# ON THE VIBRATIONS OF "SUMMED AND DIFFERENTIAL TYPES" UNDER PARAMETRIC EXCITATION

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CONTENTS

## General Introduction

In a vibratory system having multiple degree-of-freedom and under parametric excitation of frequency $\omega$, unstable vibrations of "summed and differential types" having frequencies $\omega_i$ ($\fallingdotseq p_i$), $\omega_j$ ($\fallingdotseq p_j$) satisfying the relation $\omega_i \pm \omega_j = \omega$ take place with ordinary unstable vibrations[1)~8)] of frequency $\omega$ appearing in the neighborhood of twice natural frequencies, *i.e.*, $\omega \fallingdotseq 2p_r$, when $\omega$ becomes nearly equal to sum of and difference between two natural frequencies $p_i$, $p_j$, *i.e.*, $\omega = p_i \pm p_j = p_{ij}$.

Some studies[9)~16)] only on the possibility of occurrence of these kinds of unstable vibrations have been made, and it seems that both detailed proposition of solutions and discussions of characteristics of these kinds of vibrations have not been carried out. In the present paper, through a similar procedure to Kryloff and Bogoliuboff's method[17)], properties of these kinds of vibrations, *i.e.*, solutions, frequencies, phase angles, amplitude ratios between two vibrations, negative damping coefficients, location and width of unstable region in which two unstable vibrations of frequencies $\omega_i$ and $\omega_j$ build up, and the relation between initial conditions and solutions of vibrations are studied in detail. Further, analytical results are compared with those of experiment and analog computer.

The first approximate analysis of vibrations of "summed and differential types" in vibratory system without and with damping are treated in Chapters I and II respectively. In Chapter III vibrations of "summed and differential types of higher order" appearing at $\omega \fallingdotseq (p_i \pm p_j)/s = p_{ij}/s$ ($s = 2, 3, \cdots$.) are discussed. Chapter IV is concerned with vibrations of "summed and differential types" in non-linear vibratory system. In Chapter V, forced vibrations of linear vibratory system with one degree-of-freedom under parametric excitation are discussed.

## Chapter I. Vibrations of "summed and differential types" without damping under parametric excitation[18)]

### *1.1. Introduction*

In the present chapter, vibrations of the so-called "summed and differential types" in a linear vibratory system of multiple degree-of-freedom without damping and under parametric excitation are treated, and analytical discussion of the first approximate solutions are performed, in which higher powers of small quantities are neglected. These vibrations consist of two vibrations having frequencies $\omega_i$ and $\omega_j$ which satisfy the following relations:

$$\omega_i \pm \omega_j = \omega, \quad \omega_i \fallingdotseq p_i, \quad \omega_j \fallingdotseq p_j, \tag{1.1}$$

where $\omega$ is frequency of parametric excitation and $p_i$, $p_j$ ($i \neq j$; $p_i > p_j$; $i, j = 1, 2, \cdots, h$; $h$=number of degree of freedom) are two natural frequencies of the system. Since Eq. (1.1) results in a relation $\omega \fallingdotseq p_i \pm p_j$, it is seen that vibrations of summed and differential types take place when a frequency $\omega$ of parametric excitation becomes nearly equal to the resonant frequency

$$p_{ij} = p_i \pm p_j. \tag{1.2}$$

Characteristics of these vibrations are studied theoretically and experimentally, and it is pointed out that solutions of the first approximation obtained through rather simple analysis can grasp sufficiently these vibratory phenomena. Further, it is cleared up that unstable vibrations can occur only in summed type and not in differential type.

*1.2. Equation of motion and preliminary analysis*

Vibratory system of $h$ degree-of-freedom without damping and under parametric excitation of frequency $\omega$ is governed by the following equation of motions:

$$\sum_{m=1}^{h}(a_{lm}\ddot{x}_m+\alpha_{lm}x_m)=\varepsilon_l' x_l\cos\omega t. \qquad (l=1,2,\cdots,h) \tag{1.3}$$

In a dynamic system, the left side of Eq. (1.3) are terms of inertia and spring force respectively, and the right side represents parametric excitation. In Eq. (1.3) magnitude of parametric excitation $\varepsilon_l'$ is assumed to be small quantity. Frequency equation of the system is

$$|(\alpha_{lm}-p^2 a_{lm})|=0, \tag{1.4}$$

where $p$ is natural frequency. Let cofactor of this determinant when $p=p_m$ be $A_{m,rs}$ and putting

$$d_{lm}=A_{m,lm}\Big/\sqrt{\sum_{n,r}^{h}a_{nr}A_{m,nm}A_{m,rm}}, \tag{1.5}$$

then transformation from generalized coordinate $x_l$ into normal coordinate $X_l$ is performed by

$$x_l=\sum_{m=1}^{h}d_{lm}X_m, \qquad (l=1,2,\cdots,h). \tag{1.6}$$

Substitution of Eq. (1.6) into Eq. (1.3) attains the following equation of motion expressed by normal coordinate:

$$\ddot{X}_l+p_l^2X_l=\sum_{m=1}^{h}\varepsilon_{lm}X_m\cos\omega t, \tag{1.7}$$

where

$$\varepsilon_{lm}=\sum_{n=1}^{h}d_{nl}d_{nm}\varepsilon_n'=\varepsilon_{ml}. \tag{1.8}$$

The $i$th and $j$th equations of Eq. (1.7) can be rewritten as

$$\left.\begin{aligned}\ddot{X}_i+\omega_i^2X_i&=(\omega_i^2-p_i^2)X_i+\sum_{m=1}^{h}\varepsilon_{im}X_m\cos\omega t=f_i,\\ \ddot{X}_j+\omega_j^2X_j&=(\omega_j^2-p_j^2)X_j+\sum_{m=1}^{h}\varepsilon_{jm}X_m\cos\omega t=f_j.\end{aligned}\right\} \tag{1.9}$$

When frequencies $\omega_i$, $\omega_j$ satisfy the relations of Eq. (1.1), all terms in the right side of Eq. (1.9), *i.e.*, $f_i$ and $f_j$ become small quantities, and hence approximate analysis can be carried out. In this paper, a first approximate analysis through the similar procedure to Kryloff and Bogoliuboff's method[17] is employed. Accordingly, the first step consists in taking $X_{i,j}$ and $\dot{X}_{i,j}$ in the following forms respectively:

$$X_{i,j}=a_{i,j}\sin(\omega_{i,j}t+\varphi_{i,j}),\ \dot{X}_{i,j}=a_{i,j}\omega_{i,j}\cos(\omega_{i,j}t+\varphi_{i,j}), \tag{1.10}$$

and considering amplitudes $a_{i,j}$ and phase angles $\varphi_{i,j}$ in Eq. (1.10) as functions of time $t$. Differentiation of the first equation of Eq. (1.10) results in

$$\dot{X}_{i,j}=\dot{a}_{i,j}\sin(\omega_{i,j}t+\varphi_{i,j})+a_{i,j}(\omega_{i,j}+\dot{\varphi}_{i,j})\cos(\omega_{i,j}t+\varphi_{i,j}),$$

and substitution of $\dot{X}_{i,j}$ in Eq. (1.10) into the above equation leads to

$$\dot{a}_{i,j}\sin(\omega_{i,j}t+\varphi_{i,j})+a_{i,j}\dot{\varphi}_{i,j}\cos(\omega_{i,j}t+\varphi_{i,j})=0. \tag{1.11}$$

Differentiating the second equation of Eq. (1.10) and inserting it into Eq. (1.9), we get

$$\dot{a}_{i,j}\omega_{i,j}\cos(\omega_{i,j}t+\varphi_{i,j})-a_{i,j}\omega_{i,j}\dot{\varphi}_{i,j}\sin(\omega_{i,j}t+\varphi_{i,j})=f_{i,j}. \tag{1.12}$$

Through Eqs. (1.11), (1.12), the following differential equations are derived

$$\left.\begin{aligned}\dot{a}_{i,j}&=\frac{f_{i,j}}{\omega_{i,j}}\cos(\omega_{i,j}t+\varphi_{i,j}),\\ \dot{\varphi}_{i,j}&=-\frac{f_{i,j}}{a_{i,j}\omega_{i,j}}\sin(\omega_{i,j}t+\varphi_{i,j}).\end{aligned}\right\} \tag{1.13}$$

Applying an approximate procedure of "average method" to Eq. (1.13), that is, substituting Eqs. (1.9), (1.10) into $f_{i,j}$ of the right side of Eq. (1.13) and eliminating all terms except for constant terms, Eq. (1.13) is rewritten as follows:

$$\left.\begin{aligned}\dot{a}_{i,j}&=\frac{\varepsilon_{ij}a_{j,i}}{4\omega_{i,j}}\sin(\varphi_{j,i}\pm\varphi_{i,j}),\\ \dot{\varphi}_{i,j}&=-\frac{\omega_{i,j}^2-p_{i,j}^2}{2\omega_{i,j}}\pm\frac{\varepsilon_{ij}a_{j,i}}{4\omega_{i,j}a_{i,j}}\cos(\varphi_{j,i}\pm\varphi_{i,j}).\end{aligned}\right\} \tag{1.14}$$

An assumption that solutions of vibrations are normal solutions[4] leads to

$$\dot{\varphi}_i=\dot{\varphi}_j=0. \tag{1.15}$$

Consequently Eq. (1.14) is reduced to

$$\dot{a}_i=\pm\frac{\varepsilon_{ij}a_j}{4\omega_i}\sin\varphi_{ij},\qquad \dot{a}_j=\frac{\varepsilon_{ij}a_i}{4\omega_j}\sin\varphi_{ij}, \tag{1.16}$$

$$2\Delta_i=\pm\varepsilon_{ij}\frac{a_j}{a_i}\cos\varphi_{ij},\qquad 2\Delta_j=\pm\varepsilon_{ij}\frac{a_i}{a_j}\cos\varphi_{ij}, \tag{1.17}$$

in which

$$\varphi_i\pm\varphi_j=\varphi_{ij},\quad \omega_{i,j}^2-p_{i,j}^2=\Delta_{i,j}. \tag{1.18}$$

In Eqs. (1.1), (1.2), (1.14), (1.16), (1.17) and (1.18), the upper and lower signs of $\pm$ are adopted for vibrations of summed and differential types severally. In Eq. (1.18), $\varphi_{ij}$ is phase angle and detunings $\Delta_{i,j}$ are to be small quantities because of Eq. (1.1).

*1.3. Vibrations of summed type*

Referring the following relations given from Eq. (1.17)

$$\cos^2\varphi_{ij} = 4\,\Delta_i\,\Delta_j/\varepsilon_{ij}^2, \tag{1.19}$$

$$\frac{\varepsilon_{ij}^2}{16\,\omega_i\,\omega_j}\sin^2\varphi_{ij} = \frac{\varepsilon_{ij}^2}{16\,\omega_i\,\omega_j}(1-\cos^2\varphi_{ij}) = \frac{\varepsilon_{ij}^2 - 4\,\Delta_i\,\Delta_j}{16\,\omega_i\,\omega_j} = \mu^2, \tag{1.20}$$

and adopting the upper sign of $\pm$ in Eq. (1.16), we have

$$\ddot{a}_i = \mu^2 a_i, \quad \ddot{a}_j = \mu^2 a_j, \tag{1.21}$$

which leads to

$$a_{i,j} = A_{i,j}e^{\pm\mu t}. \tag{1.22}$$

Inserting Eq. (1.22) into Eq. (1.16), amplitude ratio for two vibrations of frequencies $\omega_i$ and $\omega_j$ is given as follows:

$$a_i/a_j = A_i/A_j = \sqrt{\omega_j/\omega_i}. \tag{1.23}$$

Although we have $a_i/a_j = A_i/A_j = \pm\sqrt{\omega_j/\omega_i}$, the upper sign $+$ is employed here, because difference between both signs can be canceled by difference of $\pm\pi$ in $\varphi_{ij}$. Substituting Eq. (1.23) into Eq. (1.17) in which the upper sign of $\pm$ is adopted, we have

$$\Delta_i/\Delta_j = (\omega_i^2 - p_i^2)/(\omega_j^2 - p_j^2) = \omega_i/\omega_j, \tag{1.24}$$

and from Eq. (1.24) and the relation of $\omega = \omega_i + \omega_j$, a cubic equation determining frequencies $\omega_i$ and $\omega_j$ is given as follows:

$$2\,\omega_{i,j}^3 - 3\,\omega\omega_{i,j}^2 + (\omega^2 - p_i^2 - p_j^2)\,\omega_{i,j} + p_{i,j}^2\,\omega = 0. \tag{1.25}$$

So far as a first approximation, through Eqs. (1.2), (1.18), (1.24) we obtain the following approximate equations by which $\omega_i$ and $\omega_j$ can be determined more easily than by Eq. (1.25)

$$\omega_{i,j} = p_{i,j} + \nabla_{ij}/2, \tag{1.26}$$

in which

$$\nabla_{ij} = \omega - (p_i + p_j) = \omega - p_{ij}. \tag{1.27}$$

Similar process with the above furnish approximate values of detunings $\Delta_{i,j}$, phase angle $\varphi_{ij}$ and negative damping coefficient $\mu$ as follows:

$$\Delta_{i,j} = p_{i,j}\,\nabla_{ij}, \tag{1.28}$$

$$\cos\varphi_{ij} = \nabla_{ij}/E_{ij}, \tag{1.29}$$

$$\mu = \sqrt{E_{ij}^2 - \nabla_{ij}^2}/2, \tag{1.30}$$

where

$$E_{ij} = \varepsilon_{ij}/(2\sqrt{p_i\,p_j}). \tag{1.31}$$

*1.3.1. Vibrations of summed type within unstable region*

When the frequency $\omega$ of parametric excitation comes near to the resonant frequency $p_{ij}$, and detunings $\Delta_{i,j}$ and $\nabla_{ij}$ become so small that the relation

$$\varepsilon_{ij}^2 \geqq 4\,\Delta_i\,\Delta_j \geqq 0, \quad i.e., \quad E_{ij}^2 \geqq \nabla_{ij}^2 \tag{1.32}$$

holds, conditions of $0 \leqq \cos^2 \varphi_{ij} \leqq 1$, $1 \geqq \sin^2 \varphi_{ij} \geqq 0$ and $\mu^2 \geqq 0$ are satisfied as seen from Eqs. (1.19), (1.20) or Eqs. (1.29), (1.30) and both $\varphi_{ij}$ and $\mu$ become real numbers. Denoting $\varphi_{ij}$ with a value between 0 and $\pi$ as $\psi_{ij}$, from Eq. (1.29), we have

$$\cos \psi_{ij} = \nabla_{ij} / |E_{ij}|. \tag{1.33}$$

When $E_{ij} > 0$, $+\psi_{ij}$ and $-\psi_{ij}$ are adopted for $+\mu t$ and $-\mu t$ in Eq. (1.22) separately, and vice versa when $E_{ij} < 0$. Through Eqs. (1.10), (1.18), (1.22), (1.23), and (1.33), solutions for $X_i$, $X_j$ are given as follows:

$$\left.\begin{aligned} X_i &= Ae^{\mu t} \sin(\omega_i t + \varphi_i) + Be^{-\mu t} \sin(\omega_i t + \varphi_i'), \\ X_j &= \sqrt{\omega_i/\omega_j}\{Ae^{\mu t} \sin(\omega_j t \pm \psi_{ij} - \varphi_i) + Be^{-\mu t} \sin(\omega_j t \mp \psi_{ij} - \varphi_i')\}, \end{aligned}\right\} \tag{1.34}$$

in which the upper and lower signs correspond to conditions $E_{ij} > 0$ and $E_{ij} < 0$ respectively, and $A$, $B$, $\varphi_i$, $\varphi_i'$ are all arbitrary constants determined by initial conditions. Once $\omega$ or $\nabla_{ij}$ is given, frequencies $\omega_i$ and $\omega_j$, negative damping coefficient $\mu$ and phase angle $\psi_{ij}$ are given by Eq. (1.26), Eq. (1.30) and Eq. (1.33) severally, thus the solutions are settled. It should be noticed that both unstable vibrations of frequencies $\omega_i$ and $\omega_j$ have a common negative damping coefficient $\mu$.

By adoption of equal sign in Eq. (1.32) or putting $\mu = 0$ in Eq. (1.30), critical frequencies $\omega_{c1}$, $\omega_{c2}$ or critical detunings $\nabla_{c1}$, $\nabla_{c2}$ which decide boundaries between stable and unstable regions are derived as follows:

$$\begin{matrix}\omega_{c1} \\ \omega_{c2}\end{matrix} = p_{ij} \pm |E_{ij}| \quad : \quad \begin{matrix}\nabla_{c1} \\ \nabla_{c2}\end{matrix} = \pm |E_{ij}|, \tag{1.35}$$

in which the upper and lower signs are used for an upper boundary $\omega_{c1}$, $\nabla_{c1}$ and a lower boundary $\omega_{c2}$, $\nabla_{c2}$ separately. As seen in Eq. (1.33), when $E_{ij} > 0$ phase angle $\psi_{ij}$ takes values $\pi$, $\pi/2$ and 0 at $\omega = \omega_{c2}$, $p_{ij}$, $\omega_{c1}$ respectively, and when $E_{ij} < 0$, 0, $\pi/2$, $\pi$.

It is seen from the above discussion that when the frequency $\omega$ of parametric excitation comes near the resonant frequency $p_{ij}\,(= p_i + p_j)$ and takes a certain value between critical frequencies $\omega_{c1}$ and $\omega_{c2}$, two vibrations of frequencies $\omega_i\,(\fallingdotseq p_i)$ and $\omega_j\,(\fallingdotseq p_j)$ build up simultaneously and thus unstable vibrations of summed type take place. In the system with $h$ degree-of-freedom, there are $h(h-1)/2$ unstable regions of unstable vibrations of summed type.

Incidentally, analysis for ordinary unstable vibrations appearing in the neighborhood of $\omega = 2p_r\,(r = 1, 2, \cdots, h)$ is discussed here. Through a similar procedure with the above mentioned, we obtain the following equations of motion in place of Eqs. (1.16), (1.17):

$$\dot{a}_r = (\varepsilon_{rr}\, a_r / 2\,\omega) \sin 2\,\varphi_r, \quad \omega^2 - 4\,p_r^2 = 2\,\varepsilon_{rr} \cos 2\,\varphi_r, \tag{1.36}$$

which leads to

$$\left.\begin{aligned}
&X_r = Ae^{\mu_r t}\sin(\omega t/2 \pm \psi_r) + Be^{-\mu_r t}\sin(\omega t/2 \mp \psi_r),\\
&\quad\text{(the upper and lower signs of } \pm \text{ correspond to } \varepsilon_{rr}>0 \text{ and}\\
&\quad \varepsilon_{rr}<0 \text{ separately)}\\
&\mu_r = \sqrt{(\varepsilon_{rr}/2p_r)^2 - \nabla_r^2}/2, \quad (\nabla_r = \omega - 2p_r)\\
&\cos 2\psi_r = 2p_r\nabla_r/\varepsilon_{rr}, \quad (\pi/2 \geqq \psi_r \geqq 0)\\
&\omega_{c1} = 2p_r + |\varepsilon_{rr}|/2p_r, \quad \omega_{c2} = 2p_r - |\varepsilon_{rr}|/2p_r.
\end{aligned}\right\} \tag{1.37}$$

Results of Eq. (1.37) coincide with those of one degree-of-freedom system.

*1.3.2. Vibrations of summed type in stable region*

When $\omega$ goes far off from $p_{ij}$ and passes $\omega_{c1}$, $\omega_{c2}$, *i.e.*, the relation

$$\varepsilon_{ij}^2 < 4\Delta_i\Delta_j, \quad i.e., \quad E_{ij}^2 < \nabla_{ij}^2 \tag{1.32 a}$$

is held, $\varphi_{ij}$ and $\mu$ can be not real because of $\cos^2\varphi_{ij}>1$, $\sin^2\varphi_{ij}<0$ and $\mu^2<0$, as seen through Eqs. (1.19), (1.20). Putting

$$\nu^2 = -\mu^2 = (4\Delta_i\Delta_j - \varepsilon_{ij}^2)/(16\,\omega_i\omega_j) \fallingdotseq \frac{1}{4}(\nabla_{ij}^2 - E_{ij}^2) > 0 \tag{1.20 a}$$

leads to

$$a_{i,j} = A_{i,j}e^{\pm i\nu t}, \quad (i = \sqrt{-1}) \tag{1.22 a}$$

$$\sin\varphi_{ij} = \pm 4\,i\nu\sqrt{\omega_i\omega_j}/\varepsilon_{ij} \fallingdotseq 2\,i\nu/E_{ij}, \tag{1.38}$$

and

$$\left.\begin{aligned}
X_i &= A\sin(\omega_i+\nu)t + B\cos(\omega_i+\nu)t + C\sin(\omega_i-\nu)t + D\cos(\omega_i-\nu)t,\\
X_j &= A'\sin(\omega_j-\nu)t + B'\cos(\omega_j-\nu)t + C'\sin(\omega_j+\nu)t + D'\cos(\omega_j+\nu)t\\
&= \frac{1}{E_{ij}}\sqrt{\frac{\omega_i}{\omega_j}}[(\nabla_{ij}+2\nu)\{A\sin(\omega_j-\nu)t - B\cos(\omega_j-\nu)t\}\\
&\qquad + (\nabla_{ij}-2\nu)\{C\sin(\omega_j+\nu)t - D\cos(\omega_j+\nu)t\}],
\end{aligned}\right\} \tag{1.34 a}$$

where $A$, $B$, $C$ and $D$ are all arbitrary constants decided by initial conditions. As represented in Eq. (1.34 a), in stable region ($\omega>\omega_{c1}$, $\omega<\omega_{c2}$) free vibrations with frequencies $\omega_{i,j}\pm\nu$ take place and there is no unstable vibration. Observing Eq. (1.34 a), it is seen that two vibrations of frequencies $\omega_i+\nu$ and $\omega_j-\nu$ as well as $\omega_i-\nu$ and $\omega_j+\nu$ make a pair, and regardless of $\nu$ a sum of frequencies in pair is still equal to $\omega=\omega_i+\omega_j$. Further it is noticeable that amplitude ratios between two vibrations making a pair are fixed independently of arbitrary constants, *i.e.*, initial conditions, as shown in Eq. (1.34 a) or the following equations:

$$\left.\begin{aligned}
\frac{\sqrt{A^2+B^2}}{\sqrt{A'^2+B'^2}} &= \frac{|A|}{|A'|} = \frac{|B|}{|B'|} = \frac{|E_{ij}|}{\sqrt{\omega_i/\omega_j}\,|\nabla_{ij}+2\nu|},\\
\frac{\sqrt{C^2+D^2}}{\sqrt{C'^2+D'^2}} &= \frac{|C|}{|C'|} = \frac{|D|}{|D'|} = \frac{|E_{ij}|}{\sqrt{\omega_i/\omega_j}\,|\nabla_{ij}-2\nu|}.
\end{aligned}\right\} \tag{1.39}$$

Through the similar analysis with the above, we find solution for ordinary stable vibrations in the neighborhood of $\omega=2p_r$ as follows:

$$\left.\begin{aligned} X_r &= A\sin(\omega/2+\nu_r)t + B\cos(\omega/2+\nu_r)t \\ &\quad + A\sin(\omega/2-\nu_r)t - B\cos(\omega/2-\nu_r)t, \\ \nu_r &= \sqrt{\nabla_r^2-(\varepsilon_{rr}/2p_r)^2}/2. \end{aligned}\right\} \tag{1.37 a}$$

Furthermore, it can be concluded that unstable vibrations of summed type of higher order in the neighborhood of $\omega=(p_i+p_j)/s$ $(s=2, 3, \cdots)$ as well as ordinary unstable vibrations of higher order do not appear, in so far as a first approximation (see Chapter III).

*1.4. Vibrations of differential type*

Adopting the lower sign in Eqs. (1.16), (1.17), we have

$$\ddot{a}_{i,j} = -\mu^2 a_{i,j}, \tag{1.21 b}$$

$$a_{i,j} = A_{i,j}e^{\pm i\mu t}. \tag{1.22 b}$$

By a similar procedure with summed type, we get the following relations:

$$a_i/a_j = A_i/A_j = i\sqrt{\omega_j/\omega_i}, \tag{1.23 b}$$

$$\Delta_i/\Delta_j = (\omega_i^2-p_i^2)/(\omega_j^2-p_j^2) = -\omega_i/\omega_j, \tag{1.24 b}$$

$$\begin{aligned} &2\omega_i^3 - 3\omega\omega_i^2 + (\omega^2-p_i^2-p_j^2)\omega_i + p_i^2\omega = 0, \\ &2\omega_j^3 + 3\omega\omega_j^2 + (\omega^2-p_i^2-p_j^2)\omega_j - p_j^2\omega = 0, \end{aligned} \tag{1.25 b}$$

$$\omega_i = p_i + \nabla_{ij}/2, \quad \omega_j = p_j - \nabla_{ij}/2, \tag{1.26 b}$$

$$\nabla_{ij} = \omega - (p_i - p_j) = \omega - p_{ij}, \tag{1.27 b}$$

$$\Delta_i = p_i\nabla_{ij}, \quad \Delta_j = -p_j\nabla_{ij}, \tag{1.28 b}$$

$$\mu = \sqrt{E_{ij}^2+\nabla_{ij}^2}/2. \tag{1.30 b}$$

Referring that $\mu$ is always a real number as shown in Eq. (1.30 b) and that $\varphi_{ij}$ is not real and given by

$$\cos\varphi_{ij} = i\nabla_{ij}/E_{ij}, \tag{1.29 b}$$

solutions of differential type are written as follows:

$$\left.\begin{aligned} X_i &= A\sin(\omega_i+\mu)t + B\cos(\omega_i+\mu)t + C\sin(\omega_i-\mu)t + D\cos(\omega_i-\mu)t, \\ X_j &= A'\sin(\omega_j+\mu)t + B'\cos(\omega_j+\mu)t + C'\sin(\omega_j-\mu)t + D'\cos(\omega_j-\mu)t \\ &= -\frac{1}{E_{ij}}\sqrt{\frac{\omega_i}{\omega_j}}[(\nabla_{ij}+2\mu)\{A\sin(\omega_j+\mu)t + B\cos(\omega_j+\mu)t\} \\ &\qquad + (\nabla_{ij}-2\mu)\{C\sin(\omega_j-\mu)t + D\cos(\omega_j-\mu)t\}]. \end{aligned}\right\} \tag{1.34 b}$$

Eq. (1.34 b) is similar to Eq. (1.34 a) and there is no unstable vibration of differential type. In Eq. (1.34 b), two vibrations of frequencies $\omega_i+\mu$ and $\omega_j+\mu$ as well as $\omega_i-\mu$ and $\omega_j-\mu$ make a pair, and a difference of frequencies in pair is still equal to $\omega=\omega_i-\omega_j$, and the amplitude ratios are given by

$$\left.\begin{aligned} \frac{\sqrt{A^2+B^2}}{\sqrt{A'^2+B'^2}} = \frac{|A|}{|A'|} = \frac{|B|}{|B'|} = \frac{|E_{ij}|}{\sqrt{\omega_i/\omega_j}\,|\nabla_{ij}-2\,\mu|}, \\ \frac{\sqrt{C^2+D^2}}{\sqrt{C'^2+D'^2}} = \frac{|C|}{|C'|} = \frac{|D|}{|D'|} = \frac{|E_{ij}|}{\sqrt{\omega_i/\omega_j}\,|\nabla_{ij}+2\,\mu|}. \end{aligned}\right\} \quad (1.39\,\mathrm{b})$$

It is concluded that, in so far as a first approximation, no unstable vibration of differential type of higher order in the neighborhood of $\omega=(p_i-p_j)/s$ $(s=2, 3, \cdots)$ can appear (see Chapter III).

*1.5. Verification of analytical results through experiments and analog computer*

In this section, experiments and calculations by analog computer are performed for vibrations of summed and differential type which take place in a vibratory system of double pendulums with two degree-of-freedom as shown Fig. 1.1, where the first and second pendulums of lengthes $l_1$, $l_2$ and mass $m_1$, $m_2$, are supported at points $B_1$, $B_2$; $I_1$, $I_2$ are moments of inertia about supporting points $B_1$, $B_2$ and $b_1$, $b_2$ are distances between $B_1$ and gravitational center $G_1$ and between $B_2$ and $G_2$ respectively; there are springs having spring constants $k_1$ and $k_2$ at ends of both pendulums. When the supporting point $B_1$ vibrates vertically with small amplitude $e$ and frequency $\omega$, parametric excitation $e \cos \omega t$ is induced and the system shown in Fig. 1.1 is governed by Eq. (1.3). If only number of degrees of freedom $h$, suffixs $i$ and $j$ are replaced by 2, 1 and 2 separately, all results obtained up to now can be applied for this system. Coefficients in Eq. (1.3) for this system are given as follows:

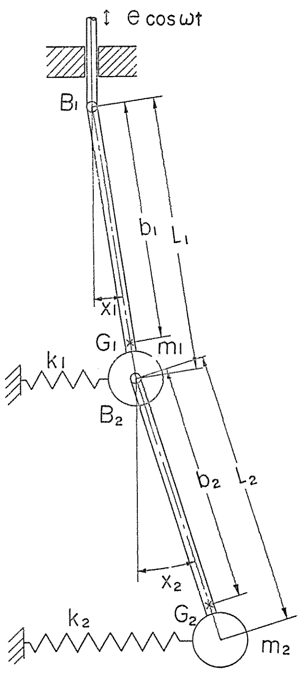



FIG. 1.1. Vibratory system of double pendulum.

$$\left.\begin{aligned} & a_{11}=I_1+m_2 l_1^2, \quad a_{12}=a_{21}=m_2 l_1 b_2, \quad a_{22}=I_2, \\ & \alpha_{11}=(k_1+k_2)\,l_1^2+(m_1 b_1+m_2 l_1)\,g, \quad \alpha_{12}=\alpha_{21}=k_2 l_1 l_2, \\ & \alpha_{22}=k_2 l_2^2+m_2 b_2 g, \quad \varepsilon_1'=e\omega^2(m_1 b_1+m_2 l_1), \\ & \varepsilon_2'=e\omega^2 m_2 b_2, \end{aligned}\right\} \quad (1.40)$$

where $g$ is gravitational acceleration. Thus frequency equation (1.4) reduces to

$$(a_{11}a_{22}-a_{12}^2)\,p^4-(a_{11}\alpha_{22}+a_{22}\alpha_{11}-2\,a_{12}\alpha_{12})\,p^2+(\alpha_{11}\alpha_{22}-\alpha_{12}^2)=0. \quad (1.41)$$

*1.5.1. Experimental apparatus and block diagram of analog computer*

Experiment are carried out by experimental apparatus shown in Fig. 1.2.

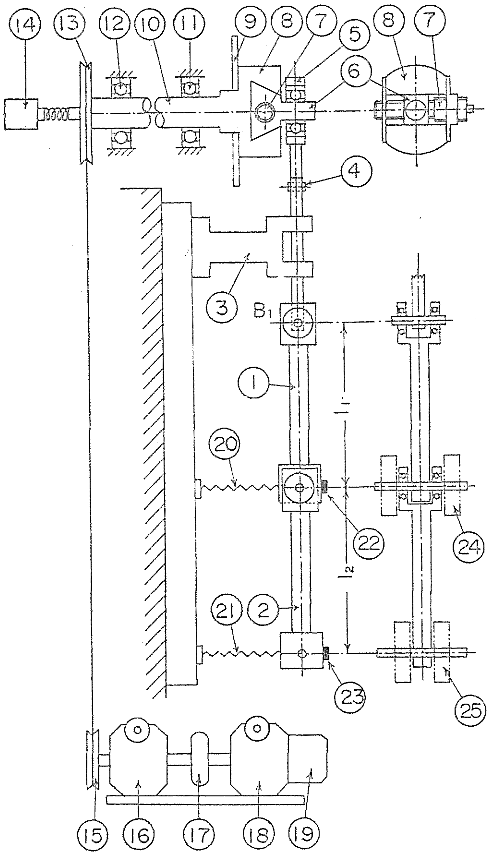
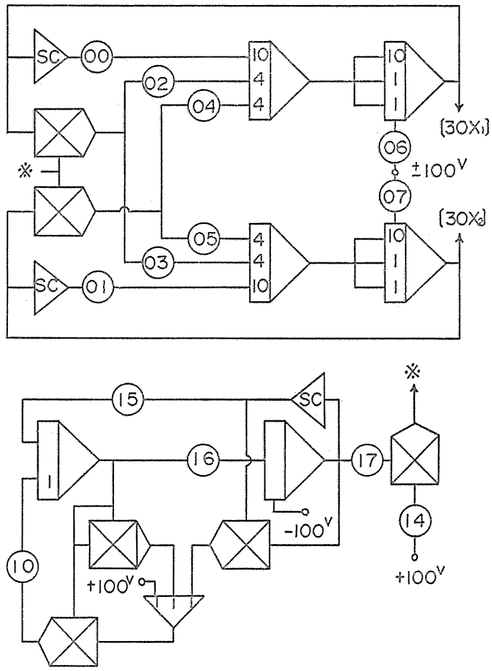



FIG. 1.2. Experimental apparatus.




| | |
|---|---|
| 00: $p_1^2/1200$ | 01: $p_2^2/1200$ |
| 02: $25\,\varepsilon_{11}/(3\,e\omega^2)$ | 03: $25\,\varepsilon_{12}/(3\,e\omega^2)$ |
| 04: $25\,\varepsilon_{21}/(3\,e\omega^2)$ | 05: $25\,\varepsilon_{22}/(3\,e\omega^2)$ |
| 06: initial condition | 07: initial condition |
| 10: 0.1 | 14: $(\omega/200)^2$ |
| 15: $\omega/10$ | 16: $\omega/10$ |
| 17: $e$ | |

FIG. 1.3. Block diagram of analog computer (SC=sign changer).

Through two stepless transmissions ⑱, ⑯, rubber coupling ⑰, pulleies ⑮, ⑬ and shaft ⑩, rotation of motor ⑲ is transmitted to rotor ⑧ which consists of eccentric shaft ⑥. Eccentricity $e$ of shaft ⑥ can be changed by screw ⑦. Rotation of the eccentric shaft ⑥ is transformed to vertical rectilinear vibration $e \cos \omega t$ of supporting point $B_1$ of the first pendulum ① through bearing ⑤, joint ④ and guide ③. Further, on both pendulum ends with mass ㉔, ㉕, coil springs ⑳, ㉑ are attached in order to adjust magnitude of natural frequencies of the system, and horizontal motions of steel edges ㉒, ㉓ at the ends of the first and second pendulums are recorded optically on oscillograph paper, thus vibrations referring to generalized coordinates $x_1$, $x_2$ are obtained experimentally. Dimensions of this experimental apparatus are as follows:

$$m_1 g = 1.910\ \mathrm{kg}, \quad m_2 g = 1.730\ \mathrm{kg}, \quad I_1 = 0.963\ \mathrm{kg \cdot cm \cdot s^2}, \quad I_2 = 0.786\ \mathrm{kg \cdot cm \cdot s^2},$$
$$l_1 = l_2 = 30.0\ \mathrm{cm}, \quad b_1 = 18.85\ \mathrm{cm}, \quad b_2 = 17.35\ \mathrm{cm}, \quad k_1 = k_2 = 6.4\ \mathrm{kg/cm},$$

and hence natural frequencies of the apparatus are

$$p_1 = 13.863\ \mathrm{c/s}, \quad p_2 = 9.850\ \mathrm{c/s}.$$

Block diagram of analog computer used to obtain calculations for vibrations referring to normal coordinates $X_1$ and $X_2$ is shown in Fig. 1.3, where sinusoidal function generator to yield excitation $e \cos \omega t$ is shown in the lower figure and it is inserted into the place shown mark ※ in the upper figure.

*1.5.2. Results of experiments and analog computer*

The $\mu-\omega$ diagram for $e=0.152$ cm ($E_{12}=0.955$ rad/s) is shown in Fig. 1.4, where magnitudes of negative damping coefficient $\mu$ as well as width of unstable region of unstable vibrations of summed type are furnished, and also $\mu$ of ordinary unstable vibrations appearing in the neighborhood of $\omega=2p_{1,2}$ are additionally illustrated for comparison. In Fig. 1.4, the resonant frequencies $\omega=p_1+p_2=p_{12}$ and $\omega=2p_{1,2}$ are indicated by vertical chain lines. Curves of broken line in Fig. 1.4 are negative damping coefficients obtained from Eqs. (1.30), (1.37), and symbols ○, ● show results of experiments and analog computer separately, which are given by the following equation

$$\mu = ln(a_t/a)/t \quad \text{rad/s}, \tag{1.42}$$

where $t$ is time and $a$, $a_t$ are amplitudes of vibratory waves obtained by experiments and analog computer when $t=0$ and $t=t$ severally. In Fig. 1.4, results of analysis agree with results of analog computer, while experimental results give rather smaller values of $\mu$ because of inevitable damping in the apparatus.

Negative damping coefficients $\mu$ of unstable vibrations of summed type for $e=0.152$ cm ($E_{12}=0.955$ rad/s) and $e=0.093$ cm ($E_{12}=0.584$ rad/s) given by Eq. (1.30) are shown in Fig. 1.5, where the larger $e$, *i.e.*, $E_{12}$ results in the larger $\mu$ and the wider unstable region.

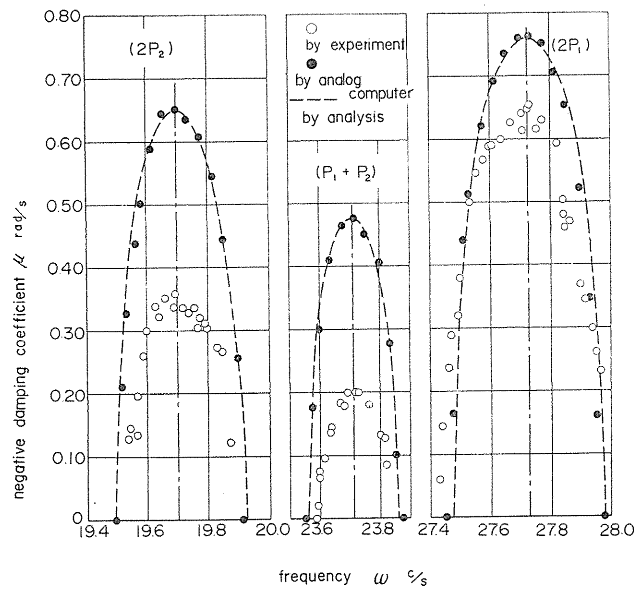



FIG. 1.4. $\mu-\omega$ diagram for unstable vibrations of summed and ordinary types ($e=0.152$ cm).

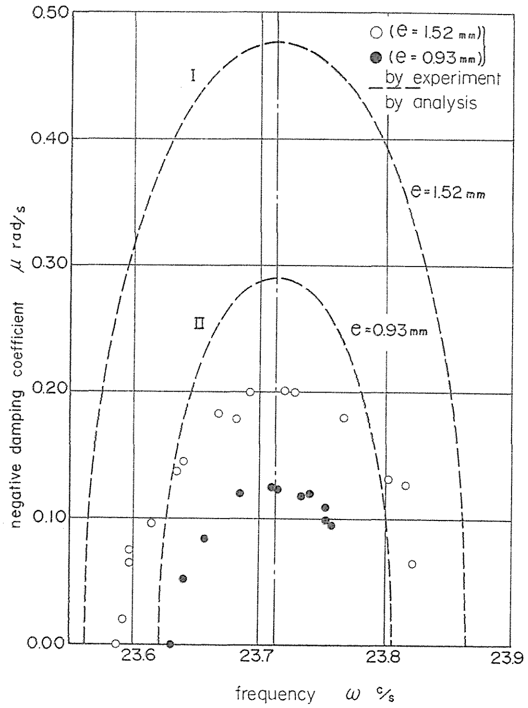



FIG. 1.5. Negative damping coefficient for unstable vibrations of summed type.

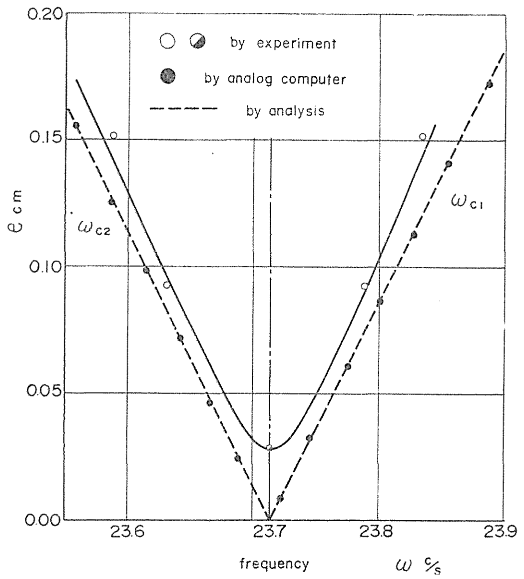



FIG. 1.6. Boundaries of unstable region of vibrations of summed type.

In Fig. 1.6, boundaries $\omega_{c1}$ and $\omega_{c2}$ of unstable region given by Eq. (1.35) are shown by broken lines, experimental results are given by symbols ○, ◍, and ◍ shows the lower limit of unstable region, and results of analog computer are indicated by symbol ●. In Fig. 1.6, difference in experimental results from those of analysis and analog computer are caused by inevitable damping in experimental apparatus.

In Fig. 1.7, frequencies $\omega_{1,2}\pm\nu$ and $\omega_{1,2}$ of stable and unstable vibrations obtained by Eqs. (1.26), (1.20 a) (curves of broken and chain line), experiments (symbols ○, ⦸) and analog computer (symbols ●, ⦸) (⦸, ⦸=stable vibrations; ○, ●=unstable vibrations) are shown in Fig. 1.7. In the stable region of the higher frequency side (the right side), only frequencies $\omega_{1,2}-\nu$ are obtained by experiments and analog computer, and vibrations of $\omega_{1,2}+\nu$ do not appear, and

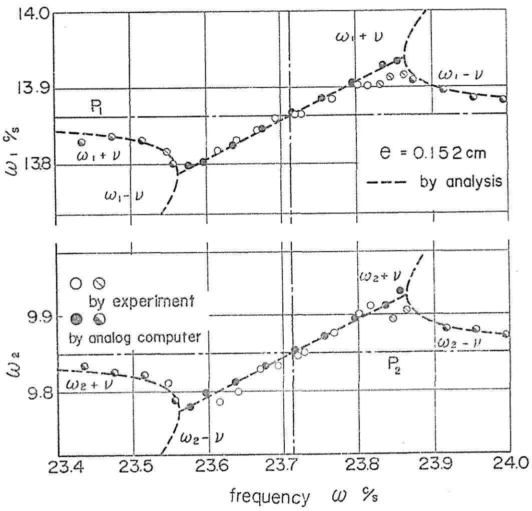



FIG. 1.7. Frequencies of stable and unstable vibrations of summed type.

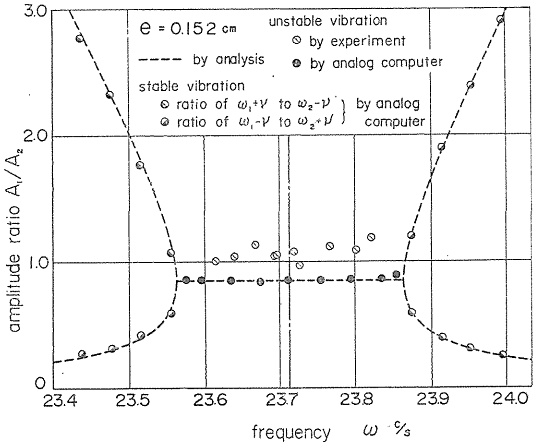



FIG. 1.8. Amplitude ratio of stable and unstable vibrations of summed type.

vice versa in the stable region of the lower frequency side (the left side), because in the higher frequency side amplitudes of $\omega_{1,2}-\nu$ are larger than those of $\omega_{1,2}+\nu$, vice versa in the lower frequency side, as shown in Fig. 1.8 and Eq. (1.39). It follows that, in the stable region, vibrations with frequency nearer to natural frequency build up remarkably. In Fig. 1.7, results of analog computer agree with curves of broken line given by analysis, and experimental results differ from them slightly.

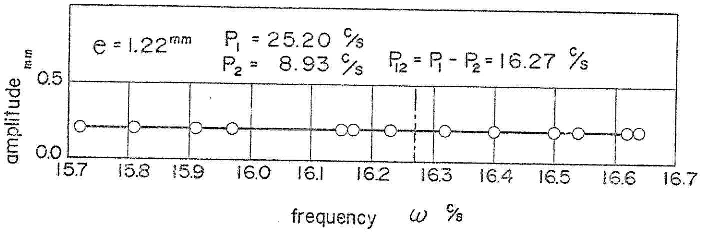



FIG. 1.9. Experimental results of no occurrence of unstable vibration of differential type.

Amplitude ratios of vibrations of summed type are given in Fig. 1.8, where curves of broken line induced by Eqs. (1.23), (1.39) coincide with results of analog computer. Inevitable damping in experimental apparatus results in some difference between curves and symbols ⦸ within unstable region.

Fig. 1.9 shows experimentally that unstable vibration of differential type cannot take place. Although there is a resonant point $p_{12}=p_1-p_2=16.27$ c/s in the experimental apparatus of Fig. 1.9, only small vibrations due to disturbance take place and no unstable vibration appears in the neighborhood of the resonant point $p_{12}$ as shown in Fig. 1.9.

In the range of $\omega$ of Fig. 1.9, analog computer furnishes only free vibrations induced by initial conditions. Frequencies and amplitude ratios of these stable free vibrations of differential type are shown in Figs. 1.10, 1.11 severally, where symbols ●, ○ indicate results of analog computer and full line curves represent analytical results through Eqs. (1.26 b), (1.30 b), (1.39 b). It is

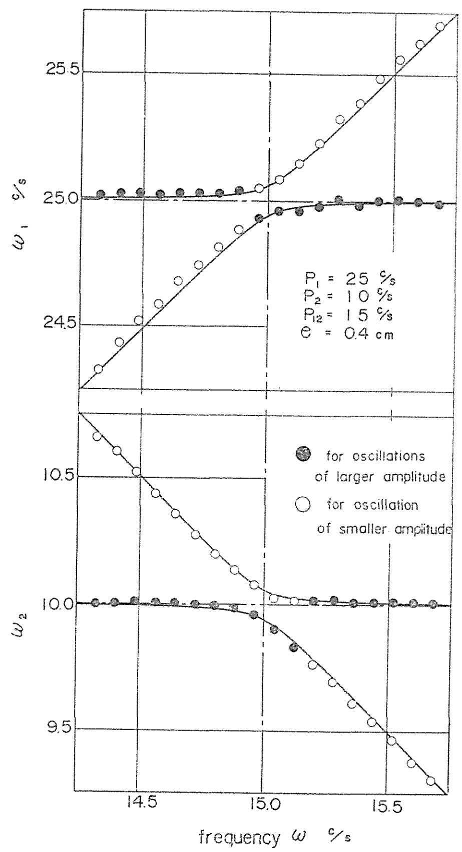



FIG. 1.10. Frequencies of stable vibrations of differential type.

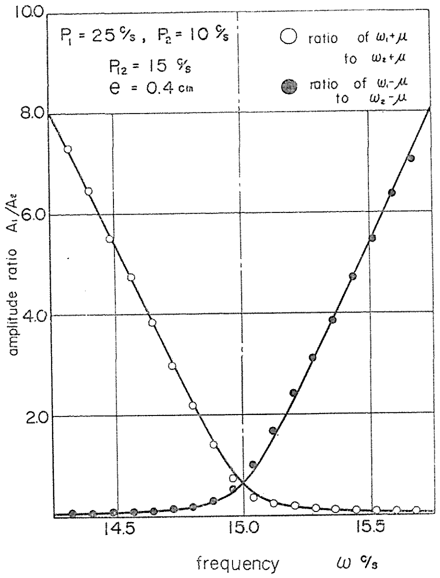



FIG. 1.11. Amplitude ratio of stable vibrations of differential type.

seen in Fig. 1.10 that the relation $\omega=\omega_1-\omega_2$ are always satisfied between two frequencies $\omega_{1,2}\pm\nu$ which make a pair.

Vibratory waves of unstable vibrations of summed type appearing in the experimental apparatus of double pendulum are illustrated in Fig. 1.12, where the upper and lower photographs give vibratory waves of the first and second pendulums respectively and frequency $\omega$ of parametric excitation is furnished by vertical black lines. Observing vibratory waves, it is seen that the relation $\omega:\omega_1:\omega_2=41:24:17$, and hence $\omega=\omega_1+\omega_2$ holds.

Vibratory waves of analog computer are shown in Fig. 1.13, in which vibrations of frequencies $\omega_1$ and $\omega_2$ appear separately because of normal coordinate.

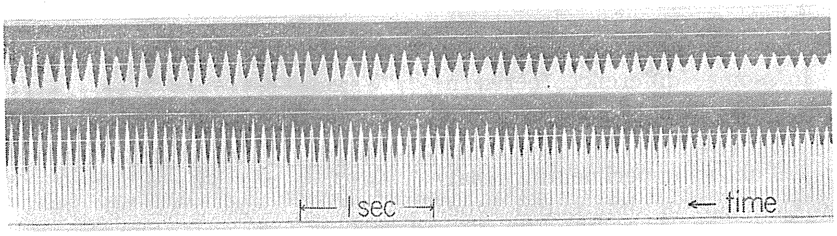



FIG. 1.12. Vibratory waves of unstable vibrations of summed type by experiment of double pendulum ($e=0.152$ cm, $\omega=23.67$ c/s, $\mu=0.19$ rad/s, $\omega:\omega_1:\omega_2=41:24:17$).

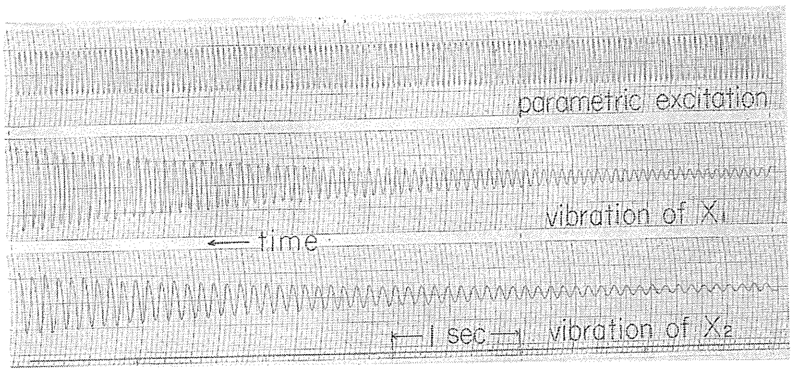



FIG. 1.13. Vibratory waves of unstable vibrations of summed type by analog computer ($e=0.152$ cm, $\omega=23.64$ c/s, $\mu=0.41$ rad/s, $\omega:\omega_1:\omega_2=41:24:17$).

### *1.6. Conclusions*

Obtained results may be summarized as follows:

(1) In a vibratory system of multiple degree-of-freedom without damping and under parametric excitation, two unstable vibrations with frequencies $\omega_i (\fallingdotseq p_i)$ and $\omega_j (\fallingdotseq p_j)$ can simultaneously take place in the neighborhood of the resonant point $p_{ij}=p_i+p_j$, that is, unstable vibrations of summed type can occur.

(2) Sum of frequencies $\omega_i$ and $\omega_j$ of unstable vibrations is equal to frequency $\omega$ of parametric excitation, *i.e.*, $\omega_i+\omega_j=\omega$.

(3) Solutions of this kind of unstable vibrations are obtained through Eq. (1.34) and amplitude ratio, frequencies, negative damping coefficient, phase angle and unstable region are given by Eqs. (1.23), (1.26), (1.30), (1.33) and (1.35) separately.

(4) In stable region of summed type, two free vibrations with frequencies $\omega_{i,j} \pm \nu$ appear, and those solutions are found by Eq. (1.34 a) and frequencies are furnished by Eqs. (1.26), (1.20 a), and amplitude ratio by Eq. (1.39).

(5) Vibrations on boundaries between stable and unstable regions are always stable.

(6) Vibrations of differential type consist of two free vibrations with frequencies $\omega_{i,j} \pm \mu$ as shown by Eq. (1.34 b), and frequencies are obtained by Eqs. (1.26 b), (1.30 b) and amplitude ratio by Eq. (1.39 b).

(7) So far as magnitude of parametric excitation is somewhat small, as in this paper, solutions of a first approximation are enough to grasp exactly vibratory phenomena due to parametric excitation and they show good agreement with results of experiment and analog computer.

(8) There is no unstable vibration of summed and differential types of higher order in so far as a first approximation, and it can not appear experimentally.

## Chapter II. Vibrations of "summed and differential types" with damping under parametric excitation [19]

### *2.1. Introduction*

In this chapter, vibrations of summed and differential types in a linear vibratory system with damping are studied in comparison with those in the previous chapter, and it is found that damping force and especially value of damping ratio explained later have considerable influences on characteristics of these vibrations, that is, contrary to expectation, damping forces do not always make smaller width of unstable region in which two unstable vibrations of frequencies $\omega_i$ and $\omega_j$ build up simultaneouly; and that for vibrations of summed type with damping properties of both unstable and stable vibrations, *i.e.*, solutions, frequencies, phase angles, amplitude ratios between two vibrations and negative damping coefficients can be represented in common forms except for a special case, while they are given separately when there is no damping. Further, theoretical results obtained by a first approximate analysis in which higher powers of small quantities are neglected, are compared with results of analog cumputer.

### *2.2. Equation of motion and preliminary analysis*

A vibratory system of $h$ degree-of-freedom with damping and parametric excitation of frequency $\omega$ is governed by the following equation of motion referring to normal coordinate $X_l(l=1, 2, \cdots, h)$ in place of Eq. (1.7):

$$\ddot{X}_l + p_l^2 X_l = \sum_{m=1}^{h} \{\varepsilon_{lm} X_m \cos \omega t - C_{lm} \dot{X}_m\}, \tag{2.1}$$

in which damping coefficient $C_{lm}$ are assumed to be small quantity as well as $\varepsilon_{lm}$ satisfying Eq. (1.8).

Through a similar procedure to Section 1.2, that is, using Kryloff and Bogoliuboff's method and assuming solutions of vibrations to be normal solutions, Eq. (2.1) reduces to the following equations corresponding to Eqs. (1.16), (1.17):

$$\dot{a}_i = \pm \frac{\varepsilon_{ij} a_j}{4\omega_i} \sin \varphi_{ij} - \frac{C_{ii}}{2} a_i, \quad \dot{a}_j = \frac{\varepsilon_{ij} a_i}{4\omega_j} \sin \varphi_{ij} - \frac{C_{jj}}{2} a_j, \tag{2.2}$$

$$2\Delta_i = \pm \varepsilon_{ij} \frac{a_j}{a_i} \cos \varphi_{ij}, \quad 2\Delta_j = \pm \varepsilon_{ij} \frac{a_i}{a_j} \cos \varphi_{ij}, \tag{2.3}$$

in which the upper and lower signs of $\pm$ are adopted for vibrations of summed and differential types respectively. Putting

$$\omega_{i,j} = p_{i,j} + \delta_{i,j}/2 \tag{2.4}$$

and neglecting higher powers of small quantities, we have

$$\Delta_{i,j} = \delta_{i,j} p_{i,j}, \tag{2.5}$$

$$\delta_i \pm \delta_j = 2(\omega - p_{ij}) = 2\nabla_{ij}, \tag{2.6}$$

and hence Eqs. (1.19) and (1.20) are rewritten by Eqs. (1.31), (2.5) as follows:

$$\cos^2 \varphi_{ij} = 4\Delta_i \Delta_j / \varepsilon_{ij}^2 = \delta_i \delta_j / E_{ij}^2, \tag{2.7}$$

$$\mu^2 = (\varepsilon_{ij}^2 / 16\,\omega_i \omega_j) \sin^2 \varphi_{ij} = (E_{ij}^2 - \delta_i \delta_j)/4. \tag{2.8}$$

### 2.3. *Vibrations of summed type*

It is assumed that amplitudes $a_{i,j}$ are represented in the form

$$a_{i,j} = A_{i,j} e^{\alpha t}. \tag{2.9}$$

Substitution of Eq. (2.9) into Eq. (2.2) adopting the upper sigh of $\pm$ yields characterisitics equation with roots of the following negative damping coefficients:

$$\alpha = \alpha_{1,2} = -n_1/2 \pm \sqrt{n_2^2 + 4\mu^2}/2 = -n_1/2 \pm \sqrt{n_2^2 + E_{ij}^2 - \delta_i \delta_j}/2, \tag{2.10}$$

where $\alpha_1$ and $\alpha_2$ correspond to the upper and lower signs respectively, and

$$n_1 = (C_{ii} + C_{jj})/2, \quad n_2 = (C_{ii} - C_{jj})/2. \tag{2.11}$$

Equating amplitude ratio $a_i/a_j$ given from Eq. (2.2) to that from Eq. (2.3), we have the following relation:

$$\frac{\Delta_i \omega_j}{\Delta_j \omega_i} = \frac{A_j^2}{A_i^2} \cdot \frac{\omega_j}{\omega_i} = \frac{\alpha + C_{ii}/2}{\alpha + C_{jj}/2} = \frac{n_2 \pm \sqrt{n_2^2 + E_{ij}^2 - \delta_i \delta_j}}{-n_2 \pm \sqrt{n_2^2 + E_{ij}^2 - \delta_i \delta_j}}. \tag{2.12}$$

On the other hand, Eqs. (2.4), (2.5) yield

$$(\Delta_i\,\omega_j)/(\Delta_j\,\omega_i)=\delta_i/\delta_j, \tag{2.13}$$

provided that higher terms of small quantities are neglected. From Eqs. (2.12), (2.13), we obtain

$$\begin{cases} \delta_i\delta_j=(E_{ij}^2+\nabla_{ij}^2+n_2^2)/2\pm\sqrt{(E_{ij}^2+\nabla_{ij}^2+n_2^2)^2-4\,\nabla_{ij}^2E_{ij}^2}/2 \quad (\text{for } n_2\neq 0), & (2.14)\\ \delta_i\delta_j=\nabla_{ij}^2 \qquad (\text{for } n_2=0). & (2.14')\end{cases}$$

*2.3.1. Vibrations of summed type when $n_2\neq 0$ $(C_{ii}\neq C_{jj})$*

Detunings $\delta_i$ and $\delta_j$ are determined through Eqs. (2.6), (2.14) and it is verified that employment of either the upper or lower sign in Eq. (2.14) results in the same conclusion; then the lower sign is adopted here for brevity. Accordingly, detunings $\delta_i$, $\delta_j$ are given as follows:

$$\delta_i=\nabla_{ij}\pm\sqrt{b-a},\quad \delta_j=\nabla_{ij}\mp\sqrt{b-a}, \tag{2.15}$$

where

$$a=(E_{ij}^2+n_2^2-\nabla_{ij}^2)/2,\quad b=\sqrt{a^2+n_2^2\nabla_{ij}^2}. \tag{2.16}$$

In Eq. (2.15), both $\delta_i$ and $\delta_j$ are real numbers because of $a<b$, and the upper and lower signs are adopted for $\alpha_1$ and $\alpha_2$ separately when $n_2\nabla_{ij}>0$, and vice versa when $n_2\nabla_{ij}<0$. Using Eq. (2.15), Eq. (2.10) is reduced to

$$\alpha_1=-n_1/2+\sqrt{b+a}/2,\quad \alpha_2=-n_1/2-\sqrt{b+a}/2, \tag{2.17}$$

where $\alpha_1$ and $\alpha_2$ are real because of $b+a>0$. Introducing amplitude ratio $A_i/A_j=\pm\sqrt{\Delta_j/\Delta_i}$ from Eq. (2.12) and employing the upper sign + by the similar reason mentioned in Chapter I, we have

$$\frac{A_i}{A_j}=\sqrt{\frac{\Delta_j}{\Delta_i}}=\sqrt{\frac{\omega_j}{\omega_i}}\sqrt{\frac{\Delta_j\,\omega_i}{\Delta_i\,\omega_j}}=\sqrt{\frac{\omega_j}{\omega_i}}\sqrt{\frac{-n_2\pm\sqrt{b+a}}{n_2\pm\sqrt{b+a}}}, \tag{2.18}$$

in which the upper and lower signs are used for $\alpha_1$ and $\alpha_2$ respectively. Since the relation $E_{ij}^2-\delta_i\delta_j>0$ is attained through Eq. (2.14), it is seen that both $\varphi_{ij}$ and $\mu$ of Eqs. (2.7), (2.8) are real. Denoting $\varphi_{ij}$ of value between 0 and $\pi$ as $\psi_{ij}$, we obtain

$$\cos\psi_{ij}=\sqrt{\nabla_{ij}^2-(b-a)}/|E_{ij}|. \tag{2.19}$$

From Eqs. (2.7), (2.15), it is seen that when $E_{ij}>0$, $+\psi_{ij}$ and $-\psi_{ij}$ are adopted for $\alpha_1$ and $\alpha_2$ separately and vice versa when $E_{ij}<0$. Accordingly through unstable and stable regions, solutions for $X_{i,j}$ are given as follows:

$$\left.\begin{aligned} X_i&=Ae^{\alpha_1 t}\sin(\omega_i t+\varphi_i)+Be^{\alpha_2 t}\sin(\omega_i' t+\varphi_i'),\\ X_j&=\sqrt{\frac{\omega_i}{\omega_j}}\left\{\sqrt{\frac{n_2+\sqrt{b+a}}{-n_2+\sqrt{b+a}}}\,Ae^{\alpha_1 t}\sin(\omega_j t\pm\psi_{ij}-\varphi_i)\right.\\ &\qquad\left.+\sqrt{\frac{n_2-\sqrt{b+a}}{-n_2-\sqrt{b+a}}}\,Be^{\alpha_2 t}\sin(\omega_j' t\mp\psi_{ij}-\varphi_i')\right\}, \end{aligned}\right\} \tag{2.20}$$

in which the upper and lower signs correspond to conditions $E_{ij}>0$ and $E_{ij}<0$

respectively and $A$, $B$, $\varphi_i$, $\varphi_i'$ are arbitrary constants determined by initial conditions. In Eq. (2.20), $\alpha_{1,2}$ and $\psi_{ij}$ are given by Eqs. (2.17) and (2.19) severally and $\omega_{i,j}$, $\omega_{i,j}'$ are derived from the following equations which are attained through Eqs. (2.4), (2.15):

$$\left.\begin{matrix}\omega_i\\ \omega_i'\end{matrix}\right\} = p_i + \frac{\nabla_{ij}}{2} \pm \frac{\sqrt{b-a}}{2}, \quad \left.\begin{matrix}\omega_j\\ \omega_j'\end{matrix}\right\} = p_j + \frac{\nabla_{ij}}{2} \mp \frac{\sqrt{b-a}}{2}, \tag{2.21}$$

where the upper and lower signs correspond to $\omega_{i,j}$ and $\omega_{i,j}'$ respectively when $n_2\nabla_{ij}>0$, and vice versa when $n_2\nabla_{ij}<0$.

Since $\alpha_1$ is larger than $\alpha_2$, as seen in Eq. (2.17), putting $\alpha_1=0$ in Eq. (2.17) results in boundaries between stable and unstable regions as follows:

$$\nabla_{\substack{c1\\c2}} = \omega_{\substack{c1\\c2}} - p_{ij} = \pm\frac{1+\lambda}{2\sqrt{\lambda}}\sqrt{E_{ij}^2-\lambda C_{ii}^2}, \tag{2.22}$$

where the upper and lower signs are used for the upper boundaries $\nabla_{c1}$, $\omega_{c1}$ and the lower boundaries $\nabla_{c2}$, $\omega_{c2}$ separately, and

$$\lambda = C_{jj}/C_{ii} \tag{2.23}$$

is damping ratio. When $E_{ij}^2<C_{ii}C_{jj}=\lambda C_{ii}^2$, critical detunings $\nabla_{c1}$, $\nabla_{c2}$ cannot now be real, and it follows that no unstable vibration can occur in the neighborhood of the resonant point $\omega=p_{ij}$. From Eq. (2.19), we are given phase angle $\psi_{ij}$ on these boundaries decided by Eq. (2.22) as follows:

$$\cos\psi_{ij} = \pm\sqrt{E_{ij}^2-C_{ii}C_{jj}}/|E_{ij}|, \tag{2.24}$$

where the upper and lower signs are adopted for $\nabla_{c1}$ and $\nabla_{c2}$ respectively, and it is found from Routh's method that vibrations on these boundaries are always stable.

It is shown by the foregoing analysis that, when frequency $\omega$ of parametric excitation comes near the resonant frequency $p_{ij}(=p_i+p_j)$ and detuning $\nabla_{ij}=\omega-p_{ij}$ takes a value between $\nabla_{c1}$ and $\nabla_{c2}$ and further the condition $E_{ij}^2\geqq C_{ii}C_{jj}$ holds, negative damping coefficient $\alpha_1$ becomes now positive and unstable vibrations of summed type take place.

*2.3.2. Vibrations of summed type when $n_2=0$ ($C_{ii}=C_{jj}$)*

In this exceptional case, detunings $\delta_{i,j}$, negative damping coefficients $\alpha_{1,2}$, amplitude ratio $A_i/A_j$ and phase angle $\psi_{ij}$ should be attained through Eqs. (2.14′), (2.6), (2.7), (2.10) and (2.12) as follows:

$$\delta_i = \delta_j = \nabla_{ij}, \tag{2.25}$$

$$\alpha_{1,2} = -n_1/2 \pm\sqrt{\mu^2} = -n_1/2 \pm \sqrt{E_{ij}^2-\nabla_{ij}^2}/2, \tag{2.26}$$

$$A_i/A_j = \sqrt{\omega_j/\omega_i}, \tag{2.27}$$

$$\cos\psi_{ij} = |\nabla_{ij}/E_{ij}|, \qquad (0\leqq\psi_{ij}\leqq\pi). \tag{2.28}$$

When detuning $\nabla_{ij}$ is so small that the relation $\nabla_{ij}^2\leqq E_{ij}^2$ holds, $\mu$, *i.e.*, $\alpha_{1,2}$ and $\psi_{ij}$ become real as seen in Eqs. (2.26), (2.28). Consequently, we obtain the following equations for $X_{i,j}$ for this exceptional case:

$$\left.\begin{aligned} X_i &= Ae^{\alpha_1 t}\sin(\omega_i t+\varphi_i)+Be^{\alpha_2 t}\sin(\omega_i t+\varphi_i') \\ &= e^{-(n_1/2)t}\{Ae^{\mu t}\sin(\omega_i t+\varphi_i)+Be^{-\mu t}\sin(\omega_i t+\varphi_i')\}, \\ X_j &= \sqrt{\omega_i/\omega_j}\{Ae^{\alpha_1 t}\sin(\omega_j t\pm\psi_{ij}-\varphi_i)+Be^{\alpha_2 t}\sin(\omega_j t\mp\psi_{ij}-\varphi_i')\} \\ &= \sqrt{\omega_i/\omega_j}e^{-(n_1/2)t}\{Ae^{\mu t}\sin(\omega_j t\pm\psi_{ij}-\varphi_i) \\ &\qquad +Be^{-\mu t}\sin(\omega_j t\mp\psi_{ij}-\varphi_i')\}. \\ &\qquad (\text{for } n_2=0, \quad \nabla_{ij}^2\leqq E_{ij}^2) \end{aligned}\right\} \tag{2.29}$$

It is seen that insertion of $n_2=0$ in Eq. (2.20) or addition of $e^{-(n_1/2)t}$ to the solutions of unstable vibrations of summed type without damping, which are treated in the previous chapter, yield the above equations. When $\alpha_1$ becomes positive, *i.e.*, $\mu$ is larger than $n_1/2$, unstable vibrations of summed type can take place, while damped vibrations with frequencies $\omega_{i,j}$ occur in case of $\alpha_1<0$, *i.e.*, $\mu<n_1/2$. Further it is obvious that boundaries between unstable and stable regions for $n_2=0$ and $\nabla_{ij}^2\leqq E_{ij}^2$ are given by putting $\lambda=1$ in Eq. (2.22).

On the other hand, when $\omega$ goes far off from $p_{ij}$ and $\nabla_{ij}^2$ becomes larger than $E_{ij}^2$, $\mu$, $\alpha_{1,2}$ and $\psi_{ij}$ are not real numbers. Putting

$$\nu^2=-\mu^2=(\nabla_{ij}^2-E_{ij}^2)/4, \tag{2.30}$$

solutions $X_{i,j}$ in this case are represented by the following equations in which term $e^{-(n_1/2)t}$ is added to solutions of stable vibration of summed type without damping:

$$\left.\begin{aligned} X_i &= e^{-(n_1/2)t}\{A\sin(\omega_i+\nu)t+B\cos(\omega_i+\nu)t+C\sin(\omega_i-\nu)t \\ &\qquad +D\cos(\omega_i-\nu)t\}, \\ X_j &= \frac{1}{E_{ij}}\sqrt{\frac{\omega_i}{\omega_j}}\,e^{-(n_1/2)t}[(\nabla_{ij}+2\nu)\{A\sin(\omega_j-\nu)t-B\cos(\omega_j-\nu)t\} \\ &\qquad +(\nabla_{ij}-2\nu)\{C\sin(\omega_j+\nu)t-D\cos(\omega_j+\nu)t\}], \\ &\qquad (\text{for } n_2=0, \quad \nabla_{ij}^2>E_{ij}^2) \end{aligned}\right\} \tag{2.31}$$

where $A, B, C, D$ are all arbitrary constants. Eq. (2.31) express simply free vibrations with damping.

Incidentally, applying the similar procedure above mentioned to ordinary unstable vibrations appearing in the neighborhood of $\omega=2p_r$ $(r=1, 2, \cdots, h)$, we have the following equations of motion in place of Eqs. (2.2), (2.3):

$$\left.\begin{aligned} \dot{a}_r &= (\varepsilon_{rr}a_r/2\omega)\sin 2\varphi_r - C_{rr}a_r/2, \\ \omega_r^2-4p_r^2 &= 2\varepsilon_{rr}\cos 2\varphi_r. \end{aligned}\right\} \tag{2.32}$$

From the above equations, solution $X_r$, negative damping coefficient $\mu_r$, phase angle $\psi_r$ and critical detunings $\nabla_{rc1}$, $\nabla_{rc2}$ of these kinds of vibrations are derived respectively as follows:

$$\left.\begin{aligned}
&X_r = e^{-(C_{rr}/2)t}\{Ae^{\mu_r t}\sin(\omega t/2 \pm \psi_r) + Be^{-\mu_r t}\sin(\omega t/2 \mp \psi_r),\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad \nabla_r^2 \leqq \varepsilon_{rr}^2/(2p_r)^2,\\
&X_r = e^{-(C_{rr}/2)t}\{A\sin(\omega/2+\nu_r)t + B\cos(\omega/2+\nu_r)t\\
&\qquad + A\sin(\omega/2-\nu_r)t - B\cos(\omega/2-\nu_r)t\}, \quad \text{for} \quad \nabla_r^2 > \varepsilon_{rr}^2/(2p_r)^2,\\
&\mu_r = \sqrt{(\varepsilon_{rr}/2p_r)^2 - \nabla_r^2}/2, \qquad (\nabla_r = \omega - 2p_r)\\
&\nu_r = \sqrt{\nabla_r^2 - (\varepsilon_{rr}/2p_r)^2}/2,\\
&\cos 2\psi_r = 2p_r\nabla_r/\varepsilon_{rr}, \qquad (\pi/2 \geqq \psi_r \geqq 0)\\
&\nabla_{rc1 \atop c2} = \omega_{rc1 \atop c2} - 2p_r = \pm\sqrt{(\varepsilon_{rr}/2p_r)^2 - C_{rr}^2}.
\end{aligned}\right\} \qquad (2.33)$$

It is seen that the results of Eq. (2.33) agree with those of one degree-of-freedom system with damping.

*2.4. Vibrations of differential type*

By the similar procedure of summed type, we have

$$\alpha_{1,2} = -n_1/2 \pm \sqrt{n_2^2 - 4\mu^2}/2 = -n_1/2 \pm \sqrt{n_2^2 - E_{ij}^2 + \delta_i\delta_j}, \qquad (2.10\text{ a})$$

$$\frac{\Delta_i\,\omega_j}{\Delta_i\,\omega_j} = \frac{A_j^2}{A_i^2}\cdot\frac{\omega_j}{\omega_i} = -\left(\frac{n_2 \pm \sqrt{n_2^2 - 4\mu^2}}{-n_2 \pm \sqrt{n_2^2 - 4\mu^2}}\right), \qquad (2.12\text{ a})$$

$$\left\{\begin{aligned}
&\delta_i\delta_j = (E_{ij}^2 - \nabla_{ij}^2 - n_2^2)/2 \pm \sqrt{(E_{ij}^2 - \nabla_{ij}^2 - n_2^2)^2 + 4E_{ij}^2\nabla_{ij}^2}/2, \quad (\text{for } n_2 \neq 0) \quad (2.14\text{ a})\\
&\delta_i\delta_j = -\nabla_{ij}^2. \qquad\qquad\qquad\qquad (\text{for } n_2 = 0) \quad (2.14'\text{ a})
\end{aligned}\right.$$

In Eqs. (2.10 a), (2.12 a) the upper and lower signs correspond to $\alpha_1$ and $\alpha_2$ respectively.

*2.4.1. Vibrations of differential type when $n_2 \neq 0$ ($C_{ii} \neq C_{jj}$)*

Adopting the upper sign in Eq. (2.14 a) by the similar reason mentioned in vibrations of summed type, we get the following equations:

$$\delta_i = \nabla_{ij} \pm \sqrt{b'-a'}, \quad \delta_j = -\nabla_{ij} \pm \sqrt{b'-a'}, \qquad (2.15\text{ a})$$

$$a' = (n_2^2 - E_{ij}^2 - \nabla_{ij}^2)/2, \quad b' = \sqrt{a'^2 + n_2^2\nabla_{ij}^2}, \qquad (2.16\text{ a})$$

$$\alpha_1 = -n_1/2 + \sqrt{b'+a'}/2, \quad \alpha^2 = -n_1/2 - \sqrt{b'+a'}/2, \qquad (2.17\text{ a})$$

$$A_i/A_j = \sqrt{\omega_j/\omega_i}\sqrt{(n_2 \mp \sqrt{b'+a'})/(n_2 \pm \sqrt{b'+a'})}, \qquad (2.18\text{ a})$$

$$\cos\psi_{ij} = \sqrt{b'-a'-\nabla_{ij}^2}/|E_{ij}|, \qquad (0 \leqq \psi_{ij} \leqq \pi) \qquad (2.19\text{ a})$$

$$\left.\begin{aligned}
&X_i = Ae^{\alpha_1 t}\sin(\omega_i t + \varphi_i) + Be^{\alpha_2 t}\sin(\omega_i' t + \varphi_i'),\\
&X_j = \sqrt{\frac{\omega_i}{\omega_j}}\left\{\sqrt{\frac{n_2 + \sqrt{b'+a'}}{n_2 - \sqrt{b'+a'}}}\,Ae^{\alpha_1 t}\sin(\omega_j t \mp \psi_{ij} + \varphi_i)\right.\\
&\qquad \left. + \sqrt{\frac{n_2 - \sqrt{b'+a'}}{n_2 + \sqrt{b'+a'}}}\,Be^{\alpha_2 t}\sin(\omega_j' t \mp \psi_{ij} + \varphi_i')\right\},
\end{aligned}\right\} \qquad (2.20\text{ a})$$

$$\begin{matrix}\omega_i\\ \omega_i'\end{matrix} = p_i + \frac{\nabla_{ij}}{2} \pm \frac{\sqrt{b'-a'}}{2}, \quad \begin{matrix}\omega_j\\ \omega_j'\end{matrix} = p_j - \frac{\nabla_{ij}}{2} \pm \frac{\sqrt{b'-a'}}{2}, \qquad (2.21\text{ a})$$

which correspond to Eqs. (2.15), (2.16), (2.17), (2.18), (2.19), (2.20) and (2.21) of summed type severally. In Eq. (2.20 a), the upper and lower signs are employed when $n_2E_{ij}>0$ and $n_2E_{ij}<0$ respectively, while in Eq. (2.21 a) they correspond to $\omega_{i,j}$ and $\omega'_{i,j}$ when $n_2\nabla_{ij}>0$ and vice versa when $n_2\nabla_{ij}<0$ separately. In Eqs. (2.15 a) ~ (2.19 a), $\delta_{i,j}$, $\alpha_{1,2}$, $A_i/A_j$ and $\varphi_{ij}$ are all real numbers because of $b'\pm a'>0$. Furthermore, since Eqs. (2.16 a), (2.17 a) result in both $\alpha_1$ and $\alpha_2$ with negative values, Eq. (2.20 a) represent simply damped vibrations.

*2.4.2. Vibrations of differential type when $n_2=0$ ($C_{ii}=C_{jj}$)*

Since this exceptional case is similar to Section 2.3.2, we obtain

$$\delta_i = -\delta_j = \nabla_{ij}, \tag{2.25 a}$$

$$\alpha_{1,2} = -n_1/2 \pm \sqrt{-\mu^2} = -n_1/2 \pm i\sqrt{E_{ij}^2+\nabla_{ij}^2}/2, \tag{2.26 a}$$

$$A_i/A_j = \sqrt{-\omega_j/\omega_i} = i\sqrt{\omega_j/\omega_i}\,, \tag{2.27 a}$$

$$\cos\varphi_{ij} = i|\nabla_{ij}/E_{ij}|, \tag{2.28 a}$$

in which $\mu$ is real and $\alpha_{1,2}$, $\varphi_{ij}$ and $A_i/A_j$ are not real. Accordingly, solutions of this case are attained as follows:

$$\left.\begin{aligned} X_i &= e^{-(n_1/2)t}\{A\sin(\omega_i+\mu)t+B\cos(\omega_i+\mu)t+C\sin(\omega_i-\mu)t \\ &\qquad +D\cos(\omega_i-\mu)t\}, \\ X_j &= -\frac{1}{E_{ij}}\sqrt{\frac{\omega_i}{\omega_j}}\,e^{-(n_1/2)t}[(\nabla_{ij}+2\mu)\{A\sin(\omega_j+\mu)t+B\cos(\omega_j+\mu)t\} \\ &\qquad +(\nabla_{ij}-2\mu)\{C\sin(\omega_j-\mu)t+D\cos(\omega_j-\mu)t\}], \end{aligned}\right\} \tag{2.31 a}$$

which are same as the equations added $e^{-(n_1/2)t}$ to solutions of differential type without damping, and represent damped vibrations of frequencies $\omega_{i,j}\pm\mu$.

Consequently, as shown in Sections 2.4.1, 2.4.2, unstable vibration of differential type cannot occur.

*2.5. Verification of analytical results through analog computer*

Since in actual experimental apparatus it is difficult to produce various damping forces exactly proportional to $\dot{X}_r$ and to freely change magnitude of damping coefficients $C_{ii}$, $C_{jj}$ and value of damping ratio $\lambda=C_{jj}/C_{ii}$, analog computer is used to verify obtained analytical results. Comparison between both results is made on a simple system with two degree-of-freedom, hence number of degrees of freedom $h$, suffixs $i$ and $j$ are replaced by 2, 1 and 2 respectively for this system. In analog computer, we employ the following same dimensions as those of the experimental apparatus in the previous chapter:

For vibrations of summed type

$$p_1=13.863\text{ c/s},\ p_2=9.850\text{ c/s},\ p_{12}=p_1+p_2=23.713\text{ c/s},$$
$$E_{12}=6.28\,e\text{ rad/s},\ \varepsilon_{11}=12.23\times10^2\,e\text{ (rad/s)}^2,\ \varepsilon_{12}=9.22\times10^2\,e\text{ (rad/s)}^2,$$
$$\varepsilon_{22}=14.33\times10^2\,e\text{ (rad/s)}^2,$$

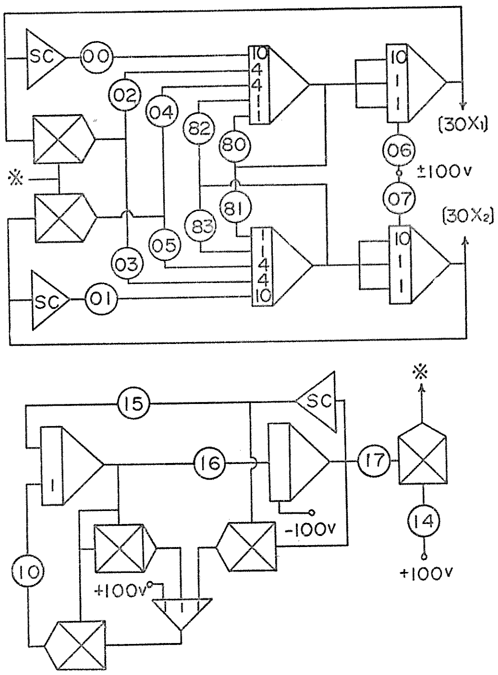
For vibrations of differential type

$$p_1=25.0\text{ c/s},\ p_2=10.0\text{ c/s},\ p_{12}=p_1-p_2=15.0\text{ c/s},$$
$$E_{12}=1.86\,e\text{ rad/s},\ \varepsilon_{11}=5.25\times10^2\,e\text{ (rad/s)}^2,\ \varepsilon_{12}=3.69\times10^2\,e\text{ (rad/s)}^2,$$
$$\varepsilon_{22}=6.15\times10^2\,e\text{ (rad/s)}^2,$$

in which $e$ with dimension $cm$ is amplitude of vertical vibration of a supporting point introducing parametric excitation into a vibratory system of double pendulum in the previous chapter. In the following figures, results of analysis are graphically shown with broken and chain or full and dotted line curves which correspond to stable and unstable vibrations without or with damping separately, and results obtained by analog computer are indicated by symbols ○, ⊖, etc. and ●, ⊖, etc., which respectively identify stable and unstable vibrations. The vertical chain lines in Figs. 2.2～2.11 except for Fig. 2.5 illustrate the location of the resonant point $\omega=p_{12}$.

Block diagram of analog computer used to solve Eq. (2.1) is shown in Fig. 2.1, where there are damping circuits.

Figs. 2.2, 2.3 and 2.4 indicate boundaries of unstable vibrations of summed type for various damping coefficients. As shown in Fig. 2.2, when $n_2=0$, *i.e.*, damping ratio $\lambda$ is equal to unity, the smaller damping results in the wider unstable region, and the situations are quite similar to those of ordinary unstable vibrations caused by parametric excitation appearing in the neighborhood of $\omega=2p_r$. On the other hand, when $\lambda\fallingdotseq 1$, the circumstance is quite different from the above as shown in Fig. 2.3, where unstable region of vibrations with damping




00: $p_1^2/1200$
01: $p_2^2/1200$
02: $25\,\varepsilon_{11}/(3\,e\omega^2)$
03: $25\,\varepsilon_{12}/(3\,e\omega^2)$
04: $25\,\varepsilon_{21}/(3\,e\omega^2)$
05: $25\,\varepsilon_{22}/(3\,e\omega^2)$
06: initial condition
07: initial condition
10: 0.1
14: $(\omega/200)^2$
15: $\omega/10$
16: $\omega/10$
17: $e$
80: $C_{11}/10$
81: $C_{12}/10$
82: $C_{21}/10$
83: $C_{22}/10$

FIG. 2.1. Block diagram of analog computer.

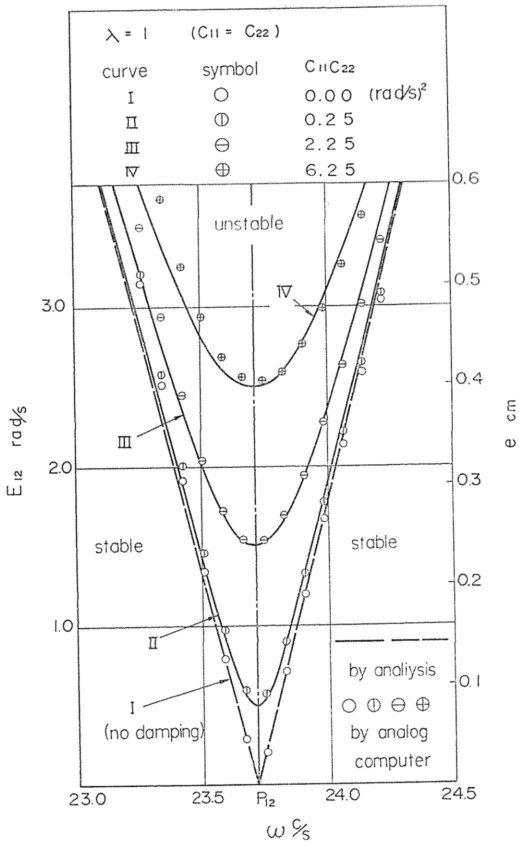



FIG. 2.2. Boundaries of unstable region for vibrations of summed type when $\lambda=1$.

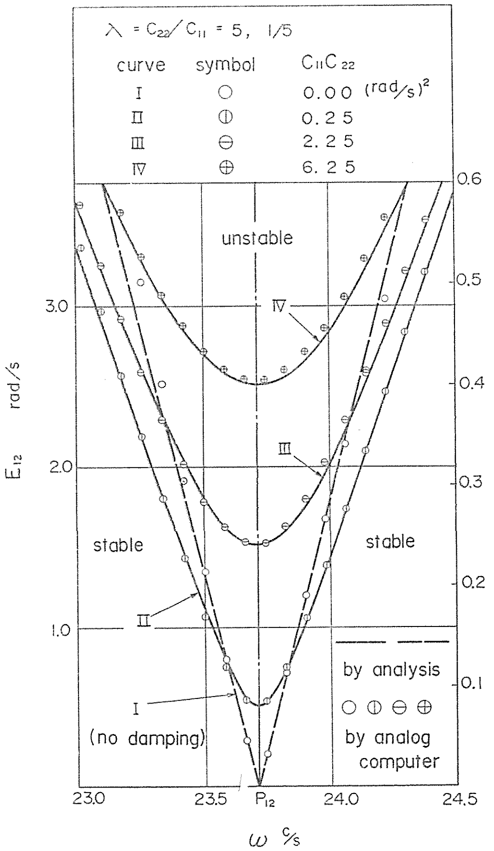



FIG. 2.3. Boundaries of unstable region for vibrations of summed type when $\lambda \fallingdotseq 1$.

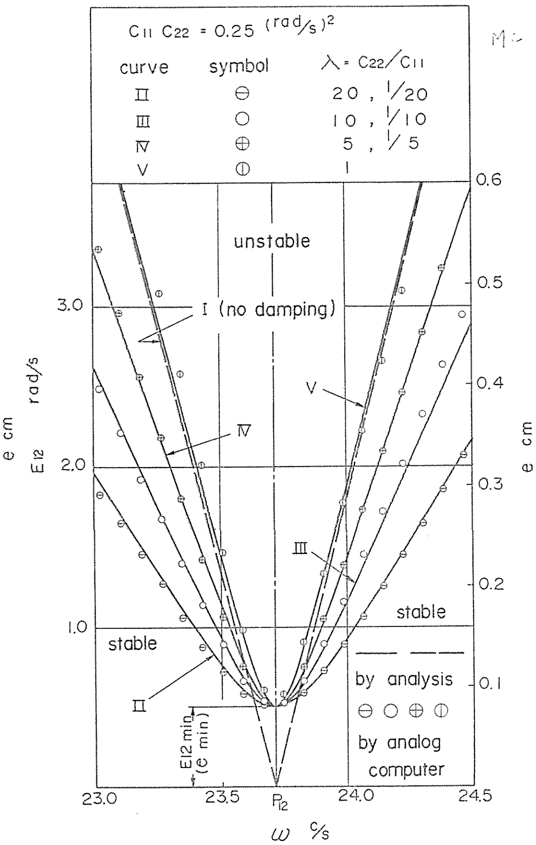



FIG. 2.4. Influence of magnitude of damping ratio $\lambda$ to boundaries of unstable region of vibrations of summed type with a common value of $E_{12\,\min}$.

is wider than that with no damping; this tendency becomes more clear when magnitude of parametric excitation $E_{12}$ takes a rather large value. Even if $E_{12\,\min} = \sqrt{C_{11}C_{22}} = \sqrt{\lambda}\,C_{11}$ derived from Eq. (2.22) is fixed, various values of damping ratio $\lambda$ result in various unstable regions, as shown in Fig. 2.4.

Influence of $\lambda$ on unstable region can be more clearly represented in Fig. 2.5 than in Figs. 2.3, 2.4. In Fig. 2.5, ratios of width $\nabla_c$ of unstable region when $C_{11} \fallingdotseq 0$, $C_{22} \fallingdotseq 0$ to $\nabla_{co}$ when $C_{11} = C_{22} = 0$ are plotted against $E_{12}/E_{12\,\min}$, where $E_{12}$ is magnitude of parametric excitation and $E_{12\,\min}$ is illustrated in Fig. 2.4. All curves corresponding to various values of parameter $\lambda$ tend to a value of $(1+\lambda)/(2\sqrt{\lambda})$ as $E_{12}$ increases, and all curves except for $\lambda = 1$ cross a horizontal line of $\nabla_c/\nabla_{co} = 1$. It means that, provided $\lambda \fallingdotseq 1$, unstable region of vibrations with damping always can be wider than one without damping in a range of larger value $E_{12}$ than $|(1+\lambda)/(1-\lambda)|E_{12\,\min}$.

Negative damping coefficients $\alpha_1$ of vibrations of summed type given by Eq. (2.17) are shown in Fig. 2.6, where a horizontal line of $\alpha_1 = 0$ is a boundary be-

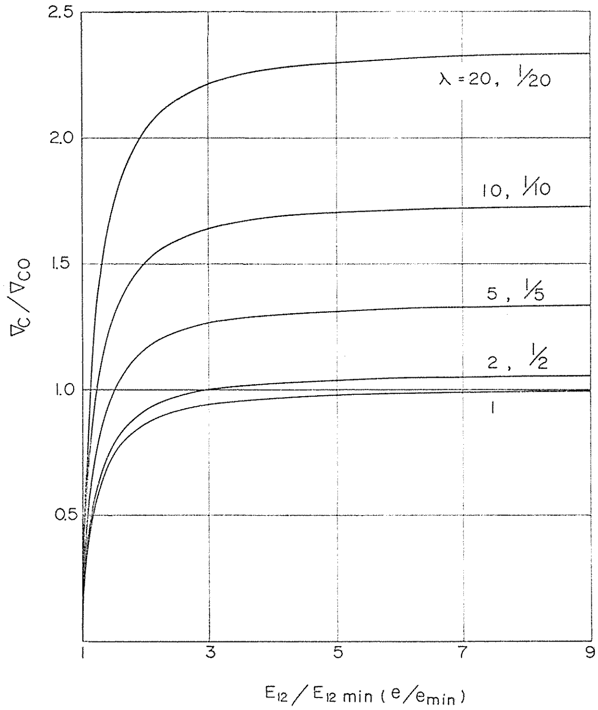



FIG. 2. 5. $\nabla_c/\nabla_{co}-E_{12}/E_{12\,\min}$ diagram
($\nabla_c$=width of unstable region for damping, $\nabla_{co}$=for no damping).

tween unstable and stable regions. Value of $\alpha_1$ is represented by a continuous curve through stable and unstable regions when there is damping, which is expressed separately when there is no damping, as shown in Chapter I.

Fig. 2. 7 shows frequencies $\omega_{1,2}$ and $\omega'_{1,2}$ of vibrations of summed type which are expressed by a continuous curve of I or II through both stable and unstable regions, while frequencies of stable and unstable vibrations are represented by broken and chain line curves separately when there is no damping. Furthermore, it is seen that the relation of $\omega_1+\omega_2=\omega'_1+\omega'_2=\omega$ always holds and $\omega_{1,2}$, $\omega'_{1,2}$ are replaced as $\lambda$ and $1/\lambda$ are exchanged. Vertical double chain lines in this figure indicate boundaries of unstable regions.

For various damping coefficients, amplitude ratio $A_1/A_2$ is shown in Fig. 2. 8, where horizontal double chain lines represent amplitude ratios $\sqrt{\lambda}\sqrt{\omega_2/\omega_1}$ on the boundaries of unstable regions and amplitude ratio for $\alpha_1$ and $\alpha_2$ are replaced as $\lambda$ and $1/\lambda$ are exchanged.

Negative damping coefficients, frequencies and amplitude ratio of vibrations of differential type are indicated in Figs. 2. 9, 2. 10 and 2. 11. It is found from Fig. 2. 9 that $\alpha_1$ is always negative, hence no unstable vibration of differential type can occur. In Fig. 2. 10, the relation $\omega_1-\omega_2=\omega'_1-\omega'_2=\omega$ holds.

Finally, vibratory waves of unstable vibrations of summed type obtained by analog computer are illustrated in Fig. 2. 12.

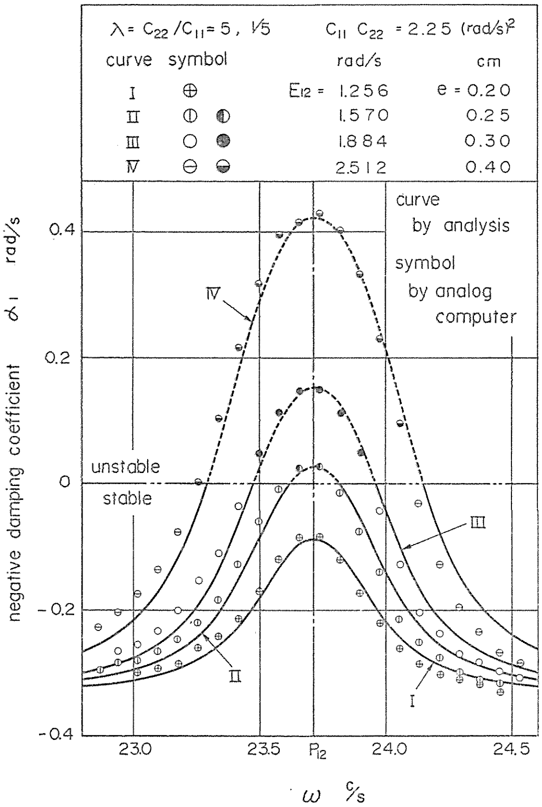



FIG. 2.6. Negative damping coefficient $\alpha_1$ of vibrations of summed type for various magnitudes of parametric excitation.

## *2.6. Conclusions*

Obtained results may be summarized as follows:

(1) In a vibratory system with multiple degree-of-freedom and under parametric excitation, two unstable vibrations of summed type with frequencies $\omega_i(\fallingdotseq p_i)$, $\omega_j(\fallingdotseq p_j)$ satisfying the relation $\omega_i+\omega_j=\omega$ can simultaneously take place in the neighborhood of the resonant point $p_{ij}=p_i+p_j$, even if damping forces exist in the system.

(2) For vibrations of summed type with damping, solutions, frequencies, negative damping coefficients, amplitude ratios and phase angle of both stable and unstable regions have common forms, as seen in Eqs. (2.20), (2.21), (2.17), (2.18) and (2.19) respectively, while in vibrations without damping, they are given separately for both regions.

(3) Magnitude of damping ratio $\lambda=C_{jj}/C_{ii}$ has considerable influences on unstable region of vibrations of summed type, that is:

(i) When $\lambda=1$, the smaller damping results in the wider unstable region (Fig. 2.2).

(ii) When $\lambda\fallingdotseq 1$, rate of expansion of unstable region to increase of magnitude

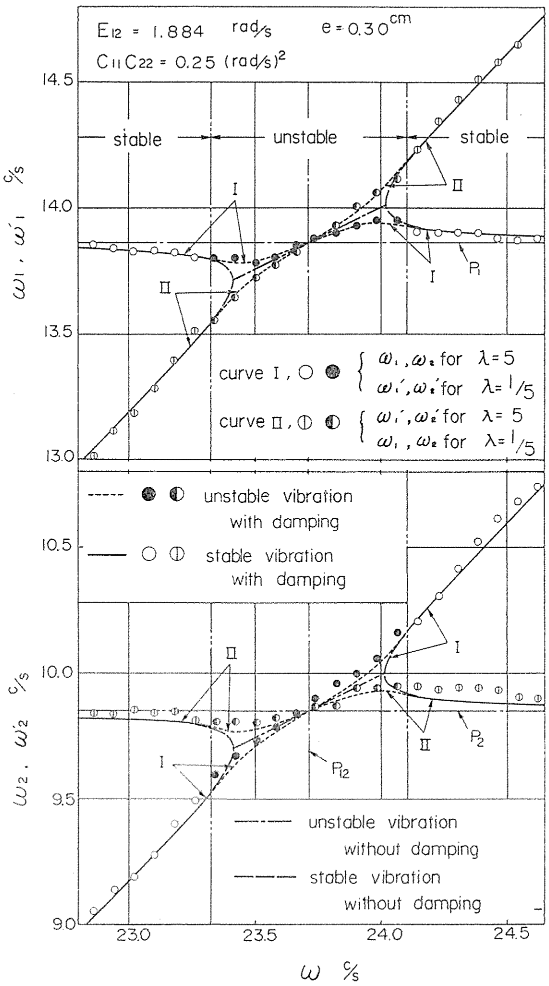



FIG. 2.7. Frequencies of vibrations of summed type.

of parametric excitation $|E_{ij}|$ becomes larger as $\lambda$ deviates from unity (Fig. 2.3).

(iii) When $\lambda \fallingdotseq 1$, unstable region of vibrations with damping can always be wider than that without damping in a range of larger values of $|E_{ij}|$ than a certain value shown in (iv) (Figs. 2.3, 2.4 and 2.5).

(iv) A certain value of $|E_{ij}|$ mentioned above is $|E_{ij\,\min}(1+\lambda)/(1-\lambda)| = \sqrt{C_{ii}C_{jj}}\,|(1+\lambda)/(1-\lambda)|$, which increases as $\lambda$ approaches unity (Fig. 2.5).

(v) If magnitude of parametric excitation $|E_{ij}|$ is smaller than $|E_{ij\,\min}| = \sqrt{C_{ii}C_{jj}}$, unstable vibration cannot occur.

(4) Vibrations on boundaries between stable and unstable regions are always stable.

(5) For both unstable and stable vibrations, the sum of or difference between two frequencies of vibrations is always equal to the frequency of parametric excitation.

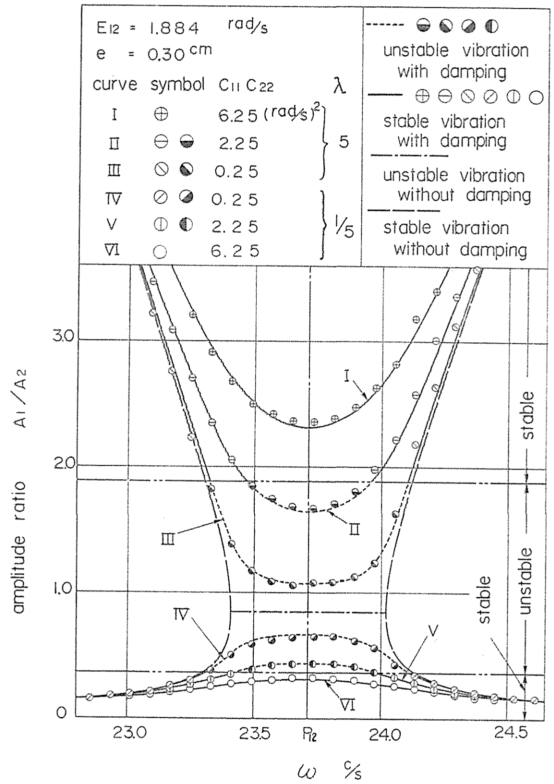



FIG. 2.8. Amplitude ratio $A_1/A_2$ of vibrations of summed type.

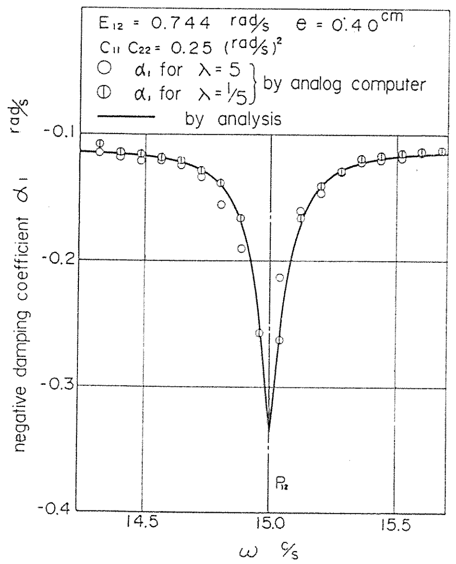



FIG. 2.9. Negative damping coefficient $\alpha_1$ of vibrations of differential type.

(6) Solutions, frequencies, negative damping coefficients, amplitude ratios and phase angle of vibrations of differential type are given by Eqs. (2.20 a), (2.21 a), (2.17 a), (2.18 a), (2.19 a); and since negative damping coefficients $\alpha_{1,2}$ are always negative, there is no unstable vibrations.

(7) So far as magnitude of parametric excitation is somewhat small as in this paper, solutions of first approximation are enough for grasping exactly vibratory phenomena induced by parametric excitation and they show good agreement with results of analog computer.

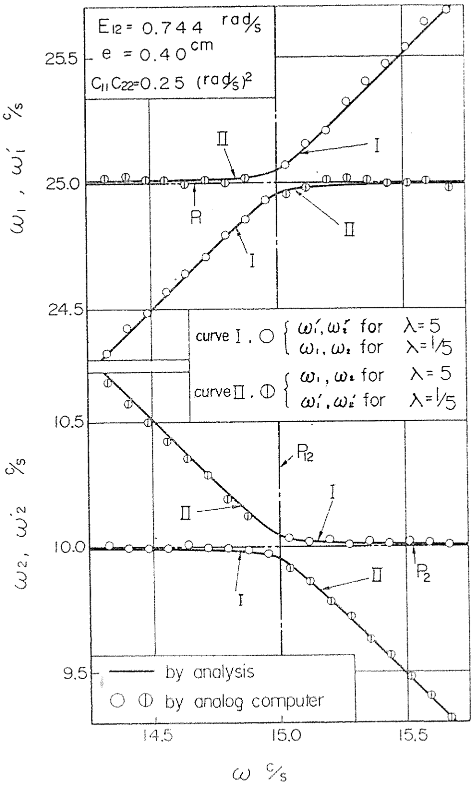



FIG. 2.10. Frequencies of vibrations of differential type.

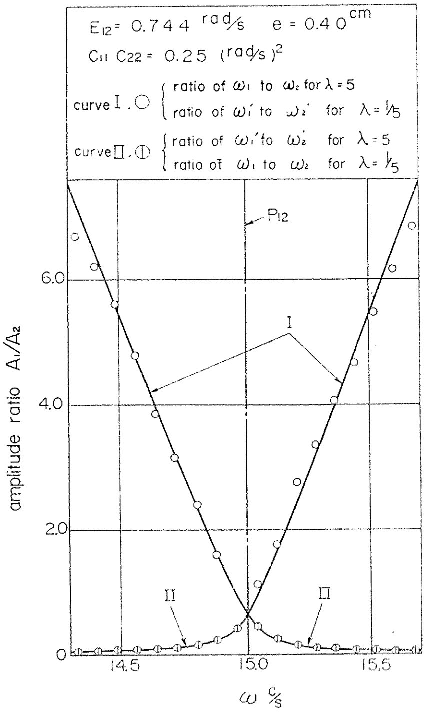



FIG. 2.11. Amplitude ratio of vibrations of differential type.

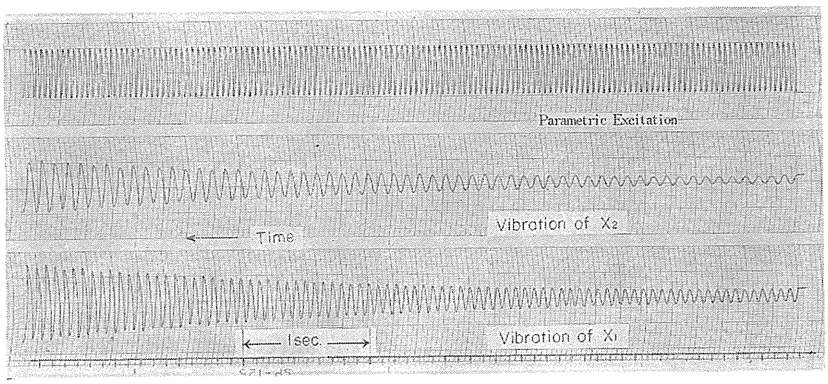



FIG. 2.12. Vibratory waves of unstable vibrations of summed type given by analog computer.

## Chapter III. Vibrations of "summed and differential types of higher order" under parametric excitation[20]

### 3.1. Introduction

The first approximate solutions of vibrations of summed and differential types appearing at $\omega \fallingdotseq p_{ij} = p_i + p_j$ have been studied in the previous chapters. In this chapter, solutions of the $n$'th approximation, in which small quantities of $\varepsilon^n$-order are taken into account, are analyzed by means of approximate method of so-called "expanded Kryloff and Bogoliuboff's method for solutions of higher approximation in non-linear vibratory systems of single degree-of-freedom to multiple degree-of-freedom", and characteristics of vibrations of "summed and differential types of higher order" of two frequencies $\omega_i (\fallingdotseq p_i)$ and $\omega_j (\fallingdotseq p_j)$ which take place in vibratory systems of multiple degree-of-freedom under parametric excitation of frequency $\omega$ are discussed. These vibrations appear when $s\omega$ $(s=2, 3, \cdots)$ becomes nearly equal to a sum of or a difference between two natural frequencies $p_i$, $p_j$, *i.e.*, $s\omega \fallingdotseq p_i \pm p_j = p_{ij}$. It is seen that the $s$'th order vibrations do not take place until the $s$'th approximation is taken up, and that there is no unstable vibration of differential type of higher order as well as that of the first order treated in the previous chapters, and that effects of damping on these vibrations are similar to those of the first order, that is, existence of damping does not always result in decrease of width of unstable regions. Further, obtained theoretical results are compared with those of analog computer, and they show good agreement with each other.

### 3.2. Equation of the $n$'th approximation

Equation of motion of system of $h$ degree-of-freedom and under parametric excitation of frequency is represented by the following equation:

$$\ddot{X}_l + p_l^2 X_l + \sum_{m=1}^{h} (c_{lm}\dot{X}_m - \varepsilon_{lm} X_m \cos \omega t) = 0, \qquad (l = 1, 2, \cdots, h) \qquad (3.1)$$

in which magnitude of parametric excitation $\varepsilon_{lm}$ is a small quantity of $\varepsilon$-order and damping coefficient $C_{lm}$ should be assumed to be small quantity of $\varepsilon^s$-order because negative damping coefficients of vibrations of $s$'th order are also $\varepsilon^s$-order as shown later. In order to treat with vibrations of summed and differential types of $s$'th order with frequencies $\omega_i$ and $\omega_j$ satisfying the following relations:

$$s\omega = \omega_i \pm \omega_j \fallingdotseq p_i \pm p_j = p_{ij} \ \ (s = 2, 3, \cdots), \quad \omega_f \fallingdotseq p_f \ \ (f = i, j), \qquad (3.2)$$

the $i$'th and $j$'th equations of Eq. (3.1) should be rewritten as

$$\ddot{X}_f + p_f^2 X_f + \sum_{m=i,j} (\varepsilon^s C_{fm}\dot{X}_m - \varepsilon\kappa_{fm} X_m \cos \omega t) + \sum_{m \neq i,j} (c_{fm}\dot{X}_m - \varepsilon_{fm} X_m \cos \omega t) = 0, \quad (f = i, j) \qquad (3.3)$$

where

$$\varepsilon_{fm} = \varepsilon\kappa_{fm} = \varepsilon\kappa_{mf}, \qquad c_{fm} = \varepsilon^s C_{fm}. \qquad (3.4)$$

Solution $X_f$ is assumed to be function of both amplitude $a_f$ and frequency $\omega_f$, it follows

$$X_f = X_f(a_f,\ \omega_f t + \varphi_f) = X_f(a_f,\ \theta_f), \tag{3.5}$$

in which

$$\theta_f = \omega_f t + \varphi_f. \tag{3.6}$$

It is also assumed that phase angle $\varphi_f$ is not function of time, and that $\dot\theta_f$, $X_f$ and $\dot a_f$ are written separately in the following power series in $\varepsilon$:

$$\left.\begin{aligned}\dot\theta_f = \omega_f = \omega_{f,0} + \varepsilon\omega_{f,1} + \varepsilon^2\omega_{f,2} + \cdots,\\ \omega_{f,0} = p_f \qquad (f = i, j),\end{aligned}\right\} \tag{3.7}$$

$$X_f = X_{f,0} + \varepsilon X_{f,1} + \varepsilon^2 X_{f,2} + \cdots, \tag{3.8}$$

$$\dot a_f = U_f = \varepsilon U_{f,1} + \varepsilon^2 U_{f,2} + \cdots. \tag{3.9}$$

Upon taking account that $X_{i,j}$ are functions of both $a_m$ and $\theta_m$ $(m=1, 2, \cdots, h)$ as shown in Eq. (3.1), and that $U_f$, $\omega_f$ should be functions of $a_m$, $\ddot X_{i,j}$ are written in the form

$$\begin{aligned}\ddot X_i &= \sum_{l,m=1}^{h}\left\{\left(\omega_l\frac{\partial}{\partial\theta_l} + U_l\frac{\partial}{\partial a_l}\right)\left(\omega_m\frac{\partial}{\partial\theta_m} + U_m\frac{\partial}{\partial a_m}\right)X_i + U_l\left(\frac{\partial\omega_m}{\partial a_l}\frac{\partial X_i}{\partial\theta_m} + \frac{\partial U_m}{\partial a_l}\frac{\partial X_i}{\partial a_m}\right)\right\}\\ &= \left\{\sum_{x=i,j}\left(\omega_x\frac{\partial}{\partial\theta_x} + U_x\frac{\partial}{\partial a_x}\right)\right\}^2 X_i + \sum_{x,y=i,j} U_x\left(\frac{\partial\omega_y}{\partial a_x}\frac{\partial X_i}{\partial\theta_y} + \frac{\partial U_y}{\partial a_x}\frac{\partial X_i}{\partial a_y}\right) + \cdots.\end{aligned} \tag{3.10}$$

The symbol $\cdots$ in the above equation lies outside in further discussion, because of containing no term concerned until suffixs $i$ and $j$. When suffixs $i$ and $j$ are exchanged in Eq. (3.10), $\ddot X_j$ is furnished. Substituting Eqs. (3.7), (3.8), (3.9), (3.10) into Eq. (3.3), the $n$'th order terms of $\varepsilon$ yield the following differential equation:

$$\begin{aligned}\left(\sum_{x=i,j} p_x\frac{\partial}{\partial\theta_x}\right)^2 X_{f,n} + p_f^2 X_{f,n} = -\sum_{x,y=i,j}\Bigg\{&\sum_{m=0}^{n-1}\sum_{l=0}^{n-m}\omega_{x,l}\,\omega_{y,n-m-l}\frac{\partial^2 X_{f,m}}{\partial\theta_x\partial\theta_y}\\ &+ 2\sum_{m=0}^{n-1}\sum_{l=0}^{n-m-1}\omega_{x,l}\,U_{y,n-m-l}\frac{\partial^2 X_{f,m}}{\partial\theta_x\partial a_y} + \sum_{m=0}^{n-2}\sum_{l=1}^{n-m-l} U_{x,l}\,U_{y,n-m-l}\frac{\partial^2 X_{f,m}}{\partial a_x\partial a_y}\\ &+ C_{fx}p_x\frac{\partial X_{f,0}}{\partial\theta_x} - \kappa_{fx}X_{x,n-1}\cos\omega t\Bigg\},\end{aligned} \tag{3.11}$$

in which notation $\Sigma$ means that all terms vanish when the lower limit in $\Sigma$ becomes larger than its upper limit. Putting $n=0$ in Eq. (3.11), solution of the zero approximation are derived:

$$X_{f,0} = a_f\sin\theta_f \tag{3.12}$$

and for $n=1, 2, \cdots$, solutions of the $n$'th approximation can be obtained successively through Eq. (3.11). When $n\leqq s$, the following relationship is given by putting coefficients of resonant terms $\sin\theta_f$ and $\cos\theta_f$ in Eq. (3.11) equal to zero:

$$U_{f,1} = U_{f,2} = \cdots = U_{f,n-1} = 0 \tag{3.13}$$

and hence

$$\left.\begin{array}{l}\text{(i) when } n=2r \text{ (even)}\\ \qquad \omega_{f,1}=\omega_{f,3}=\cdots=\omega_{f,n-1}=0,\\ \qquad \omega_{f,2}\neq 0,\ \omega_{f,4}\neq 0,\ \ldots,\ \omega_{f,n-2}\neq 0,\\ \text{(ii) when } n=2r+1 \text{ (odd)}\\ \qquad \omega_{f,1}=\omega_{f,3}=\cdots=\omega_{f,n-2}=0,\\ \qquad \omega_{f,2}\neq 0,\ \omega_{f,4}\neq 0,\ \ldots,\ \omega_{f,n-1}\neq 0.\end{array}\right\} \tag{3.14}$$

Upon use of Eqs. (3.13), (3.14), Eq. (3.11) is reduced to

(i) when $n=2r$ (even)

$$\left(\sum_{x=i,j} p_x \frac{\partial}{\partial \theta_x}\right)^2 X_{f,2r}+p_f^2 X_{f,2r}=-\sum_{m=0}^{r-1}\sum_{l=0}^{r-m}\sum_{x,y=i,j}\omega_{x,2l}\,\omega_{y,2(r-m-l)}\frac{\partial^2 X_{f,2m}}{\partial\theta_x\partial\theta_y}$$
$$-2p_f U_{f,2r}\cos\theta_f-\sum_{x=i,j}C_{fx}p_x\frac{\partial X_{f,0}}{\partial\theta_x}+\sum_{x=i,j}\kappa_{fx}X_{x,2r-1}\cos(\theta_i\pm\theta_j-\varphi_{ij})/s, \tag{3.15}$$

(ii) when $n=2r+1$ (odd)

$$\left(\sum_{x=i,j} p_x \frac{\partial}{\partial \theta_x}\right)^2 X_{f,2r+1}+p_f^2 X_{f,2r+1}=-\sum_{m=0}^{r-1}\sum_{l=0}^{r-m}\sum_{x,y=i,j}\omega_{x,2l}\,\omega_{y,2(r-m-l)}\frac{\partial^2 X_{f,2m+1}}{\partial\theta_x\partial\theta_y}$$
$$+2p_f\omega_{f,2r+1}a_f\sin\theta_f-2p_f U_{f,2r+1}\cos\theta_f-\sum_{x=i,j}C_{fx}p_x\frac{\partial X_{f,0}}{\partial\theta_x}$$
$$+\sum_{x=i,j}\kappa_{fx}X_{x,2r}\cos(\theta_i\pm\theta_j-\varphi_{ij})/s, \tag{3.16}$$

in which

$$\varphi_{ij}=\varphi_i\pm\varphi_j. \tag{3.17}$$

Introducing the following two symbols

$$S_{f,g}=\sin\{\theta_f+q(\theta_i\pm\theta_j-\varphi_{ij})/s\},\qquad (q=\text{integer}) \tag{3.18}$$
$${}_fD_{f'+q}=[p_f^2-\{p_{f'}+q(p_i\pm p_j)/s\}^2]^{-1},\qquad (f,f'=i,j) \tag{3.19}$$

in which ${}_fD_{f'+q}$ is magnification factor of $S_{f,q}$, solutions of the $n$'th approximation are obtained through Eqs. (3.15), (3.16) as follows:

(i) when $n=2r$ (even)

$$X_{f,n}=\sum_{l=-r}^{r}\sum_{f'=i,j}a_{f'}\cdot F(f,n,f',2l)\cdot S_{f',2l}, \tag{3.20}$$

(ii) when $n=2r+1$ (odd)

$$X_{f,n}=\sum_{l=-(r+1)}^{r}\sum_{f'=i,j}a_{f'}\cdot F(f,n,f',2l+1)\cdot S_{f',2l+1}, \tag{3.21}$$

where coefficients $F(f,n,f',2l)$ and $F(f,n,f',2l+1)$ containing symbol ${}_fD_{f'+q}$ are successively determined through Eqs. (3.12), (3.15), (3.16) and (3.19). For example, for $n=0$, $n=1$ coefficient $F$ is given by

$$\left.\begin{aligned} F(f,\,0,\,f',\,0) &= \begin{cases} 1 \ (\text{when } f = f'), \\ 0 \ (\text{when } f \neq f'), \end{cases} \\ F(f,\,1,\,f',\,\pm 1) &= \kappa_{ff'} \cdot {}_f D_{f' \pm 1}/2. \end{aligned}\right\} \tag{3.22}$$

Substituting Eqs. (3.20), (3.21) into Eqs. (3.15), (3.16) severally and putting coefficients of resonant terms $\sin\theta_f$, $\cos\theta_f$ equal to zero, we get equation of motion for vibrations of $s$'th order. It is, however, readly seen from Eqs. (3.15), (3.16) that occurrence of vibrations of summed and differential types of higher order requires existence of $\cos\theta_f$ in term of parametric excitation, *i.e.*, requires that $U_{f,n}$ does not vanish. Consequently vibrations of summed and differential types of $s$'th order can appear when the $s$'th approximate analysis is performed.

*3.3. Vibrations of summed type of s'th order*

Substituting Eqs. (3.20), (3.21) into Eqs. (3.15), (3.16), and putting $n=s$ and further putting coefficients of resonant terms equal to zero, equations of motion are written in the following forms:

$$\left.\begin{aligned} \dot{a}_i &= \varepsilon^s U_{i,s} = \varepsilon^s\left(\frac{Ma_j}{4p_i}\sin\varphi_{ij} - \frac{C_{ii}}{2}a_i\right), \\ \dot{a}_j &= \varepsilon^s U_{j,s} = \varepsilon^s\left(\frac{Ma_i}{4p_j}\sin\varphi_{ij} - \frac{C_{jj}}{2}a_j\right), \end{aligned}\right\} \tag{3.23}$$

$$\frac{a_j}{a_i}\cos\varphi_{ij} = \frac{2P_0}{M}, \quad \frac{a_i}{a_j}\cos\varphi_{ij} = \frac{2Q_0}{M}, \tag{3.24}$$

in which

$$M = \sum_{f=i,j}\kappa_{if}F(f,\,s-1,\,j,\,-s+1) = \sum_{f=i,j}\kappa_{jf}F(f,\,s-1,\,i,\,-s+1), \tag{3.25}$$

$$\left.\begin{aligned} &P_0 = \sum_{l=0}^{r}\omega_{i,2l}\omega_{i,2(r-l)} + I_s, \quad Q_0 = \sum_{l=0}^{r}\omega_{j,2l}\omega_{j,2(r-l)} + J_s \quad (\text{when } s = 2r), \\ &P_0 = 2p_i\omega_{i,s}, \quad Q_0 = 2p_j\omega_{j,s} \quad (\text{when } s = 2r+1), \end{aligned}\right\} \tag{3.26}$$

$$\left.\begin{aligned} I_s &= \sum_{f=i,j}\kappa_{if}\{F(f,\,2r-1,\,i,\,-1) + F(f,\,2r-1,\,i,\,1)\}/2, \\ J_s &= \sum_{f=i,j}\kappa_{jf}\{F(f,\,2r-1,\,j,\,-1) + F(f,\,2r-1,\,j,\,1)\}/2. \end{aligned}\right\} \tag{3.27}$$

*3.3.1. Vibrations of summed type of s'th order without damping referring the following relations*

$$\mu^2 = \frac{\varepsilon^{2s}}{16p_ip_j}M^2\sin^2\varphi_{ij} = \frac{\varepsilon^{2s}}{4}\left(E_{ij}^2 - \frac{P_0}{p_i}\frac{Q_0}{p_j}\right), \tag{3.28}$$

$$E_{ij} = M/(2\sqrt{p_ip_j}), \tag{3.29}$$

which are given from Eq. (3.24), and putting $C_{ii}=C_{jj}=0$ in Eq. (3.23), we have

$$a_{i,j} = A_{i,j}e^{\pm\mu t}. \tag{3.30}$$

Through Eqs. (3.23), (3.24), (3.30), amplitude ratio is furnished by

$$A_i/A_j = \sqrt{Q_0/P_0} = \sqrt{p_j/p_i}\,. \tag{3.31}$$

Now, putting

$$\left.\begin{aligned} P \equiv P_0 - 2p_i\omega_{i,s} &= \begin{cases} \sum_{l=1}^{r-1} \omega_{i,2l}\omega_{i,2(r-l)} + I_s & (\text{for } s=2r), \\ 0 & (\text{for } s=2r+1), \end{cases} \\ Q \equiv Q_0 - 2p_j\omega_{j,s} &= \begin{cases} \sum_{l=1}^{r-1} \omega_{j,2l}\omega_{j,2(r-l)} + J_s & (\text{for } s=2r), \\ 0 & (\text{for } s=2r+1), \end{cases} \end{aligned}\right\} \quad (3.32)$$

$$\frac{G_s}{H_s} \equiv \sum_{l=1}^{\gamma} \varepsilon^{2l}(\omega_{i,2l} \pm \omega_{j,2l}) - \frac{\varepsilon^s}{2}\left(\frac{P}{p_i} \pm \frac{Q}{p_j}\right) \quad (3.33)$$

(the upper and lower signs correspond to $G_s$ and $H_s$ respectively, and $\gamma=r-1$ when $s=2r$ and $\gamma=r$ when $s=2r+1$),

$$\nabla_{ij} \equiv \omega - (p_i + p_j)/s = \omega - p_{ij}/s, \quad (3.34)$$

$$\Delta \equiv (s\nabla_{ij} - G_s)/\varepsilon^s, \quad (3.35)$$

then frequencies $\omega_{i,j}$ are derived from Eqs. (3.7), (3.14), (3.26), (3.31) and $P_0/p_i$, $Q_0/p_j$ are determined from Eq. (3.31) as follows:

$$\omega_i = p_i + (s\nabla_{ij} + H_s)/2, \quad \omega_j = p_j + (s\nabla_{ij} - H_s)/2, \quad (3.36)$$

$$P_0/p_i = Q_0/p_j = \Delta. \quad (3.37)$$

Using Eq. (3.37), Eqs. (3.28), (3.24) are led to

$$\mu^2 = \varepsilon^{2s}(E_{ij}^2 - \Delta^2)/4, \quad (3.38)$$

$$\cos\psi_{ij} = \Delta/|E_{ij}|, \quad (3.39)$$

in which, when $E_{ij}>0$, $+\psi_{ij}$ and $-\psi_{ij}$ are adopted for $+\mu t$ and $-\mu t$ in Eq. (3.30) separately, and vice versa when $E_{ij}<0$. Accordingly solutions of $X_{i,j}$ are written in the following equations:

$$\left.\begin{aligned} X_i &= Ae^{\mu t}\sin(\omega_i t + \varphi_i) + Be^{-\mu t}\sin(\omega_i t + \varphi_i') + \cdots, \\ X_j &= \sqrt{p_i/p_j}\{Ae^{\mu t}\sin(\omega_j t \pm \psi_{ij} - \varphi_i) + Be^{-\mu t}(\omega_j t \mp \psi_{ij} - \varphi_i')\} + \cdots, \end{aligned}\right\} \quad (3.40)$$

in which the upper and lower signs correspond to conditions $E_{ij}>0$ and $E_{ij}<0$ severally, and $A$, $B$, $\varphi_i$, $\varphi_i'$ are all arbitrary constants determined by initial conditions. It is shown in Eq. (3.40) unstable vibrations of summed type of $s$'th order take place, only when negative damping coefficient $\mu$ of $\varepsilon^s$-order becomes real number, *i.e.*, $|E_{ij}|>|\Delta|$.

Critical frequencies $\omega_{c1}$, $\omega_{c2}$ ($\omega_{c1}>\omega_{c2}$) which decide boundaries between unstable and stable regions, are derived by putting $\mu=0$ in Eq. (3.38) as follows:

$$\omega_{c1} = (p_{ij} + G_s + \varepsilon^s|E_{ij}|)/s, \quad \omega_{c2} = (p_{ij} + G - \varepsilon^s|E_{ij}|)/s. \quad (3.41)$$

It can be said from the above equation that unstable region ($\omega_{c1}-\omega_{c2}$) has width of $\varepsilon^s$-order and its center $(\omega_{c1}+\omega_{c2})/2$ locates at the distance of $\varepsilon^2$ order from the resonant point $p_{ij}/s$, regardless of number of order $s$.

On the other hand, in the stable region of $\omega>\omega_{c1}$ or $\omega<\omega_{c2}$, $\mu$ and $\varphi_{ij}$ can be not real numbers, and hence the following $\nu$ is introduced:

$$\nu^2=-\mu^2=\varepsilon^{2s}(\varDelta^2-E_{ij}^2)/4>0. \tag{3.42}$$

Through a similar procedure to vibrations of summed type of a first order without damping in Chapter I, we attain the following equations for solutions $X_{i,j}$ in stable region:

$$\left.\begin{aligned} X_i&=A\sin(\omega_i+\nu)t+B\cos(\omega_i+\nu)t+C\sin(\omega_i-\nu)t+D\cos(\omega_i-\nu)t+\cdots,\\ X_j&=\frac{1}{E_{ij}}\sqrt{\frac{p_i}{p_j}}\Big[\Big(\varDelta+\frac{2\nu}{\varepsilon^s}\Big)\{A\sin(\omega_j-\nu)t-B\cos(\omega_j-\nu)t\}\\ &\qquad+\Big(\varDelta-\frac{2\nu}{\varepsilon^s}\Big)\{C\sin(\omega_j+\nu)t-D\cos(\omega_j+\nu)t\}\Big]+\cdots, \end{aligned}\right\} \tag{3.43}$$

in which $A$, $B$, $C$, $D$ are all arbitrary constants decided by initial conditions. It is seen that Eq. (3.43) represents simply free vibrations in which frequencies $\omega_i+\nu$ and $\omega_j-\nu$ as well as $\omega_i-\nu$ and $\omega_j+\nu$ make a pair.

### *3.3.2. Vibrations of summed type of s'th order with damping*

It is assumed that amplitudes $a_{i,j}$ are expressed in the form

$$a_{i,j}=A_{i,j}e^{\alpha t}. \tag{3.44}$$

Substitution of Eq. (3.44) into Eq. (3.23) yields the following negative damping coefficients:

$$\alpha=\alpha_{1,2}=-\varepsilon^s n_1/2\pm\sqrt{\varepsilon^{2s}n_1^2+4\mu^2}/2, \tag{3.45}$$

in which the upper and lower signs are adopted for $\alpha_1$ and $\alpha_2$ severally, and $n_1$, $n_2$ are indicated in Eq. (2.11). By a similar procedure to Section 2.3, the following equations are obtained upon use of $(P_0/p_i)/(Q_0/p_j)$ and $(P_0/p_i)+(Q_0/p_j)$ which are given from Eqs. (3.23), (3.24), (3.44), (3.45) and Eqs. (3.7), (3.32), (3.34), (3.35) respectively:

$$\begin{cases} P_0/p_i=\varDelta\pm\sqrt{b-a},\quad Q_0/p_j=\varDelta\mp\sqrt{b-a}, & (\text{when } n_2\neq 0) \quad (3.46)\\ P_0/p_i=Q_0/p_j=\varDelta, & (\text{when } n_2=0), \quad (3.46') \end{cases}$$

where

$$a=(E_{ij}^2+n_2^2-\varDelta^2)/2,\qquad b=\sqrt{a^2+n_2^2\varDelta^2}. \tag{3.47}$$

In Eq. (3.46), both $P_0/p_i$ and $Q_0/p_j$ are real numbers because of $a<b$, and when $n_2\varDelta>0$ the upper and lower signs are employed for $\alpha_1$ and $\alpha_2$ separately, and vice versa when $n_2\varDelta<0$.

At first, the case of $n_2\neq 0$ $(C_{ii}\neq C_{jj})$ is discussed. In this case, Eq. (3.45) is reduced to

$$\alpha_1=\varepsilon^s(-n_1+\sqrt{a+b})/2,\quad \alpha_2=\varepsilon^s(-n_1-\sqrt{a+b})/2, \tag{3.48}$$

in which both $\alpha_1$ and $\alpha_2$ are real because of $a+b>0$. Maximum negative damping coefficient $\alpha_{1\max}$ is obtained by putting $\Delta=0$ in Eqs. (3.47), (3.48) as follows:

$$\alpha_{1\max}=\varepsilon^s(-n_1+\sqrt{E_{ij}^2+n_2^2})/2. \tag{3.49}$$

From Eqs. (3.23), (3.48), amplitude ratio is rewritten as

$$A_i/A_j=\sqrt{p_j/p_i}\sqrt{(-n_2\pm\sqrt{a+b})/(n_2\pm\sqrt{a+b})}, \tag{3.50}$$

where the upper and lower signs correspond to $\alpha_1$ and $\alpha_2$ separatly. Denoting phase angle $\varphi_{ij}$ having magnitude between 0 and $\pi$ as $\psi_{ij}$, we have

$$\cos\psi_{ij}=\sqrt{\Delta^2+a-b}/|E_{ij}|, \tag{3.51}$$

in which, when $E_{ij}>0$, $+\psi_{ij}$ and $-\psi_{ij}$ are employed for $\alpha_1$ and $\alpha_2$ severally and vice versa when $E_{ij}<0$. Thus solutions for $X_{i,j}$ through unstable and stable regions are written as follows:

$$\left.\begin{aligned}X_i&=Ae^{\alpha_1 t}\sin(\omega_i t+\varphi_i)+Be^{\alpha_2 t}\sin(\omega_i' t+\varphi_i')+\cdots,\\ X_j&=\sqrt{\frac{p_i}{p_j}}\left\{\sqrt{\frac{n_2+\sqrt{a+b}}{-n_2+\sqrt{a+b}}}Ae^{\alpha_1 t}\sin(\omega_j t\pm\psi_{ij}-\varphi_i)\right.\\ &\left.\quad+\sqrt{\frac{n_2-\sqrt{a+b}}{-n_2-\sqrt{a+b}}}Be^{\alpha_2 t}\sin(\omega_j' t\mp\psi_{ij}-\varphi_i')\right\}+\cdots.\end{aligned}\right\} \tag{3.52}$$

In Eq (3.52), the upper and lower signs are adopted for $E_{ij}>0$ and $E_{ij}<0$ respectively, and frequencies $\omega_{i,j}$, $\omega_{i,j}'$ are decided by the following equations which are attained by Eqs. (3.32), (3.46):

$$\left.\begin{aligned}\begin{matrix}\omega_i\\ \omega_i'\end{matrix}&=p_i+(s\nabla_{ij}+H_s)/2\pm\varepsilon^s\sqrt{b-a}/2,\\ \begin{matrix}\omega_j\\ \omega_j'\end{matrix}&=p_j+(s\nabla_{ij}-H_s)/2\mp\varepsilon^s\sqrt{b-a}/2,\end{aligned}\right\} \tag{3.53}$$

where the upper and lower signs correspond to $\omega_{i,j}$ and $\omega_{i,j}'$ separately when $n_2\Delta>0$ and vice versa $n_2\Delta<0$.

Putting $\alpha_1=0$ results in boundaries between stable and unstable regions as follows:

$$\begin{matrix}\omega_{c1}\\ \omega_{c2}\end{matrix}=\frac{1}{s}\left\{p_{ij}+G_s\pm\frac{\varepsilon_s}{2}\frac{(1+\lambda)}{\sqrt{\lambda}}\sqrt{E_{ij}^2-\lambda C_{ii}^2}\right\}, \tag{3.54}$$

in which the upper and lower signs are used for $\omega_{c1}$ and $\omega_{c2}$ respectively, and $\lambda$ is damping ratio written by

$$\lambda=C_{jj}/C_{ii}=c_{jj}/c_{ii}. \tag{3.55}$$

It is seen from the above discussion that, when frequency $\omega$ of parametric excitation comes near the resonant frequency $p_{ij}/s$ and takes a value between $\omega_{c1}$ and $\omega_{c2}$ and further the condition $E_{ij}^2\geqq C_{ii}C_{jj}$ holds, negative damping coefficient $\alpha_1$ of $\varepsilon^s$-order takes positive value and unstable vibrations of summed type of $s$'th order occur.

For the case of $n_2=0$ $(C_{ii}=C_{jj})$, we get the following equations from Eq. (3.46′):

$$\alpha_{1,2}=-\varepsilon^s n_1/2\pm\sqrt{\mu^2}=\varepsilon^s(-n_1\pm\sqrt{E_{ij}^2-\Delta^2})/2, \tag{3.48′}$$

$$A_i/A_j=\sqrt{p_j/p_i}, \tag{3.50′}$$

$$\cos\psi_{ij}=\Delta/|E_{ij}|. \tag{3.51′}$$

It is obvious from Eqs. (3.48′), (3.51′) that both $\alpha_{1,2}$ and $\psi_{ij}$ are real when $E_{ij}^2\geqq\Delta^2$, and hence solutions $X_{i,j}$ in this case are of the form which is obtained by putting $n_2=0$ in Eq. (3.52), or by adding $e^{-(\varepsilon^s n_1/2)t}$ to solution of Eq. (3.40) of unstable vibrations of summed type without damping. When $\alpha_1$ takes positive value, *i.e.*, $\mu$ is larger than $\varepsilon^s n_1/2$, unstable vibrations can take place, while damped vibrations with frequencies $\omega_{i,j}$ occur when $\alpha_1<0$, *i.e.*, $\mu<\varepsilon^s n_1/2$. On the other hand, $\mu$, $\alpha_{1,2}$ and $\psi_{ij}$ are not real numbers, when $\Delta^2$ becomes larger than $E_{ij}^2$. Accordingly solutions $X_{i,j}$ in the latter case are found by addition of $e^{-(\varepsilon^s n_1/2)t}$ to solutions of Eq. (3.43) of stable vibrations of summed type without damping and take the form of damped vibrations with frequencies $\omega_{i,j}\pm\nu$.

*3.4. Vibrations of differential type of s'th order*

In this case, we have the following equations in place of Eqs. (3.23), (3.24):

$$\left.\begin{aligned}\dot a_i&=\varepsilon^s\left(-\frac{M'a_j}{4p_i}\sin\varphi_{ij}-\frac{C_{ii}}{2}a_i\right),\\ \dot a_j&=\varepsilon^s\left(-\frac{M'a_i}{4p_j}\sin\varphi_{ij}-\frac{C_{jj}}{2}a_j\right),\end{aligned}\right\} \tag{3.23 a}$$

$$\frac{a_j}{a_i}\cos\varphi_{ij}=-\frac{2P_0}{M'},\quad \frac{a_i}{a_j}\cos\varphi_{ij}=-\frac{2Q_0}{M'}, \tag{3.24 a}$$

in which

$$M'\sum_{f=i,j}\kappa_{if}\cdot F(f,s-1,j,s-1)=\sum_{f=i,j}\kappa_{jf}\cdot F(f,s-1,i,-s+1). \tag{3.25 a}$$

*3.4.1. Vibrations of differential type of s'th order without damping*

Through Eqs. (3.23 a), (3.24 a) in which we put $C_{ii}=C_{jj}=0$, the following equations corresponding to Eqs. (3.29), (3.30), (3.31), (3.34), (3.35), (3.36), (3.37), (3.38), (3.39) respectively are obtained by a similar procedure to Section 3.3.1:

$$E'_{ij}=M'/(2\sqrt{p_ip_j}), \tag{3.29 a}$$

$$a_{i,j}=A_{i,j}e^{\pm i\mu t}, \tag{3.30 a}$$

$$A_i/A_j=\sqrt{Q_0/P_0}=i\sqrt{p_j/p_i}, \tag{3.31 a}$$

$$\nabla_{ij}=\omega-(p_i-p_j)/s=\omega-p_{ij}/s, \tag{3.34 a}$$

$$\Delta'=(s\nabla_{ij}-H_s)/\varepsilon^s, \tag{3.35 a}$$

$$\omega_i=p_i+(s\nabla_{ij}+G_s)/2,\quad \omega_j=p_j+(-s\nabla_{ij}+G_s)/2, \tag{3.36 a}$$

$$P_0/p_i=-Q_0/p_j=\Delta', \tag{3.37 a}$$

$$\mu^2=\varepsilon^{2s}(E'^2_{ij}+\Delta'^2)/4, \tag{3.38 a}$$

$$\cos\varphi_{ij}=-i\Delta'/E'_{ij}. \tag{3.39 a}$$

Referring the fact that $\mu$ is always a real number as shown in Eq. (3.38 a), solutions of differential type are written as follows:

$$\left.\begin{aligned} X_i &= A\sin(\omega_i+\mu)t + B\cos(\omega_i+\mu)t + C\sin(\omega_i-\mu)t + D\cos(\omega_i-\mu)t + \cdots, \\ X_j &= \frac{1}{E'_{ij}}\sqrt{\frac{p_i}{p_j}}\Big[\Big(-\Delta' - \frac{2\mu}{\varepsilon^s}\Big)\{A\sin(\omega_j+\mu)t + B\cos(\omega_j+\mu)t\} \\ &\qquad + \Big(-\Delta' + \frac{2\mu}{\varepsilon^s}\Big)\{C\sin(\omega_j-\mu)t + D\cos(\omega_j-\mu)t\}\Big] + \cdots. \end{aligned}\right\} \tag{3.43 a}$$

The above equation represents simply free vibrations, frequencies of which $\omega_i+\mu$ and $\omega_j+\mu$ as well as $\omega_i-\mu$ and $\omega_j-\mu$ make a pair.

*3.4.2. Vibrations of differential type of s'th order with damping*

By a similar procedure shown in Section 3.3.2, we get the following equations for $n_2 \neq 0$ $(C_{ii} \neq C_{jj})$:

$$P_0/p_i = \Delta' \pm \sqrt{b'-a'}, \quad Q_0/p_j = -\Delta' \pm \sqrt{b'-a'}, \tag{3.46 a}$$

$$a' = (n_2^2 - E'^2_{ij} - \Delta'^2)/2, \quad b' = \sqrt{a'^2 + n_2^2\Delta'^2}, \tag{3.47 a}$$

$$\alpha_1 = \varepsilon^s(-n_1 + \sqrt{a'+b'})/2, \quad \alpha_2 = \varepsilon^s(-n_1 - \sqrt{a'+b'})/2, \tag{3.48 a}$$

$$A_i/A_j = \sqrt{p_j/p_i}\sqrt{(n_2 \mp \sqrt{a'+b'})/(n_2 \pm \sqrt{a'+b'})}, \tag{3.50 a}$$

$$\cos\psi_{ij} = \sqrt{b'-a'-\Delta'^2}/|E'_{ij}|, \quad (0 \leqq \psi_{ij} \leqq \pi) \tag{3.51 a}$$

$$\left.\begin{aligned} X_i &= Ae^{\alpha_1 t}\sin(\omega_i t + \varphi_i) + Be^{\alpha_2 t}\sin(\omega'_i t + \varphi'_i) + \cdots, \\ X_j &= \sqrt{\frac{p_i}{p_j}}\Big\{\sqrt{\frac{n_2+\sqrt{a'+b'}}{n_2-\sqrt{a'+b'}}}Ae^{\alpha_1 t}\sin(\omega_j t \mp \psi_{ij} + \varphi_i) \\ &\qquad + \sqrt{\frac{n_2-\sqrt{a'+b'}}{n_2+\sqrt{a'+b'}}}Be^{\alpha_2 t}\sin(\omega'_j t \mp \psi_{ij} + \varphi'_i)\Big\} + \cdots, \end{aligned}\right\} \tag{3.52 a}$$

$$\left.\begin{aligned} \begin{matrix}\omega_i \\ \omega'_i\end{matrix} &= p_i + (s\nabla_{ij} + G_s)/2 \pm \varepsilon^s\sqrt{b'-a'}/2, \\ \begin{matrix}\omega_j \\ \omega'_j\end{matrix} &= p_j + (-s\nabla_{ij} + G_s)/2 \pm \varepsilon^s\sqrt{b'-a'}/2, \end{aligned}\right\} \tag{3.53 a}$$

which correspond to Eqs. (3.46), (3.47), (3.48), (3.50), (3.51), (3.52), (3.53) respectively. In Eq. (3.52 a), the upper and lower signs are employed when $n_2E'_{ij} > 0$ and $n_2E'_{ij} < 0$ severally, while in Eq. (3.53 a) they are adopted for $\omega_{i,j}$ and $\omega'_{i,j}$ separately when $n_2\Delta' > 0$, and vice versa when $n_2\Delta' < 0$. In Eqs. (3.46 a) ~ (3.51 a), $P_0/p_i$, $Q_0/p_j$, $\alpha_{1,2}$, $A_i/A_j$ and $\psi_{ij}$ are all real numbers because of $b' \pm a' > 0$. Furthermore, since Eqs. (3.47 a), (3.48 a) results in that both $\alpha_1$ and $\alpha_2$ are always negative, Eq. (3.52 a) represents simply damped vibrations with frequencies $\omega_{i,j}$, $\omega'_{i,j}$.

For a special case of $n_2 = 0$ $(C_{ii} = C_{jj})$ we attain

$$P_0/p_i = -Q_0/p_j = \Delta', \tag{3.46' a}$$

$$\alpha_{1,2} = -\varepsilon^s n_1/2 \pm \sqrt{-\mu^2} = \varepsilon^s(-n_1 \pm i\sqrt{E'^2_{ij} + \Delta'^2})/2, \tag{3.48' a}$$

$$A_i/A_j = \sqrt{Q_0/P_0} = i\sqrt{p_j/p_i}, \tag{3.50' a}$$

$$\cos\varphi_{ij} = -i\Delta'/E'_{ij}. \qquad (3.51'a)$$

Accordingly solutions for this case are given by addition of $e^{-(\varepsilon^s n_1/2)t}$ to solutions Eq. (3.43 a) of differential type without damping and they are always damped vibrations with frequencies $\omega_{i,j}+\mu$.

Consequently, as seen from Sections 3.4.1 and 3.4.2, unstable vibration of differential type cannot occur.

Incidentally, it is found that results derived by $s=1$ in Sections 3.2~3.4 coincide with those of vibrations of summed and differential types at a first approximation in Chapters I, II.

Furthermore, it can be verified that any other vibrations of summed and differential types but those satisfying Eq. (3.2), for example, vibrations of $[p_i \pm p_j \pm p_k)/s]$ or $[(2p_i \pm p_j)/s]$ ($s=1, 2, 3, \cdots$), etc., could not take place, because there is no resonant term $\cos\theta_f$ in parametric excitation of Eqs. (3.15), (3.16).

*3.5. Verification of analytical results through analog computer*

Since in actual experimental apparatus it is difficult to treat with the smaller quantity than $\varepsilon^2$-order and to furnish viscous damping forces exactly proportional to velocity, analog computer is used to verify obtained analytical results. Comparison of analytical results with those through analog computer is performed in a system with two degree-of-freedom, and only vibrations of summed and differential types of second order are examined by means of analog computer, because they can be most easily investigated among all higher order vibrations. If only number of degrees of freedom $h$, number of orders of this vibrations $s$, suffixs $i$, $j$ are replaced by 2, 2, 1 and 2 respectively in Sections 3.1~3.4, all results obtained in the preceding sections can be applied for the system treated in this section. In order to connect with results of vibrations of summed and differential types of a first approximation in Chapters I, II, we adopt the following dimensions in analog computer:

For vibrations of summed type of second order

$p_1=13.863$ c/s, $p_2=9.850$ c/s, $p_{12}=23.713$ c/s,

$\varepsilon_{11}=3.06\times10^2 e$ (rad/s)$^2$, $\varepsilon_{12}=2.31\times10^2 e$ (rad/s)$^2$, $\varepsilon_{22}=3.58\times10^2 e$ (rad/s)$^2$,

For vibrations of differential type of second order

$p_1=25.000$ c/s, $p_2=10.000$ c/s, $p_{12}=15.000$ c/s,

$\varepsilon_{11}=1.31\times10^2 e$ (rad/s)$^2$, $\varepsilon_{12}=0.92\times10^2 e$ (rad/s)$^2$, $\varepsilon_{22}=1.54\times10^2 e$ (rad/s)$^2$,

in which $e$ is eccentricity explained in Section 2.5. In the following figures except for Fig. 3.1, analytical results are graphically shown by broken and chain or full and dotted line curves which correspond to stable and unstable vibrations without or with damping severally, and results obtained by analog computer are indicated by symbols ●, ⊖, etc. and ○, ⊖, etc., which are employed for unstable and stable vibrations separately. A vertical chain line illustrates location of the resonant point $\omega=p_{12}/2$.

Boundaries between stable and unstable regions of vibrations of summed type of the first, second and third orders without damping are shown in Fig. 3.1, where the abscissa is detuning $\nabla_{12}$ from the resonant frequencies $p_{12}/s$ $(s=1, 2, 3)$. It is seen that width of unstable regions are $\varepsilon^s$-order and centers of unstable regions of higher order deviate always at distance of $\varepsilon^2$-order from the resonant point $p_{12}/s$.

Unstable regions of vibrations of summed type of second order with damping are indicated in Figs. 3.2, 3.3; in Fig. 3.2 damping ratio $\lambda = c_{22}/c_{11}$ is constant, and in Fig. 3.3 product of damping coefficients $c_{11}c_{22} = \lambda c_{11}^2$ is fixed. The smaller damping force results in the wider unstable region when $\lambda = 1$ $(n_2 = 0)$, while when $\lambda \fallingdotseq 1$ $(n_2 \fallingdotseq 0)$, unstable region of vibrations with damping can be wider than that without damping in a range of larger magnitude of parametric excitation $\varepsilon_{12}$, and ratio of width of unstable region to magnitude of parametric excitation $\varepsilon_{12}$ becomes larger as $\lambda$ goes off from unity; this fact is also held in case of vibrations of the first order in Chapter II.

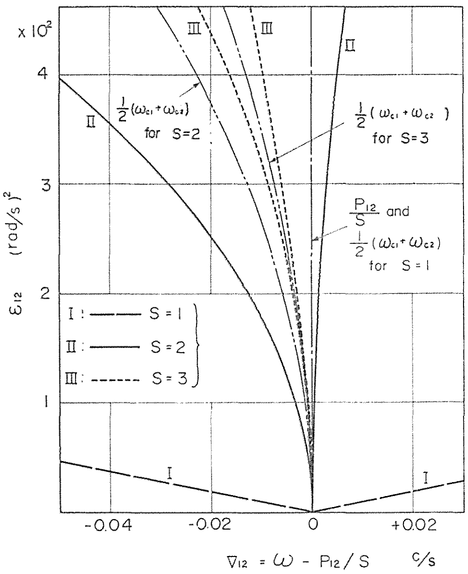



FIG. 3.1. Effect of order $s$ on the unstable regions of vibrations of summed type without damping.

Figs. 3.4, 3.5 illustrate negative damping coefficients $\mu$ and $\alpha_1$ of vibrations

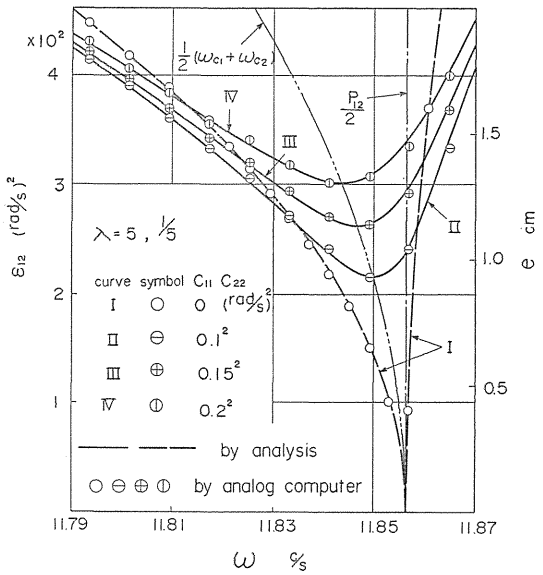



FIG. 3.2. Boundaries of unstable region for vibrations of summed type of second order when $\lambda \fallingdotseq 1$.

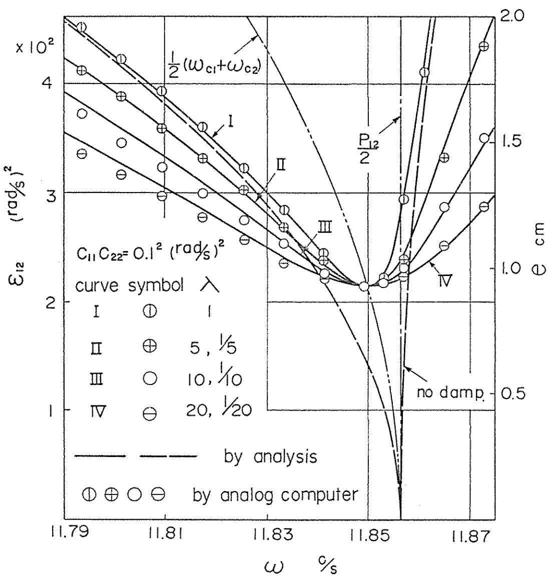



FIG. 3.3. Boundaries of unstable region for vibrations of summed type of second order when $c_{11}c_{22}=$ constant.

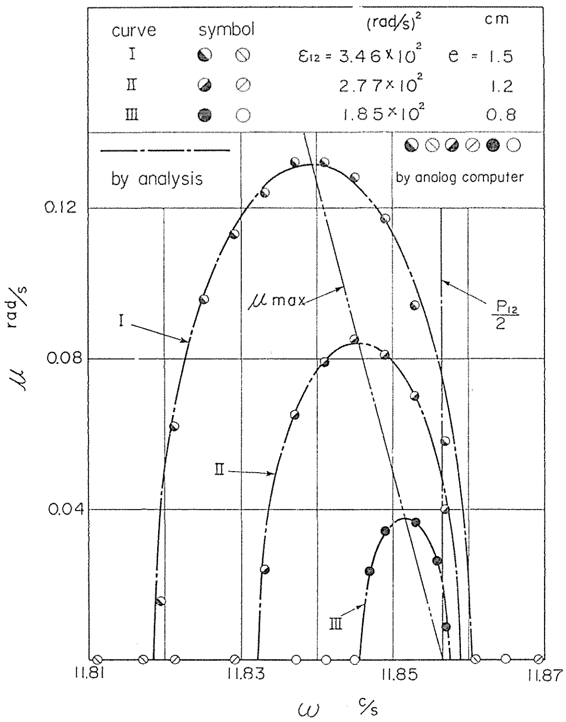



FIG. 3.4. Negative damping coefficient $\mu$ for vibrations of summed type of second order without damping.

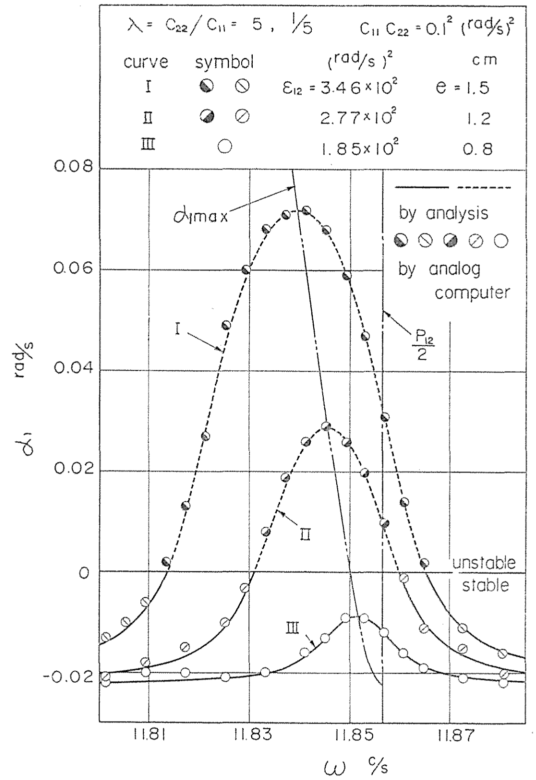



FIG. 3.5. Negative damping coefficient $\alpha_1$ for vibrations of summed type of second order with damping.

without and with damping severally, and frequency $\omega$ at which $\mu$ and $\alpha_1$ take the maximum values $\mu_{max}$, $\alpha_{1max}$ goes off from the resonant point $p_{12}/2$ as $\varepsilon_{12}$ increases. Negative damping coefficient of unstable and stable vibrations without damping is represented by Eqs. (3.38), (3.42) separately, while $\alpha_1$ of vibrations with damping is expressed by a continuous curve through both stable and unstable regions as shown in Eq. (3.48).

Fig. 3.6 show frequencies $\omega_{1,2}$ (curves I, II) and $\omega_{1,2}\pm\nu$ (curves I′, II′) of unstable and stable vibrations without damping, which are obtained by Eqs. (3.36), (3.42), and frequencies $\omega_{1,2}$ (curve I), $\omega'_{1,2}$ (curve II) of vibrations with damping are expressed in a common equation (3.53) through both stable and unstable regions, as shown in Fig. 3.7. Further, it is seen that a sum of two frequincies of vibrations is equal to twice frequency $\omega$ in Figs. 3.6, 3.7, and $\omega_{1,2}$ and $\omega'_{1,2}$ are replaced when $\lambda$ and $1/\lambda$ are exchanged in Fig. 3.7. Double chain lines in Fig. 3.7 indicate boundaries of unstable region.

Amplitude ratios of vibrations without and with damping are illustrated in Fig. 3.8, where curves I, II given by Eq. (3.31) show amplitude ratios of unstable without damping and curves I′, II′ through Eq. (3.43) are amplitude ratios between two stable vibrations without damping having frequencies $\omega_1+\nu$ and $\omega_2-\nu$, and curves I″, II″ from Eq. (3.43) express those of vibrations with frequencies

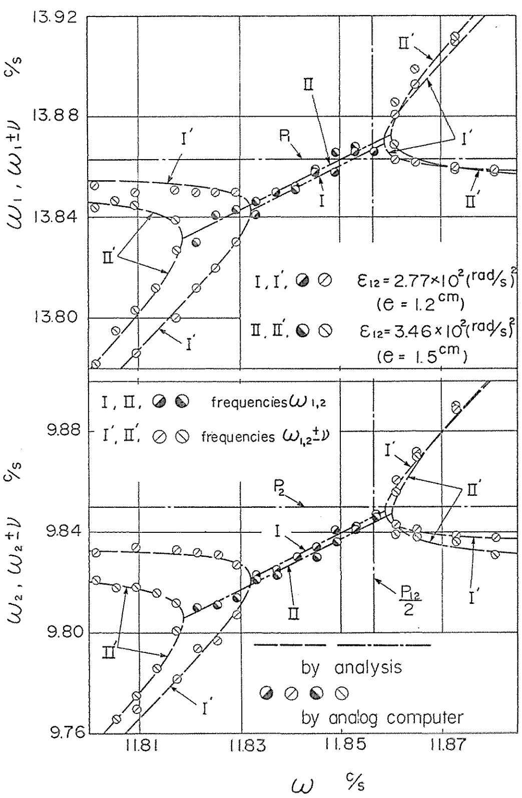



FIG. 3.6. Frequencies for vibrations of summed type of second order without damping.

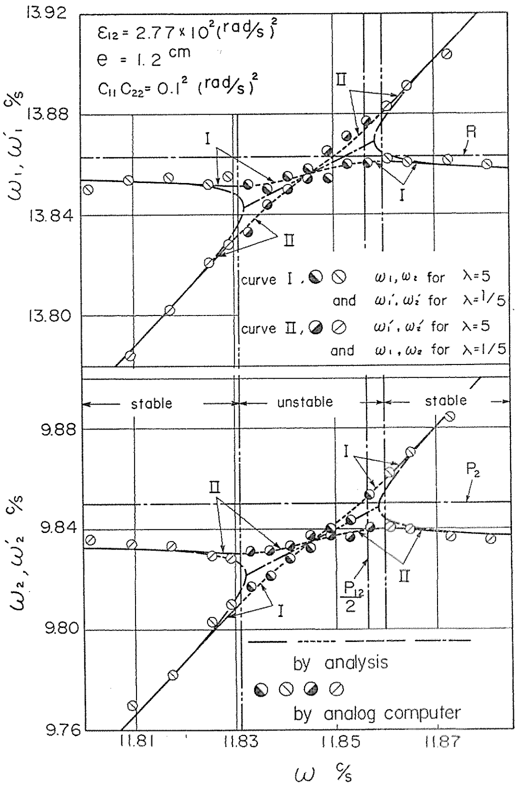



FIG. 3.7. Frequencies for vibrations of summed type of second order with damping.

$\omega_1-\nu$ and $\omega_2+\nu$, while curves III~VI with damping obtained by Eq. (3.50) are continuous through stable and unstable regions, boundaries of which are illustrated by two horizontal chain lines, *i.e.*, lines of amplitude ratios $\sqrt{p_2/p_1}\sqrt{\lambda}$. Amplitude ratios for $\alpha_1$ are $\alpha_2$ as well as frequencies and are replaced, if $\lambda$ and $1/\lambda$ are exchanged.

Negative damping coefficient $\alpha_1$ and frequencies $\omega_{1,2}$, $\omega_{1,2}'$ of vibrations of differential type of second order are indicated in Figs. 3.9, 3.10. It is seen from Fig. 3.9 that $\alpha_1$ is always negative and hence any unstable vibration cannot occur in differential type. Relation $\omega_1-\omega_2=\omega_1'-\omega_2'=2\,\omega$ holds in Fig. 3.10. Vertical double chain line in Figs. 3.9, 3.10 represent frequency $\omega$ satisfying $\Delta'=0$, at which negative damping coefficient takes its minimum value and discontinuity in frequency takes place.

Finally, vibratory waves of unstable vibrations of summed type of second order with damping obtained by analog computer are illustrated in Fig. 3.11.

*3.6. Conclusions*

Obtained results may be summarized as follows:

(1) In vibratory system with multiple degree-of-freedom and under parametric

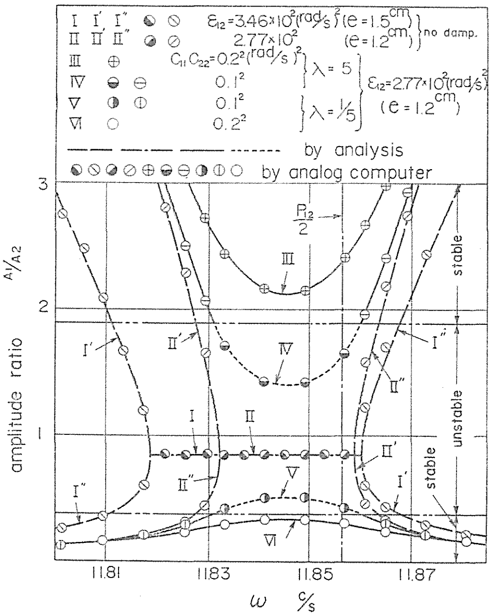



FIG. 3.8. Amplitude ratio for vibrations of summed type of second order.

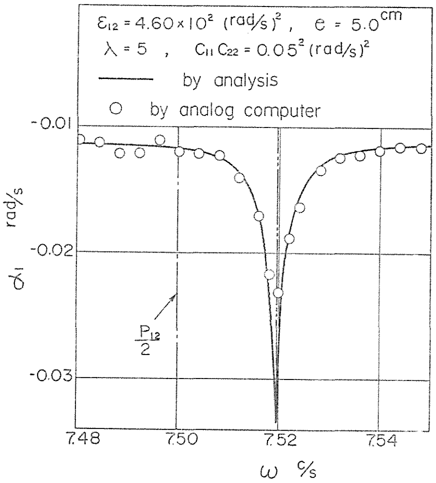



FIG. 3.9. Negative damping coefficient $\alpha_1$ for vibrations of differential type of second order.

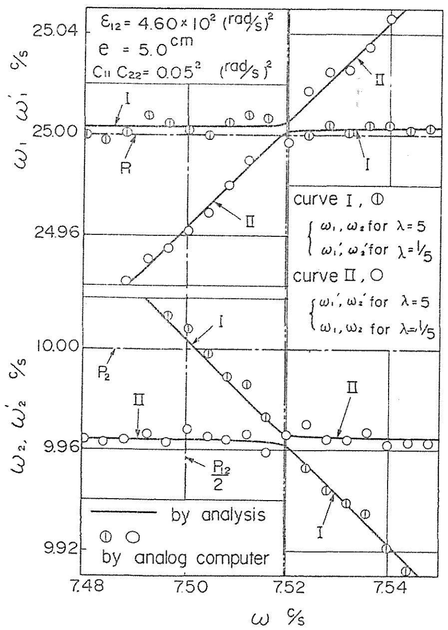



FIG. 3.10. Frequencies for vibrations of summed type of second order with damping.

excitation of frequency $\omega$, vibrations of summed and differential types of higher order with frequencies $\omega_i\,(\doteqdot p)$, $\omega_j\,(\doteqdot p_j)$ satisfying the relation $\omega_i \pm \omega_j = s\omega$ can take place in the neighborhood of the resonant point $(p_i \pm p_j)/s = p_{ij}/s$, and cannot appear in the neighborhood of $(p_i \pm p_j \pm p_k)/s$, $(2p+p_j)/s$, etc..

(2) Vibrations of summed and differential types of $s$'th order can appear at the $s$'th approximation, and there are unstable vibrations of higher order only in summed type and not in differential type.

(3) If magnitude of damping is larger than $\varepsilon^s$-order, any unstable vibration of summed type of $s$'th order cannot take place.

(4) For unstable vibrations of summed type of $s$'th order, width of unstable region is $\varepsilon^s$-order and its center locates at a $\varepsilon^2$-order distance from the resonant point $p_{ij}/s$.

(5) When magnitude of damping

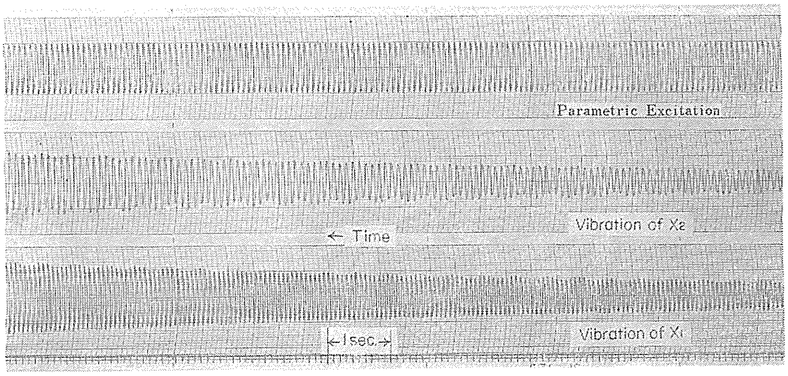



FIG. 3.11. Vibratory waves of unstable vibrations of summed type obtained by analog computer.
($\varepsilon_{12}=3.46\times10^2$ rad$^2$/s$^2$, $e=1.5$ cm, $\lambda=5$, $c_{11}c_{22}=0.01$ rad$^2$/s$^2$, $\omega=11.841$ c/s, $\omega:\omega_1:\omega_2=53:62:44$, $\alpha_1=0.072$ rad/s).

ratio $\lambda$ is not unity, *i.e.*, $\lambda\fallingdotseq1$, unstable region of vibrations with damping can always be wider than that without damping in a range of larger value of $\varepsilon_{12}$ than a certain value.

(6) For vibrations of summed type of higher order with damping properties of both unstable and stable vibrations, that is, form of solutions, frequencies, negative damping coefficients, amplitude ratios, phase angles can be expressed by common forms except for a special case $n_2=0$, while they are given separately for both regions in vibrations without damping.

(7) The center of vibrations of differential type of $s$'th order, *i.e.*, frequency $\omega$ of parametric excitation at which $\Delta'$ vanishes, is also $\varepsilon^2$-order distant from the resonant point $p_{ij}/s$.

(8) Putting $s=1$ in theoretical results for vibrations of summed and differential types of higher order, the results of Chapters I, II are derived.

(9) Obtained theoretical results show a good agreement with results of analog computer.

## Chapter IV. Vibrations of "summed and differential types" under parametric excitation in non-linear vibratory systems[22]

### *4.1. Introduction*

In the preceding chapter, vibrations of "summed and differential types" in linear vibratory system of multiple degree-of-freedom under parametric excitation have been studied in detail. In the present chapter, a first approximate solution of these kinds of vibrations in non-linear vibratory systems both without and with damping are treated through a similar procedure with that in Chapters I, II, and properties of these vibrations, *i.e.*, solutions, frequencies, phase angles, amplitude ratios between two vibrations, negative damping coefficients and unstable regions (amplitude—frequency diagrams) are discussed in detail. It is seen that all of these properties are analytically represented as functions of amplitude, and that unstable vibrations can occur only in summed type and not in differential type, and these unstable vibrations have so called "limit cycle", that is, vibrations appearing in unstable region grow up at the start and are settled in steady state, in which they have constant amplitudes. On the other hand, they increase without limitation in linear systems. Further, it is concluded that effects of damping on these vibrations are similar to those of linear systems, that is, width and shape of unstable region are modified by magnitude of damping ratio and existence of damping does not always result in decrease of width of unstable region. Theoretical results obtained by a first approximate analysis are ascertained by analog computer.

*4.2. Equation of motion and preliminary analysis*

Vibratory system of $h$ degree-of-freedom with non-linear spring characteristics and under parametric excitaion is governed by the following equation of motion referring to normal coordinate $X_l$ $(l=1, 2, \cdots, h)$:

$$\ddot{X}_l+p_l^2 X_l=\sum_{m=1}^{h}\{\varepsilon_{lm}X_m\cos\omega t-C_{lm}\dot{X}_m\}-\sum_{m,\,m'=1}^{h}{}_l\beta_{mm'}X_m^2X_{m'},\tag{4.1}$$

in which magnitude of parametric excitation $\varepsilon_{lm}$, damping coefficient $C_{lm}$ and coefficient of non-linear term ${}_l\beta_{mm'}$ are assumed to be small. Now we treat with vibrations of summed and differential types with frequencies $\omega_i$ and $\omega_j$ satisfying Eqs. (1.1), (1.2), in like manner of Chapters I, II. The $i$'th and $j$'th equations of Eq. (4.1) can be rewritten as

$$\begin{aligned}\ddot{X}_f+p_f^2X_f=&\sum_{m=i,\,j}\{\varepsilon_{fm}X_m\cos\omega t-C_{fm}\dot{X}_m\}-\beta(X_i+X_j)^3\\&+\sum_{m\neq i,\,j}\{\varepsilon_{fm}X_m\cos\omega t-C_{fm}\dot{X}_m\}-\sum_{m,\,m'\neq i,\,j}{}_f\beta_{mm'}X_m^2X_{m'},\quad(f=i,\,j)\end{aligned}\tag{4.2}$$

where the following relation is introduced without failing generality:

$$\beta={}_f\beta_{ii}=3\,{}_f\beta_{ij}=3\,{}_f\beta_{ji}={}_f\beta_{jj}.\tag{4.3}$$

Through a similar procedure to Chapters I, II, Eq. (4.2) reduces to the following equations of motion:

$$\dot{a}_i=\pm\frac{\varepsilon_{ij}a_j}{4\,\omega_i}\sin\varphi_{ij}-\frac{C_{ii}}{2}a_i,\quad \dot{a}_j=\frac{\varepsilon_{ij}a_i}{4\,\omega_j}\sin\varphi_{ij}-\frac{C_{jj}}{2}a_j,\tag{4.4}$$

$$\left.\begin{aligned}2\,\Delta_i&=\pm\,\varepsilon_{ij}\frac{a_j}{a_i}\cos\varphi_{ij}+\frac{3}{2}\beta(a_i^2+2\,a_j^2),\\2\,\Delta_j&=\pm\,\varepsilon_{ij}\frac{a_i}{a_j}\cos\varphi_{ij}+\frac{3}{2}\beta(a_j^2+2\,a_i^2),\end{aligned}\right\}\tag{4.5}$$

in which the upper and lower signs correspond to vibrations of summed and differential types respectively.

*4.3. Vibrations of summed type*

Assumnig $\alpha$ to be negative damping coefficient, we have

$$\alpha=\dot{a}_{i,j}/a_{i,j}.\tag{4.6}$$

*4.3.1. Vibrations of summed type without damping*

Substitution of Eq. (4.6) into Eq. (4.4) in which the upper sign of $\pm$ is adopted and both $C_{ii}$ and $C_{jj}$ are neglected, leads to the following amplitude ratio:

$$a_i/a_j = \sqrt{\omega_j/\omega_i}\,. \tag{4.7}$$

By using Eqs. (2.5), (2.6), (4.7), (4.6) and eliminating phase angle $\varphi_{ij}$, then detunings $\delta_{i,j}$ are determined as follows:

$$\left.\begin{aligned} \delta_i &= \nabla_{ij} + \frac{3}{8}\frac{\beta}{p_j}\Big(\frac{p_j}{p_i} - \frac{p_i}{p_j}\Big)a_i^2, \\ \delta_j &= \nabla_{ij} - \frac{3}{8}\frac{\beta}{p_j}\Big(\frac{p_j}{p_i} - \frac{p_i}{p_j}\Big)a_j^2. \end{aligned}\right\} \tag{4.8}$$

Inserting Eq. (4.7) into Eqs. (4.4), (4.5) in which phase angle are eliminated, negative damping coefficient $\alpha$ is derived as follows:

$$\alpha^2 = \frac{1}{4}\Big[E_{ij}^2 - \Big\{\nabla_{ij} - \frac{3}{8}\frac{\beta}{p_j}\Big(\frac{p_j}{p_i} + \frac{p_i}{p_j} + 4\Big)a_i^2\Big\}^2\Big], \tag{4.9}$$

where

$$E_{ij} = \varepsilon_{ij}/\sqrt{4p_ip_j}\,. \tag{4.10}$$

Further, phase angle $\varphi_{ij}$ is obtained through Eqs. (4.5), (2.5) and (4.8) as follows:

$$\cos\varphi_{ij} = \frac{1}{E_{ij}}\Big\{\nabla_{ij} - \frac{3}{8}\frac{\beta}{p_j}\Big(\frac{p_j}{p_i} + \frac{p_i}{p_j} + 4\Big)a_i^2\Big\}. \tag{4.11}$$

(i) Vibrations within unstable region

When frequency $\omega$ of parametric excitation comes near to the resonant frequency $p_{ij}$, and detuning $\nabla_{ij}$ becomes so small that the relation

$$E_{ij}^2 \geqq \Big\{\nabla_{ij} - \frac{3}{8}\frac{\beta}{p_j}\Big(\frac{p_i}{p_j} + \frac{p_j}{p_i} + 4\Big)a_i^2\Big\}^2 \tag{4.12}$$

holds, negative damping coefficient $\alpha$ and phase angle $\varphi_{ij}$ become real numbers as seen from Eqs. (4.9), (4.11), and hence amplitudes $a_{i,j}$ increase exponentially. It is found in Eq. (4.9) that $\alpha$ decreases with increase of $a_{i,j}$ until $\alpha$ vanishes, that is, $a_i$ and $a_j$ are settled in steady state amplitudes $a_{i0}$ and $a_{j0}$ respectively. By adoption of equal sign in Eq. (4.12) or putting $\alpha=0$ in Eq. (4.9), the critical detunings $\nabla_{c1}$, $\nabla_{c2}$ ($\nabla_{c1}>\nabla_{c2}$) are derived as follows:

$$\begin{matrix}\nabla_{c1} \\ \nabla_{c2}\end{matrix} = \pm|E_{ij}| + \frac{3}{8}\frac{\beta}{p_j}\Big(\frac{p_i}{p_j} + \frac{p_j}{p_i} + 4\Big)a_{i0}^2. \tag{4.13}$$

Detunings $\delta_{i,j}$ and phase angle $\varphi_0$ of steady state vibrations on these boundaries are determined by the following equations which are attained by Eqs. (4.8), (4.11), (4.13):

$$
\left.\begin{aligned}
\delta_i &= \pm |E_{ij}| + \frac{3}{4}\frac{\beta}{p_j}\left(\frac{p_j}{p_i} + 2\right)a_{i0}^2, \\
\delta_j &= \pm |E_{ij}| + \frac{3}{4}\frac{\beta}{p_j}\left(\frac{p_i}{p_j} + 2\right)a_{i0}^2,
\end{aligned}\right\} \tag{4.14}
$$

$$
\cos\varphi_0 = \pm 1, \tag{4.15}
$$

where the upper and lower signs in Eq. (4.14) correspond to $\nabla_{c1}$ and $\nabla_{c2}$ respectively and in Eq. (4.15) they are used for $\nabla_{c1}$ and $\nabla_{c2}$ when $E_{ij}>0$ and vice versa when $E_{ij}<0$ separately. Consequently, steady state solution of $X_{i,j}$ are given as follows:

$$
\left.\begin{aligned}
X_i &= a_{i0}\sin(\omega_i t + \varphi_i), \\
X_j &= \pm\sqrt{\omega_i/\omega_j}\, a_{i0}\sin(\omega_j t - \varphi_i),
\end{aligned}\right\} \tag{4.16}
$$

in which the upper and lower signs are adopted for $\nabla_{c1}$ and $\nabla_{c2}$ when $E_{ij}>0$ and vice versa when $E_{ij}<0$.

We discuss here stability problem of steady state solutions Eq. (4.16) by mean of Routh's method. Let

$$
a_i = a_{i0} + \xi,\ \ a_j = a_{j0} + \eta,\ \ \varphi_i = \varphi_{i0} + \zeta_i,\ \ \varphi_j = \varphi_{j0} + \zeta_j \tag{4.17}
$$

be solutions which differ slightly from steady state solutions $a_{i0}$, $a_{j0}$, $\varphi_{i0}$ and $\varphi_{j0}$ ($\varphi_{i0}+\varphi_{j0}=\varphi_0$). Substituting Eq. (4.17) into Eq. (4.2) without damping and neglecting all but linear terms in $\xi$, $\eta$, $\zeta_i$, $\zeta_j$, we obtain

$$
\left.\begin{aligned}
\dot{\xi} &= (\varepsilon_{ij}/4\,\omega_i)\{a_{j0}(\zeta_i+\zeta_j)\cos\varphi_0 + \eta\sin\varphi_0\}, \\
\dot{\eta} &= (\varepsilon_{ij}/4\,\omega_j)\{a_{i0}(\zeta_i+\zeta_j)\cos\varphi_0 + \xi\sin\varphi_0\}, \\
\dot{\zeta}_i &= \frac{\varepsilon_{ij}}{4\,\omega_i}\frac{a_{j0}}{a_{i0}}\left(\frac{\eta}{a_{j0}} - \frac{\xi}{a_{i0}}\right)\cos\varphi_0 - \frac{\varepsilon_{ij}}{4\,\omega_i}\frac{a_{j0}}{a_{i0}}(\zeta_i+\zeta_j)\sin\varphi_0 \\
&\quad + \frac{3}{4}\frac{\beta}{\omega_i}(\xi a_{i0} + 2\,\eta a_{j0}), \\
\dot{\zeta}_j &= \frac{\varepsilon_{ij}}{4\,\omega_j}\frac{a_{i0}}{a_{j0}}\left(\frac{\xi}{a_{i0}} - \frac{\eta}{a_{j0}}\right)\cos\varphi_0 - \frac{\varepsilon_{ij}}{4\,\omega_j}\frac{a_{i0}}{a_{j0}}(\zeta_i+\zeta_j)\sin\varphi_0 \\
&\quad + \frac{3}{4}\frac{\beta}{\omega_j}(\eta a_{j0} + 2\,\xi a_{i0}).
\end{aligned}\right\} \tag{4.18}
$$

Inserting the assumed solutions

$$
\xi = \xi_0 e^{zt},\ \ \eta = \eta_0 e^{zt},\ \ \zeta_i = \zeta_{i0} e^{zt},\ \ \zeta_j = \zeta_{j0} e^{zt} \tag{4.19}
$$

into Eq. (4.18), the following characteristic equation is attained:

$$
E_4 z^4 + E_3 z^3 + E_2 z^2 + E_1 z + E_0 = 0, \tag{4.20}
$$

in which

$$
\left.\begin{aligned}
&E_4 = 1,\ \ E_3 = E_1 = E_0 = 0, \\
&E_2 = -\frac{3\,\varepsilon_{ij}\beta}{16\,\omega_i\omega_j}\left(\frac{\omega_i}{\omega_j} + \frac{\omega_j}{\omega_i} + 4\right)a_{i0}a_{j0}\cos\varphi_0.
\end{aligned}\right\} \tag{4.21}
$$

It follows from Routh's theorem that steady state solutions for $\nabla_{c1}$ and $\nabla_{c2}$ are unstable and stable severally when $\beta>0$, and vice versa when $\beta<0$ and that boundary condition between stability and instability of steady state solution is $E_2=0$, *i.e.*, $\varepsilon_{ij}=0$. Accordingly the boundary line derived by this boundary condition coincides with the so-called "back bone curve" which is represented by

$$\nabla_{ij}=\frac{3}{8}\frac{\beta}{p_j}\left(\frac{p_i}{p_j}+\frac{p_j}{p_i}+4\right)a_{i0}^2. \tag{4.22}$$

The curve of the above equation is also the curve furnishing the largest value of negative damping coefficient

$$\alpha_{\max}=|E_{ij}|/2. \tag{4.23}$$

(ii) Vibrations in stable region

When $\nabla_{ij}>\nabla_{c1}$ or $\nabla_{ij}<\nabla_{c2}$, $\alpha$ and $\varphi_{ij}$ are real. Putting

$$\nu^2=-\alpha^2=\frac{1}{4}\left[\left\{\nabla_{ij}-\frac{3}{8}\frac{\beta}{p_j}\left(\frac{p_i}{p_j}+\frac{p_j}{p_i}+4\right)a_i^2\right\}^2-E_{ij}^2\right], \tag{4.24}$$

we attain the following equations for solutions $X_{i,j}$ in stable region:

$$\left.\begin{aligned} X_i&=A\sin(\omega_i+\nu)t+B\cos(\omega_i+\nu)t+C\sin(\omega_i-\nu)t+D\cos(\omega_i-\nu)t,\\ X_j&=\frac{1}{E_{ij}}\sqrt{\frac{\omega_i}{\omega_j}}\left[(\pm\sqrt{4\nu^2+E_{ij}^2}+2\nu)\{A\sin(\omega_j-\nu)t-B\cos(\omega_j-\nu)t\}\right.\\ &\qquad\left.+(\pm\sqrt{4\nu^2+E_{ij}^2}-2\nu)\{C\sin(\omega_j+\nu)t-D\cos(\omega_j+\nu)t\}\right], \end{aligned}\right\} \tag{4.25}$$

where the upper and lower signs correspond to $\nabla_{ij}>\nabla_{c1}$ and $\nabla_{ij}<\nabla_{c2}$ separately. It is seen that Eq. (4.25) represents simply free vibrations in which frequencies $\omega_i+\nu$ and $\omega_j-\nu$ as well as $\omega_i-\nu$ and $\omega_j+\nu$ make a pair.

*4.3.2. Vibrations of summed type with damping*

From Eqs. (4.4), (4.6) amplitude ratio is of the form

$$(a_i/a_j)^2=(\alpha+C_{jj}/2)\,\omega_j/(\alpha+C_{ii}/2)\,\omega_i=K\omega_j/\omega_i, \tag{4.26}$$

in which

$$K=(\alpha+C_{jj}/2)/(\alpha+C_{ii}/2). \tag{4.27}$$

By a similar procedure to the previous section, we obtain

$$\left.\begin{aligned} \delta_i&=\frac{1}{1+K}\left\{2\nabla_{ij}+\frac{3}{4}\frac{\beta}{p_j}\left(K\frac{p_j}{p_i}-\frac{1}{K}\frac{p_i}{p_j}\right)a_i^2\right\},\\ \delta_j&=\frac{K}{1+K}\left\{2\nabla_{ij}-\frac{3}{4}\frac{\beta}{p_jK}\left(K\frac{p_j}{p_i}-\frac{1}{K}\frac{p_i}{p_j}\right)a_i^2\right\}, \end{aligned}\right\} \tag{4.28}$$

$$\cos\varphi_{ij}=\frac{2}{E_{ij}}\frac{\sqrt{K}}{1+K}\left[\nabla_{ij}-\frac{3}{8}\frac{\beta}{Kp_j}\left\{K\left(\frac{p_j}{p_i}+2\right)+\left(\frac{p_i}{p_j}+2\right)\right\}a_i^2\right]. \tag{4.29}$$

Eliminating phase angle $\varphi_{ij}$ in Eqs. (4.4), (4.5), the following fifth degree equation for negative damping coefficient $\alpha$ is attained from Eq. (3.26):

$$\frac{K}{(1+K)^2}\left[2\,\nabla_{ij} - \frac{3}{4}\frac{\beta}{Kp_j}\left\{K\left(\frac{p_j}{p_i}+2\right)+\frac{p_i}{p_j}+2\right\}a_i^2\right]^2$$
$$+\,4(\alpha + C_{ii}/2) + (\alpha + C_{jj}/2) = E_{ij}^2, \qquad (4.30)$$

which cannot be solved analytically. It is seen from Eqs. (4.26), (4.30) that real roots of the above equation should satisfy the following relations:
For $\lambda > 1$

$$(-n_1 - \sqrt{n_2^2 + E_{ij}^2})/2 < \alpha < -C_{jj}/2, \quad -C_{ii}/2 < \alpha < (-n_1 + \sqrt{n_2^2 + E_{ij}^2})/2,$$

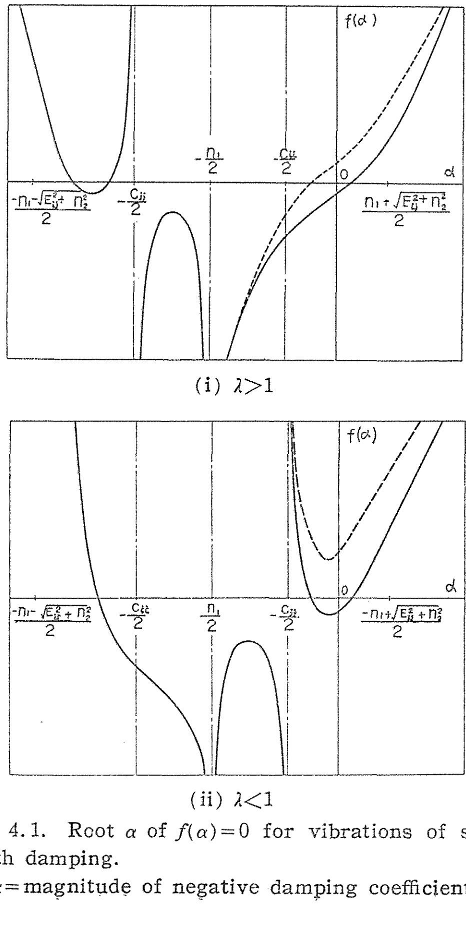
For $\lambda < 1$

$$(-n_1 - \sqrt{n_2^2 + E_{ij}^2})/2 < \alpha < -C_{ii}/2, \quad -C_{jj}/2 < \alpha < (-n_1 + \sqrt{n_2^2 + E_{ij}^2})/2,$$

Hence the function

$$f(\alpha) = \frac{K}{(1+K)^2}\left[2\,\nabla_{ij} - \frac{3}{4}\frac{\beta}{Kp_j}\left\{K\left(\frac{p_j}{p_i}+2\right)+\frac{p_i}{p_j}+2\right\}a_i^2\right]^2$$
$$+\,4(\alpha + C_{ii}/2)(\alpha + C_{jj}/2) - E_{ij}^2 = 0$$

has at least one real root and others, real parts of which are negative, as shown




(i) $\lambda > 1$

(ii) $\lambda < 1$

FIG. 4.1. Root $\alpha$ of $f(\alpha) = 0$ for vibrations of summed type with damping.
($\alpha$ = magnitude of negative damping coefficient)

in Fig. 4.1, and it follows that unstable vibrations of summed type take place when the largest real root becomes positive, and damped vibrations appear when it is negative.

Putting $\alpha=0$ in Eq. (4.30) results in boundaries between stable and unstable regions as follows:

$$\begin{matrix}\nabla_{c1}\\ \nabla_{c2}\end{matrix} = \pm \frac{1+\lambda}{2\sqrt{\lambda}}\sqrt{E_{ij}^2-\lambda C_{ii}^2}+\frac{3}{8}\frac{\beta}{\lambda p_j}\left\{\lambda\left(\frac{p_j}{p_i}+2\right)+\frac{p_i}{p_j}+2\right\}a_{i0}^2, \tag{4.31}$$

in which the upper and lower signs are used for $\nabla_{c1}$ and $\nabla_{c2}$ ($\nabla_{c1}>\nabla_{c2}$) respectively. Inserting Eq. (4.31) into Eqs. (4.28), (4.29), detunings $\delta_{i,j}$, phase angle $\varphi_0$ on these boundaries of unstable regions are derived as follows:

$$\left.\begin{aligned}\delta_i &= \frac{1}{\lambda}\left\{\pm\sqrt{\lambda}\sqrt{E_{ij}^2-\lambda C_{ii}^2}+\frac{3}{4}\frac{\beta}{p_j}\left(\lambda\frac{p_j}{p_i}+2\right)a_{i0}^2\right\},\\ \delta_j &= \left\{\pm\sqrt{\lambda}\sqrt{E_{ij}^2-\lambda C_{ii}^2}+\frac{3}{4}\frac{\beta}{p_j}\left(\frac{1}{\lambda}\frac{p_i}{p_j}+2\right)a_{i0}^2\right\},\end{aligned}\right\} \tag{4.32}$$

$$\cos\varphi_0 = \pm\sqrt{E_{ij}^2-\lambda C_{ii}^2}/|E_{ij}|, \qquad (0\leqq\varphi_0\leqq\pi) \tag{4.33}$$

where the upper and lower signs correspond to $\nabla_{c1}$ and $\nabla_{c2}$ separately. Accordingly steady state solutions of $X_{i,j}$ on these boundaries are obtained as follows:

$$\left.\begin{aligned}X_i &= a_{i0}\sin(\omega_i t+\varphi_i),\\ X_j &= \sqrt{\omega_i/(\lambda\omega_j)}\,a_{i0}\sin(\omega_j t+\varphi_0-\varphi_i).\end{aligned}\right\} \tag{4.34}$$

Stability of $X_i$, $X_j$ of Eq. (4.34) will be studied. By a similar process with the previous section, it is known that, for Eq. (4.34), coefficients of the characteristic equation (4.20) are given by the following equations:

$$\left.\begin{aligned}&E_4=1,\ E_3=C_{ii}+C_{jj},\ E_0=0,\\ &E_2=n_1^2+\frac{n_1^2(E_{ij}^2-C_{ii}C_{jj})}{C_{ii}C_{jj}}-\frac{3}{16}\frac{\varepsilon_{ij}\beta}{\omega_i\omega_j}\left(\frac{\omega_j}{\omega_i}+\frac{\omega_i}{\omega_j}+4\right)a_{i0}a_{j0}\cos\varphi_0,\\ &E_1=-\frac{3\,\varepsilon_{ij}\beta}{16\,\omega_i\omega_j}\left(C_{ii}\frac{\omega_i}{\omega_j}+C_{jj}\frac{\omega_j}{\omega_i}+4\,n_1\right)a_{i0}a_{j0}\cos\varphi_0,\end{aligned}\right\} \tag{4.35}$$

and it follows through Routh's theorem that steady state solutions for $\nabla_{c1}$ and $\nabla_{c2}$ are unstable and stable separately when $\beta>0$, and vice versa when $\beta<0$. Boundary between stability and instability of these steady state solutions is decided by the following equation which agree with the so-called "back bone curve" for vibrations with damping:

$$\nabla_{ij} = \frac{3}{8}\frac{\beta}{\lambda p_j}\left\{\lambda\left(\frac{p_j}{p_i}+2\right)+\frac{p_i}{p_j}+2\right\}a_{i0}^2. \tag{4.36}$$

The curve furnishing the largest value of negative damping coefficient is represented from Eq. (4.30) as follows:

$$\nabla_{ij} = \frac{3}{8}\frac{\beta}{K_0 p_j}\left\{K_0\left(\frac{p_j}{p_i}+2\right)+\frac{p_i}{p_j}+2\right\}a_{i0}^2, \tag{4.37}$$

in which

$$K_0 = (\sqrt{E_{ij}^2 + n_2^2} - n_2)/(\sqrt{E_{ij}^2 + n_2^2} + n_2). \tag{4.38}$$

Substitution of Eq. (4.37) into Eq. (4.30) leads to the following largest value of negative damping coefficient:

$$\alpha_{\max} = (-n_1 + \sqrt{E_{ij}^2 + n_2^2})/2. \tag{4.39}$$

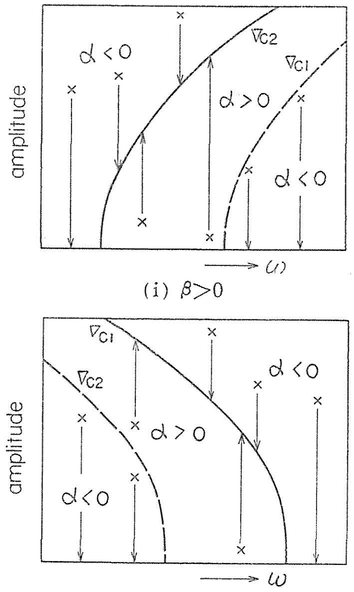



FIG. 4.2. Change of amplitude of vibrations of summed type. ( × = initial amplitude)

Consequently, when detuning $\nabla_{ij}$ takes a value between $\nabla_{c1}$ and $\nabla_{c2}$ and further the relation $E_{ij}^2 \geqq C_{ii} C_{jj}$ holds, negative damping coefficient $\alpha$ takes a positive value and unstable vibrations of summed type occur. However, they are settled in steady state vibrations having a finite amplitude $a_{i0}$, because $\alpha$ decreases with increase of amplitude as shown in Eq. (4.30) and it vanishes finally. On the other hand, when $\nabla_{ij} > \nabla_{c1}$, or $\nabla_{ij} < \nabla_{c2}$, damped vibrations take place, amplitude of which reach to either steady state amplitude or zero. If magnitude of parametric excitation $|E_{ij}|$ is smaller than $\sqrt{C_{ii} C_{jj}}$, unstable vibration cannot occur because of $\alpha < 0$. Furthermore, the discussion above mentioned is illustrated graphycally in Fig. 4.2, where full and dotted line curves are employed for stable and unstable steady state solutions respectively.

### *4.4. Vibrations of differential type*

#### *4.4.1. Vibrations of differential type without damping*

Putting $C_{ii} = C_{jj} = 0$ in Eq. (4.4) where the lower sign is adopted, we have the following amplitude ratio by a similar procedure to summed type:

$$a_i / a_j = i \sqrt{\omega_j / \omega_i}\,, \tag{4.7 a}$$

and through Eqs. (4.5), (2.6) in which the lower sign is adopted, the following equations corresponding to Eqs. (4.8), (4.9), (4.11) are given:

$$\left.\begin{aligned} \delta_i &= \nabla_{ij} + \frac{3}{8}\frac{\beta}{p_j}\left(\frac{p_j}{p_i} - \frac{p_i}{p_j}\right) a_i^2, \\ \delta_j &= -\nabla_{ij} + \frac{3}{8}\frac{\beta}{p_j}\left(\frac{p_j}{p_i} - \frac{p_i}{p_j}\right) a_i^2, \end{aligned}\right\} \tag{4.8 a}$$

$$\alpha^2 = -\frac{1}{4}\left[E_{ij}^2 + \left\{\nabla_{ij} - \frac{3}{8}\frac{\beta}{p_j}\left(\frac{p_j}{p_i} + \frac{p_i}{p_j} - 4\right) a_i^2\right\}^2\right], \tag{4.9 a}$$

$$\cos\varphi_{ij} = -i\frac{1}{E_{ij}}\left\{\nabla_{ij} \mp \frac{3}{8}\frac{\beta}{p_j}\left(\frac{p_j}{p_i} + \frac{p_i}{p_j} - 4\right) a_i^2\right\}^2. \tag{4.11 a}$$

Referring the fact that $\alpha$ and $\varphi_{ij}$ as well as $a_i/a_j$ are not real as shown in the above equations, and introducing the similar relation to Eq. (4.24)

$$\mu^2 = -\alpha^2 = \frac{1}{4}\left[E_{ij}^2 + \left\{\nabla_{ij}^2 - \frac{3}{8}\frac{\beta}{p_j}\left(\frac{p_j}{p_i} + \frac{p_i}{p_j} - 4\right)a_i^2\right\}^2\right], \qquad (4.24\,a)$$

the solutions of differential type without damping are written as

$$\left.\begin{aligned} X_i &= A\sin(\omega_i+\mu)t + B\cos(\omega_i+\mu)t + C\sin(\omega_i-\mu)t + D\cos(\omega_i-\mu)t,\\ X_j &= -\frac{1}{E_{ij}}\sqrt{\frac{\omega_i}{\omega_j}}[(\pm\sqrt{4\mu^2-E_{ij}^2}+2\mu)\{A\sin(\omega_j+\mu)t + B\cos(\omega_j+\mu)t\}\\ &\quad + (\pm\sqrt{4\mu^2-E_{ij}^2}-2\mu)\{C\sin(\omega_j-\mu)t + D\cos(\omega_j-\mu)t\}], \end{aligned}\right\} \qquad (4.25\,a)$$

where the upper and lower signs are employed for solutions of the higher and lower frequency sides. The above equation which is similar to Eq. (4.25) reprsents simply free vibrations, frequencies of which $\omega_i+\mu$ and $\omega_j+\mu$ as well as $\omega_i-\mu$ and $\omega_j-\mu$ make a pair, and there is no unstable vibration.

*4.4.2. Vibrations of differential type with damping*

By a similar procedure shown in Section 4.3.2, we get the following equations:

$$(a_i/a_j)^2 = -(\alpha + C_{jj}/2)\,\omega_j/(\alpha + C_{ii}/2)\,\omega_i = -K\omega_j/\omega_i, \qquad (4.26\,a)$$

$$\left.\begin{aligned} \delta_i &= \frac{1}{K+1}\left\{2\nabla_{ij} + \frac{3}{4}\frac{\beta}{p_j}\left(K\frac{p_j}{p_i} - \frac{1}{K}\frac{p_i}{p_j}\right)a_i^2\right\},\\ \delta_j &= \frac{K}{K+1}\left\{-2\nabla_{ij} + \frac{3}{4}\frac{\beta}{p_jK}\left(K\frac{p_j}{p_i} - \frac{1}{K}\frac{p_i}{p_j}\right)a_i^2\right\}, \end{aligned}\right\} \qquad (4.28\,a)$$

$$\cos\varphi_{ij} = \frac{2}{E_{ij}}\frac{\sqrt{-K}}{1+K}\left[-\nabla_{ij} + \frac{3}{8}\frac{\beta}{p_jK}\left\{K\left(\frac{p_j}{p_i}-2\right) + \frac{p_i}{p_j} - 2\right\}a_i^2\right], \qquad (4.29\,a)$$

$$\frac{K}{(1+K)^2}\left[2\nabla_{ij} - \frac{3}{4}\frac{\beta}{Kp_j}\left\{K\left(\frac{p_j}{p_i}-2\right) + \frac{p_i}{p_j} - 2\right\}a_i^2\right]^2 + 4(\alpha + C_{ii}/2)(\alpha + C_{jj}/2) = -E_{ij}^2, \qquad (4.30\,a)$$

which correspond to Eqs. (4.26), (4.28), (4.29), (4.30) separately. Taking account that the real root $\alpha$ of Eq. (4.30 a) exists in the following regions:

for $\lambda>1$

$$-C_{jj}/2 < \alpha < (-n_1 - \sqrt{n_2^2 - E_{ij}^2}), \quad (-n_1 + \sqrt{n_2^2 - E_{ij}^2}) < \alpha < -C_{ii}/2,$$

for $\nu<1$

$$-C_{ii}/2 < \alpha < (-n_1 - \sqrt{n_2^2 - E_{ij}^2})/2, \quad (-n_1 + \sqrt{n_2^2 - E_{ij}^2}) < \alpha < -C_{jj}/2,$$

it is seen in Fig. 4.3 that neither real root or real parts of the other roots is positive, and it follows that vibrations are always simply damped vibrations.

Consequently, as shown in Sections 4.4.1, 4.4.2, unstable vibration of differential type cannot occur.

### *4. 5. Verification of analytical results through analog computer*

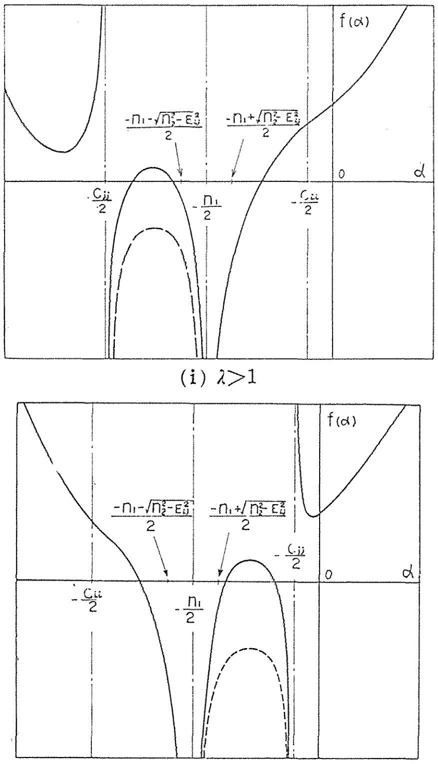
Since in actual experimental appratus it is difficult to produce non-linearity shown in Eq. (4.3) and viscous damping forces exactly proportional to velocity, analog computer is used to ascertain obtained analytical results. Comparison of analytical results with those of analog computer is performed in a simple system of two dgree-of-freedom, and hence number of degrees of freedom $h$, suffixs $i, j$ are 2, 1, 2 respectively in this case.

Block diagram of analog computer simulating Eq. (4.2) is shown in Fig. 4.4, where the following dimensions are adopted in order to connect with results of vibrations in linear systems treated before.

$$p_1 = 13.863\ \mathrm{c/s},\ p_2 = 9.850\ \mathrm{c/s},\ p_{12} = p_1 + p_2 \doteqdot 23.713\ \mathrm{c/s},$$

$$\varepsilon_{11} = 12.23 \times 10^2\, e\ (\mathrm{rad/s})^2,\ \varepsilon_{12} = 9.22 \times 10^2\, e\ (\mathrm{rad/s})^2,\ \varepsilon_{22} = 14.33 \times 10^2\, e\ (\mathrm{rad/s})^2,\ \beta = -0.312\ \mathrm{kg/cm^3}.$$

In the following figures, results given by analysis are graphycally shown by chain and dotted or full and broken line curves which correspond to stable and unstable steady state vibrations without or with damping separately, and results by analog computer are indicated by symbols ○, ⊖, etc. and ●, ⊖, etc., which are employed for stable and unstable steady state vibrations respectively. A double chain line curve illustrates boundary of stability of steady state solution.

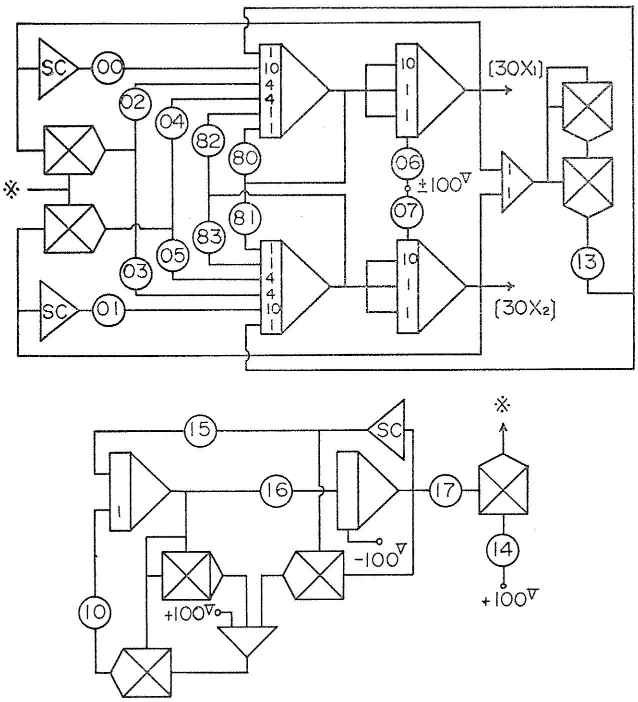



(i) $\lambda > 1$

(ii) $\lambda < 1$

FIG. 4.3. Root $\alpha$ of $f(\alpha) = 0$ for vibrations of differential type with damping ($\alpha$ = magnitude of negative damping coefficient).

Amplitude-frequency diagrams of vibrations of summed type for various magnitudes of damping coefficient are indicated in Figs. 4.5, 4.6, 4.7, where magnitude of amplitude of unstable steady state solutions takes a value between both initial amplitudes which begin to increase and to decrease. As shown in Fig. 4.5, when $C_{ii} = C_{jj}$, *i.e.*, damping ratio $\lambda$ is equal to unity, the smaller damping of the larger magnitude of parametric excitation results in the wider unstable region. On the other hand, when $\lambda \doteqdot 1$ ($C_{ii} \doteqdot C_{jj}$), width of unstable region of vibrations with damping can be larger than that without damping in a range of larger magnitude of parametric excitation $|E_{12}|$, and degree of inclination of unstable region differ from that for $\lambda = 1$, as shown in Fig. 4.6. It is seen in Fig. 4.7 that magnitude of damping ratio $\lambda$ has considerable influences on degree of inclination of both unstable region and back bone curve, that is, inclinations for $\lambda < 1$ and $\lambda > 1$ are more and less than for $\lambda = 1$ respectively.




00: $p_1^2/1200$
01: $p_2^2/1200$
02: $25\,\varepsilon_{11}/(3\,e\omega^2)$
03: $25\,\varepsilon_{12}/(3\,e\omega^2)$
04: $25\,\varepsilon_{21}/(3\,e\omega^2)$
05: $25\,\varepsilon_{22}/(3\,e\omega^2)$
06: initial condition
07: initial condition
10: 0.1
13: $\beta$
14: $(\omega/200)^2$
15: $\omega/10$
16: $\omega/10$
17: $e$
80: $C_{11}/10$
81: $C_{12}/10$
82: $C_{21}/10$
83: $C_{22}/10$

FIG. 4.4. Block diagram of analog computer.

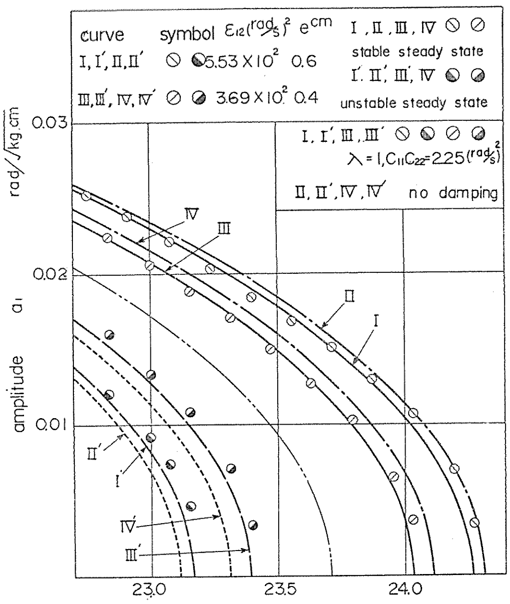



FIG. 4.5. Amplitude-frequency diagrams of summed type vibrations without and with damping ($\lambda=1$).

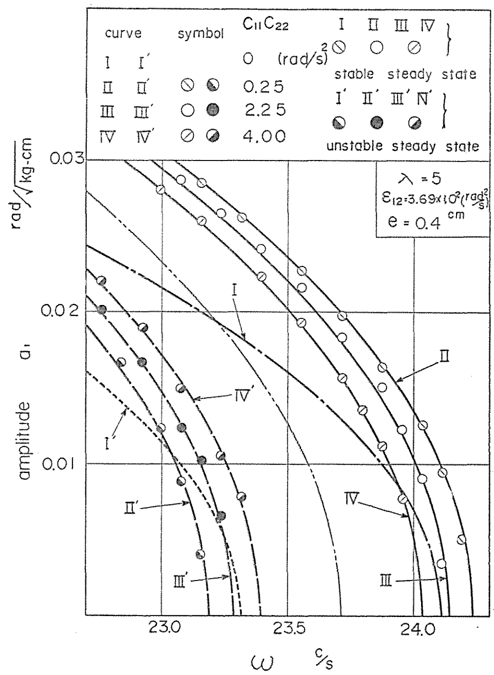


FIG. 4. 6. Amplitude-frequency diagrams of summed type vibrations with damping ($\lambda \doteqdot 1$).

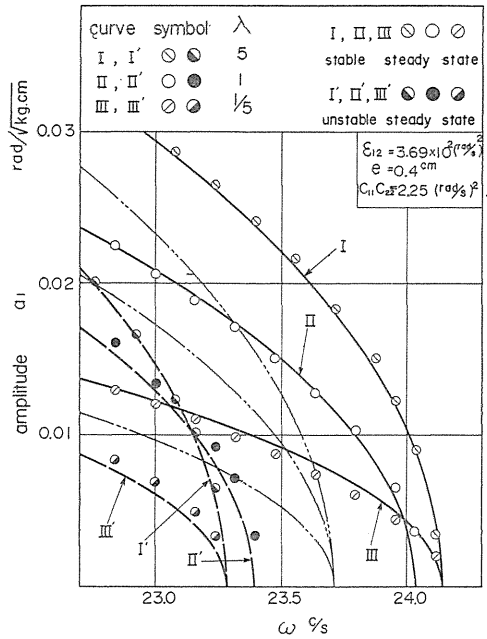


FIG. 4. 7. Influence of magnitude of damping ratio $\lambda$ to amplitude-frequency diagrams of summed type vibrations.

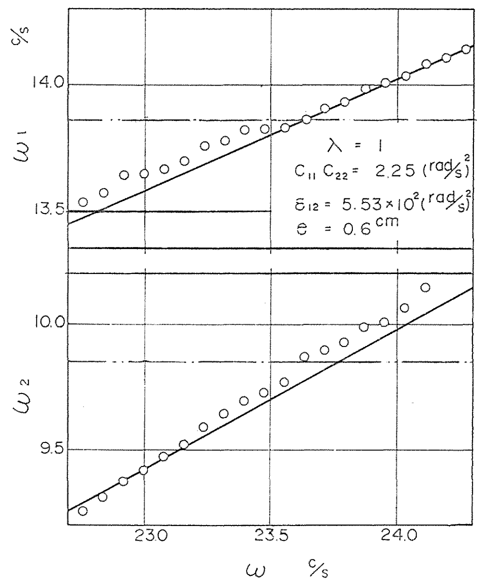


FIG. 4. 8. Frequencies of stable steady state vibrations of summed type with damping.

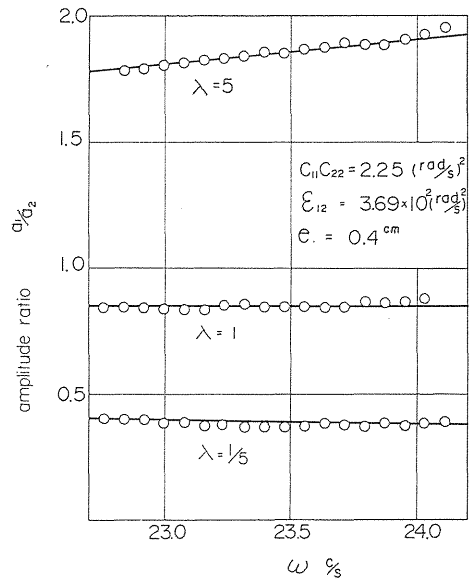


FIG. 4. 9. Amplitude ratio of stable steady state vibrations of summed type with damping.

Frequencies $\omega_1$ and $\omega_2$ of stable steady state vibrations of summed type with damping ($\lambda=1$) is shown in Fig. 4.8, where the relation of $\omega_1+\omega_2=\omega$ always holds.

Amplitude ratio of two steady state vibrations of summed type with damping for various magnitudes of damping ratio are indicated in Fig. 4.9, in which amplitude ratio increases with $\lambda$.

Negative damping coefficient—amplitude diagram of summed type vibrations with damping is represented in Fig. 4.10, where frequency of parametric excitation is fixed ($\omega=23.47$ c/s), and full line curve shows analytical results obtained by Eq. (4.30). It is obvious that amplitude continues to increase until $\alpha$ becomes equal to zero.

Finally, vibratory waves of unstable vibrations of summed type obtained by analog computer are illustrated in Fig. 4.11, in which unstable vibrations are settled in stable steady vibrations, in contrast with linear systems.

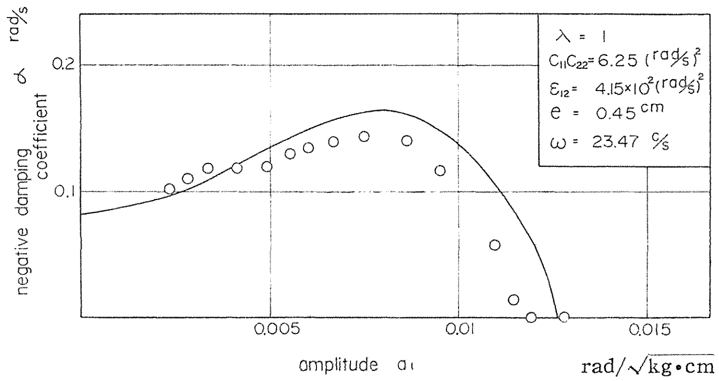



FIG. 4.10. Negative damping coefficient-amplitude daigrams of summed type vibrations with damping.

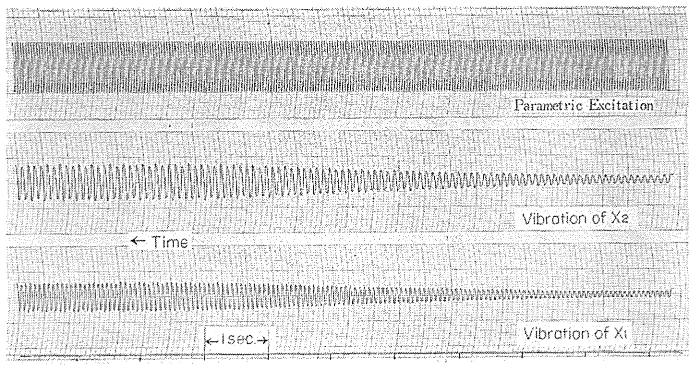



FIG. 4.11. Vibratory waves of unstable vibrations of summed type with damping given by analog computer.
($\varepsilon_{12}=3.69\times10^2$ rad$^2$/s$^2$, $e=0.4$ cm, $\omega=23.715$ c/s, $\lambda=5$, $c_{11}c_{22}=4.00$ rad$^2$/s$^2$).

*4.6. Conclusions*

Obtained results in the present chapter may be summarized as follows:

(1) In non-linear vibratory system with multiple degree-of-freedom and under parametric excitation as well as in linear system, two unstable vibrations of summed type with frequencies $\omega_i (\fallingdotseq p_i)$, $\omega_j (\fallingdotseq p_j)$ satisfying the relation $\omega_i + \omega_j = \omega$ can simultaneously take place when frequency $\omega$ of parametric excitation becomes nearly equal to the resonant point $p_{ij} = p_i + p_j$.

(2) These unstable vibrations could not increase indefinitely but are settled in stable steady state vibrations having a certain finite amplitude.

(3) Sum of two frequencies of summed type vibrations is always equal to frequency $\omega$ of parametric excitation.

(4) Influences of damping on characteristics of summed type vibrations are similar to those of linear system, that is, existence of damping does not always result in small width of unstable regions and degree of inclination of unstable region is varied considerably by magnitude of damping ratio.

(5) When coefficient of non-linear term $\beta$ is positive, steady state vibrations of summed type on the higher and lower frequency side are unstable and stable respectively and vice versa when $\beta$ is negative.

(6) Vibrations of differential type both without and with damping are simply free vibrations, thus there is no unstable vibration of differential type.

(7) Difference in two frequencies of vibrations of differential type is always equal to frequency $\omega$.

(8) Theoretical results obtained by a first approximate analysis in which higher powers of small quantities are rejected, show good agreement with results of analog computer.

## Chapter V. Forced vibrations of linear vibratory system under parametric excitation[23]

*5.1. Introduction*

It is well-known that the so-called "combination tones" of summed and differential types take place[24][25] when two periodic disturbing forces excite a vibratory system with unsymmetrical non-linear spring characteristics. On the other hand, in linear vibratory system under parametric excitation of frequency $\Omega$ and periodic disturbing force of frequency $\Omega_2$, forced vibrations having frequency $\Omega_1 (= |\Omega \pm \Omega_2| \fallingdotseq p)$ which are similar to combination tones can occur when a sum of and difference in two frequencies $\Omega$ and $\Omega_2$, *i.e.*, $|\Omega \pm \Omega_2|$ becomes nearly equal to a natural frequency $p$. These vibrations appearing in the system with flat shaft carrying a symmetrical rotor have been studied[26], while they take place in a special case when frequency $\Omega_2$ of periodic disturbing force is equal to a half of frequency $\Omega$ of parametric excitation, that, is, ratio of frequencies $\kappa = \Omega / \Omega_2$ is equal to 2.

In the present chapter, this kind of forced vibrations appearing in a single pendulum system excited by both parametric excitation and disturbing force, the frequencies of which are able to be given arbitrarily, is discussed analytically and experimentally, and it is found that characteristics of response curves are influenced by magnitudes of parametric excitation and disturbing force, damping coefficient and frequency ratio $\kappa = \Omega/\Omega_2$. For example, response curves of differential type $[\Omega_1 = \Omega - \Omega_2]$ when $\kappa > 2$ and $\kappa < 2$ are quite different from each other. Theoretical results are compared with those of experiment and analog computer, and they show good agreement.

*5. 2. Equation of motion and preliminary analysis*

A vibratory system of single pendulum with one degree-of-freedom is shown in Fig. 5. 1, where pendulum with length $l_1$ (cm) and mass $m_1$ (kg·s²/cm) is supported at the point $B_1$; $I_1$ is moment of inertia about the supporting point $B_1$ and $b_1$ (cm) is a distance between $B_1$ and gravitational center $G_1$; there is a spring of spring constant $k_1$ (kg/cm) at the end of pendulum and it is assumed that a damping force having small damping coefficient $c$ (kg·s/cm) exists in this system. The supporting point $B_1$ is furnished both vertical vibration $e \cos \Omega t$ and horizontal one $d_1 \cos (\Omega_2 t + \varphi)$ ($\varphi$ = phase angle) with small amplitudes $e$, $d_1$ (cm) and frequencies $\Omega$, $\Omega_2$ (rad/s) respectively. Motion of a point $K$ which is at a distance $l_0$ (cm) from the supporting point $B_1$ is recorded, and it is given by the following equation:

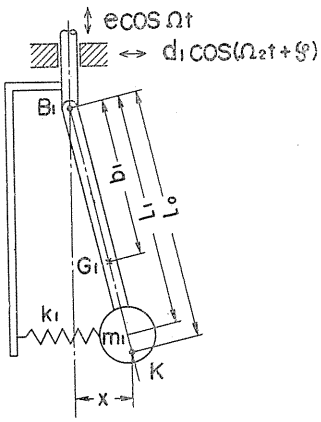



FIG. 5. 1. Vibratory system of single pendulum.

$$m_0 \ddot{x}_1 + c_0 \dot{x}_1 + k_0 x_1 = \varepsilon_0 x_1 \cos \Omega t + d_0 \cos (\Omega_2 t + \varphi), \tag{5.1}$$

in which $x_1$ (cm) is a distance between $K$ and its equilibrium point, and

$$\left. \begin{aligned} & m_0 = I_1/l_0, \quad c_0 = c l_1^2/l_0^2, \quad k_0 = (k_1 l_1^2 + m_1 b_2 g)/l_0^2, \\ & \varepsilon_0 = e m_1 \Omega^2 b_1/l_0^2, \quad d_0 = d_1 m_1 \Omega_2^2 b_1/l_0. \end{aligned} \right\} \tag{5.2}$$

For brevity, we introduce dimensionless quantities

$$\left. \begin{aligned} & t' = p_1 t, \quad X = x_1/x_0, \quad c' = c_0/\sqrt{m_0 k_0}, \quad \Omega' = \Omega/p_1, \quad \Omega_1' = \Omega_1/p_1, \\ & \Omega_2' = \Omega_2/p_1, \quad \varepsilon_1' = \varepsilon_0/k_0 = h_1 e \Omega_1^2, \quad d' = d_0/(k_0 x_0) = h_2 d_1 \Omega_2^2, \end{aligned} \right\} \tag{5.3}$$

in which $p_1 (= \sqrt{k_0/m_0})$ is natural frequency of the system, and $x_0$ is a length of unity, *i.e.*, $x_0 = 1$ cm, and $h_1 (= m_1 b_1/(k_1 l_1^2 + m_1 b_1 g))$, $h_2 (= h_1 l_0)$ are constants. Substituting Eq. (5.3) into Eq. (5.1) and neglecting primes, the following equation of motion expressed by dimensionless quantities can be obtained:

$$\ddot{X} + c\dot{X} + X = \varepsilon_1 X \cos \Omega t + d \cos (\Omega_2 t + \varphi). \tag{5.4}$$

Upon use of the above equation, forced vibrations haveing frequency $\Omega_1$ which satisfy the following relation:

$$\Omega_1 = |\Omega \pm \Omega_2| \fallingdotseq 1 \tag{5.5}$$

are treated in the present chapter.

*5.3. Response curves*

When frequency $\Omega_1$ is nearly equal to natural frequency $p_1=1$, it is expected that forced vibrations builds up and magnitude of its amplitude grows up remarkably, and becomes comparable to that of harmonic vibration with frequency $\Omega_2$. Accordingly, a first approximate solution of Eq. (5.4) should be assumed as follows:

$$X = A\cos(\Omega_1 t+\varphi_1) + B\cos(\Omega_2 t+\varphi_2). \tag{5.6}$$

In the above equation, the first and second terms are vibration statisfying the relation of Eq. (5.5) and harmonic vibration severally. Inserting Eq. (5.6) into Eq. (5.4), referring the relation of Eq. (5.5) and comparing coefficients of terms with frequencies $\Omega_1$ and $\Omega_2$ respectively, the following equations are attained in so far as $\Omega_1 \fallingdotseq \Omega_2$:

$$\left.\begin{aligned} -2c\Omega_2 B &= \varepsilon_1 A\sin(\varphi_2\mp\varphi_1) + 2d\sin(\varphi_2-\varphi), \\ 2(1-\Omega_2^2)B &= \varepsilon_1 A\cos(\varphi_2\mp\varphi_1) + 2d\cos(\varphi_2-\varphi), \\ -2c\Omega_1 A &= \varepsilon_1 B\sin(\varphi_1\mp\varphi_2), \\ 2(1-\Omega_1^2)A &= \varepsilon_1 B\cos(\varphi_1\mp\varphi_2). \end{aligned}\right\} \tag{5.7}$$

Elimination of phase angle $\varphi$, $\varphi_1$ and $\varphi_2$ in Eq. (5.7) yields the following equations which furnish response curves of amplitudes $A$ and $B$:

$$\left.\begin{aligned} A^2 &= [4\{c^2\Omega_1^2+(1-\Omega_1^2)^2\}/\varepsilon_1^2]d^2/F, \\ B^2 &= [4\{c^2\Omega_1^2+(1-\Omega_1^2)^2\}/\varepsilon_1^2]d^2/F, \end{aligned}\right\} \tag{5.8}$$

in which

$$F = c^2\Big[\Omega_1 \pm \frac{4\,\Omega_2}{\varepsilon_1^2}\{c^2\Omega_1^2+(1-\Omega_1^2)^2\}\Big]^2 + \Big[(1-\Omega_1^2) - \frac{4}{\varepsilon_1^2}(1-\Omega_2^2)\{c^2\Omega_1^2+(1-\Omega_1^2)^2\}\Big]^2. \tag{5.9}$$

Let frequency ratio $\Omega/\Omega_2$ be $\kappa$, we have

$$\kappa = \Omega/\Omega_2,\ \ \Omega = \kappa\Omega_1/|\kappa\pm1|,\ \ \Omega_2 = \Omega_1/|\kappa\pm1|. \tag{5.10}$$

It is seen in Eq. (5.8) that amplitudes $A$ and $B$ are functions of four quantities, *i.e.*, magnitudes $\varepsilon_1$, $d$ of parametric excitation and disturbing force, frequency ratio $\kappa$ and damping coefficient $c$. From Eq. (5.7), however, phase angle is given as follows:

$$\varphi_1\mp\varphi_2 = -\tan^{-1}\{c\Omega_1/(1-\Omega_1^2)\}. \tag{5.11}$$

The right side of Eq. (5.11) is the same form as phase angle in ordinary forced vibrations. In Eqs. (5.5) and (5.7) ~ (5.11), the upper and lower signs of $\pm$ or $\mp$ are adopted for forced vibrations of summed type $[\Omega_1=\Omega+\Omega_2]$ and differential type $[\Omega_1=\Omega-\Omega_2]$ severally.

Incidentally, analysis for forced vibrations in a special case $\Omega_1=\Omega_2$, *i.e.*, $\kappa=2$ is discussed here. Eq. (5.10) is rewritten as follows:

$$\Omega_1 = \Omega_2 = \Omega/2. \tag{5.12}$$

Thus, assuming that a first approximate solution of Eq. (5.4) in this case is given by

$$X = C\cos(\Omega_1 t + \varphi_3) \tag{5.13}$$

in place of Eq. (5.6), and comparing coefficients of terms with frequency $\Omega_1$, we have the following equations furnishing the same response curve as that obtained in the paper (26):

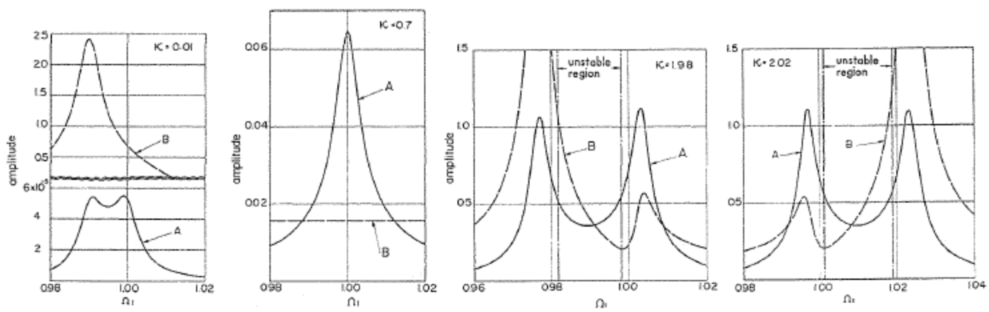
$$\left.\begin{aligned} (1-\Omega_1^2)C &= \frac{\varepsilon_1 C}{2}\cos 2\varphi_3 + d\cos(\varphi_3-\varphi), \\ -cC &= \frac{\varepsilon_1 C}{2}\sin 2\varphi_3 + d\sin(\varphi_3-\varphi), \end{aligned}\right\} \tag{5.14}$$

which are functions of phase angle $\varphi$.

*5.3.1. Forced vibrations of differential type* $[\Omega_1 = \Omega - \Omega_2]$

In the first place, influence of frequency ratio $\kappa = \Omega/\Omega_2$ on response curve of vibrations of differential type is examined for the system of $e=0.175$ cm, $d_1=0.015$ cm, $c=0.0059$, $h_1=0.0512$ s$^2$/cm, $h_2=0.9476$ s$^2$/cm. In Fig. 5.2 amplitudes $A$ and $B$ in the neighborhood of resonant point $\Omega_1=1$, *i.e.*, response curves for various values of $\kappa$ are shown. In Fig. 5.3, the maximum value $A_{max}$ of amplitude $A$, frequency $\Omega_{1\,max}$ and amplitude $B$ at $\Omega_{1\,max}$ are represented against frequency ratio $\kappa$.

Response curves when $\kappa$ is nearly equal to zero, *i.e.*, $\kappa \ll 1$, are shown in Fig. 5.2 (a) where $\kappa=0.01$. It is seen from Eqs. (5.3), (5.10) that parametric excitation $\varepsilon_1$ as well as $\Omega$ take very small values when $\kappa \ll 1$. It results in small amplitude $A$. The relationships Eq. (5.5) and $\kappa \ll 1$ lead to $\Omega_2 \fallingdotseq 1$ where amplitude $B$ of harmonic vibration of frequency $\Omega_2$ grows up remarkably as shown in Fig. 5.2 (a), because of resonance. When $\kappa \ll 1$, there are two $\Omega_{1\,max}$ as shown in Fig. 5.3. Accordingly there are two peaks of $A_{max}$ as shown in Fig. 5.2 (a), one appears in the neighborhood of $\Omega_1 \fallingdotseq 1$, the other takes place at $\Omega_1 \fallingdotseq 0.99$, *i.e.*, in the neighborhood of $\Omega_2=1$ where amplitude $B$ of vibration of frequency $\Omega_2$ builds up with




(a) (b) (c) (d)

FIG. 5.2. Response curves of forced vibrations of differential type for various values of frequency ratio $\kappa$.

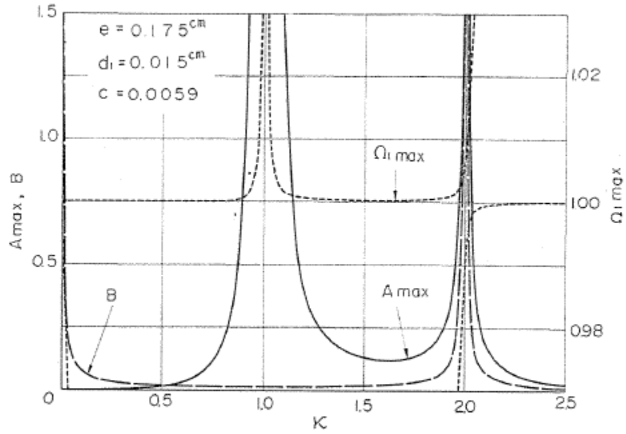



FIG. 5.3. $\kappa - A_{max}$, $\Omega_{1\,max}$, $B$ curves in differential type ($\varepsilon_1$ and $d$ are variable).

amplitude $A$. Eq. (5.8) shows that the magnitude of $A_{max}$ of the former is always larger than that of the latter. When small $\kappa$ increases gradually its magnitude, $\Omega_{1\,max}$ of $A_{max}$ of the latter goes rapidly far off the resonant point $\Omega_1=1$. It should be noticed that only $A_{max}$ and $B$ corresponding to $\Omega_{1\,max}$ of the former, *i.e.*, to $\Omega_{1\,max}$ appearing at $\Omega_1 \fallingdotseq 1$ are illustrated in Fig. 5.3.

As $\kappa$ increases further, $A_{max}$ increases gradually its magnitude, and the resonant point $\Omega_2=1$ of $B$ goes far off the point $\Omega_1=1$ because of $\Omega_2=\Omega_1/|K-1|$. It follows that $A_{max}$ becomes larger than $B$, as shown in Fig. 5.2 (b) where $\kappa=0.7$.

When $\kappa$ becomes nearly equal to unity, magnitudes of $\varepsilon_1$, $d$ of parametric excitation and disturbing force as well as $\Omega$, $\Omega_2$ take considerably large values, as is seen from Eqs. (5.3), (5.10). Conclusion that large values of $\varepsilon_1$ and $d$ in the neighborhood of $\kappa=1$ results in remarkably large values of $A_{max}$ and $\Omega_{1\,max}$, as is seen in Fig. 5.3, can be derived from investigation of influence of magnitudes

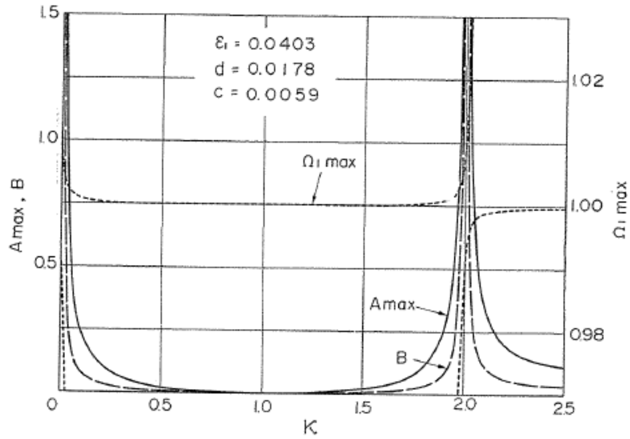



FIG. 5.4. $\kappa - A_{max}$, $\Omega_{1\,max}$, $B$ curves in differential type ($\varepsilon_1$ and $d$ are fixed).

of $\varepsilon_1$, $d$ to response curves, which is discussed later. Incidentally, $\kappa - A_{max}$, $\kappa - \Omega_{1\,max}$ curves when $\varepsilon_1$, $d$=constant is shown in Fig. 5.4, where there is no peak of $A_{max}$, $\Omega_{1\,max}$ in $\kappa \fallingdotseq 1$. It shows that small $\varepsilon_1$, $d$ do not result in large $A_{max}$, $\Omega_{1\,max}$ even if $\kappa \fallingdotseq 1$, and large $\varepsilon_1$, $d$ result in large $A_{max}$, $\Omega_{1\,max}$ as shown in Fig. 5.3, because $A_{max}$, $\Omega_{1\,max}$ are functions of $\varepsilon_1$, $d$.

As $\kappa$ approaches to 2, $A_{max}$ increases gradually and $\Omega_{1\,max}$ shifts to the higher frequency as seen in Figs. 5.3 and 5.4, and further the resonant point $\Omega_2=1$ of $B$ comes near to the point $\Omega_1=1$ as shown in Fig. 5.2 (c) ($\kappa=1.98$), because of the relationship $\Omega_1 \fallingdotseq \Omega_2 \fallingdotseq 1$ can again holds. It is seen in Fig. 5.2 (c) that $A_{max}$ in the higher frequency side is larger than that of the lower side. Unstable region of ordinary unstable vibration under parametric excitation furnished by

$$-\sqrt{\varepsilon_1^2/4-c^2} \leqq \Omega - 2 \leqq \sqrt{\varepsilon_1^2/4-c^2} \tag{5.15}$$

approaches to the resonant point $\Omega_1=1$, because $\kappa$ is nearly equal to 2, that is, $\Omega_2 \fallingdotseq 2$.

Fig. 5.2 (d) ($\kappa=2.02$) in which $\kappa$ is slightly larger than 2, illustrates different characteristics of the response curve from Fig. 5.2 (c), and the resonant point $\Omega_2=1$ of $B$ is higher than $\Omega_1=1$ and $A_{max}$ for the lower frequency side is larger than that of the higher side.

As $\kappa$ increases further beyond 2, $A_{max}$ decreases and both $\Omega_{1\,max}$ being larger than unity and unstable region of free vibration under parametric excitation shift rapidly to the higher frequency side. While $\Omega_{1\,max}$ in the lower frequency side comes furthermore close to the resonant point $\Omega_1=1$.

It can be said from the above mentioned discussion that, when $\kappa$ is small, amplitude $A$ of vibration of frequency $\Omega_1$ induced by small disturbing force $\varepsilon_1$ is also small, and when $\kappa=1$, amplitude $A$ becomes simply large because $\varepsilon_1$, $d$, and unstable vibration does not appear, which can take place only when $\kappa \fallingdotseq 2$.

Influence of magnitude of $e$, *i.e.*, $\varepsilon_1$ of parametric excitation on response curve is shown in Fig. 5.5, where as $\varepsilon_1$ increases, both $A$ and $B$ grow up and $\Omega_{1\,max}$ goes far off from the resonant point $\Omega_1=1$. Influence of magnitude $d$ of disturbing force is same as in case of ordinary forced vibrations, that is, both $A$ and $B$ are represented by linear functions of $d$, and hence amplitudes $A$ and $B$ are proportional to $d$ and $\Omega_{1\,max}$ is independent of $d$.

The larger damping results in the smaller amplitude as shown in Fig. 5.6. Frequency $\Omega_{1\,max}$ decreases as damping increases when $\kappa<2$ and vice veras when $\kappa>2$. Furthermore, curves and marks in Figs. 5.5 and 5.6 illustrate results

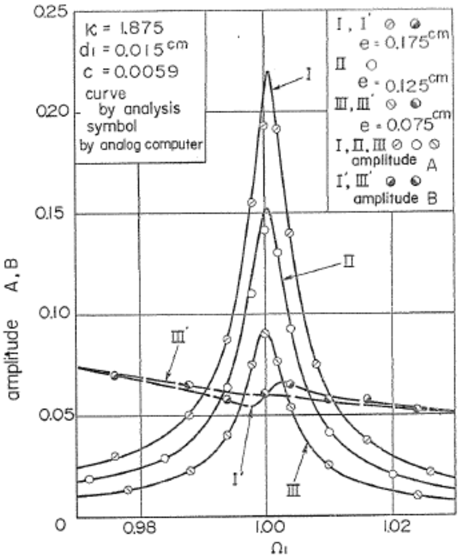



FIG. 5.5. Effect cf magnitude $e$ of parametric excitation on the response curves in differential type.

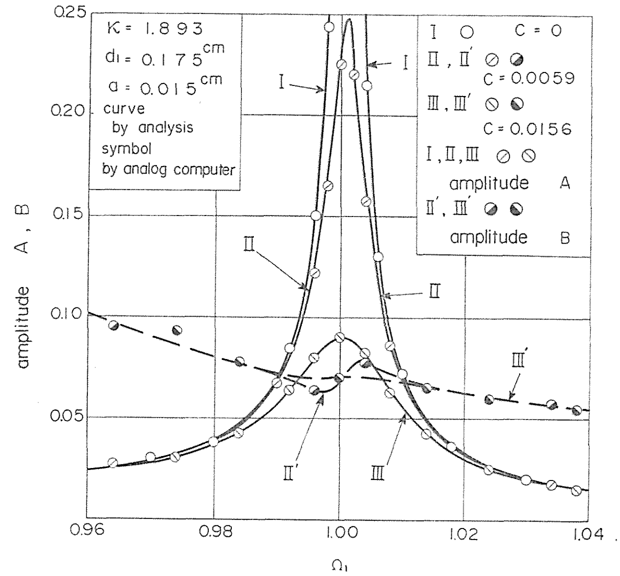



FIG. 5.6. Effect of damping coefficient $c$ on the response curves in differential type.

obtained by Eq. (5.8) and by analog computer respectively.

*5.3.2. Forced vibrations of summed type* $[\Omega_1=\Omega+\Omega_2]$

Since it is obvious from Eq. (5.10) that both frequencies $\Omega$ and $\Omega_2$ are always smaller than unity, large magnitudes of $e$ and $d_1$ are required to obtain comparable amplitudes with those of differential type. Accordingly somewhat large values of $e=2.5$ cm and $d_1=0.25$ cm are adopted for vibrations of summed type.

The $\kappa-A_{max}$, $\Omega_{1\,max}$, $B$ curves in summed type are shown in Fig. 5.7. In the neighborhood of $\kappa=0$ there is another $\Omega_{1\,max}$ induced by resonant amplitude $B$ for

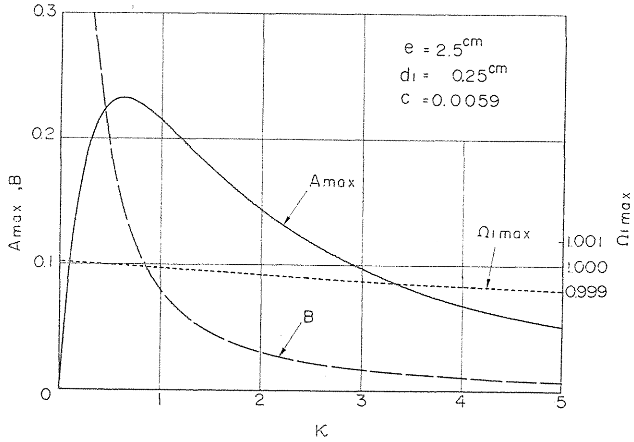



FIG. 5.7. $\kappa-A_{max}$, $\Omega_{1\,max}$, $B$ curves in summed type.

$\Omega_2 \fallingdotseq 1$, which goes away to the higher frequency side with increase of $\kappa$ and does not approach again to the resonant point $\Omega_1=1$, unlike in case of differential type. It shows that resonant frequency of $B$ is always larger than that of $A$ in case of summed type. As $\kappa$ increases, $A_{\max}$ increases and $B$ decreases, and finally $A_{\max}$ becomes larger than $B$. Amplitude $A_{\max}$ begins decreasing monotonously after it reaches its maximum value, because frequency $\Omega_2$ decreases as frequency $\Omega$ increases, as is seen from Eq. (5.10). While $\Omega_{1\max}$ decreases monotonously as $\kappa$ increases. If both magnitudes of parametric excitation $\varepsilon_1$ and disturbing force $d$ are not functions of frequency $\Omega_1$ and they are constants, $A_{\max}$ is always larger than $B$ and $A_{\max}$ being infinity at $\kappa=0$ decreases manotonously with increment of $\kappa$.

Response curves for various values of magnitude $\varepsilon_1$ of parametric excitation and damping coefficient $c$ are shown in Figs. 5.8, 5.9 severally, where the larger magnitude $\varepsilon_1$ or the smaller damping coefficient $c$ results in the larger amplitude $A$ as well as the smaller frequency $\Omega_{1\max}$. Further marks in these figures illustrate results of analog computer.

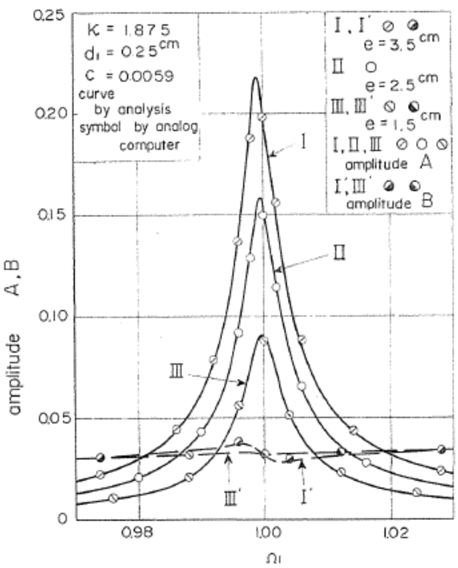



FIG. 5.8. Effect of magnitude $e$ of parametric excitation on the response curves in summed type.

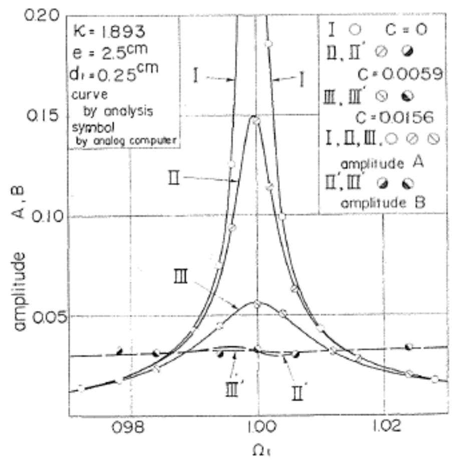



FIG. 5.9. Effect of damping coefficient $c$ on the response curves in summed type.

### *5.4. Experimental results*

Experimental apparatus consists of a single pendulum, supporting point of which is excited by both vertical vibration $e \cos \Omega t$ and horizontal vibrations $d_1 \cos \Omega_2 t$. If two frequencies $\Omega$ and $\Omega_2$ are choosen so as to satisfy relation $|\Omega \pm \Omega_2| \fallingdotseq p_1$ through two stepless transmissions, pendulum begins vibrating, and its motion is recorded optically on oscillograph paper. Experimental apparatus has the following dimensions:

$$I_1=0.273 \text{ kg}\cdot\text{cm}\cdot\text{s}^2,\ m_1g=0.711 \text{ kg},\ k=2.01 \text{ kg/cm},$$
$$b_1=17.0 \text{ cm},\ l_1=15.8 \text{ cm},\ l_0=18.5 \text{ cm},\ p_1=7.41 \text{ c/s},$$

and hence $\varepsilon_1$ and $d$ in Eq. (5.4) are found by

$$\varepsilon_1=h_1e\Omega^2=0.0512\ e\Omega^2,\ d=h_2d_1\Omega_2^2=0.9476\ d_1\Omega_2^2.$$

In experimental apparatus, vibrations can occur in differential type $[\Omega_1=\Omega-\Omega_2]$ and not in summed type $[\Omega_1=\Omega+\Omega_2]$. Since in summed type, as mentioned in Section 5.3.2, both frequencies $\Omega$ and $\Omega_2$ are always smaller than unity, and hence large eccentricities $e$ and $d_1$ enough to furnish rather large magnitudes of $\varepsilon_1$ and $d$ are obtained only by the larger values of eccentricities $e$ and $d_1$ which could not be realized in our apparatus. Accordingly, comparison between both results of experiment and analysis in this chapter is performed only in differential type.

Block diagram of analog computer used in the previous section is shown in

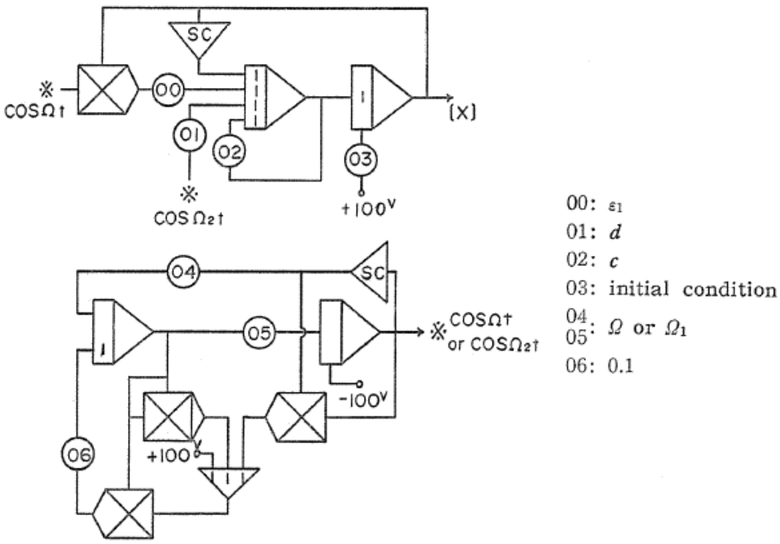



FIG. 5.10. Block diagram of analog computer.

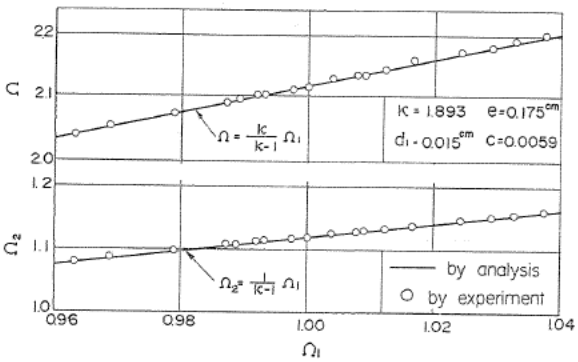



FIG. 5.11. Frequencies in differential type.

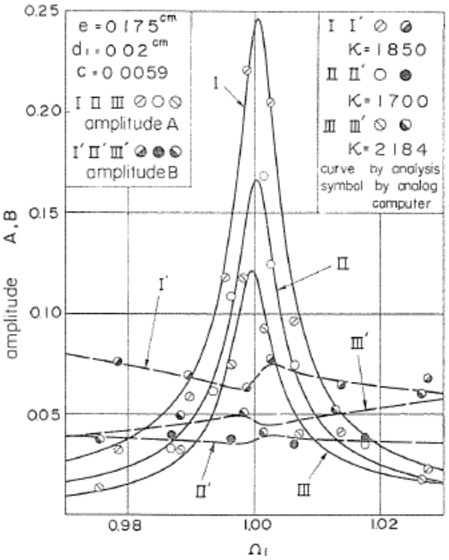



FIG. 5.12. Effect of frequency ratio on the response curves in differential type (experimental results).

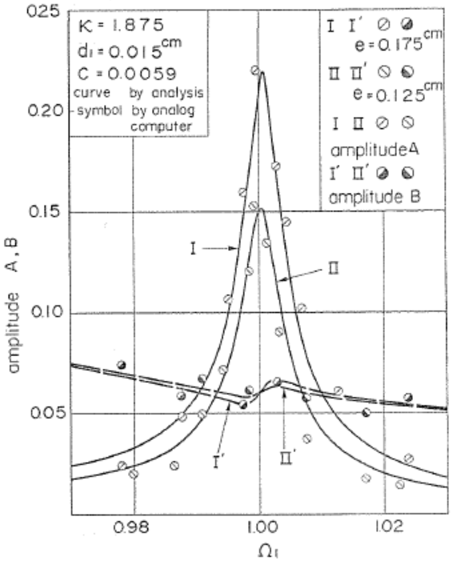



FIG. 5.13. Effect of magnitude $e$ of parametric excitation on the response curves in differential type (experimental results).

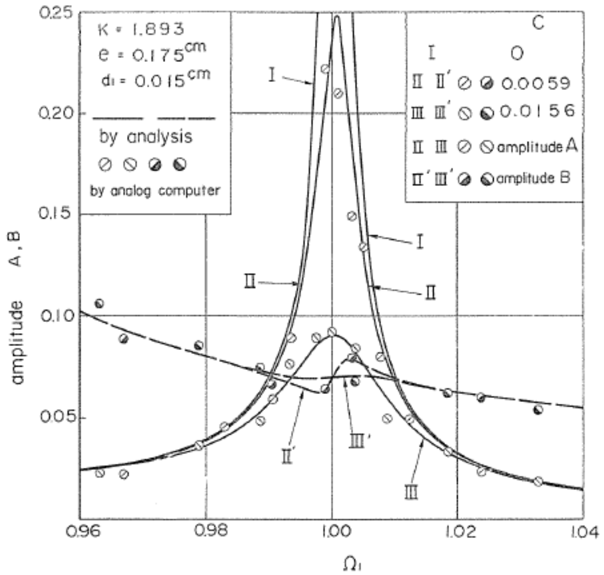



FIG. 5.14. Effect of damping coefficient $c$ on the response curves in differential type (experimental results).

Fig. 5.10, where the lower figure is sinusoidal function generator to yield two periodic excitations of frequencies $\Omega$ and $\Omega_2$.

Both frequencies $\Omega$ and $\Omega_2$ are shown against frequency $\Omega_1$ in Fig. 5.11, where the relation $\Omega_1=\Omega-\Omega_2$ holds always. In Fig. 5.11, full line curves indicate analytical results obtained by Eq. (5.10). Response curves for various values of frequency ratio $\kappa$, magnitude $e$ of parametric excitation and damping coefficient $c$ are illustrated in Figs. 5.12, 5.13 and 5.14 respectively, where experimental results agree with analytical results represented by curves and ascertain the discussion in Section 5.3.1.

Vibratory waves of differential type appeared on experimental apparatus of single pendulum are given in Fig. 5.15, in which frequencies $\Omega$ and $\Omega_2$ are known through the upper and lower vertical white lines severally, and vibratory waves change periodically their shapes at intervals of marks $A$. At an interval of mark $A$, there are 11 and 6 periods of parametric excitation and disturbing force, while forced vibration of frequency $\Omega_1$ vibrates 5 times, as shown in Fig. 5.15. It follows that the relation $\Omega_1=\Omega-\Omega_2=11-6=5$ is satisfied. Fig. 5.16 shows vibratory waves of differential type obtained by analog computer, in which the relation $\Omega_1:\Omega:\Omega_2=58:107:49$ is realized.

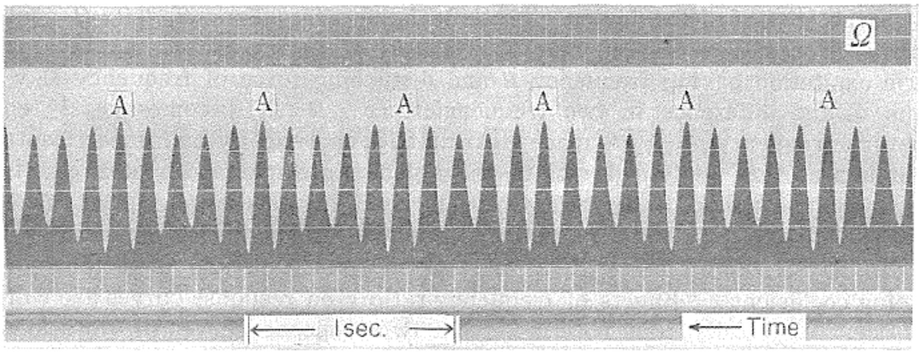



FIG. 5.15. Vibratory waves of forced vibration in differential type by experiment of single pendulum.
($\kappa=1.833$, $e=0.175$ cm, $d_1=0.015$ cm, $c=0.0059$, $\Omega_1=0.997$, $\Omega_1:\Omega:\Omega_2=5:11:6$)

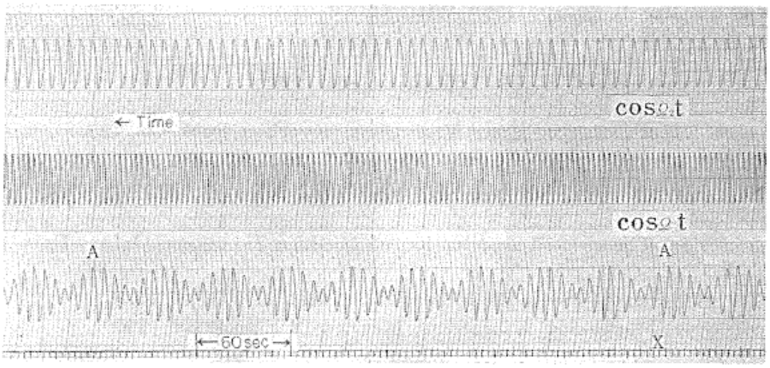



FIG. 5.16. Vibratory waves of forced vibration in differential type by analog computer. ($\kappa=2.184$, $e=0.175$ cm, $d_1=0.02$ cm, $c=0.0059$, $\Omega_1=1.002$, $\Omega_1:\Omega:\Omega_2=58:107:49$).

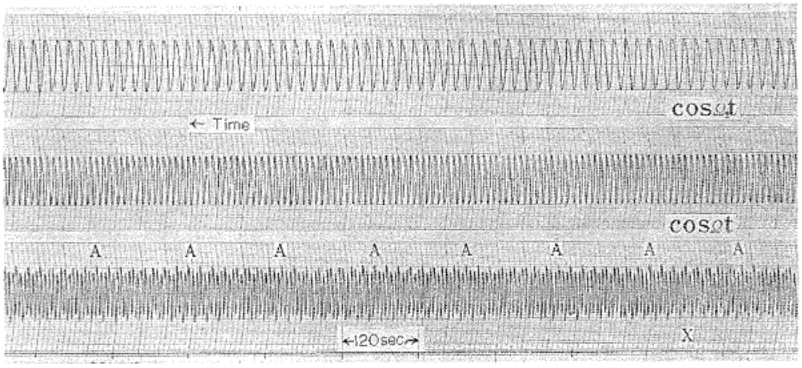



FIG. 5.17. Vibratory waves of forced vibration in summed type by analog computer. ($\kappa=1.875$, $e=3.5$ cm, $d_1=0.25$ cm, $c=0.0059$, $\Omega_1=0.994$, $\Omega_1 : \Omega : \Omega_2=23 : 15 : 8$)

Further, vibratory waves of summed type given by analog computer are illustrated in Fig. 5.17, where the relation $\Omega_1 : \Omega : \Omega_2=23 : 15 : 8$ and hence $\Omega_1 =\Omega+\Omega_2$ holds.

*5.5. Conclusions*

Characteristics of the forced vibrations having frequency $\Omega_1=|\Omega\pm\Omega_2|$, which occurs in a vibratroy system with one degree-of-freedom excited by both parametric excitation having frequency $\Omega$ and disturbing force of frequency $\Omega_2$ when a sum of and difference in two frequencies, *i.e.*, $|\Omega\pm\Omega_2|$ becomes nearly equal to natural frequency $p_1=1$, are made clear up both analytically and experimentally, and it is seen that analytical results show good agreement with both results of experiment and analog computer. Characteristics of response curves of this kind of forced vibrations are influenced by the following four factors, *i.e.*, frequency ratio $\kappa=\Omega/\Omega_2$, magnitudes $\varepsilon$, $d_1$ of parametric excitation and disturbing force, damping coefficient $c$. Effects of frequency ratio $\kappa$ on response curves are worthy of notice. Especially, it is noticeable that two response curves of vibrations of differential type when $\kappa$ is somewhat smaller and larger than 2, are quite different from each other. Further, it is seen that in vibratory system, treated in this chapter, in which both magnitudes of parametric excitation $\varepsilon_1$ and disturbing force $d$ are functions of frequency $\Omega$, vibrations of summed type [$\Omega_1=\Omega+\Omega_2$] is difficult to grow up in comparison with those of differential type [$\Omega_1=\Omega-\Omega_2$].

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