now, the characteristic equation for A is given by

$$(1-\lambda)^2(1+\lambda+\lambda^2)=0$$

So, λ_{11} and λ_{12} must be mutually distinct numbers which are not equal to one and zero in order to satisfy the required conditions in theorem 3.

The equation (4–6) becomes

$$(t_{1j},\ \tilde{b}_{1j})\begin{pmatrix}A-\lambda_{1j}\\-c_1A\end{pmatrix}=0,\qquad j=1,\ 2$$

and solving this by substituting A and c_1 gives

$$\begin{cases} t_{1j}=\dfrac{\{(1-\lambda_{1j})(1+\lambda_{1j}+\lambda_{1j})^2,\ -\lambda_{1j}^2,\ -\lambda_{1j}^3,\ -\lambda_{1j}\}}{(1-\lambda_{1j})^2(1+\lambda_{1j}+\lambda_{1j}^2)}\\ \tilde{b}_{1j}=1 \qquad j=1,\ 2.\end{cases}$$

To proceed further the desired zeros are assumed to be -1 and -2, *i.e.*, $\lambda_{11}=-1$ and $\lambda_{12}=-2$. Then

$$t_{11}=(2\ \ -1\ \ 1\ \ 1)/4,\qquad t_{12}=(9\ \ -4\ \ 8\ \ 2)/27$$

Therefore all the required conditions in theorem 3 are satisfied. The transformation matrix T given by eq. (4-4) becomes

$$T=\begin{bmatrix} c_1 \\ t_{11} \\ t_{12} \\ c_2 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & -1/4 & 1/4 & 1/4 \\ 1/3 & -4/27 & 8/27 & 2/27 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Finally $\overline{B}$ can be obtained by solving equation (4-13), and becomes

$$\overline{B}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 7/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}.$$

Consequently, desired system can be represented by

$$\dot{x}=\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}x+\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}u,$$

$$y=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}x.$$

Substituting the decoupling control law (4-5) into the above reveals that the first subsystem has two zeros at $s=-1$ and -2, and three poles which can be arbitrarily assigned by suitable vector η_1, and the second subsystem has nozeros and one pole at arbitraly place.

6. Concluding remarks

In this paper, the conditions for the existence of the input matrix B with which the system

$$\dot{x}=Ax+Bu$$

$$y=Cx$$

can be decoupled by state feedback and has zeros at required positions are discussed. Theorem 2 and Theorem 3 give such conditions, which can easily be verified by vector and matrix calculations.

Several assumptions on the zeros are made, *i.e.*, they must be mutually distinct real nonzero numbers which are not equal to the eigenvalues of A, but except for requiring to be real they would not be too strict restrictions in applications. The extension to the case of complex zeros is straightforward.

If the input matrix B can not be selected arbitrarily, but must obey some constraints, the decoupling control stated in theorem 3 is possible if and only if the equation (4-13) can be solved subject to these constraints.

Using a simple example, an algolirism for obtaining the input matrix is given and the pole-zero compensation of the decoupled system is illustrated.

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