OPTIMAL CONTROL OF DISTRIBUTED PARAMETER SYSTEMS

YASUYUKI FUNAHASHI, TAKUMI NOMURA, and KAHEI NAKAMURA

Automatic Control Laboratory

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CONTENTS*

1. Introduction

The purpose of this article is to construct the general theory of optimal control for distributed parameter systems through "Functional Analysis" approach.

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* Chapter 2 is due to Funahashi.
Chapters 3 and 4 are due to Nomura.

Optimal control theory initially born in engineering interest has been much developed in the increasing need to perform with high degree complex control actions such as the automatic landing of airplane, the guidance control of flying rocket, the attitude control of space viecle and the optimum control of atomic reactor, etc. There are many studies contributed largely to the development of optimal control for the lumped parameter systems described by ordinary differential equations. But for the distributed parameter systems, there are not so many remarkable results owing to the wall of mathematical difficulty. Recently, advanced control engineers have great interest in the optimal control of distributed parameter systems. In fact, the great progress in modern industry has caused the need to develop sophisticated techniques for controlling complex and large-scale systems with the highest efficiency and the largest profits. These systems are, in general, hybrid ones composed of lumped parameter systems and distributed parameter systems. Hence the optimum control problems for distributed parameter systems have come in light in control engineering, and the unified research on optimal control theory for distributive system has now put in urgent need.

The authors, restricting themselves to the most fundamental control problems such as the optimal pursuit problem and the optimal regulator problem, develop the general optimal control theory for linear distributive systems. These problems were treated by several methods: the extended Pontryagin's maximum principle by Egorov[1)] and Wang[2)], nonlinear programming by Sakawa[3)] and variational method by authors[4)]. In this article the authors grasp the problem as the one of minimum distance problem in Hilbert spaces and solve it by the orthogonal pojection. This method is lucid and appeals to the geometrical intuition. By noticing that the feedback controller is the transformation from the state space to input space, the feedback control theory is established. These developments become possible only through Functional Analysis.

When it is desired to realize the optimal control for practical systems in real time, it is inevitable to rely on the aid of digital computer. Digital computers can not solve the problem described by the continuous-time function and the infinite dimensional state. From this point of view, discrete-time version of the optimal control theory established in chapter 2 is developed further in chapter 3. This discrete-time regulator theory is the extension of Tou's theory for lumped parameter systems. In chapter 4 an approximate computational method is developed by approximating the distributed parameter system by the finite dimensional ones. The control calculated for this new system is valuable to take as suboptimal control for the original system. The validity of this approximation method is assured by proving that the sub-optimal control converges to the true optimal one if the dimension of the approximate system approaches to infinity.

2. Optimal Pursuit Control Problems—Continuous-time Case

In this chapter the unified optimal control theory is developed for the optimal pursuit problem, the special case of which is the optimal regulator problem. In 2–1 the problem is described and formulated in mathematical terms. The solution to the problem is obtained by the orthogonal projection method. It is in the form of open loop. In control technology it is desired to design a controller in the form of closed loop. From this standpoint, in 2–3 the optimal feedback controller is designed for the regulator problem and is demonstrated that the feedback operator is characterized by the nonlinear differential equation of Riccati-type. In 2–4 the different approach is taken and the optimal feedback controller is designed for the optimal pursuit problem. In this approach the feedback operator is characterized by the linear integral equation of Fredholm type.

2–1. Control Problem

The aim of this article is to design a controller for the distributed parameter system in order to acheive desired action in some optimal way. But, for the purpose of demonstrating the generality of the theory developed in 2–2, control problems are considered for the general linear systems not necessarily described by dynamical equations. The theory is then applied to the dynamical systems including distributed parameter systems.

The input-state relation of a non-anticipatory linear system is separated in zero-input response A_i and zero-state response A_s[1].

$$x(t) = A_i(x(0) \,:\, t) + A_s(u_{(0,t]} \,;\, t) \tag{2-1-1}$$

where $x(t)$ is state at time t and belongs to a Hilbert space X.

$u_{(0,t]}$ is a function defined on the time interval $(0, t]$. $u(t)$ is control at time t and belongs to a Hilbert space U.

Often it is desired to drive the system (2–1–1) to approach the specified state $x^d(T)$ at the final time T or, to follow the specified trajectory $x^d(t)$ over the interval $(0, T)$. The difference between the state of the system (2–1–1) and the desired state is assumed to be measured by the norm of the space X. The control problem is loosely stated as the minimization of

$$\|x(T) - x^d(T)\|_X^2, \tag{2-1-2}$$

or

$$\int_0^T \|x(t) - x^d(t)\|_X^2 dt. \tag{2-1-3}$$

While, on the way of restricting control energy, two cases are obtained.

(a) The available control energy is limited and is subject to the constraint.

$$\int_0^T \|u(t)\|^2 dt \leq k \,:\, \text{const.} \tag{2-1-4}$$

The problem is to find the control u° that minimizes (2–1–2) or (2–1–3) under the restriction (2–1–4).

By letting λ be a Lagrangian multiplier, the performance index is expressed by

$$\left.\begin{aligned} J_1(u) &= \|x(T) - x^d(T)\|_X^2 + \lambda\int_0^T \|u(t)\|_U^2 dt \\ &\text{or} \\ J_2(u) &= \int_0^T \|x(t) - x^d(t)\|_X^2 dt + \lambda\int_0^T \|u(t)\|_U^2 dt \end{aligned}\right\} \tag{2-1-5}$$

(b) Though the available energy is not limited it is desired to minimize (2–1–2) or (2–1–3) by using the least control energy possible. In this case the performance index is expressed, letting r a weighting factor, by

$$\left.\begin{aligned} J_1(u) &= \|x(T)-x^d(T)\|_X^2 + r\int_0^T \|u(t)\|_U^2 dt \\ &\text{or} \\ J_2(u) &= \int_0^T \|x(t)-x^d(t)\|_X^2 dt + r\int_0^T \|u(t)\|_U^2 dt \end{aligned}\right\}. \tag{2-1-6}$$

Replacing $\sqrt{\lambda}\,u(t)$ in (2-1-5) and $\sqrt{r}\,u(t)$ in (2-1-6) by new $u(t)$, the above two problems are formulated in the same form and the problem is to find out the optimal control which minimizes the performance index

$$J_1(u) = \|x(T)-x^d(T)\|_X^2 + \int_0^T \|u(t)\|_U^2 dt, \tag{2-1-7}$$

or

$$J_2(u) = \int_0^T \|x(t)-x^d(t)\|_X^2 dt + \int_0^T \|u(t)\|_U^2 dt. \tag{2-1-8}$$

Define the operators K_1 and K_2 by

$$\left.\begin{aligned} K_1 u &= A_s(u_{(0,T]}\ ;\ T), \\ (K_2 u)(t) &= A_s(u_{(0,T]}\ ;\ t), \qquad 0 \le t \le T. \end{aligned}\right\} \tag{2-1-9}$$

and c_1 and c_2 by

$$\left.\begin{aligned} c_1 &= x^d(T) - A_i(x(0)\ ;\ T), \\ c_2(t) &= x^d(t) - A_i(x(0)\ ;\ t), \qquad 0 \le t \le T. \end{aligned}\right\} \tag{2-1-10}$$

respectively. By using these notations, Eqs (2-1-7) and (2-1-8) can be written as

$$J_1(u) = \|K_1 u - c_1\|_X^2 + \int_0^T \|u(t)\|_U^2 dt \tag{2-1-11}$$

and

$$J_2(u) = \int_0^T \|(K_2 u)(t) - c_2(t)\|_X^2 dt + \int_0^T \|u(t)\|_U^2 dt \tag{2-1-12}$$

Let $L^2(0, T : X)$ be a linear space of X-valued square integrable functions defined on $(0, T)$, *i.e.* a linear space of functions $y(t)$ such that

$$y(t) \in X \text{ a.e. in } t,$$

and

$$\int_0^T \|y(t)\|_X^2 dt < +\infty.$$

Introduce an inner product into this space by

$$\langle x, y\rangle_{L^2(0,T:X)} = \int_0^T \langle x(t), y(t)\rangle_X dt,$$

and the norm deduced from inner product by

$$\|x\|^2_{L^2(0, T : X)} = \int_0^T \|x(t)\|^2_X dt.$$

This $L^2(0, T ; X)$ is a Hilbert space. By using this notation Eqs. (2-1-11) and (2-1-12) can be once more written

$$J_1(u) = \|K_1 u - c_1\|^2_X + \|u\|^2_{L^2(0, T ; U)}, \tag{2-1-13}$$

$$J_2(u) = \|K_2 u - c_2\|^2_{L^2(0, T ; X)} + \|u\|^2_{L^2(0, T ; U)}. \tag{2-1-14}$$

By definding $H_1 = L^2(0, T ; U)$ and $H_2 = X$ or $L^2(0, \mathrm{T} ; X)$, Eqs. (2-1-13) and (2-1-14) are written in the same equation.

$$J(u) = \|Ku - c\|^2_{H_2} + \|u\|^2_{H_1}. \tag{2-1-15}$$

So far the operator K is assumed only a linear operator in H_1 into H_2. But the operator of physical system is always a closed operator. In the sequel, it is assumed that the operator K is a closed operator with domain $\mathfrak{D}(K)$ dense in H_1. Thus the optimal control is finally phrased as follows: Find the control which gives

$$\inf_{u \in \mathfrak{D}(K)} J(u). \tag{2-1-16}$$

In a special case where Eq. (2-1-1) is the integral form of a dynamical system *)

$$\frac{d}{dt} x(t) = Ax(t) + Bu(t), \tag{2-1-17}$$

the operators K_1 and K_2, and C_1 and C_2 are given as follows:

$$K_1 u = \int_0^T S(T - \sigma) Bu(\sigma) d\sigma \tag{2-1-18}$$

$$(K_2 u)(t) = \int_0^t S(t - \sigma) Bu(\sigma) d\sigma, \qquad 0 \le t \le T \tag{2-1-19}$$

$$c_1 = x^d(T) - S(T) x(0) \tag{2-1-20}$$

$$c_2(t) = x^d(t) - S(t) x(0), \qquad 0 \le t \le T. \tag{2-1-21}$$

And the general theory developed in the next section is applicable to the system (2-1-17).

2-2. Solution by Orthogonal Projection[4]

2-2-1. Problem Reformulation

Following J. von Neumman[5], define the operator K by its graph $\mathfrak{G}(K)$ and give a geometric image to the closed operator. This makes it possible to view the control problem in a different standpoint and to demonstrate the existence and uniqueness of the optimal control.

Define the product space of input space H_1 and the state space H_2 by

*) Concerning the representation of Eq. (2-1-17) and the fact that Eq. (2-1-17) represents the general control system, see reference 2 and 3.

$$H = H_1 \times H_2$$
$$= \{(u,\ x)\ ;\ \ u \in H_1,\ \ x \in H_2\}.$$

Introduce into H scalar multiplication and addition in a natural way as

$$\alpha(u,\ x) = (\alpha u,\ \alpha x) \tag{2-2-1}$$

$$(u_1,\ x_1) + (u_2,\ x_2) = (u_1 + u_2,\ x_1 + x_2) \tag{2-2-2}$$

and inner product by

$$\langle (u_1,\ x_1),\ (u_2,\ x_2)\rangle_H = \langle u_1,\ u_2\rangle_{H_1} + \langle x_1,\ x_2\rangle_{H_2}. \tag{2-2-3}$$

Induce on H a norm determined by the inner product.

$$\|(u,\ x)\|^2 = \langle (u,\ x),\ (u,\ x)\rangle$$
$$= \|u\|^2_{H_1} + \|x\|^2_{H_2}. \tag{2-2-4}$$

Then H is complete with respect to this norm, *i.e.* H is a Hilbert space.

Definition: The graph $\mathfrak{G}(K)$ of an operator K is a subspace of H defined by

$$\mathfrak{G}(K) = \{(u,\ Ku)\ ;\ u \in \mathfrak{D}(K)\}.$$

In particular, if the operator K is a closed operator if and only if $\mathfrak{G}(K)$ is a closed subspace.

Recalling that the distance between a point $(u,\ Ku)$ on $\mathfrak{G}(K)$ and a point $(0,\ c)$ is given by

$$\|(u,\ Ku) - (0,\ c)\|_H = \|(u,\ Ku - c)\|_H$$
$$= \sqrt{J(u)}, \tag{2-2-5}$$

it is found that the control problem is equivalent to finding the minimum distance between $\mathfrak{G}(K)$ and $(0,\ c)$. As K is assumed to be closed, $\mathfrak{G}(K)$ is closed. From the well known theorem[5] that in a Hilbert space any closed convex set contains a unique minimum norm element, the existence of $u^\circ \in \mathfrak{D}(K)$ such that gives the infinimum of Eq. (2-2-5) and its uniqueness are assured. Thus the existence and uniqueness of the optimal control u° for the problem are proved.

Moreover, from the orthogonal projection theorem[7] $(u^\circ,\ Ku^\circ)$ is given by the orthogonal projection onto $\mathfrak{G}(K)$ of $(0,\ c)$.

2-2-2. Optimal Control

Let P be an orthogonal projection onto $\mathfrak{G}(K)$. Then, as shown in the previous section, the optimal control u° is given by

$$(u^\circ,\ Ku^\circ) = P(0,\ c). \tag{2-2-6}$$

In the following, the explicit representation of the operator P will be given.

Define an orthogonal complement of $\mathfrak{G}(K)$ by

$$\mathfrak{G}(K)^\perp = \{h \in H\ ;\ \langle g,\ h\rangle = 0,\ {}^\forall g \in \mathfrak{D}(K)\}$$

From the orthogonal projection theorem, H is decomposed orthogonally as

$$H=\mathfrak{G}(K)\oplus\mathfrak{G}(K)^{\perp},$$

This means that any h in H is represented uniquely by

$$h=h_1+h_2 \qquad h_1\in\mathfrak{G}(K), \qquad h_2\in\mathfrak{G}(K)^{\perp} \tag{2-2-7}$$

The $\mathfrak{G}(K)^{\perp}$ is, as shown in the following, just the graph $\mathfrak{G}(K^*)$ of the adjoint operator K^*. Let us define the adjoint operator.

Definition: Let K be a closed operator with dense domain in H_1. If for some $x\in H_2$, there exists a $v\in H_1$ satisfying

$$\langle Ku,\ x\rangle_{H_2}=\langle u,\ v\rangle_{H_1} \quad \text{for} \quad {}^{\forall}u\in\mathfrak{D}(K), \tag{2-2-8}$$

the operator defined by this correspondence is called an adjoint operator of K and is denote by K^*, *i.e.*

$$v=K^*x.$$

The domain $\mathfrak{D}(K^*)$ of K^* is given by

$$\mathfrak{D}(K^*)=\{x\in H_2;\ \text{for this } x, \text{ there exists } v \text{ satisfying (2-2-8)}\}.$$

By rewriting Eq. (2–2–8),

$$\langle Ku,\ x\rangle_{H_2}=\langle u,\ K^*x\rangle_{H_1}$$
$$-\langle u,\ K^*x\rangle_{H_1}+\langle Ku,\ x\rangle_{H_2}=0,$$

and recalling Eq. (2–2–3), the following relation is obtained.

$$\langle (u,\ Ku),\ (-K^*x,\ x)\rangle_H=0 \qquad {}^{\forall}u\in\mathfrak{D}(K),\ {}^{\forall}x\in\mathfrak{D}(K^*). \tag{2-2-9}$$

That is, $\{(-K^*x,\ x)\ ;\ x\in\mathfrak{D}(K^*)\}$ is orthogonal to $\mathfrak{G}(K)$.

Introduce a continuous linear operator V on $H_2\times H_1$ into $H=H_1\times H_2$ by

$$V(x,\ u)=(-u,\ x). \tag{2-2-10}$$

Then

$$\begin{aligned}&\{(-K^*x,\ x)\ ;\ x\in\mathfrak{D}(K^*)\}\\ &=\{V(x,\ K^*x)\ ;\ x\in\mathfrak{D}(K^*)\}\\ &=V\mathfrak{G}(K^*).\end{aligned}$$

It is shown that $\mathfrak{G}(K)$ and $V\mathfrak{G}(K^*)$ are orthogonal to each other, *i.e.*

$$H=\mathfrak{G}(K)\oplus V\mathfrak{G}(K^*). \tag{2-2-11}$$

Now we are in a position to represent the othogonal projection $P(u,\ x)$ of $(u,\ x)$ onto $\mathfrak{G}(K)$. By the linearity of P and

$$(u,\ x)=(u,\ 0)+(0,\ x), \tag{2-2-12}$$

it follows that

$$P(u,\ x)=P(u,\ 0)+P(0,\ x). \tag{2-2-13}$$

First, $P(0, x)$ is to be obtained. By Eq. (2-2-11), $(0, x)$ is uniquely represented through suitable $v \in \mathfrak{D}(K)$ and $y \in \mathfrak{D}(K^*)$ by

$$(0, x) = (v, Kv) + (-K^* y, y) \tag{2-2-14}$$

i.e.

$$0 = v - K^* y$$
$$x = Kv + y$$

From these relations,

$$[I + KK^*] y = x. \tag{2-2-15}$$

$I + KK^*$ is self-adjoint, positive definite operator on X, and hence is invertible.

$$y = [I + KK^*]^{-1} x,$$

Thus the orthogonal projection of $(0, x)$ is obtained.

$$P(0, x) = (v, Kv)$$
$$= (K^*[I + KK^*]^{-1} x,\ KK^*[I + KK^*]^{-1} x). \tag{2-2-16}$$

In the same way as above, $P(u, 0)$ can be obtained.

$$P(u, 0) = ([I + K^*K]^{-1} u,\ K[I + K^*K]^{-1} u) \tag{2-2-17}$$

$$\therefore\ P(u, x) = ([I + K^*K]^{-1} u + K^*[I + KK^*]^{-1} x,$$
$$K[I + K^*K]^{-1} u + KK^*[I + KK^*]^{-1} x). \tag{2-2-18}$$

For the optimal control problem at hand, the orthogonal projection of $(0, c)$ is wanted. From Eq. (2-2-6),

$$(u^\circ, Ku^\circ) = P(0, c)$$
$$= (K^*[I + KK^*]^{-1} c,\ KK^*[I + KK^*]^{-1} c). \tag{2-2-19}$$

That is, the optimal control u° is given by

$$u^\circ = K^*[I + KK^*]^{-1} c. \tag{2-2-20}$$

In particular, if c belongs to $\mathfrak{D}(K^*)$ or if K is a bounded operator, Eq. (2-2-20) can be written in another form.

$$u^\circ = [I + K^*K]^{-1} K^* c. \tag{2-2-21}$$

2-2-3. Examples

In order to show the application of the theory developed so far, optimal controllers will be designed for some illustrative systems. Ex. 1 is for the purpose of demonstrating the width of applicability of the theory. Ex. 2 and Ex. 3 treats the control problems for the distributed parameter systems.

Ex. 1. Optimal Pursuit Problem for a Differentiator

Let us consider a differentiator whose input-output relation is given by the following equation:

$$y(t) = \frac{d}{dt}u(t) \qquad t>0. \tag{2-2-22}$$

Here it is desired to control the differentiator such that the performence index given by Eq. (2-2-23) is minimized.

$$J(u) = \int_0^T |y(t) - c(t)|^2 dt + \int_0^T |u(t)|^2 dt, \tag{2-2-23}$$

where $c(t)$ is the desired output which is to be followed and T is the fixed time. It is required from Eq. (2-2-23) that both the input space U and the output space Y be $L^2(0, T)$. Define an operator K from U into Y by

$$y = Ku \quad i.e. \quad y(t) = u'(t), \tag{2-2-24}$$

where the domain $\mathfrak{D}(K)$ of K is given by

$$\mathfrak{D}(K) = \{u \in L^2(0, T);\ u \text{ is absolutely continuous on } (0, T) \text{ and } u' \text{ belongs to } L^2(0, T)\}.$$

The operator K is easily proved to be a closed operator and $\mathfrak{D}(K)$ is dense in $U = L^2(0, T)$. Then from the general theory it is assured that the optimal control exists uniquely.

The adjoint of K is defined, from Eq. (2-2-8), by

$$\langle Ku, z\rangle_Y = \langle u, K^*z\rangle_U \qquad \forall u \in \mathfrak{D}(K). \tag{2-2-25}$$

By defining the domain of K^* by

$$\mathfrak{D}(K^*) = \{z \in L^2(0, T);\ z \text{ is absolutely continuous and } z' \text{ belongs to } L^2(0, T) \text{ and } z(T) = 0\},$$

the adjoint operator K^* is found to be

$$(K^*z)(t) = -\frac{d}{dt}z(t). \tag{2-2-26}$$

Put $g = [I + KK^*]^{-1}c$, then the optimal control u is given by $u^\circ = K^*g$ from Eq. (2-2-20). From the relation

$$[I + KK^*]g = c, \tag{2-2-27}$$

it must hold that g belongs to $\mathfrak{D}(K^*)$ and K^*g to $\mathfrak{D}(K)$, *i.e.*,

$$g(T) = 0 \quad \text{and} \quad (K^*g)(0) = 0. \tag{2-2-28}$$

The solution of Eq. (2-2-27) under the condition Eq. (2-2-28) is

$$g(t) = \frac{\cosh t}{\cosh T}\int_0^T c(s) \sinh (T-s)\, ds - \int_0^t c(s) \sinh (T-s)\, ds. \tag{2-2-29}$$

The optimal control $u^\circ = K^*g$ is

$$u^\circ(t) = \int_0^t c(s) \cosh (t-s)\, ds - \frac{\cosh t}{\cosh T}\int_0^T c(s) \sinh (T-s)\, ds. \tag{2-2-30}$$

Ex. 2. Final Value Problem—Distributed Parameter Systems with Spatially Distributed Control

Consider the system described by the partial differential equation:

$$\frac{\partial}{\partial t} x(t, \alpha) = \frac{\partial^2}{\partial \alpha^2} x(t, \alpha) + u(t, \alpha), \qquad 0<\alpha<1, \tag{2-2-31}$$

with boundary conditions $x(t, 0)=x(t, 1)=0$.

In the case of heat conduction system, for example, $x(t, \alpha)$ denotes the temperature at time t and at spatial coordinate α, and $u(t, \alpha)$ denotes the heating source at t and α.

Given the specified temperature distribution $x^d(T, \alpha)$, it is desired to bring the $x(T, \alpha)$ of the system (2-2-31) to $x^d(T, \alpha)$ as nearly as possible with the least control energy. That is, it is desired to minimize the performance index

$$J_1(u) = \int_0^1 |x(T, \alpha) - x^d(T, \alpha)|^2 d\alpha + \int_0^T \int_0^1 |u(t, \alpha)|^2 d\alpha\, dt. \tag{2-2-32}$$

By defining

$$x(t) = x(t, \cdot)$$
$$u(t) = u(t, \cdot)$$
$$\mathfrak{D}(A) = \{y \in L^2(0, 1) | y(\alpha) \text{ and } y'(\alpha) \text{ are absolutely continuous and } y''(\alpha) \in L^2(0, 1)\}$$
$$(Ay)(\alpha) = \frac{d^2}{d\alpha^2} y(\alpha) \quad \text{on } \mathfrak{D}(A),$$

Eq. (2-2-3) can be written as

$$\frac{d}{dt} x(t) = Ax(t) + u(t). \tag{2-2-33}$$

For any control $u \in L^2(0, T; L^2(\Omega))$, $\Omega=(0, 1)$, the trajectory of Eq. (2-2-33) is given by

$$x(t) = S(t)\, x(0) + \int_0^t S(t-\sigma)\, u(\sigma)\, d\sigma \tag{2-2-34}$$

where $S(t)$ is the semi-group generated by A and is given by

$$S(t)\, x = \sum_{n=1}^{\infty} e^{\lambda_n t} x_n \phi_n. \tag{2-2-35}$$

Eq. (2-2-34) is explicitly written as

$$x(t, \alpha) = \sum_{n=1}^{\infty} e^{\lambda_n t} x_n(0)\, \phi_n(\alpha) + \sum_{n=1}^{\infty} \int_0^t e^{\lambda_n (t-\sigma)} u_n(\sigma)\, d\sigma\, \phi_n(\alpha), \tag{2-2-36}$$

where,

$$\left.\begin{aligned}
&\lambda_n = -(n\pi)^2 \\
&\phi_n(\alpha) = \sqrt{2}\sin n\pi\alpha \\
&x_n(t) = \int_0^1 x(t,\alpha)\,\phi_n(\alpha)\,d\alpha \\
&u_n(t) = \int_0^1 u(t,\alpha)\,\phi_n(\alpha)\,d\alpha \qquad n=1,2,\dots
\end{aligned}\right\} \tag{2-2-37}$$

The operator defined by

$$Ku = \int_0^T S(T-\sigma)\,u(\sigma)\,d\sigma, \tag{2-2-38}$$

is a bounded operator on $L^2(0, T; L^2(\Omega))$ into $L^2(\Omega)$. The adjoint operator K^* of K is a bounded operator on $L^2(\Omega)$ into $L^2(0, T; L^2(\Omega))$, and is given by

$$(K^*x)(t) = S(T-t)x \qquad 0\leqq t\leqq T. \tag{2-2-39}$$

By putting $c=x'(T)-S(T)x(0)$ and $y=[I+KK^*]^{-1}c$, the optimal control u° is, from Eq. (2-2-20), given by

$$u^\circ = K^*y. \tag{2-2-40}$$

From the relation,

$$[I+KK^*]y = c \tag{2-2-41}$$

y must be determined.

Expansion in Fourier series of both sides of Eq. (2-2-41) leads to

$$y_n + \frac{1}{2}\int_0^{2T} e^{\lambda_n\sigma}\,d\sigma\, y_n = c_n,$$

$$i.e.\quad y_n = \frac{c_n}{1+\dfrac{1-e^{-2(n\pi)^2T}}{2(n\pi)^2}}. \tag{2-2-42}$$

where

$$y_n = \langle y, \phi_n\rangle$$
$$c_n = \langle c, \phi_n\rangle.$$

The optimal control u° is thus obtained.

$$\begin{aligned}
u^\circ(t) &= (K^*y)(t) \\
&= \sum_{n=1}^{\infty} \frac{c_n}{1+\dfrac{1-e^{-2(n\pi)2T}}{2(n\pi)^2}}\, e^{-(n\pi)^2(T-t)}\phi_n
\end{aligned} \tag{2-2-43}$$

where

$$c_n = \langle x^d, \phi_n\rangle - e^{-(n\pi)^2T}x_n(0). \tag{2-2-44}$$

Ex. 3. Optimal Pursuit Problem

Consider the same system as Ex. 2 and notations are the same as in Ex. 2.

It is desired now for the system to follow the specified trajectory $x^d(t, \cdot)$ with the least control energy possible, *i.e.* to minimize the performance index,

$$J(u) = \int_0^T\int_0^1 |x(t, \alpha) - x^d(t, \alpha)|^2 d\alpha\, dt + \int_0^T\int_0^1 |u(t, \alpha)|^2 d\alpha\, dt. \qquad (2\text{-}2\text{-}45)$$

Define the operator L on $L^2(0, T;\ L^2(\Omega))$ into itself, by

$$(Lu)(t) = \int_0^t S(t-\sigma)\, u(\sigma)\, d\sigma, \qquad 0 \leq t \leq T. \qquad (2\text{-}2\text{-}46)$$

The adjoint operator L^* of L is

$$(L^*x)(t) = \int_t^T S(\sigma - t)\, x(\sigma)\, d\sigma. \qquad (2\text{-}2\text{-}47)$$

Put $c(t) = x^d(t) - S(t)x(0)$. The optimal control u° is characterized, from Eq. (2-2-21), by

$$[I + L^*L]u = L^*c. \qquad (2\text{-}2\text{-}48)$$

Expand both sides of Eq. (2-2-48) in Fourier sieres.

$$u_n(t) + \int_t^T de \int_0^\sigma e^{\lambda_n(2\sigma - t - \tau)} u_n(\tau)\, dt = \int_t^T e^{\lambda_n(\sigma - t)} c_n(\sigma)\, d\sigma. \qquad (2\text{-}2\text{-}49)$$

If $c(t)$ is differentiable almost everywhere, Eq. (2-2-49) is twice differentiable and is equivalent to

$$\left.\begin{aligned} &\ddot{u}_n(t) - (1+\lambda_n^2)\, u_n(t) = \lambda_n u_n(t) - \dot{u}_n(t), \\ &\dot{u}_n(0) + \lambda_n u_n(0) = -c_n(0), \\ &u_n(T) = 0. \end{aligned}\right\}. \qquad (2\text{-}2\text{-}50)$$

The solution of Eq. (2-2-50) is, by letting $\beta_n = \sqrt{1+\lambda_n^2}$,

$$u_n(t) = \frac{\cosh \beta_n t - \dfrac{\lambda_n}{\beta_n}\sinh \beta_n t}{\cosh \beta_n T - \dfrac{\lambda_n}{\beta_n}\sinh \beta_n T} \int_0^T c_n(\sigma)\left[\cosh \beta_n(T-\sigma) - \frac{\lambda_n}{\beta_n}\sinh \beta_n(T-\sigma)\right] d\sigma$$

$$- \int_0^t c_n(\sigma)\left[\cosh \beta_n(t-s) - \frac{\lambda_n}{\beta_n}\sinh \beta_n(t-s)\right] d\sigma. \qquad (2\text{-}2\text{-}51)$$

The optimal control u° is thus obtained.

$$u^\circ(t) = \sum_{n=1}^\infty u_n(t)\, \phi_n, \qquad (2\text{-}2\text{-}52)$$

where $u_n(t)$ is given by Eq. (2-2-51).

2-3. Optimal Feedback Control—Riccati-type Differential Equation

In this section we restrict the control problem to the one in which the specified trajectory or state are zero. This problem is called regulator problem and one of the most important control problems. The optimal feedback controllers will be designed for this problem.

For the case of dynamical systems with finite dimensional state space, Kalman gave a feedback as the solution of nonlinear differential equation of Riccati-type[9]. He derived the result from Hamilton-Jacobi equation. There are some extension of Kalman's theory to distributed parameter systems. Wang and Tung[10] developed their theory for a special distributed parameter system with spatially distributed control. Falb and Kleinman[11] extend Kalman's theory to systems in Hilbert space. But the theory is restricted to systems whose infinitesimal generators are bounded operators and is not applicable to distributed systems.

The derivation of the theory developed here is quite different from Kalman's. The key point is the fact that the feedback operator is the mapping from the state space into itself. The feedback operator is defined geometrically and its properties are examined. One of them shows that the feedback operator satisfies the nonlinear differential equation of Riccati-type in Hilbert space. This equation corresponds to the one which Kalman has obtained for the finite dimensional dynamic systems. Another property shows that the value of the performance index under the optimal control is given by the quadratic form in its initial state. This property is just the assumption on which Kalman and others have constructed their theory. But the validity of the assumption is not clear. The theory in this chapter needs no assumption and reveals the meaning of the feedback operator.

2-3-1. Control Problems

We restrict the control problem in 2-2 to the following: the system is given by

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) \tag{2-3-1}$$

and the desired trajectory or state be zero, *i.e.* the performance indices of Eq. (2-1-7) or (2-1-8) are

$$J_1(u) = \|x(T)\|_X^2 + \int_0^T \|u(t)\|_U^2\, dt \tag{2-3-2}$$

$$J_2(u) = \int_0^T \|x(t)\|_X^2\, dt + \int_0^T \|u(t)\|_U^2\, dt \tag{2-3-3}$$

respectively.

The problem is to design a feedback controller whose action at time t is expressed by the state at time t only.

Define

$$\left.\begin{aligned} Lu &= \int_0^T S(T-\sigma)Bu(\sigma)\, d\sigma, \\ d &= -S(T)x(0), \\ (Ku)(t) &= \int_0^t S(t-\sigma)Bu(\sigma)\, d\sigma, \\ c(t) &= -S(t)x(0). \end{aligned}\right\} \tag{2-3-4}$$

Then the optimal control for the problem of Eq. (2-3-2) is given, from the theory in 3.2, by

$$u^\circ = [I+L^*L]^{-1}L^*d, \tag{2-3-5}$$

and in the case of Eq. (2-3-3), by

$$u^\circ = [I+K^*K]^{-1}K^*c, \tag{2-3-6}$$

where

$$(L^*u)(t) = B^*S^*(T-t)\,u, \qquad 0 \leq t \leq T, \tag{2-3-7}$$

$$(K^*u)(t) = \int_t^T B^*S^*(\sigma - t)\,u d\sigma, \qquad 0 \leq t \leq T. \tag{2-3-8}$$

2-3-2. Optimal Feedback Controller for Final Value Problem

Rewrite Eq. (2-3-5) as

$$[I+L^*L]\,u = L^*d$$

or

$$u = L^*(d-Lu) \tag{2-3-9}$$

where

$$\begin{aligned} d - Lu^\circ &= -S(T)\,x(0) - \int_0^T S(T-\sigma)\,Bu^\circ(\sigma)\,d\sigma \\ &= -x^\circ(T) \end{aligned}$$

$x^\circ(T)$ denotes the state at the final time T under the optimal control. Thus the optimal control is given by

$$u^\circ(t) = -B^*S^*(T-t)\,x^\circ(T). \tag{2-3-10}$$

If there exists an operator such that

$$Q(t)\,x^\circ(t) = S^*(T-t)\,x^\circ(T), \tag{2-3-11}$$

the optimal control can be expressed in the form of feedback control.

$$u^\circ(t) = -B^*Q(t)\,x^\circ(t). \tag{2-3-12}$$

To get the geometric image of the operator $Q(t)$, consider the system

$$q(t) = A^*q(t). \tag{2-3-13}$$

A^* denotes the adjoint operator of A. Since A is an infinitesimal generator of a strongly continuous semi-group $S(t)$, $t \geqq 0$, A^* also generates a strongly continuous semi-group. The generated semi-group is the same as the adjoint of $S(t)$, *i.e.* $S^*(t)$.

$$\left.\begin{aligned} &S^*(t)\,S^*(\tau) = S^*(t+\tau) && {}^\forall t,\ \tau \geqq 0 \\ &\lim_{t\downarrow 0} S^*(t)\,x = x && {}^\forall x \in X \\ &\frac{d}{dt}S^*(t)\,x = A^*S^*(t)\,x && {}^\forall x \in \mathfrak{D}(A^*) \end{aligned}\right\} \tag{2-3-14}$$

The solution of Eq. (2-3-13) under the initial condition $q(0)\in\mathfrak{D}(A^*)$ is $q(t)=S^*(t)q(0)$. Let $z(t)=q(T-t)$. $z(t)$ satisfies

$$\frac{d}{dt}z(t) = -A^*z(t). \tag{2-3-15}$$

This system represents the backward response of the adjoint system of Eq. (2-3-13). Given the initial state $x^\circ(T)$, the state at time t of Eq. (2-3-15) is

$$\begin{aligned} z(t) &= S^*(T-t)z(T) \\ &= S^*(T-t)x^\circ(T). \end{aligned} \tag{2-3-16}$$

Thus, the operator $Q(t)$ defined by Eq. (2-3-11) can be considered an operator mapping the state $x^\circ(t)$ at time t under the optimal control into the state $S^*(T-t)x^\circ(T)$ attained at time t by Eq. (2-3-15) under free motion.

In the following, the various properties of $Q(t)$ are examined and the feedback controller will be designed.

Property 1. The operator defined by Eq. (2-3-11) is linear and unique on $\mathfrak{D}(A)$.

Since the right-hand side $S^*(T-t)x^\circ(T)$ of Eq. (2-3-11) is strongly continuous in t and differentiable, the $Q(t)$ must be differentiable.

$$\begin{aligned} \left[\frac{d}{dt}Q(t)\right]x^\circ(t) + Q(t)\frac{d}{dt}x^\circ(t) &= -A^*S^*(T-t)x^\circ(T) \\ &= -A^*Q(t)x^\circ(t). \end{aligned} \tag{2-3-17}$$

On the other hand, Eq. (2-3-1) becomes under the optimal feedback control

$$u^\circ(t) = -B^*Q(t)x^\circ(t),$$

$$\frac{d}{dt}x^\circ(t) = Ax^\circ(t) - BB^*Q(t)x^\circ(t).$$

Substituting this relation into Eq. (2-3-17), we obtain

$$\left[\frac{d}{dt}Q(t) + A^*Q(t) + Q(t)A - Q(t)BB^*Q(t)\right]x^\circ(t) = 0.$$

Since this equation must hold for all the optimal trajectories, $Q(t)$ satisties

$$\frac{d}{dt}Q(t) + A^*Q(t) + Q(t)A - Q(t)BB^*Q(t) = 0. \tag{2-3-18}$$

The boundary condition for $Q(t)$ is obtained, by letting $t=T$ in Eq. (2-3-11) as

$$Q(T)x^\circ(T) = S^*(0)x^\circ(T)$$

i.e.

$$Q(T) = I.$$

It is thus proved that

Property 2. The operator $Q(t)$ defined by Eq. (2-3-11) is strongly differentiable and satisfies the differential equation of Riccati-type.

$$\left.\begin{aligned} \frac{d}{dt}Q(t) + A^*Q(t) + Q(t)A - Q(t)BB^*Q(t) = 0 \\ Q(T) = I \end{aligned}\right\} \tag{2-3-19}$$

Let $Q(t)$ be the solution of Eq. (2–3–19). The adjoint operator $Q^*(t)$ is easily seen to satisgy Eq. (2–3–19). Combining this with Property 1, the following is obtained.

Property 3. The operator $Q(t)$ is symmetric, *i.e.*

$$Q^*(t) = Q(t). \tag{2-3-20}$$

In the following, it is shown that the value of the performance index under the optimal control is given by $\langle x(0),\ Q(0)x(0)\rangle_X$.

Differentiate the scalar $\langle x^\circ(t),\ Q(t)x^\circ(t)\rangle_X$ with respect to time t.

$$\begin{aligned} &\frac{d}{dt}\langle x^\circ(t),\ Q(t)x^\circ(t)\rangle_X \\ &= -\|u^\circ(t)\|_U^2. \end{aligned}$$

Integrate the above equation from 0 to T.

$$\int_0^T \frac{d}{dt}\langle x^\circ(t),\ Q(t)x^\circ(t)\rangle_X dt = -\int_0^T \|u^\circ(t)\|_U^2 dt.$$

Remembering $Q(T)=I$, $x^\circ(0)=x(0)$, we obtain

$$\|x^\circ(\tau)\|_X^2 + \int_0^T \|u^\circ(t)\|_U^2 dt = \langle x(0),\ Q(0)x(0)\rangle_X.$$

Thus we have

Property 4. The feedback control $u^\circ(t) = -B^*Q(t)x^\circ(t)$, where $Q(t)$ is determined from Eq. (2–3–11), minimizes the performance index

$$I(u) = \|x(T)\|_X^2 + \int_0^T \|u(t)\|_U^2 dt,$$

and the minimum value of $I(u)$ is given by

$$I(u^\circ) = \langle x(0),\ Q(0)x(0)\rangle_X. \tag{2-3-21}$$

2-3-3. Optimum Feedback Control for Regulator Problem

The optimal control is given by Eq. (2–3–6).

$$u^\circ = [I + K^*K]^{-1}K^*c$$

Rewrite this equation as

$$u^\circ = K^*(c - Ku^\circ)$$

Where

$$(c-Ku^\circ)(t) = c(t) - (Ku^\circ)(t)$$
$$= -S(t)x(0) - \int_0^t S(t-\sigma)Bu^c(\sigma)\,d\sigma$$
$$= -x^\circ(t).$$

That is, the optimal control is given by,

$$u^\circ(t) = -\int_t^T B^*S^*(\sigma-t)x^\circ(\sigma)\,d\sigma. \tag{2-3-22}$$

If there exists an operator such that

$$P(t)x^\circ(t) = \int_t^T S^*(\sigma-t)x^\circ(\sigma)\,d\sigma, \tag{2-3-23}$$

then the optimal feedback control is designed by

$$u(t) = -B^*P(t)x^\circ(t). \tag{2-3-24}$$

In the same way as the previous section, the property of $P(t)$ is examined and feedback controller will be designed.

Property 5. The operator $P(t)$ defined by Eq. (2-3-23) is linear and unique. The right-hand side $\int_t^T S^*(\sigma-t)x^\circ(\sigma)\,d\sigma$ is differentiable with respect to time t and thus $P(t)$ must be differentiable. Differentiate both sides of Eq. (2-3-23) yields the following property.

Property 6. The operator $P(t)$ defined by Eq. (2-3-23) is differentiable and satisfies the differential equation of Reccati-type.

$$\frac{d}{dt}P(t) + A^*P(t) + P(t)A + I - P(t)BB^*P(t) = 0 \tag{2-3-25}$$

$$P(T) = 0. \tag{2-3-26}$$

Same reasoning as in 3-3-2, the following two properties are obtained.

Property 7. The operator $P(t)$ is symmetric, *i.e.*

$$P(t)^* = P(t). \tag{2-3-27}$$

Property 8. The optimal feedback control $u(t) = -B^*P(t)x^\circ(t)$ minimizes the performance index

$$J(u) = \int_0^T \|x(t)\|_X^2\,dt + \int_0^T \|u(t)\|_U^2\,dt$$

and the minimum value of $J(u)$ is given by

$$J(u^\circ) = \langle x(0), P(0)x(0)\rangle_X. \tag{2-3-28}$$

5-3-4. Examples

Feedback controllers will be designed for the same systems considered in 2-3-4.

Ex. 1. Optimal Final Value Problem

Let us consider the system

$$\left.\begin{aligned}&\frac{\partial}{\partial t}x(t,\alpha)=\frac{\partial^2}{\partial x^2}x(t,\alpha)+u(t,\alpha)\qquad 0<\alpha<1\\&x(t,0)=x(t,1)=0\end{aligned}\right\}\tag{2-3-29}$$

and the performance index

$$I(u)=\int_0^1|x(T)\alpha|^2d\alpha+\int_0^T\int_0^1|u(t,\alpha)|^2d\alpha\,dt.\tag{2-3-30}$$

Let it be desired to construct the optimal feedback controller which minimizes (2-3-30).

From Eq. (2-3-30), $x(t,\alpha)$ and $u(t,\alpha)$ are required to be square integrable in α, *i.e.* $x(t,\cdot)$, $u(t,\cdot)\in L^2(0,1)$. $L^2(0,1)$ is isomorphic to the Hilbert space l^2 of infinite sequences of square summable. By expanding $x(t,\alpha)$ and $u(t,\alpha)$ in Fourier series, Eq. (2-3-29) becomes

$$\frac{d}{dt}\begin{bmatrix}x_1(t)\\x_2(t)\\\cdot\\\cdot\\\cdot\end{bmatrix}=\begin{bmatrix}\lambda_1&0&&\\0&\lambda_2&&\\&&\cdot&\\&&&\cdot\\&&&&\cdot\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\\cdot\\\cdot\\\cdot\end{bmatrix}+\begin{bmatrix}u_1(t)\\u_2(t)\\\cdot\\\cdot\\\cdot\end{bmatrix}\tag{2-3-31}$$

where

$$x_n(t)=\int_0^1x(t,\alpha)\sqrt{2}\sin n\pi\alpha\,d\alpha\tag{2-3-32}$$

$$u_n(t)=\int_0^1u(t,\alpha)\sqrt{2}\sin n\pi\alpha\,d\alpha\tag{2-3-33}$$

$$\lambda_n=-(n\pi)^2,\qquad n=1,2,\ldots$$

Put

$$\boldsymbol{x}(t)=\begin{bmatrix}x_1(t)\\x_2(t)\\\cdot\\\cdot\\\cdot\end{bmatrix}\quad \boldsymbol{u}(t)=\begin{bmatrix}u_1(t)\\u_2(t)\\\cdot\\\cdot\\\cdot\end{bmatrix}\quad \boldsymbol{A}=\begin{bmatrix}\lambda_1&&0&\\&\lambda_2&&\\&&\cdot&\\&0&&\cdot\\&&&&\cdot\end{bmatrix}\tag{2-3-34}$$

Eq. (2-3-31) is written as

$$\frac{d}{dt}\boldsymbol{x}(t)=\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{u}(t)\tag{2-3-35}$$

$\boldsymbol{A}$ is a linear operator from l^2 into itself and its domain of definition is given by

$$\mathfrak{D}(\boldsymbol{A})=\{z=(z_1\cdots)\in l^2\,|\,\sum_{n=1}^{\infty}|\lambda_nz_n|^2<+\infty\}$$

$D(\boldsymbol{A})$ is dense in l^2. $\boldsymbol{A}$ is not a bounded oerator, but is a closed one. $\boldsymbol{A}$ generates a semi-group $S(t)$ and $\boldsymbol{S}(t)$ is an infinite dimensional matrix given by

$$S(t)=\mathrm{diag}\,[e^{\lambda_1t},\ e^{\lambda_2t},\ \ldots].\tag{2-3-36}$$

The feedback matrix $Q(t)$ is an infinite dimensional matrix and is characterized by Eq. (2-3-19).

$$\left.\begin{aligned} \frac{d}{dt}Q(t)+A^*Q(t)+Q(t)A-Q(t)Q(t)=0 \\ Q(T)=I \end{aligned}\right\} \tag{2-3-37}$$

By rewriting Eq. (2-3-37) componentwise

$$\left.\begin{aligned} \frac{d}{dt}q_{nm}(t)+(\lambda_n+\lambda_m)q_{nm}(t)-\sum_{e=1}^{\infty}q_{ne}(t)q_{em}(t)=0 \\ q_{nm}(T)=\delta_{nm}=\begin{cases}1 & m=n\\ 0 & m\neq n\end{cases} \end{aligned}\right\} \tag{2-3-38}$$

When $m\neq n$

$$q_{nm}(t)=0 \qquad 0\leq t\leq T. \tag{2-3-39}$$

When $m=n$

$$q_{nn}(t)=q_n(t)=\frac{2(n\pi)^2}{[1+2(n\pi)^2]e^{2(n\pi)^2(T-t)}-1}. \tag{2-3-40}$$

From Eq. (2-3-10), the optimal feedback control $u^\circ(t)$ is given by

$$u^\circ(t)=-Q(t)x^\circ(t)$$

i.e.

$$u_n(t)=-\frac{2(n\pi)^2}{[1+2(n\pi)^2]e^{2(n\pi)^2(T-t)}-1}x_n(t). \tag{2-3-41}$$

The value of the performance index under the optimal feedback control is

$$I(u^\circ)=\sum_{n=1}^{\infty}\frac{2(n\pi)^2}{[1+2(n\pi)^2]e^{2(n\pi)^2T}-1}x_n(0)^2, \tag{2-3-42}$$

where

$$x_n(0)=\int_0^1 x(0,\alpha)\sqrt{2}\sin n\pi\alpha\, d\alpha. \tag{2-3-43}$$

Ex. 2. Regulator Problem

Let us consider the system of Eq. (2-3-30) and the performance index

$$J(u)=\int_0^T\int_0^1|x(t,\alpha)|^2d\alpha\,dt+\int_0^T\int_0^1|u(t,\alpha)|^2d\alpha dt. \tag{2-3-44}$$

Let it be desired to design optimal feedback controller for the system. The notation is the same used in Ex. 1. The feedback matrix $P(t)$ is an infinite dimensional matrix and is characterized by Eq. (2-3-25).

$$\left.\begin{aligned} \frac{d}{dt}P(t)+AP(t)+AP(t)+I-P(t)P(t)=0 \\ P(T)=0 \end{aligned}\right\}. \tag{2-3-49}$$

Rewritting componentwise, when $m\neq n$,

$$\left.\begin{aligned} \frac{d}{dt} p_{nm}(t) + (\lambda_n + \lambda_m) p_{nm}(t) - \sum_{e=1}^{\infty} p_{ne}(t) p_{em}(t) = 0 \\ p_{nm}(T) = I \end{aligned}\right\} \tag{2-3-46}$$

It follows

$$p_{nm}(t) = 0, \qquad 0 \leq t \leq T.$$

When $m=n$

$$\left.\begin{aligned} \frac{d}{dt} p_{nn}(t) + 2\lambda_n p_{nn}(t) + 1 - p_{nn}^2(t) = 0 \\ p_{nn}(T) = 0 \end{aligned}\right\}. \tag{2-3-47}$$

The solution of Eq. (2-3-47) is given by

$$\begin{aligned} p_{nn}(t) &= p_n(t) \\ &= \frac{1}{\sqrt{1+(n\pi)^4} \cosh \sqrt{1+(n\pi)^4}\,(T-t) + (n\pi)^2}. \end{aligned}$$

The feedback control is given by

$$u^\circ(t) = -\boldsymbol{P}(t)\, x^\circ(t)$$

that is, componentwise

$$\begin{aligned} u_n(t) &= -p_n(t)\, x_n(t) \\ &= -\frac{1}{\sqrt{1+(n\pi)^4} \cosh \sqrt{1+(n\pi)^4}\,(T-t) + (n\pi)^2}\, x_n^0(t). \end{aligned} \tag{2-3-48}$$

And the value of the performance index under the optimal feedback control is given by

$$J(u^\circ) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{1+(n\pi)^4} \cosh \sqrt{1+(n\pi)^4}\, T + (n\pi)^2}\, x_n(0)^2. \tag{2-3-49}$$

2-4. *Optimal Feedback Control Fredholm Intergral Equation* [13]

In 2-3 we solved the regulator problem for distributed parameter systems in the form of feedback control and derived a nonlinear differential equation corresponding to the Riccati equation obtained by Kalman. The regulator problem is the one in which we wish the state of the system approaches to the zero state as near as possible over the time interval (0, T]. In this section, feedback controller is designed for the optimal pursuit problem in Hilbert spaces, in which it is desired to bring the trajectory of the system state to the specified one not necessarily zero over (0, T].

This problem was considered by Merriam III for finite dimensional dynamical systems [14]. He assumed that the value of the performance index under the optimal control is given by the quadric (not quadratic) form of initial state and he characterized parameters by nonlinear time-varying equations. On the other hand, Balakrishnan gave a feedback control law for the final value problem without derivation [15].

In this section it is shown first that the feedback control law for the final value problem given by Balakrishnan can be derived directly from the results of

2–2 and then the feedback solution for the optimal pursuit problem is obtained referring to the derivation for the final value problem. Since control is expressed explicitly with the present state and the desired trajectory in future, the result makes clear the structure of feedback control. Next, the value of the performance index under optimal control is obtained in the explicit form. In the case of final value control problem, it is given by the quadratic form of the difference between the desired state and the system state at time T under no control. And in the case of the pursuit problem it is expressed as the quadratic form of the difference between the desired trajectory and the system trajectory under free motion. In this section, we did derive quite analytically the above results which were the very assumptions on which Merriam III developed his theory.

Letting the desired state be zero in the above result, we can obtain the solution for regulator problem. Then the solution satisfies the nonlinear differential equation of Riccati type. Thus we have solved the Riccati equation analytically. It is noticeable that our theory contains no nonlinear analysis and simplifies the solution of the optimal control.

2–4–1. Control Problem

We now consider the control problems where the desired trajectory or state may not necessarily be zero, *i.e.* the system is given by

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \qquad (2\text{-}4\text{-}1)$$

and the performance index is given by

$$J_1(u) = \|x(T) - x^d(T)\|_X^2 + \int_0^T \|u(t)\|_U^2 dt, \qquad (2\text{-}4\text{-}2)$$

or

$$J_2(u) = \int_0^T \|x(t) - x^d(t)\|_X^2 dt + \int_0^T \|u(t)\|_U^2 dt. \qquad (2\text{-}4\text{-}3)$$

The control problem is now to design for the system of Eq. (2–4–1) a feedback controller which minimizes (2–4–2) or (2–4–3).

Using the same notation as in 2–3–1, with only exeption that c and d are replaced by

$$\left.\begin{aligned} c(t) &= x^d(t) - S(t)x(0), \\ d &= x^d(T) - S(T)x(0), \end{aligned}\right\} \qquad (2\text{-}4\text{-}4)$$

the optimal feedforward control is given by Eq. (2–3–5) or (2–3–6).

2–4–2. Optimal Feedback Controller for Final Value Problem

Proceeding in the same way as in 2–3–2, $u^\circ(t)$ can be expressed by

$$u^\circ(t) = B^*S^*(T-t)[x^d(T) - x^\circ(T)]. \qquad (2\text{-}4\text{-}5)$$

where $x^\circ(T)$ denotes the state at the final time T when the optimal control is applied to the system.

$x^\circ(T)$ can be expressed by the state $x^\circ(t)$ at time t and the optimal control on $(t, T]$, *i.e.*

$$x^\circ(T) = S(T-t)\,x^\circ(t) + \int_0^T S(T-\sigma)\,Bu^\circ(\sigma)\,d\sigma.$$

Combining this relation with Eq. (2-4-5), we obtain

$$\begin{aligned} x^d(T) - x^\circ(T) &= x^d(T) - S(T-t)\,x^\circ(t) - \int_t^T S(T-\sigma)\,Bu^\circ(\sigma)\,d\sigma \\ &= x^d(T) - S(T-t)x^\circ(t) - \int_t^T S(T-\sigma)\,BB^*S^*(T-\sigma)\,[x^d(T) \\ &\qquad\qquad - x(T)]\,d\sigma. \end{aligned}$$

Then

$$[I + \int_t^T S(T-\sigma)\,BB^*S^*(T-\sigma)\,d\sigma][x^d(T) - x^\circ(T)] = x^d(T) - S(T-t)\,x^\circ(t). \tag{2-4-6}$$

Define the operator $G(t)$ by

$$G(t) = I + \int_t^T S(T-\sigma)\,BB^*S^*(T-\sigma)\,d\sigma. \tag{2-4-7}$$

The operator $G(t)$ is a bounded linear operator on U into itself, and is self-adjoint and positive-definite. The straightforward calculation assures the existence of the inverse operator $G(t)$ and the boundedness of $G(t)$. Eq. (2-4-6) is solved for $x^d(T) - x^\circ(T)$ as

$$x^d(T) - x^\circ(T) = G(t)^{-1}[x^d(T) - S(T-t)\,x^\circ(t)]. \tag{2-4-8}$$

Substitution of Eq. (2-4-8) into Eq. (2-4-5) yields the feedback control law,

$$u^\circ(t) = B^*S^*(T-t)\,G(t)^{-1}[x^d(T) - S(T-t)\,x^\circ(t)], \tag{2-4-9}$$

which is desired. Eq. (2-4-9) states that the optimal control $u^\circ(t)$ at time t is determined only by the state $x^\circ(t)$ at the same time and the desired final state $x^d(T)$ at time T.

Now let us evaluate the value of the performance index under the optimal feedback control law given by Eq. (2-4-9). We consider the scalar,

$$m(t) = \langle x^\circ(t),\ S^*(s-t)\,B(t)[x^d(\cdot) - S(\cdot - t)\,x^\circ(t)]\rangle_X$$

and differentiate it with respect to time t. After straightforward calculation, we obtain the value of the performance index under the optimal control as

$$\begin{aligned} I(u^\circ) &= \|x^\circ(T) - x^d(T)\|_X^2 + \int_0^T \|u^\circ(t)\|_V^2 dt \\ &= \langle x^d(T) - S(T)\,x(0),\ G(0)^{-1}[x^d(T) - S(T)\,(0)]\rangle_X, \end{aligned} \tag{2-4-10}$$

that is, the value of the performance index under the optimal control is expressed in the quadratic form of $x^d(T) - S(T)\,x(0)$, which is the difference between the desired state and the final state under no control.

In a special case where $x^d(T) = 0$, Eq. (2-4-9) becomes

$$u^{\circ}(t) = -B^{*}S^{*}(T-t)G(t)^{-1}S(T-t)x^{\circ}(t), \qquad (2\text{-}4\text{-}11)$$

and Eq. (4–4–10) becomes

$$\begin{aligned} I(u^{\circ}) &= \langle S(T)x(0),\ G(0)^{-1}S(T)x(0)\rangle_{X} \\ &= \langle x(0),\ S^{*}(T)G(0)^{-1}S(T)x(0)\rangle_{X}. \end{aligned} \qquad (2\text{-}4\text{-}12)$$

Let us define a bounded linear operator on X by

$$Q(t) = S^{*}(T-t)G(t)^{-1}S(T-t). \qquad (2\text{-}4\text{-}13)$$

The operator $Q(t)$ is strongly differntiable with respect to time t and satisfies the following Riccati-type nonlinear differential equation:

$$\frac{d}{dt}Q(t) + A^{*}Q(t) + Q(t)A - Q(t)BB^{*}Q(t) = 0,\ Q(T) = I. \qquad (2\text{-}4\text{-}14)$$

The notation $Q(t)$ simplifies the expression of Eqs. (2–4–11) and (2–4–12) as follows.

$$u^{\circ}(t) = -B^{*}Q(t)x(t). \qquad (2\text{-}4\text{-}15)$$

$$I(u^{\circ}) = \langle x(0),\ Q(0)x(0)\rangle_{X}. \qquad (2\text{-}4\text{-}16)$$

Comparing the results obtained in 2–3–2, we find that the $Q(t)$ given by Eq. (2–4–13) is the analytical solution of the Riccati equation of Eq. (2–4–14).

2–4–3. Optimal Feedback Controller for Pursuit Problem

Just as in 2–3–3, we obtain

$$u^{\circ}(t) = \int_{t}^{T} B^{*}S^{*}(\sigma - t)[x^{d}(\sigma) - x^{\circ}(\sigma)]d\sigma. \qquad (2\text{-}4\text{-}17)$$

Now write $x^{\circ}(s)$, $s>t$ with the state $x^{\circ}(t)$ at time t and control u° on $(t, s]$.

$$x^{\circ}(s) = S(s-t)x(t) + \int_{t}^{s} S(s-\sigma)Bu^{\circ}(\sigma)d\sigma.$$

Substitute Eq. (2–4–17) into the above.

$$x^{\circ}(s) = S(s-t)x^{\circ}(t) + \int_{t}^{s} S(s-\sigma)B\Big[\int_{t}^{T} B^{*}S^{*}(\tau-\sigma)(x^{d}(\tau) - x^{\circ}(\tau))d\tau\Big]d\sigma$$

$$[x^{d}(s) - x^{\circ}(s)] + \int_{t}^{s} S(s-\sigma)B\int_{\sigma}^{T} B^{*}S^{*}(\tau-\sigma)[x^{d}(\tau) - x^{\circ}(\tau)]d\tau d\sigma$$
$$= x^{d}(s) - S(s-\tau)x^{\circ}(t). \qquad (2\text{-}4\text{-}18)$$

Define an operator for arbitrary $h \in L^{2}(t, T; U)$ by

$$(H(t)h)(s) = h(s) + \int_{t}^{s} d\sigma \int_{\sigma}^{T} (s-\sigma)BB^{*}S^{*}(\tau-\sigma)h(\tau)d\tau, \qquad s \leq t \leq T. \quad (2\text{-}4\text{-}19)$$

$H(t)$ is a bounded, self-adjoint and positive definite linear operator on $L^{2}(t, T; U)$ into itself. The inverse $H^{-1}(t)$ of $H(t)$ exists and is bounded. Use of $H(t)$ reduces Eq. (2–4–18) into the following equation.

$$H(t)[x^d(\cdot)-x^\circ(\cdot)](s)=x^d(s)-S(s-t)x^\circ(t)$$

i.e.

$$x^d(s)-x^\circ(s)=H(t)^{-1}[x^d(\cdot)-S(\cdot-t)x^\circ(t)](s). \tag{2-4-20}$$

Substitution of Eq. (2–4–20) into Eq. (2–4–17) yields the feedback control law,

$$u^\circ(t)=\int_t^T B^*S^*(s-t)(H(t)^{-1}[x^d(\cdot)-S(\cdot-t)x^\circ(t)])(s)\,ds, \tag{2-4-21}$$

which is just the desired one. Eq. (2–4–21) states that the optimal control $u^\circ(t)$ at time t is determined by the state $x^\circ(t)$ at the present time and the desired trajectory on $(t, T]$.

Now evaluate the value of the performance index under the optimal feedback control law given by Eq. (2–4–21). We consider the scalar,

$$n(t)=\langle x^\circ(t),\ \int_t^T S^*(s-t)(H(t)^{-1}[x^d(\cdot)-S(\cdot-t)x^\circ(t)])(s)\,ds\rangle \tag{2-4-22}$$

and differentiate it with respect to time t and integrate from 0 to T.

Then the optimal value of the performance index is obtained.

$$\int_0^T \|x(t)-x^d(t)\|_X^2\,dt+\int_0^T\|u^\circ(t)\|_U^2\,dt$$
$$=\int_0^T\langle x^d(t)-S(t)x(0),\ (H(0)^{-1}[x^d(\cdot)-S(\cdot)x(0)])(t)\rangle_X\,dt. \tag{2-4-23}$$

Eq. (2–4–23) indicates that the value of the performance index under the optimal control is expressed as the quadratic form of $x^d(t)-S(t)x(0)$, $0\le t\le T$.

In a special case where

$$x^d(t)=0 \qquad 0\le t\le T,$$

that is the regulator problem, Eq. (2–4–21) reduces to

$$u^\circ(t)=-B^*\int_t^T S^*(s-t)(H(t)^{-1}S(\cdot-t)x(t))(s)\,ds, \tag{2-4-24}$$

and Eq. (2–4–33) to

$$J(u^\circ)=\int_0^T\langle S(t)x(0),\ (H(0)^{-1}S(\cdot)x(0))(t)\rangle_X\,dt$$
$$=\int_0^T\langle x(0),\ S^*(t)(H(0)^{-1}S(\cdot))(t)x(0)\rangle_X\,dt. \tag{2-4-25}$$

Define an operator on U into itself by

$$P(t)u=\int_t^T S^*(s-t)(H(t)^{-1}S(\cdot-t)x)(s)\,ds. \tag{2-4-26}$$

$P(t)$ is a bounded linear operator, strongly defferentiable with respect to time t and satisfies the following Riccati-type nonlinear defferential equations:

$$\left.\begin{aligned} \frac{d}{dt}P(t)+A^{*}P(t)+P(t)A+I-P(t)BB^{*}P(t)=0 \\ P(T)=0 \end{aligned}\right\}. \qquad (2\text{-}4\text{-}27)$$

Rewriting Eqs. (2–4–24) and (2–4–25) with use of $P(t)$, we obtain the following equation:

$$u^{\circ}(t) = -B^{*}P(t)x(t), \qquad (2\text{-}4\text{-}28)$$

$$J(u^{\circ}) = \langle x(0),\ P(0)x(0)\rangle_X. \qquad (2\text{-}4\text{-}29)$$

Which agree with the results in 2–3–3. Thus the feedback matrix which Kalman obtained as the solution of Riccati equation is given by Eq. (2–4–26) in an analytical form.

2–4–4. Examples

We now take up the same example of distributed parameter system as in 2–3–4 and construct the optimal feedback control law.

Ex 1. Optimal Final Value Problem

We consider the same distributed prameter system as in 5–3–4,

$$\frac{\partial}{\partial t}x(t,\ \alpha) = \frac{\partial^2}{\partial\alpha^2}x(t,\ \alpha)+u(t,\ \alpha), \qquad 0<\alpha<1 \qquad (2\text{-}4\text{-}30)$$

with boundary condition $x(t,\ 0)=x(t,\ 1)=0$.

Given the initial state $x(0,\ \alpha)$ and the desired state $x^d(T,\ \alpha)$, we design the optimal feedback control law which minimizes

$$I(u) = \int_0^1 |x(T,\ \alpha)-x^d(T,\ \alpha)|^2 d\alpha + \int_0^T\int_0^1 |u(t,\ \alpha)|^2 d\alpha\, dt \qquad (2\text{-}4\text{-}31)$$

and evaluate $I(u)$ under the optimal feedback law.

The optimal feedback control is given by Eq. (2–4–9) as

$$u^{\circ}(t) = S(T-t)\left[I+\int_t^T S(T-\sigma)S(T-\sigma)\,d\sigma\right]^{-1}[x^d(T)-S(T-t)x(t)]. \quad (3\text{-}4\text{-}32)$$

Put

$$y(t) = x^d(T)-S(T-t)x(t), \qquad (2\text{-}4\text{-}33)$$

$$z(t) = \left[I+\int_t^T S(T-\sigma)S(T-\sigma)\,d\sigma\right]^{-1} y(t), \qquad (2\text{-}4\text{-}34)$$

and evaluate Eq. (2–4–34).

Now,

$$\left[I+\int_t^T S(T-\sigma)S(T-\sigma)\,d\sigma\right]z(t) = y(t).$$

Fourier expansion of $z(t)$ and $y(t)$ with respect to $\{\phi_n\}$ yields

$$z_n(t) = \frac{1}{1+\dfrac{1-e^{-2\lambda_n(T-t)}}{2\lambda_n}}\,y_n(t). \qquad (2\text{-}4\text{-}35)$$

Where $z_n(t)$ and $y_n(t)$ are Fourier coefficients of $z(t)$ and $y(t)$ respectively.

Let $u_n(t)$, $x_n(t)$ and $x_n^d(T)$ the Fourier coefficients of $u(t)$, $x(t)$ and $x^d(T)$ with respect to $\{\phi_n\}$ respectively. $y_n(t)$ and $u_n^\circ(t)$ are written as follows:

$$y_n(t) = x_n^d(T) - e^{-\lambda_n(T-t)} x_n(t)$$
$$u_n^\circ(t) = e^{-\lambda_n(T-t)} z_n(t).$$

So the optimal feedback control is given by

$$u_n^\circ(t) = \frac{x_n^d(T) - e^{-\lambda_n(T-t)} x_n(t)}{1 + \dfrac{1 - e^{-2\lambda_n(T-t)}}{2\lambda_n}} e^{-\lambda_n(T-t)}. \qquad (2\text{-}4\text{-}36)$$

The value of the performance index under Eq. (2–4–36) is given by Eq. (2–4–10) as

$$I(u^\circ) = \langle x^d(T) - S(T)x(0),\ G(0)^{-1}[x(T) - S(T)x(0)]\rangle_X$$
$$= \sum_{n=1}^{\infty} \frac{1}{1 + \dfrac{1 - e^{-2\lambda_n T}}{2\lambda_n}} [x_n^d(T) - e^{-\lambda_n T} x_n(0)]^2. \qquad (2\text{-}4\text{-}37)$$

Ex. 2. Optimal Pursuit Problem

We again consider the system in Ex 1. Given the desired trajectory $x^d(t, \alpha)$, it is desired to construct the optimal feedback control law $u^\circ(t, \alpha)$ which minimizes the following performance index.

$$J(u) = \int_0^T \int_0^1 | x(t, \alpha) - x^d(t, \alpha) |^2 d\alpha\, dt + \int_0^T \int_0^1 | u(t, \alpha) |^2 d\alpha\, dt. \qquad (2\text{-}4\text{-}38)$$

Optimal feedback control is written, from Eq. (2–4–21), as

$$u^\circ(t) = \int_t^T S(s-t)(H(t)^{-1}[x^d(\cdot) - S(\cdot - t)x(t)])(s)\, ds \qquad (2\text{-}4\text{-}39)$$

Put

$$y(s, t) = x^d(s) - S(s-t)x(t), \qquad (2\text{-}4\text{-}40)$$
$$h(s, t) = (H(t)^{-1} y(\cdot, t))(s). \qquad (2\text{-}4\text{-}41)$$

Reference to Eq. (2–4–19) reveals that $h(s, t)$ satisfies the Fredholm integral equation of the second kind.

$$h(s, t) + \int_t^s S(s-\sigma)\left[\int_\sigma^T S(\tau - \sigma) h(\tau, t) d\tau\right] d\sigma = y(s, t). \qquad (2\text{-}4\text{-}42)$$

Fxpand $h(s, t)$ and $v(s, t)$ into Fourier series and compare the coefficients of both sides of Eq. (2–4–42)

$$h_n(s, t) + \int_t^s d\sigma \int_\sigma^T e^{-\lambda_n(s+\tau-2\sigma)} h_n(\tau, t)\, d\tau = y_n(s, t). \qquad (2\text{-}4\text{-}43)$$

If $x^d(t)$ is twice differentiable almost everywhere, then Eq. (2–4–43) is equivalent to the following equation.

$$\left.\begin{aligned} \frac{d^2}{ds^2}h_n(s, t) - (1+\lambda_n^2)h_n(s, t) &= \frac{d^2}{ds^2}y_n(s, t) - \lambda_n^2 y_n(s, t) \\ h_n'(T, t) - \lambda_n h_n(T, t) &= y_n'(T, t) - \lambda_n y_n(T, t) \\ h_n(t, t) &= y_n(t, t) \end{aligned}\right\} \tag{2-4-44}$$

Solution of Eq. (2-4-44) is expressed, by letting $\beta_n = \sqrt{1+\lambda_n^2}$, as

$$h_n(s, t) = y_n(s, t) + \frac{1}{\beta_n}\int_t^s y_n(\sigma, t)\sinh\beta_n(s-\sigma)\,d\sigma$$
$$-\frac{\sinh\beta_n(s-t)}{\beta_n}\int_t^T y_n(\sigma, t)\frac{\beta_n\cosh\beta_n(T-\sigma)-\lambda_n\sinh\beta_n(T-\sigma)}{\beta_n\cosh\beta_n(T-t)-\lambda_n\sinh\beta_n(T-t)}\,d\sigma. \tag{2-4-45}$$

Referring to Eqs. (2-4-39) and (3-4-41), we can write $u^\circ(t)$ as

$$u_n^\circ(t) = \int_t^T e^{-\lambda_n(\sigma-t)}h_n(\sigma, t)\,d\sigma.$$

Some straight-forward calculation after substitution of Eq. (2-4-45) into the above equation yields

$$u_n^\circ(t) = -\frac{\sinh\beta_n(T-t)}{\beta_n\cosh\beta_n(T-t)-\lambda_n\sinh\beta_n(T-t)}x_n(t)$$
$$+\int_t^T x_n^d(\sigma)\frac{\beta_n\cosh\beta_n(T-\sigma)-\lambda_n\sinh\beta_n(T-\sigma)}{\beta_n\cosh\beta_n(T-t)-\lambda_n\sinh\beta_n(T-t)}\,d\sigma. \tag{2-4-46}$$

Now let us evaluate the value of the performance index under the control given by Eq. (2-4-46).

$$J(u^\circ) = \sum_{n=1}^{\infty}\left\{\int_0^T y_n^2(t, 0)\,dt + \frac{1}{\beta_n}\frac{\beta_n\sinh\beta_nT-\lambda_n\cosh\beta_nT}{\beta_n\cosh\beta_nT-\lambda_n\sinh\beta_nT}\right.$$
$$\times\left[\int_0^T y_n(t, 0)\sinh\beta_nt\,dt\right]^2$$
$$\left.-\frac{2}{\beta_n}\int_0^T y_n(t, 0)\cosh\beta_nt\left(\int_0^t y_n(\sigma, 0)\sinh\beta_n\sigma\,d\sigma\right)dt\right\} \tag{2-4-47}$$

where $y(t, 0) = x_n^d(t) - e^{-\lambda_nt}x_n(0)$.

Let $x^d(t)=0$ $0\le t\le T$ in Eq. (2-4-38). The problem is the so-called regulator problem for the distributed parameter system and Eqs. (2-4-46) and (2-4-47) agree with Eqs. (2-3-48) and (2-3-49) in 2-3-4.

3. Discrete-Time Regulator of Infinite-Dimemsional System

In this chapter the optimal and sub-optimal discrete-time regulators of countably infinite-dimensional systems are studied. The performance indices of regulator problems are weighted sums of squares of both states and control inputs. But in most practical control problems, systems are randomly perturbed by external disturbances, and the measured outputs are contaminated by random noises. Then the states of systems must be estimated, and as function of these estimated values of states, the control inputs must be generated so that the quadratic performance indices are minimized. Consequently regulators consist of feedback controllers and state-estimators. For finite-dimensional systems the above regulator problems are investigated[1)2)], but there has been no report for discrete-time infinite-dimensional systems. The parabolic and hyperbolic partial differential systems are transformed into the infinite-dimensional systems by eigen-function expansion. The dimensions of these systems are countably infinite and state-transition equations are described in terms of infinite-dimensional matrices. It is convenient to study the optimal regulator problems of these infinite-dimensional systems with Functional Analysis. Then the inner product will be defined on the state spaces and so they become Hilbert spaces, and the infinite-dimenisonal matrices are usually treated as linear operators.

In Secs. 3.2 and 3.3, are shown the recurrence relations which are satisfied by optimal controllers and state-estimators. These relations are extensions of Tou's ones[1)2)]. The extensions are heuristically achieved by changing finite-dimensional matrices with infinite-dimensional ones (operators) and conjugate matrices with normed conjugate operators. But the covariance operator of infinite-dimensional random vectors must be re-defined. The definitions of Falb[3)] then will be used. By Functional Analysis, it is shown the relations obtained heuristically are correct.

At first the systems with no disturbance and no measurement noise are studied. The minimal value of the performance index is shown by the inner product as

$$\langle \zeta, P\zeta \rangle$$

where ζ is the initial state of the system, and P is proved to be positive semi-definite and self-adjoint. With this result the optimal feedback controllers are shown to be bounded operators from the infinite-dimensional state space to the control input space. Next it is shown that these optimal controllers are also optimal for systems with independent random disturbances. At last the optimal estimators are calculated by the orthogonal projection of the infinite-dimensional state-space to the output space and estimators are shown to be bounded operators. Furthermore the seperation theorem is proved for the infinite-dimensional system, so that the following relation holds

$$\hat{u} = F\hat{\zeta},$$

where $\hat{\zeta}$ is the optimal estimate of the state, $\hat{u}$ is the optimal control and F is the optimal feedback controller obtained in the cases where there are no disturbance and no measurement noise.

3-1. Formulation of Discrete-Time Regulator Problems

At first the following simple control problem, *i.e.*, the control process of one-sided heating of metal in a furnace is considered, and discrete-time regulator problems of distributed-parameter systems will be formulated. Let $x(\cdot)$ and $\underset{\sim}{x}(\cdot)$ be the temperature of the material and the heating gas medium respectively, and $u(\cdot)$ is the fuel flow which is the control input. The system is described by the linear parabolic equation, and the controller is governed by the first-order ordinary differential equation.

$$\left.\begin{aligned}&\frac{\partial}{\partial t}x(t,\alpha)=A\frac{\partial^2}{\partial\alpha^2}x(t,\alpha),\quad 0<\alpha<l,\quad t>0\\&\frac{\partial x}{\partial t}(t,0)=0,\ \left[(1-k)\frac{\partial}{\partial\alpha}x(t,\alpha)+kx(t,\alpha)\right]_{\alpha=l}=Bu(t)+Dd(t),\ 1>k>0,\\&\underline{u}(t)=H\underline{x}(t),\\&\frac{d}{dt}\underline{x}(t)=F\underline{x}(t)+Gu(t)+\underline{D}\underline{d}(t),\end{aligned}\right\}\tag{3-1-1}$$

where A, B, D, H, F, G and $\underline{D}$ are nonzero scalers.

λ_i and $\psi_i(\cdot)$ are defined as the eigen-value and the normal eigen-function repectively of the boundary value problem

$$\left.\begin{aligned}&A\frac{\partial^2}{\partial\alpha^2}\psi(\alpha)-\lambda\psi(\alpha)=0,\\&\frac{\partial\psi}{\partial\alpha}(0)=0,\ \left[(1-k)\frac{\partial}{\partial\alpha}\psi(\alpha)+k\psi(\alpha)\right]_{\alpha=l}=0,\end{aligned}\right\}\tag{3-1-2}$$

where λ_i, $i=1,2,\ldots$ are negative and ordered such that $|\lambda_1|<|\lambda_2|<\cdots$. We obtain the following equation by the Fourier transformation of Eq. (3–1–1) with $\psi_i(\cdot)$.

$$\frac{d}{dt}\xi_i(t)=\lambda_i\xi_i(t)+B_i\underline{u}(t)+D_id(t),$$

where

$$B_i=\frac{\psi_i(l)}{1-k}B,\quad D_i=\frac{\psi_i(l)}{1-k}D.$$

Letting

$$\xi(t)\equiv(\xi_1(t),\ \xi_2(t),\ \ldots)'^{*1)},\ \Phi(t)\equiv\operatorname*{diag}_i e^{\lambda_i t},\ \phi(t)\equiv e^{tF},$$
$$\overline{B}\equiv[B_1,\ B_2,\ \ldots]',\ \overline{D}=[D_1,\ D_2,\ \ldots]'$$

then the input-state relation of the system (1) is equivalent to the following equations.

$$\xi(t)=\Phi(t)\xi(0)+\int_0^t\Phi(t-\tau)(\overline{B}H\underline{x}(\tau)+\overline{D}d(\tau))\,d\tau,$$
$$\underline{x}(t)=\phi(t)\underline{x}(0)+\int_0^t\phi(t-\tau)(Gu(\tau)+\underline{D}\underline{d}(\tau))\,d\tau.$$

It is assumed that the input $u(\cdot)$ and disturbances $d(\cdot)$, $\underline{d}(\cdot)$ are piecewise constant in time, *i.e.*,

$$u(\tau)=u(i),\ d(\tau)=d(i),\ \underline{d}(\tau)=\underline{d}(i),\ iT<\tau\le\overline{i+1}\,T.$$

---

*1) The prime denotes the transpose of vectors and matrices.

Moreover let $\xi(i) \equiv \xi(iT)$ and $\underline{x}(i) \equiv \underline{x}(iT)$.

Then the discrete-time input-state relation becomes

$$\begin{bmatrix} x(i+1) \\ \xi(i+1) \end{bmatrix} = \begin{bmatrix} \phi(T) & 0 \\ \int_0^T \Phi(T-t)\overline{B}H\phi(t)dt & \Phi(T) \end{bmatrix} \begin{bmatrix} \underline{x}(i) \\ \xi(i) \end{bmatrix} + \begin{bmatrix} \int_0^T \phi(T-t)\,Gdt \\ \int_0^T \Phi(T-t)\overline{B}H\int_0^t \phi(t-\tau)\,Gd\tau dt \end{bmatrix} u(i)$$

$$+ \begin{bmatrix} \int_0^T \phi(T-t)\,\underline{D}dt & 0 \\ \int_0^T \Phi(T-t)\overline{B}H\int_0^t \phi(t-\tau)\,\underline{D}d\tau\, dt & \int_0^T \Phi(T-t)\overline{D}dt \end{bmatrix} \begin{bmatrix} \underline{d}(i) \\ d(i) \end{bmatrix}.$$

By the following new definitions of $\zeta(i)$ and $d(i)$

$$\zeta(i) \equiv \begin{bmatrix} \underline{x}(i) \\ \xi(i) \end{bmatrix}, \quad d(i) \equiv \begin{bmatrix} \underline{d}(i) \\ d(i) \end{bmatrix},$$

the input-state relation can be rewritten by

$$\zeta(i+1) = \Phi\zeta(i) + \boldsymbol{B}u(i) + \boldsymbol{D}d(i), \tag{3-1-3}$$

where Φ, $\boldsymbol{B}$ and $\boldsymbol{D}$ are

$$\Phi \equiv \begin{bmatrix} \phi(T) & 0 \\ \int_0^T \Phi(T-t)\overline{B}H\phi(t)\,dt & \Phi(T), \end{bmatrix}, \quad \boldsymbol{B} \equiv \begin{bmatrix} \int_0^T \phi(T-t)\,Gdt \\ \int_0^T \Phi(T-t)\overline{B}H\int_0^t \phi(t-\tau)\,Gd\tau\, dt \end{bmatrix},$$

$$\boldsymbol{D} \equiv \begin{bmatrix} \int_0^T \phi(T-t)\,\underline{D}dt & 0 \\ \int_0^T \Phi(T-t)\,\overline{B}H\int_0^t \phi(t-\tau)\,\underline{D}d\tau\, dt & \int_0^T \Phi(T-t)\overline{D}dt \end{bmatrix},$$

and clearly they are bounded linear operators.

The output $m(i)$ of the system is assumed as follows:

$$m(i) \equiv x(i,\, l) + n(i). \tag{3-1-4}$$

The output measurement operator $\boldsymbol{M}$ is defined by

$$m(i) = \boldsymbol{M}\zeta(i) + n(i). \tag{3-1-5}$$

Since

$$x(i,\, l) = \Sigma_k \xi_k(i)\,\psi_k(l),$$

we have

$$\boldsymbol{M} = [0,\, \psi_1(l),\, \psi_2(l),\, \ldots]. \tag{3-1-6}$$

The norm of the state $\zeta(i)$ is defined by $\|\zeta(i)\|^2 = |\underline{x}(i)|^2 + \|\xi(i)\|^2$, where

$$\|\xi(i)\|^2 = \Sigma_k(p_1^2 + \lambda_k^2 p_2^2)\,\xi_k^2(i), \quad p_1,\ p_2 > 0.$$

If we set

$$\|x(\cdot)\|^2=\int_0^l\left\{p_1^2x^2(l)+p_2^2\left(A\frac{\partial^2}{\partial\alpha^2}x(\alpha)\right)^2\right\}d\alpha,\quad p_1,p_2>0,$$

then $\|x\|=\|\xi\|$, when $x(\cdot)=\Sigma_k\xi_k\phi_k(\cdot)$.

And the topology determined by the norm $\|x(\cdot)\|$ is equivalent to the one determined by the norm $\|x(\cdot)\|'$

$$\|x(\cdot)\|'=\sup_{0<\alpha<l}|x(\alpha)|^{4)}.$$

Hence M is bounded and linear.

By these discussions the dicrete-time regulator problems are formulated, *i.e.*, the state of the system is governed by Eq. (3–1–3) and control inputs are generated by linear feedbacks of measured outputs which are obtained by Eq. (3–1–5). For this system, find the optimal control sequence so that the following quadratic performance index is minimized

$$J_{N-j}[\xi(j)]=E[\textstyle\sum_{k=j+1}^N\langle\zeta(k),\ Q(k)\zeta(k)\rangle+\langle u(k-1),\ H(k-1)u(k-1)\rangle],\quad (3\text{-}1\text{-}7)$$

where E stands for the expected-value operator over the sample space of $d(\cdot)$, and $Q(1)$, $Q(2)$, ..., $Q(N)$ are positive semi-definite self-adjoint operators and $H(0)$, $H(1)$, ..., $H(N-1)$ are positive constants, and $\langle a, b\rangle$ denotes the inner product of a and b.

Henceforth we generalize the operators Φ, D, B and M as

$$\Phi=[\Phi_{ij}],\quad D=[D_1',\ D_2',\ \ldots]',$$
$$B=[B_1',\ B_2',\ \ldots]',\quad M=[M_1,\ M_2,\ \ldots]',$$

where D_i, B_i, M_i are $1\times d$, $1\times u$, $m\times 1$ matrices, respectively. We urther suppose that $H(k)$, $k=0, 1, 2, \ldots, N-1$ are $u\times u$ positive definite diagonal matrices, and

$$\zeta(k)\equiv(\zeta_1(k),\ \zeta_2(k),\ \ldots)',$$
$$u(k)\equiv(u_1(k),\ u_2(k),\ \ldots,\ u_u(k))',$$
$$d(k)\equiv(d_1(k),\ d_2(k),\ \ldots,\ d_d(k))',$$
$$m(k)\equiv(m_1(k),\ m_2(k),\ \ldots,\ m_m(k))',$$
$$n(k)\equiv(n_1(k),\ n_2(k),\ \ldots,\ n_n(k))',$$
$$E(d(k))=0,\ E(d(j)d'(k))=\delta_{jk}R(k),$$
$$E(n(k))=0,\ E(n(j)n'(k))=\delta_{jk}N(k),$$
$$E(d(j)n'(k))=0,$$

where δ_{jk} is the Kronecker's delta and $R(k)$, $N(k)$ are both positive definite diagonal matrices.

Since the hyperbolic differential systems are transformed to the system (3–1–3) similarly as parabolic differential systems, our formulation of infinite-dimensional discrete-time regulator problems are sufficiently general.

3-2. Optimal Feedback Controllers

3-2-1. Systems with all state variables accesible for measurement

In this section it is assumed that all the state variables are measureable with no measurement noise. The optimal feedback controller will be calculated by the variational methods.

The performance index is the following quadratic form

$$J_{N-j}(\zeta(j)) = E[\textstyle\sum_{k=j+1}^{N}\{\langle\zeta(k),\ Q(k)\zeta(k)\rangle + \langle u(k-1),\ H(k-1)u(k-1)\rangle\}],$$

which is defined in Sec. 3.1.

We can show the following lemma.

Lemma Let $F(i)$, $i=0, 1, 2, \dots, N-1$, be optimal feedback controllers of the system (3-1-3) with respect to the performance index (3-1-7). And similarly let $\bar{F}(i)$, $i=0, 1, 2, \dots, N-1$, be optimal feedback controllers of the system

$$\bar{\zeta}(i+1) = \Phi\bar{\zeta}(i) + Bu(i),$$

with respect to the performance index

$$\bar{J}_{N-j}[\bar{\zeta}(j)] = \textstyle\sum_{k=j+1}^{N}\{\langle\bar{\zeta}(k),\ Q(k)\bar{\zeta}(k)\rangle + \langle u(k-1),\ H(k-1)u(k-1)\rangle\}.$$

Then $F(i) = \bar{F}(i)$, $i = 0, 1, 2, \dots, N-1$.

Proof; Let $f_{N-j}[\zeta(j)]$ and $\bar{f}_{N-j}[\bar{\zeta}(j)]$ be

$$f_{N-j}[\zeta(j)] \equiv \min_{u(j)\cdots u(N-1)} E[\textstyle\sum_{k=j+1}^{N}\{\langle\zeta(k),\ Q(k)\zeta(k)\rangle + \langle u(k-1),\ H(k-1)u(k-1)\rangle\}],$$

$$\bar{f}_{N-j}[\bar{\zeta}(j)] \equiv \min_{u(j)\cdots u(N-1)} [\textstyle\sum_{k=j+1}^{N}\{\langle\bar{\zeta}(k),\ Q(k)\bar{\zeta}(k)\rangle + \langle u(k-1),\ H(k-1)u(k-1)\rangle\}].$$

$$f_1[\zeta(N-1)] = \min_{u(N-1)} E\{\langle\zeta(N),\ Q(N)\zeta(N)\rangle + \langle u(N-1),\ H(N-1)u(N-1)\rangle\}$$

$$\bar{f}_1[\zeta(N-1)] = \min_{u(N-1)} \{\langle\Phi\zeta(N-1) + Bu(N-1),\ Q(N)\Phi\zeta(N-1) + Q(N)Bu(N-1)\rangle + \langle u(N-1),\ H(N-1)u(N-1)\rangle\}.$$

By Eq. (3-1-3),

$$\begin{aligned} f_1[\zeta(N-1)] &= \min_{u(N-1)} [\langle\Phi\zeta(N-1) + Bu(N-1),\ Q(N)\Phi\zeta(N-1) + Q(N)Bu(N-1)\rangle \\ &\quad + \langle u(N-1),\ H(N-1)u(N-1)\rangle] + E\langle Dd(N-1),\ Q(N)Dd(N-1)\rangle \\ &= \bar{f}_1[\zeta(N-1)] + E\langle Dd(N-1),\ Q(N)Dd(N-1)\rangle, \end{aligned}$$

where $e(N-1) \equiv E\langle Dd(N-1),\ Q(N)Dd(N-1)\rangle$ is independent of the states

$$\zeta(i),\ i = 0, 1, 2, \dots, N.$$

Since it is proved later by Eq. (3-2-5) that $\bar{f}_{N-k}[\zeta(k)]$ can be witten as

$$\bar{f}_{N-k}[\zeta(k)] = \langle\zeta(k),\ P(N-k)\zeta(k)\rangle,\quad k = 0, 1, 2, \dots, N,$$

where $P(N-k)$ is a positive semi-definite linear operator and clearly $P(0)=0$, then by the principle of optimality and Eq. (3-1-3)

$$f_{N-k}[\zeta(k) = \bar{f}_{N-k}[\zeta(k)] + e(k+1) + E\langle Dd(k),\ \{Q(k+1) + P(N-\overline{k+1})\}Dd(k)\rangle.$$

where

$$e(k) \equiv e(k+1) + E\langle Dd(k),\ \{Q(k+1) + P(N-\overline{k+1})\}Dd(k)\rangle$$

is also independent of

$$\zeta(i),\ i=0,\ 1,\ 2,\ \ldots,\ N.$$

Therefore we obtain the relation

$$f_{N-j}[\zeta(j)] = \bar{f}_{N-j}[\zeta(j)] + e(j), \quad j=0,\ 1,\ 2,\ \ldots,\ N-1,$$

where $e(j)$ is the constant independent of $\zeta(i)$, $i=0, 1, \ldots, N$. Since

$$f_{N-j}[\zeta(j)] = \min_{u(j)} [\langle \Phi\zeta(j) + Bu(j),\ Q(j+1)\Phi\zeta(j) + Q(j+1)Bu(j)\rangle + \langle u(j),\ H(j)u(j)\rangle + \bar{f}_{N-\overline{j+1}}[\bar{\zeta}(j+1)] + e(j),$$

the optimal control $F(j)\zeta(j)$ which minimizes the P.I.,

$$\langle \Phi\zeta(j) + Bu(j),\ Q(j+1)\Phi\zeta(j) + Q(j+1)Bu(j)\rangle + \langle u(j),\ H(j)u(j)\rangle + \bar{f}_{N-\overline{j+1}}[\bar{\zeta}(j+1)],$$

equals to the optimal control $F(j)\zeta(j)$ which minimizes the P.I.,

$$E\langle \zeta(j+1),\ Q(j+1)\zeta(j+1)\rangle + \langle u(j),\ H(j)u(j)\rangle + f_{N-\overline{j+1}}[\zeta(j+1)]$$

Hence $F(j) = \bar{F}(j)$, $j=0, 1, 2, \ldots, N-1$. Q.E.D.

By this lemma, we can neglect random disturbances $d(i)$, $i=0, 1, 2, \ldots$ in this section and define $P(i)$ by the following equation.

$$f_i[\zeta(N-i)] = \langle \zeta(N-i),\ P(i)\zeta(N-1)\rangle,\ i=1,\ 2,\ \ldots,\ N-1. \qquad (3\text{-}2\text{-}1)$$

Since $[B^*Q(N)B + H(N-1)]$[*1)] is a positive definite matrix, the optimal feedback controller can be easily calculated as

$$F(N-1) = -[B^*Q(N)B + H(N-1)]^{-1}B^*Q(N)\Phi. \qquad (3\text{-}2\text{-}2)$$

By Eqs. (3–2–1) and (3–2–2)

$$P(1) = [\Phi + BF(N-1)]^*Q(N)[\Phi + BF(N-1)] + F^*(N-1)H(N-1)F(N-1).$$

This is a self-adjoint positive semi-definite operator. Next we suppose that k is a fixed integer such that $1 \le k \le N-1$ and $P(k)$ is self-adjoint positive semi-definite. Further we put

$$S(k) = P(k) + Q(N-k). \qquad (3\text{-}2\text{-}3)$$

So $S(k)$ is a self-adjoint positive semi-definite operator. Using Eq. (3–1–3),

*1) The asterisk of an operator denotes the its normed conjugate operator, *i.e.*, A^* denotes the normed conjugate operator of A.

$$f_{k+1}[\zeta(N-\overline{k+1})] = \min_{u(N-\overline{k+1})} \{\langle \Phi\zeta(N-\overline{k+1}),\ S(k)\,\Phi\zeta(N-\overline{k+1})\rangle + \langle u(N-\overline{k+1}),\ [B^*S(k)\,B + H(N-\overline{k+1})]u(N-\overline{k+1}) + 2\,B^*S(k)\,\Phi\zeta(N-\overline{k+1})\rangle\}.$$

Since $[B^*S(k)B + H(N-\overline{k+1})]$ is a positive definite matrix, the optimal feedback controller $F(N-\overline{k+1})$ is calculated as follows:

$$F(N-\overline{k+1}) = -[B^*S(k)\,B + H(N-\overline{k+1})]^{-1}\,B^*S(k)\,\Phi. \tag{3-2-4}$$

This implies that $F(N-\overline{k+1})$ is a bounded linear operator, because $[B^*S(k)\,B + H(N-\overline{k+1})]$ is a positive definite matrix. Furthemore we will show that $P(k+1)$ is a self-adjoint positive semi-definite operator. By Eqs. (3-1-3) and (3-2-1)

$$\begin{aligned} f_{k+1}[\zeta(N-\overline{k+1})] &= \langle \zeta(N-\overline{k+1}),\ P(k+1)\,\zeta(N-\overline{k+1})\rangle \\ &= \langle \zeta(N-\overline{k+1}),\ \{[\Phi + BF(N-\overline{k+1})]^*S(k)[\Phi + BF(N-\overline{k+1})] \\ &\quad + F^*(N-\overline{k+1})\,H(N-\overline{k+1})\,F(N-\overline{k+1})\}\zeta(N-\overline{k+1})\rangle. \end{aligned}$$

From these two equations it follows

$$P(k+1) = [\Phi + BF(N-\overline{k+1})]^*S(k)[\Phi + BF(N-\overline{k+1})] + F^*(N-\overline{k+1})\,H(N-\overline{k+1})\,F(N-\overline{k+1}). \tag{3-2-5}$$

Hence $P(k+1)$ is a self-adjoint positive semi-definite operator. As we have shown by the mathematical induction, the optimal feedback controllers $F(j), j=0, 1, 2, \ldots, N-1$ are linear bounded, $P(k), k=1, 2, \ldots$ are self-adjoint and positive semi-definite.

By putting $P(0)=0$, from recurrence equations (3-2-4) and (3-2-5) we can obtain successively $S(0)$, $F(N-1)$, $P(1)$, $S(1)$, $F(N-2)$, ...

3-2-2. Systems with inaccessible state variables for measurement

We must use the output-feedback controller instead of the state-feedback controller in the cases where there are some inaccesible state variables for measurement. Let $\hat{u}(k|j)$ be the control input which is generated by the linear feedback of measured output components $m_1(0), m_2(0), \ldots, m_1(j), \ldots, m_m(j)$, and $\hat{u}^*(k|j)$ be the optimal control which minimizes the performance index

$$J^*_{N-k}[\zeta(k)] = E\{\langle \zeta(k+1),\ Q(k+1)\,\zeta(k+1)\rangle + \langle \hat{u}(k|j),\ H(k)\,\hat{u}(k|j)\rangle + f_{N-\overline{k+1}}[\zeta(k+1)]\} \tag{3-2-6}$$

As in Sec. 3-2-1, we see that random disturbanced $(\cdot)$ can be neglected in this section.

Let M_k be the linear space of all components of output vectors $m(1), \ldots, m(k)$ and so M_k is a space of random variables. $\hat{u}^\circ(k|k)$ is called the orthogonal projection of $u^\circ(k)$ to M_k where $u^\circ(k)$ is the optimal control when the states $\zeta(i)$, $i=0, 1, 2, \ldots$ are completely measurable. And it is defined as

$$\hat{u}^\circ(k|k) \equiv (\hat{u}^\circ_1(k|k),\ \hat{u}^\circ_2(k|k),\ \ldots,\ \hat{u}^\circ_u(k|k))', \tag{3-2-7}$$

where $\hat{u}_i^\circ(k|k)$ is the orthogonal projection of $u_i^\circ(k)$ to M_k in the following sense, *i.e.*, two random variables $a(\omega)$ and $b(\omega)$ are orthogonal if

$$E(ab) = \int_\Omega a(\omega)\,b(\omega)\,d\omega = 0, \tag{3-2-8}$$

where Ω is the probability space and $d\omega$ is the probability measure.

$$\left.\begin{aligned} &\text{We put} && \check{u}_i^\circ(k|k) \equiv u_i^\circ(k) - \hat{u}_i^\circ(k|k), \\ &\text{and} && \check{u}^\circ(k|k) \equiv (\check{u}_1^\circ(k|k),\ \check{u}_2^\circ(k|k),\ \dots,\ \check{u}_n^\circ(k|k))'. \end{aligned}\right\} \tag{3-2-9}$$

These equations imply

$$E(\check{u}_i^\circ(k|k)\cdot m) = \int_\Omega \check{u}_i^\circ(k|k)\,m\,d\omega = 0,\quad {}^\forall m \in M_k.$$

Next we show that

$$\hat{u}^*(k|k) = \hat{u}^\circ(k|k). \tag{3-2-10}$$

With Eqs. (3-2-1) and (3-2-6), it is easily calculated that

$$\begin{aligned} J_{N-k}^*[\zeta(k)] - f_{N-k}[\zeta(k)] = E\{&2\langle\hat{u}(k|k) - u^\circ(k),\ B^*S(N-\overline{k+1})\Phi\zeta(k)\rangle \\ &+ \langle\hat{u}(k|k),\ [B^*S(N-\overline{k+1})B + H(k)]\hat{u}(k|k)\rangle \\ &- \langle u^\circ(k),\ [B^*S(N-\overline{k+1})B + H(k)]u^\circ(k)\rangle\}. \end{aligned}$$

By applying Eq. (3-2-4) and using the relation $E\langle\hat{u}^\circ(k|k),\ \check{u}^\circ(k|k)\rangle = 0$, it follows:

$$\begin{aligned} J_{N-k}^*[\zeta(k)] - f_{N-k}[\zeta(k)] = E\{&\langle\check{u}^\circ(k|k),\ [B^*S(N-\overline{k+1})B + H(k)]\check{u}^\circ(k|k)\rangle \\ &+ \langle\hat{u}(k|k) - \hat{u}^\circ(k|k),\ [B^*S(N-\overline{k+1})B + H(k)](\hat{u}(k|k) - \hat{u}^\circ(k|k))\rangle\}. \end{aligned}$$

Futhermore every component of $\hat{u}(k|k)$ and $\hat{u}^\circ(k|k)$ belongs to M_k, and every component of $\check{u}^\circ(k|k)$ is orthogonal to M_k. Hence the optimal control $\hat{u}^*(k|k)$ which minimizes $J_{N-k}^*[\zeta(k)]$ is the orthogonal projection $\hat{u}^\circ(k|k)$ of $u^\circ(k)$ to M_k.

In the next section it is shown that $\hat{u}^\circ(k|k) = F(k)\zeta^\circ(k|k)$, where $\zeta^\circ(k|k)$ is the optimal estimate of $\zeta(k)$ with the measured outputs $m(0), m(1), \dots, m(k)$. This is the seperation theorem, *i.e.*, we can synthesize the optimal feedback controller and the optimal estimator seperately.

3–3. Optimal State-Estimators

Generally the measured output is contaminated by the measurement noise as Eq. (3–1–5). We then estimate the state $\zeta(k)$ with the linear combination of the measured outputs $m(0), m(1), \dots, m(j)$. And we calculate the a° which minimizes the following mean-square error

$$E\langle\zeta(k) - a^\circ,\ \zeta(k) - a^\circ\rangle. \tag{3-3-1}$$

This a° is the optimal estimate $\zeta^\circ(k|j)$ of the state $\zeta(k)$ with $m(0), m(1), \dots, m(j)$. The components of a° are defined as follows:

$$a^\circ \equiv (a_1, a_2, a_3, \dots)'.$$

Let $\hat{\zeta}_i(k|j)$ be the orthogonal projection of $\zeta(k)$ to M_j and we define $\hat{\zeta}(k|j)$ and $\check{\zeta}(k|j)$ as follows:

$$\left.\begin{aligned}\hat{\zeta}(k|j) &\equiv (\hat{\zeta}_1(k|j),\ \hat{\zeta}_2(k|j),\ \dots)',\\ \check{\zeta}(k|j) &\equiv \zeta(k) - \hat{\zeta}(k|j).\end{aligned}\right\} \tag{3-3-2}$$

Since $E\langle\hat{\zeta}(k|j) - a^\circ,\ \check{\zeta}(k|j)\rangle = 0$, then

$$E\langle\zeta(k) - a^\circ,\ \zeta(k) - a^\circ\rangle = E\langle\hat{\zeta}(k|j) - a^\circ,\ \hat{\zeta}(k|j) - a^\circ\rangle + E\langle\check{\zeta}(k|j),\ \check{\zeta}(k|j)\rangle.$$

Hence it follows

$$\zeta^\circ(k|j) = \hat{\zeta}(k|j). \tag{3-3-3}$$

It is assumed in Sec. 2 that the measurement noise has the following stochastic properties $E(n(k)) = 0$, $E[n(j)n'(k)] = \delta_{jk}N(k)$, where $N(k)$ is a positive definite diagonal matrix, and that the measurement noise $n(\cdot)$ and the disturbance $d(\cdot)$ are statistically independent. We can obtain the following results by the methods which are extensions of Tou's ones to the infinite-dimensional system (3–1–3).

Let

$$\Psi(k) \equiv \Phi + \boldsymbol{B}\boldsymbol{F}(k),\quad k = 0, 1, 2, \dots, N-1. \tag{3-3-4}$$

Since $\zeta(k+1) = \Psi(k)\zeta(k) + \boldsymbol{D}d(k)$, when the system (3–1–3) is optimally controlled, then every component of $[\zeta(k+1) - \Psi(k)\hat{\zeta}(k|k)]$ is orthogonal to M_k, and every component of $\Psi(k)\hat{\zeta}(k|k)$ belongs to M_k.

Therefore

$$\hat{\zeta}(k+1|k) = \Psi(k)\hat{\zeta}(k|k). \tag{3-3-5}$$

Let $\hat{m}_i(k|j)$ be the orthogonal projection of $m_i(k)$ to M_j, and

$$\left.\begin{aligned}\check{m}_i(k|j) &\equiv m_i(k) - \hat{m}_i(k|j),\\ \hat{m}(k|j) &\equiv (\hat{m}_1(k|j),\ \hat{m}_2(k|j),\ \dots,\ \hat{m}_m(k|j))',\\ \check{m}(k|j) &\equiv (\check{m}_1(k|j),\ \check{m}_2(k|j),\ \dots,\ \check{m}_m(k|j))'.\end{aligned}\right\} \tag{3-3-6}$$

Then

$$\left.\begin{aligned}\hat{m}(k|j) &= \boldsymbol{M}\hat{\zeta}(k|j),\\ \check{m}(k|j) &= \boldsymbol{M}\check{\zeta}(k|j) + n(k).\end{aligned}\right\} \tag{3-3-7}$$

$Z(k+1)$ is defined as the linear space of $\check{m}_1(k+1|k), \dots, \check{m}_m(k+1|k)$ and let

$$\eta(k+1) \equiv \hat{\zeta}(k+1|k+1). \tag{3-3-8}$$

Since it is easily seen that M_{k+1} consists of M_k and $Z(k+1)$ which are orthogonal each other, then $\hat{\eta}(k+1|k)$ which is the orthogonal projection of $\eta(k+1)$ to M_k is calculated as follows:

$$\hat{\eta}(k+1|k) = \hat{\zeta}(k+1|k). \tag{3-3-9}$$

And every component of $\check{\eta}(k+1|k)$ which is equal to $\eta(k+1) - \hat{\eta}(k+1|k)$ belongs to $Z(k+1)$ and is orthogonal to M_k. Hence we can obtain the linear bounded operator $K(k+1)$ such that

$$\check{\eta}(k+1|k)=K(k+1)\check{m}(k+1|k). \tag{3-3-10}$$

From Eqs. (3-3-5), (3-3-9), and (3-3-10), it follows

$$\hat{\zeta}(k+1|k+1)=\Psi(k)\hat{\zeta}(k|k)+K(k+1)\check{m}(k+1|k). \tag{3-3-11}$$

We assume that Y is any inner product space. Let ξ and η be the X valued random variable and the Y valued random variable respectively. The covariance operator*1) $[\xi\cdot\eta]$ is defined as follows:

$$[\xi\cdot\eta]\zeta\equiv E[\langle\eta,\ \zeta\rangle\xi],\ \zeta\in Y.^{[3]} \tag{3-3-12}$$

Since

$$\|[\zeta\cdot\eta]\zeta\|^2\leq\|\xi\|^2\int_\Omega\|\xi\|^2d\omega\int_\Omega\|\eta\|^2d\omega,$$

$[\xi\cdot\eta]$ is a bounded linear operator from Y to X. We define the operator $C(k)$ as

$$C(k)\xi=[\check{\zeta}(k|k-1)\cdot\check{\zeta}(k|k-1)]\xi,\ \xi\in X. \tag{3-3-13}$$

Cleary $C(k)$ is a bounded linear operator. Furthermore

$$\begin{aligned}\langle C(k)\xi,\ \eta\rangle&=\langle E[\langle\check{\zeta}(k|k-1),\ \xi\rangle\check{\zeta}(k|k-1)],\ \eta\rangle\\&=\langle\xi,\ E[\langle\check{\zeta}(k|k-1),\ \eta\rangle\check{\zeta}(k|k-1)]\rangle\\&=\langle\xi,\ C(k)\eta\rangle.\end{aligned}$$

This implies that $C(k)$ is self-adjoint and positive semi-definite.

Every component of $\check{m}(k+1|k)$ belongs to $Z(k+1)$ and the random variable $\langle\zeta(k+1)-K(k+1)\check{m}(k+1|k),\ \zeta\rangle$ is orthogonal to $Z(k+1)$. Then for any $\zeta\in X$,

$$E(\langle\zeta(k+1),\zeta\rangle\check{m}(k+1|k))-E(\langle K(k+1)\check{m}(k+1|k),\ \zeta\rangle\check{m}(k+1|k))=0. \tag{3-3-14}$$

The first term of the above equation is calculated by Eq. (3-3-7) as follows:

$$\begin{aligned}&E(\langle\zeta(k+1),\ \zeta\rangle\check{m}(k+1|k))\\&\quad=ME(\langle\hat{\zeta}(k+1|k)+\check{\zeta}(k+1|k),\ \zeta\rangle\check{\zeta}(k+1|k))+E(\langle\zeta(k+1),\ \zeta\rangle n(k+1).\end{aligned}$$

Since $\langle\hat{\zeta}(k+1|k),\ \zeta\rangle$ is orthogonal to every component of $\check{\zeta}(k+1|k)$ and

$$E(\langle\zeta(k+1),\ \zeta\rangle n(k+1))=(E\langle\zeta(k+1),\ \zeta\rangle)(En(k+1))=0,$$

then

$$E(\langle\zeta(k+1),\ \zeta\rangle\check{m}(k+1|k))=MC(k+1)\zeta. \tag{3-3-15}$$

On the other hand, the second term of Eq. (3-3-14) is calculated by Eq. (3-3-7) as

*1) $[\xi\cdot\eta]$ equals to the covariance matrix $E[\xi\eta']$ when ξ and η are both finite-dimensional random vectors.

$$E(\langle K(k+1)\check{m}(k+1|k),\ \zeta\rangle \check{m}(k+1|k))$$
$$= ME(\langle\check{\zeta}(k+1|k),\ M^*K^*(k+1)\zeta\rangle\check{\zeta}(k+1|k)+E(\langle n(k+1),\ K^*(k+1)\zeta\rangle n(k+1))$$
$$+ME(\langle n(k+1),\ K^*(k+1)\zeta\rangle\check{\zeta}(k+1|k)+E(\langle\zeta(k+1|k),\ M^*K^*(k+1)\zeta\rangle n(k+1)).$$

Since for any m-dimensional vector n

$$E(\langle n(k+1),\ n\rangle\check{\zeta}(k+1|k))=E(\langle n(k+1),\ n\rangle\zeta(k+1))-E(\langle n(k+1),\ n\rangle\hat{\zeta}(k+1|k))=0,$$

then

$$E(\langle K(k+1)\check{m}(k+1|k),\ \zeta\rangle\check{m}(k+1|k))=[MC(k+1)M^*+N(k+1)]K^*(k+1)\zeta, \qquad (3\text{-}3\text{-}16)$$

where

$$N(k+1)=[n(k+1)\cdot n(k+1)].$$

By Eqs. (3-3-15) and (3-3-16),

$$\left.\begin{aligned} MC(k+1)&=[MC(k+1)M^*+N(k+1)]K^*(k+1),\\ K(k+1)[MC(k+1)M^*+N(k+1)]&=C(k+1)M^*. \end{aligned}\right\} \qquad (3\text{-}3\text{-}17)$$

Since $[MC(k+1)M^*+N(k+1)]$ is a positive definite matrix, then the inverse matrix of this exists and

$$K(k+1)=C(k+1)M^*[MC(k+1)M^*+N(k+1)]^{-1}. \qquad (3\text{-}3\text{-}18)$$

By Eqs. (3-3-2) and (3-3-5)

$$\check{\zeta}(k+1|k)=\Psi(k)\zeta(k)+Dd(k)-\Psi(k)\hat{\zeta}(k|k). \qquad (3\text{-}3\text{-}19)$$

From Eqs. (3-3-7), (3-3-8), (3-3-9) and (3-3-10)

$$\check{\zeta}(k+1|k)$$
$$=\Psi(k)\zeta(k)+Dd(k)-\Psi(k)\{\hat{\zeta}(k|k-1)+K(k)M\check{\zeta}(k|k-1)+K(k)n(k)\}$$
$$=\Psi(k)\check{\zeta}(k|k-1)-\Psi(k)K(k)M\check{\zeta}(k|k-1)+Dd(k)-\Psi(k)K(k)n(k).$$

The for any $\xi\in X$,

$$C(k+1)\xi$$
$$=E[\langle\Psi(k)(I-K(k)M)\check{\zeta}(k|k-1)+Dd(k)-\Psi(k)K(k)n(k),\ \xi\rangle$$
$$\times\{\Psi(k)(I-K(k)M)+\check{\zeta}(k|k-1)+Dd(k)-\Psi(k)K(k)n(k)\}]$$
$$=E[\langle\Psi(k)(I-K(k)M)\check{\zeta}(k|k-1),\ \xi\rangle\Psi(k)(I-K(k)M)\check{\zeta}(k|k-1)]$$
$$+E[\langle\Psi(k)K(k)n(k),\ \xi\rangle\Psi(k)K(k)n(k)]+E[\langle Dd(k),\ \xi\rangle Dd(k)].$$

In these calculations we used the fact that $\check{\zeta}(k|k-1)$, $d(k)$ and $n(k)$ are orthogonal each other. By the last equation, it follows that

$$C(k+1)=\Psi(k)[I-K(k)M]C(k)[I-K(k)M]^*\Psi^*(k)+DR(k)D^*$$
$$+\Psi(k)K(k)N(k)K^*(k)\Psi^*(k), \qquad (3\text{-}3\text{-}20)$$

With these recurrence relations (3–3–18) and (3–3–20) we can calculate $K(0)$, $C(1), \ldots$, succesively from the initial value $C(0)$, which is the covariance operator of the estimate error $\check{\zeta}(0|-1)$ of $\hat{\zeta}(0|-1)$. By Eqs. (3–3–7) and (3–3–11) we obtain the optimal estimate $\check{\zeta}(k|k)$ from the measured output $m(k)$. This is shown as the following block diagram, which is an extension of the Tou's result for finite-dimensional cases to the infinite-dimensional cases.

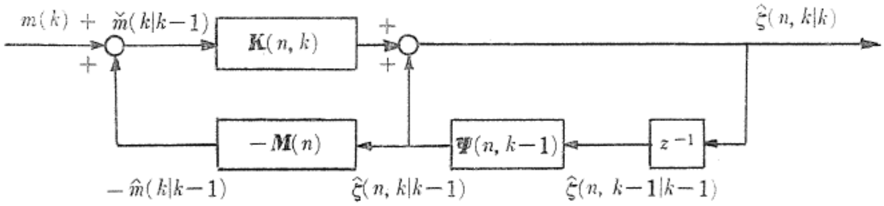

FIG. 1. Optimal Estimator for Infinite-Dimensional System (3-1-3).

Next we derive the relation between $\hat{u}^\circ(k|k)$ and $\hat{\zeta}(k|k)$. From the definitions of $u^\circ(k)$ and $F(k)$, $u^\circ(k)=F(k)\zeta(k)$. Since

$$u^\circ(k)=\hat{u}^\circ(k|k)+\check{u}^\circ(k|k)$$

and $\zeta(k)=\check{\zeta}(k|k)+\check{\zeta}(k|k)$, it follows

$$\hat{u}^\circ(k|k)-F(k)\hat{\zeta}(k|k)=F(k)\check{\zeta}(k|k)-\hat{u}^\circ(k|k).$$

The components of the left hand vectors of the above equation belong to M_k and the components of the right hand vectors are orthogonal to M_k.

Then

$$\left.\begin{aligned}\hat{u}^\circ(k|k)&=F(k)\hat{\zeta}(k|k),\\ \check{u}^\circ(k|k)&=F(k)\check{\zeta}(k|k),\end{aligned}\right\} \qquad (3\text{-}3\text{-}21)$$

where $F(k)$ is the optimal feedback controller obtained in Sec. 3–2–1.

4. Sub-Optimal Controls and Their Convergence

In Chaps, 2 and 3 the equations to be satisfied by the optimal control and the optimal feedback controller of the infinite-dimensional system are obtained. But these equations are difficult to be solved, because these equations are infinite-dimensional. Therefore in this Chapter are proposed the methods by which approximate solutions are obtained.

The methods proposed are modal control methods[1] by which the distributed-parameter system is approximated by the n-dimensional system composed of n main modes of the infinite-dimensional system and is controlled by the input which is optimal for this n-dimensional system. This control input can be easily calculated and is called the sub-optimal control. But there remains an important question whether this sub-optimal control converges to the optimal control when n tends to infinity, and the convergence of the sub-optimal control to the optimal control is equivalent to the convergence of the value of the performance index of the sub-optimal control to the one of the optimal control. Therefore it is necessary to be shown the convergence of the sub-optimal control to the optimal control. In Sec. 4–1 it is concerned with the continuous-time system and in Sec. 4-2 concerned with the discrete-time system.

4–1. Continuous-Time Requlator

In this section the control inputs are continuous-time functions and the performance index contains the spatial derivatives of the state because in heating or cooling problems of metal and glass the concentration of the thermal stress must be avoided. For this purpose the W^2 topology (Sovolev's topology)[2] will be introduced in the state space. And the performance index contains the time-derivatives of the control input, because in this case the sub-optimal control which will be defined later converges to the optimal control uniformly in time. In Sec. 4–1–1 the optimal control for this performance index will be derived by the similar methods which are used in Chap. 2, and in Sec. 4–1–2 the convergence of sub-optimal control to the optimal one will be shown.

4–1–1. Parabolic Differential System

System 1

We investigate the following system.

$$\left.\begin{aligned} &\frac{\partial}{\partial t}x(t,\alpha)=A_1x(t,\alpha)+D(\alpha)u(t),\ \alpha\in\Sigma,\ t>0,\\ &A_1=\textstyle\sum_{i,j=1}^{n}a_{ij}\dfrac{\partial^2}{\partial\alpha_i\partial\alpha_j}+b,\ \ a_{ij}=a_{ji},\\ &k(\sigma)\frac{\partial}{\partial n}x(t,\sigma)+(1-k(\sigma))x(t,\sigma)=0,\ \ \sigma\in\partial\Sigma, \end{aligned}\right\} \tag{4-1-1}$$

where $D(\cdot)=(D^1(\cdot),\ D^2(\cdot),\ ...,\ D^r(\cdot)),\ u(\cdot)=(u_1(\cdot),u_2(\cdot),\ ...,\ u_r(\cdot))'$ $D_i(\cdot)$ is square integrable in Σ, Σ is a bounded set in R_n and has a smooth boundary $\partial\Sigma$, $\partial/\partial n$ is an outward normal differential operator on $\partial\Sigma$, $0\le k(\cdot)\le 1$, and $k(\cdot)$ is continuous on $\partial\Sigma$.

The other cases where controllers have dynamics or the systems are controlled by boundary controllers can be investigated similiarly as the above system, and only the results will be shown.

Let A be the distributional extension of A[2], and $\mathscr{H}(\Sigma)$ be the following inner product space which is topologically equivalent to a Sobolev space $W^2(\Sigma)$.

$$\left.\begin{aligned} &x,y\in\mathscr{H}(\Sigma),\\ &\langle x,y\rangle_{\mathscr{H}}=\int_\Sigma(p_1^2xy+p_2^2AxAy)\,d\alpha,\\ &\|x\|_{\mathscr{H}}^2=\int_\Sigma\{p_1^2x^2+p_2^2(Ax)^2\}\,d\alpha, \end{aligned}\right\} \tag{4-1-2}$$

where

$$p_1>0,\quad p_2>0.$$

Clerly $\mathscr{H}(\Sigma) \subset L_2(\Sigma)$ (set of square integrable functions).

Let λ_n, $\psi_n(\cdot)$ be the eigenvalue and the normal eigenfunction in $\mathscr{H}(\Sigma)$ respectively of the following boundary value problem, and m_n be the multiplicity of the eigenvalue λ_n.

$$\left.\begin{aligned} &A\psi = \lambda\psi, \\ &k(\sigma)\frac{\partial}{\partial n}\psi(\sigma) + (1-k(\sigma))\psi(\sigma) = 0. \end{aligned}\right\} \tag{4-1-3}$$

By these normal eigenfunctions $\{\psi_{ni}\}$, we can obtain the input-state relation of the system (4-1-1) as follows.

$$x(T, \alpha) = \textstyle\sum_{n=1}^{\infty}\sum_{i=1}^{m_n}\xi_{ni}(T)\psi_{ni}(\alpha), \quad T>0, \tag{4-1-4}$$

$$\xi_{ni}(T) = e^{\lambda_n T}\xi_{ni}(0) + \int_0^T e^{\lambda_n(T-t)}D_{ni}u(t)\,dt, \tag{4-1-5}$$

$$(n=1, 2, \ldots, \; i=1, 2, \ldots, m_n),$$

where

$$\xi_{ni}(0) = \int_\Sigma \psi_{ni}(\alpha)x_0(\alpha)\,d\alpha, \quad x_0(\cdot)$$

is the initial state,

$$D_{ni} = (D_{ni}^1, \; D_{ni}^2, \ldots, D_{ni}^r),$$

$$D_{ni}^j = \int_\Sigma D^j(d)\psi_{ni}(\alpha)\,d\alpha, \quad j=1, 2, \ldots, r.$$

By defining

$$\xi(t) \equiv (\xi_{11}(t), \; \xi_{12}(t), \ldots, \xi_{21}(t), \ldots, \xi_{31}(t), \ldots)',$$

$$D \equiv [D_{11}', \; D_{12}', \ldots, D_{21}', \ldots, D_{31}', \ldots]',$$

$$\Phi(t) \equiv \operatorname*{diag}_n\,[e^{\lambda_n t}I_n],$$

I_n is the $m_n \times m_n$ unit matrix.

Eq. (4-1-5) can be rewritten as follows:

$$\xi(T) = \Phi(T)\xi(0) + \int_0^T \Phi(T-t)Du(t)\,dt. \tag{4-1-6}$$

Let $\mathfrak{h}$ be the space of infinite-dimensional vectors which has the following inner product.

$$\left.\begin{aligned} &\xi \equiv (\xi_{11}, \xi_{12}, \ldots, \xi_{21}, \ldots)', \\ &\eta \equiv (\eta_{11}, \eta_{12}, \ldots, \eta_{21}, \ldots)', \\ &\xi, \; \eta \in \mathfrak{h}, \\ &\langle\xi, \eta\rangle_{\mathfrak{h}} = \Sigma_n\Sigma_i\xi_{ni}\eta_{ni}(p_1^2 + p_2^2\lambda_n^2), \\ &\|\xi\|_{\mathfrak{h}}^2 = \Sigma_n\Sigma_i\xi_{ni}^2(p_1^2 + p_2^2\lambda_n^2). \end{aligned}\right\} \tag{4-1-7}$$

It can be seen that $\mathscr{H}(\Sigma)$ is congruent to $\mathfrak{h}$, and

$$\|x\|^2_{\mathscr{X}} = \|\xi\|^2_{\mathfrak{h}}, \tag{4-1-8}$$

by equating x and ξ as follows.

$$x = \Sigma_n \Sigma_i \xi_{ni} \psi_{ni} \tag{4-1-9}$$

where

$$\xi = (\xi_{11}, \xi_{12}, \ldots)'.$$

And $\xi(T)$ corresponds to $x(T)$ uniquely by Eq. (4-1-4), therefore $\mathfrak{h}$ and Eq. (4-1-6) can be the state space X and the input-state relation respectively.

The performance index is

$$J(u) = \int_0^T \left\{ \|u(t)\|^2 + q^2 \left\| \frac{d}{dt} u(t) \right\|^2 + p_1^2 \|x(t) - x_d(t)\|^2 + p_2^2 \|A(x(t) - x_d(t))\|^2 \right\} dt \tag{4-1-10}$$

where $(d/dt)\, u(t)$ is the distributional time-derivative of $u(t)$, T and $x_d(\cdot)$ are specified,

$$\|u(t)\|^2 = \sum_{i=1}^r u_r^2(t), \text{ and } \|x(t)\|^2 = \int_0^l |x(t, \alpha)|^2 d\alpha.$$

The operator $K(\cdot)$ and the error function $e(\cdot)$ are defined as follows:

$$\left.\begin{aligned} K(t)u &= \int_0^t \Phi(t-\tau)\, Du(\tau)\, d\tau, \\ e(t) &= x_d(t) - \Phi(t)\,\xi(0). \end{aligned}\right\} \tag{4-1-11}$$

$C_0^\infty(T)$ is the set of infinitely differentiable functions which are defined on $(0, T)$ and have compact support. $H^1(T)$ is the completion of $C_0^\infty(T)$ by the following norm.

$$\|f(\cdot)\|^2_{H^1} = \int_0^T \left\{ f^2(t) + q^2 \left(\frac{d}{dt} f(t) \right)^2 \right\} dt, \tag{4-1-12}$$

where d/dt is the distributional time-derivative. And $H^1(T)$ is also the inner product space which has the following inner product.

$$\left.\begin{aligned} & f(\cdot),\ g(\cdot) \in H^1(T), \\ & \langle f(\cdot),\ g(\cdot) \rangle_{H^1} = \int_0^T \left\{ f(t)g(t) + q^2 \frac{df(t)}{dt} \frac{dg(t)}{dt} \right\} dt. \end{aligned}\right\} \tag{4-1-13}$$

$f(\cdot) \in H^1(T)$ is continuous and bounded, and

$$\sup_{t \in (0, T)} |f(t)| \leq C \|f\|_{H^1}. \tag{4-1-14}$$

Therfore we can extend the domain of $H^1(T)$ to $(0, T]$ by setting

$$f(T) \equiv \lim_{t \uparrow T} f(t),\ f(\cdot) \in H^1(T),$$

We define $\boldsymbol{H}^1(T)$ as the r-fold product space $H^1(T)\times H^1(T)\times\cdots\times H^1(T)$ and define the following inner product in $\boldsymbol{H}^1(T)$.

$$\left.\begin{aligned}&f\equiv(f_1, f_2, \ldots, f_r)',\quad g\equiv(g_1, g_2, \ldots, g_r)'\in \boldsymbol{H}^1(T),\\&\langle f, g\rangle_{\boldsymbol{H}^1}=\textstyle\sum_{i=1}^{r}\langle f_i, g_i\rangle_{H^1},\\&\|f\|_{\boldsymbol{H}^1}^2=\textstyle\sum_{i=1}^{r}\|f_i\|_{H^1}^2.\end{aligned}\right\}\qquad(4\text{-}1\text{-}15)$$

Let $L_2(T, \mathfrak{h})$ be the set of square integrable $\mathfrak{h}$-valued functions defined on the time interval $(0, T]$ and has the following inner product and the norm.

$$\left.\begin{aligned}&e, f\in L_2(T, \mathfrak{h}),\\&\langle e, f\rangle_{L_2(T,\mathfrak{h})}=\int_0^T\langle e(t), f(t)\rangle_{\mathfrak{h}}\,dt,\\&\|e\|_{L_2(T,\mathfrak{h})}^2=\int_0^T\|e(t)\|_{\mathfrak{h}}^2\,dt.\end{aligned}\right\}\qquad(4\text{-}1\text{-}16)$$

By Eqs. (4–1–8), (4–1–11) and (4–1–15) the performance index (4–1–10) can be written as follows:

$$J(u)=\|u\|_{\boldsymbol{H}^1}^2+\|K(\cdot)u-e(\cdot)\|_{L_2(T,\mathfrak{h})}^2.$$

Furthermore we define the following inner product space $\boldsymbol{H}^1(T)\times L_2(T, \mathfrak{h})$.

$$\left.\begin{aligned}&(u, e),\ (v, f)\in \boldsymbol{H}^1(T)\times L_2(T, \mathfrak{h}),\\&\langle(u, e),\ (v, f)\rangle=\langle u, v\rangle_{\boldsymbol{H}^1}+\langle e, f\rangle_{L_2(T,\mathfrak{h})},\\&\|(u, e)\|^2=\|u\|_{\boldsymbol{H}^1}^2+\|e\|_{L_2(T,\mathfrak{h})}^2.\end{aligned}\right\}\qquad(4\text{-}1\text{-}17)$$

Then the performance index can be rewritten as following form.

$$J(u)=\|(u, Ku)-(0, e)\|^2.\qquad(4\text{-}1\text{-}18)$$

Since K is a bounded linear operator from the complete space $\boldsymbol{H}^1(T)$ to the complete space $L_2(T, \mathfrak{h})$, the set $\mathfrak{G}=\{(u, Ku)\,|\,u\in \boldsymbol{H}^1(T)\}$ is a closed linear set in $\boldsymbol{H}^1(T)\times\mathfrak{h}$. Therefore the control u_0 which minimizes $J(u)$ is obtained by the projection of $(0, e)$ to $\mathfrak{G}$, similarly as Chap. 2. That is

$$\left.\begin{aligned}&(0, e)=(u_0, Ku_0)+(-u_0, e-Ku_0),\\&(-u_0, e-Ku_0)\perp\mathfrak{G}.\end{aligned}\right\}\qquad(4\text{-}1\text{-}19)$$

Then

$$u_0=K^*[I+KK^*]^{-1}e=[I+K^*K]^{-1}K^*e,\qquad(4\text{-}1\text{-}20)$$

where K^* is the norm conjugate operator of K.

System II

We show the optimal control input of the following system, whose controller has dynamics.

$$\left.\begin{array}{l}\dfrac{\partial}{\partial t}x(t,\alpha)=A_1x(t,\alpha)+D(\alpha)\underline{u}(t), \quad \alpha\in\Sigma,\\ k(\sigma)\dfrac{\partial}{\partial n}x(t,\sigma)+(1-k(\sigma))x(t,\sigma)=0, \ \sigma\in\partial\Sigma,\\ \dfrac{d}{dt}\underline{x}(t)=F\underline{x}(t)+Gu(t),\\ \underline{u}(t)=H\underline{x}(t),\end{array}\right\} \tag{4-1-21}$$

where $\underline{x}(t)\in E_k$, $u(t)\in E_l$, and F, G, H are $k\times k$, $k\times l$, $r\times k$ matrices respectively.

We correspond $x(\cdot)$ and $\xi(\cdot)$ by Eq. (4–1–4), then the state of the system (4–1–21) is $(\underline{x}(\cdot), \zeta(\cdot))$ and the input-state relation can be written as

$$\begin{bmatrix}\underline{x}(t)\\ \xi(t)\end{bmatrix} = \begin{bmatrix}\phi(t)\underline{x}(0)+\int_0^t\phi(t-\tau)Gu(\tau)d\tau\\ \Phi(t)\xi(0)+\int_0^t\Phi(t-\tau)DH\phi(t)\underline{x}(0)d\tau+\int_0^t\Phi(t-\tau)DH\int_0^\tau\phi(\tau-u)Gu(\mu)\,d\mu d\tau\end{bmatrix} \tag{4-1-22}$$

where $\phi(t)\equiv e^{tF}$. The state space will be written as $\underline{X}\times X$ where $\underline{X}$ is the k-dimensional Euclidian space E_k.

The performance index is

$$J(u)=\int_0^T\Big\{\|u(t)\|^2+q^2\|\frac{d}{dt}u(t)\|^2+p_1^2\|x(t)-x_d(t)\|^2 + p_2^2\|A(x(t)-x_d(t))\|^2+p_3^2\|\underline{x}(t)-\underline{x}_d(t)\|^2\Big\}dt,$$

where $\|\underline{x}(t)\|^2=\sum_{i=1}^k\underline{x}_i^2(t)$, $p_1>0$, $p_2>0$, $p_3>0$, $q>0$.

We define the operator $L(t)$ and the error function $f(t)$ as follows:

$$\left.\begin{array}{l} L(t)u\equiv\begin{bmatrix}\int_0^t\phi(t-\tau)Gu(\tau)\,d\tau\\ \int_0^t\Phi(t-\tau)DH\int_0^\tau\phi(\tau-\mu)\,Gu(\mu)\,d\mu d\tau\end{bmatrix},\\ f(t)\equiv\begin{bmatrix}-\phi(t)\underline{x}(0)+\underline{x}_d(t)\\ -\Phi(t)\xi(0)-\int_0^t\Phi(t-\tau)\,DH\phi(t)\,\underline{x}(0)d\tau+x_d(t)\end{bmatrix}.\end{array}\right\} \tag{4-1-23}$$

Moreover we define the inner product in the state space $\underline{X}\times X$ as follows:

$$\left.\begin{array}{l}(\underline{x},\xi),\ (\underline{y},\eta)\in\underline{X}\times X,\\ \langle(\underline{x},\xi),\ (\underline{y},\eta)\rangle_{\underline{X}\times X}=\langle\underline{x},\underline{y}\rangle_{\underline{X}}+\langle\xi,\eta\rangle_{\mathfrak{h}},\\ \|(\underline{x},\xi)\|^2_{\underline{X}\times X}=\|\underline{x}\|^2_{\underline{X}}+\|\xi\|^2_{\mathfrak{h}}.\end{array}\right\} \tag{4-1-24}$$

where $\langle\underline{x},\underline{y}\rangle_{\underline{X}}=p_3^2\Sigma\underline{x}_i\underline{y}_i$, $\langle\xi,\eta\rangle_{\mathfrak{h}}$ is defined by Eq. (4–1–7), and let $L_2(T,\underline{X}\times X)$ be the set of square integrable $\underline{X}\times X$-valued functions defined on the time interval

$(0, T]$ and has the following inner product.

$$\left.\begin{aligned} & e(\cdot),\ f(\cdot) \in L_2(T, \underline{X} \times X), \\ & \langle e(\cdot),\ f(\cdot)\rangle_{L_2(T, \underline{X}\times X)} = \int_0^T \langle e(t),\ f(t)\rangle_{\underline{X}\times X}\, dt, \\ & \|e(\cdot)\|^2_{L_2(T, \underline{X}\times X)} = \int_0^T \|e(t)\|^2_{\underline{X}\times X}\, dt. \end{aligned}\right\} \tag{4-1-25}$$

Consequently we can obtain the following inner product space $H^1(T) \times L_2(T, \underline{X} \times X)$.

$$\left.\begin{aligned} & (u, e),\ (v, f) \in H^1(T) \times L_2(T, \underline{X} \times X), \\ & \langle (u, e),\ (v, f)\rangle = \langle u, v\rangle_{H^1} + \langle e, f\rangle_{L_2(T, \underline{X}\times X)}, \\ & \|(u, e)\|^2 = \|u\|^2_{H^1} + \|e\|^2_{L_2(T, \underline{X}\times X)}. \end{aligned}\right\} \tag{4-1-26}$$

By Eq. (4–1–4), (4–1–8), (4–1–23) and (4–1–26), the performance index can be rewritten by the following equation.

$$J(u) = \|(u, Lu) - (0, f)\|^2. \tag{41-2-7}$$

Therfore the optimal control u_0 can be calculated in the same way as Eq. (4–1–20) so that

$$u_0 = L^*[I + LL^*]^{-1}f = [I + L^*L]^{-1}L^*f. \tag{4-1-28}$$

System III

The systems which have boundary control inputs will be concerned with. The systems are described by the following equation.

$$\left.\begin{aligned} & \frac{\partial}{\partial t} x(t, \alpha) = A_2 x(t, \alpha), \quad \alpha \in \Sigma, \\ & k(\sigma) \frac{\partial}{\partial n} x(t, \sigma) + (1 - (\sigma)) x(t, \sigma) = B(\sigma)\, \underline{u}(t), \quad \sigma \in \partial\Sigma, \\ & \frac{d}{dt} \underline{x}(t) = F\underline{x}(t) + Gu(t), \\ & \underline{u}(t) = H\underline{x}(t), \end{aligned}\right\} \tag{4-1-29}$$

where $A_2 = \sum_{i=1}^{\eta} \frac{\partial_2}{\partial \alpha_i^2} + b$, $B(\cdot) = (B^1(\cdot),\ B^2(\cdot),\ \ldots,\ B^r(\cdot))$, $B^i(\cdot)$ is square integroble on $\partial\Sigma$.

We extend the differential operator A_2 in the distributional sense and denote it A. Furthermore we can consider the boundary volue problem (4–1–3), and can use λ_n, ϕ_n as the eigenvalue and the normal eigenfunction respectively.

By the Fourier transformation with these eigenfunctions, we can derive the input-state relation of the system (4–1–29) in the sequel.

Similary as the system (II), the state and state space are $(\underline{x}(t), \xi(t))$ and $\underline{X} \times X$ respectively. The input-state relation is

$$\begin{bmatrix} \underline{x}(t) \\ \zeta(t) \end{bmatrix} = \begin{bmatrix} \phi(t)\,\underline{x}(0) + \int_0^t \phi(t-\tau) Gu(\tau)\,d\tau \\ \Phi(t)\,\xi(0) + \int_0^t \Phi(t-\tau)\,\boldsymbol{B}H\phi(t)\,\underline{x}(0)d\tau + \int_0^t \Phi(t-\tau)\boldsymbol{B}H\int_0^\tau \phi(\tau-\mu)\,Gu(\mu)\,d\mu d\tau \end{bmatrix}, \tag{4-1-30}$$

where

$$\boldsymbol{B} \equiv [B'_{11}, B'_{12}, \ldots, B'_{21}, \ldots]',$$
$$B_{ni} \equiv (B^1_{ni}, B^2_{ni}, \ldots, B^r_{ni}),$$
$$B^j_{ni} \equiv \int_{\partial\Sigma} B^j\Big(\frac{\partial}{\partial n}\psi_{ni} - \psi_{ni}\Big)\,d\sigma, \quad j = 1, 2, \ldots, r.$$

This operator $\boldsymbol{B}$ is an unbounded operator, but $\Phi(t)\boldsymbol{B}$, $t>0$ is a bounded operator. Then we can obtain the optimal control in the same way as the system (II), by changing the operator $\boldsymbol{D}$ by operator $\boldsymbol{B}$.

4-1-2. Convergence of Sub-Optimal Control of Continuous-Time Systems

At first it is concerned with the sub-optimal control of the system (I) and next with the one of systems (II) and (III).

We approximate the system (I) by the following finite-dimensional system which has finite main modes.

$$\xi_n(t) = \Phi_n(t)\,\xi_n(0) + \int_0^t \Phi_n(t-\tau)\,D_n u(\tau)\,d\tau, \tag{4-1-31}$$

where

$$\xi_n(t) = (\xi_{11}(t), \xi_{12}(t), \ldots, \xi_{nm_n}(t))',$$
$$\Phi_n(t) = [\operatorname*{diag}_{1\leq i\leq n} e^{\lambda_i t} I_i],$$
$$D_n = [D'_{11}, D'_{12}, \ldots, D'_{21}, \ldots, D'_{nm_n}]'.$$

This is the n-th approximate system of the system (I). Futhermore $K_n(t)$ and $e_n(t)$ are defined by

$$\left.\begin{aligned} K_n(t)\,u &= \int_0^t \Phi_n(t-\tau)\,D_n u(\tau)\,d\tau, \\ e_n(t) &= \xi_{dn}(t) - \Phi_n(t)\,\xi_n(0), \end{aligned}\right\} \tag{4-1-32}$$

$\widetilde{K}_n(t)$ and $\widetilde{e}_n(t)$ are defined by

$$\left.\begin{aligned} \widetilde{K}_n(t) &= \begin{bmatrix} K_n(t) \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \\ \widetilde{e}_n(t) &= (e'_n(t), 0, 0, \ldots)', \end{aligned}\right\} \tag{4-1-33}$$

$\xi_{dn}(t)$ is the n-th approximation of $\xi_d(t)$ and is defined by the following equation.

$$\xi_{dn}(t) = (\xi_{d11}(t),\ \xi_{d12}(t),\ \ldots,\ \xi_{dnm_n}(t))',$$

where

$$\xi_d(t) = (\xi_{d11}(t),\ \xi_{d12}(t),\ \ldots)',$$

The performance index $J_n(u)$ for the system (4–1–31) is

$$J_n(u) = \|u(\cdot)\|^2_{H^1} + \|K_n u - e_n\|_{L_2(T,\,\mathfrak{y}_n)}, \tag{4-1-34}$$

where $\mathfrak{y}_n$ and $L_2(T, \mathfrak{y}_n)$ are the following inner product spaces.

$$\left.\begin{aligned}
&\xi_n,\ \eta_n \in \mathfrak{y}_n,\\
&\xi_n = (\xi_{11},\ \xi_{12},\ \ldots,\ \xi_{nm_n})',\\
&\eta_n = (\eta_{11},\ \eta_{12},\ \ldots,\ \eta_{nm_n})',\\
&\langle \xi_n, \eta_n\rangle_{\mathfrak{y}_n} = \textstyle\sum_{i=1}^{n}\sum_{j=1}^{m_n} \xi_{ij}\eta_{ij}(p_1^2 + p_2^2\lambda_i^2),\\
&\|\xi_n\|^2_{\mathfrak{y}_n} = \textstyle\sum_{i=1}^{n}\sum_{j=1}^{m_n} \xi_{ij}^2(p_1^2 + p_2^2\lambda_i^2),
\end{aligned}\right\} \tag{4-1-35}$$

$$\left.\begin{aligned}
&e_n(\cdot),\ f_n(\cdot) \in L_2(T, \mathfrak{y}_n),\\
&\langle e_n, f_n\rangle_{L_2(T,\,\mathfrak{y}_n)} = \int_0^T \langle e_n(t),\ f_n(t)\rangle_{\mathfrak{y}_n}\,dt\\
&\|e_n\|^2_{L_2(T,\,\mathfrak{y}_n)} = \int_0^T \|e_n(t)\|^2_{\mathfrak{y}_n}\,dt.
\end{aligned}\right\} \tag{4-1-36}$$

Let $H^1(T) \times L_2(T, \mathfrak{y}_n)$ be the following product space,

$$\left.\begin{aligned}
&(u_n,\ e_n),\ (v_n,\ f_n) \in H^1(T) \times L_2(T, \mathfrak{y}_n),\\
&\langle (u_n,\ e_n),\ (v_n,\ f_n)\rangle = \langle u_n, v_n\rangle_{H^1} + \langle e_n, f_n\rangle_{L_2(T,\,\mathfrak{y}_n)},\\
&\|(u_n,\ e_n)\|^2 = \|u_n\|^2_{H^1} + \|e_n\|^2_{L_2(T,\,\mathfrak{y}_n)},
\end{aligned}\right\} \tag{4-1-37}$$

then the performance index (4–1–34) can be rewritten in the following form.

$$J_n(u) = \|(u,\ K_n u) - (0,\ e_n)\|^2. \tag{4-1-38}$$

For this performance index $J_n(u)$ the optimal control u_n of the system (4–1–31) is

$$u_n = K_n^*[I + K_n K_n^*]^{-1} e_n = [I + K_n^* K_n]^{-1} K_n^* e_n, \tag{4-1-39}$$

which is obtained similarly as Eq. (4–1–20).

Next we show that u_n converges to u_0 in the norm topology, *i.e.* u_n converges to u_0 uniformly in the time interval $(0, T]$ when n tends to infinity.

From Eq. (4–1–20) $u_0 + K^*Ku_0 = K^*e$, and from Eq. (4–1–39) $u_n + K_n^* K_n u_n = K_n^* e_n$.

Then

$$u_0 - u_n = (K^*e - K_n^* e_n) + (K_n^* K_n u_n - K^* K u_0), \tag{4-1-40}$$

and

$$\|u_0 - u_n\| \le \|K^*e - K_n^* e_n\| + \|K_n^* K_n u_n - K^* K u_0\|.$$

Since $K_n^* e_n = (\widetilde{K}_n)^* \widetilde{e}_n$ and $\|\widetilde{e}_n\| \le \|e\|$, then

$$\begin{aligned}\|K^*e - K_n^* e_n\| &= \|K^*e - (\widetilde{K}_n)^* \widetilde{e}_n\| \\ &\le \|(\widetilde{K}_n)^* \widetilde{e}_n - K^* \widetilde{e}_n\| + \|K^* \widetilde{e}_n - K^* e\| \\ &\le \|(\widetilde{K}_n)^* - K^*\| \, \|\widetilde{e}_n\| + \|K^*\| \, \|\widetilde{e}_n - e\| \\ &\le \|(\widetilde{K}_n)^* - K^*\| \, \|e\| + \|K^*\| \, \|\widetilde{e}_n - e\|.\end{aligned}$$

we define P_n as the following orthogonal projection,

$$P_n(\xi_{11}, \xi_{12}, \ldots, \xi_{21}, \ldots)' = (\xi_{11}, \xi_{12}, \ldots, \xi_{nm_n}, 0, \ldots)',$$

then

$$(\widetilde{K}_n)^* = K^* P_n.$$

Hence

$$\lim_n \|K^*e - K_n^* e_n\| = 0, \tag{4-1-41}$$

because

$$\lim_n \|\widetilde{e}_n - e\| = 0, \ \lim_n \|(\widetilde{K}_n)^* - K^*\| = \lim_n \|K^*(P_n - I)\| = 0.$$

We can obtain the following inequalities.

$$\begin{aligned}&\|(K_n^* K_n u_n - K^* K u_0) - K^* K(u_n - u_0)\| \\ &= \|(\widetilde{K}_n)^* \widetilde{K}_n u_n - K^* K u_n\| \\ &\le \|K^*\| \, \|K u_n - \widetilde{K}_n u_n\| + \|K^* - (\widetilde{K}_n)^*\| \, \|\widetilde{K}_n u_n\| \\ &\le \|K^*\| \, \|K - \widetilde{K}_n\| \, \|u_n\| + \|K^* - (\widetilde{K}_n)^*\| \, \|\widetilde{K}_n\| \, \|u_n\|.\end{aligned} \tag{4-1-42}$$

As $\|u_n\|^2 \le J_n(u_n) \le J(u_0)$ and $\|\widetilde{K}_n\| \le \|K\|$, then both $\|u_n\|^2$ and $\|\widetilde{K}_n\|$ are uniformly bounded in n. Hence by Eq. (4–1–42),

$$\lim_n (K_n^* K_n u_n - K^* K u_0) = \lim_n K^* K(u_n - u_0). \tag{4-1-43}$$

From these Eq. (4–1–40), (4–1–41) and (4–1–43), we obtain the following relation.

$$\lim_n (I + K^*K)(u_0 - u_n) = (I + K^*K)[\lim_n (u_n - u_0)] = 0. \tag{4-1-44}$$

Since the null space of $[I + K^*K]$ is composed of only $\{0\}$, Eq. (4–1–44) implies

$$\lim_n u_n = u_0. \tag{4-1-45}$$

Therefore

$$\lim_n \sup_{t \in (0, T]} \|u_n(t) - u_0(t)\| = 0. \tag{4-1-46}$$

Moreover there exist the following relations among the values of

$$J_1(u_1), J_2(u_2), \ldots, J_n(u_n), J(u_0),$$
$$J_1(u_1) \le J_2(u_2) \le \cdots \le J_n(u_n) \le J(u_0) \le J(u_i), \ i = 1, 2, \ldots, n.$$

Then the error of u_n is evaluated by

$$0 \leq J(u_n) - J(u_0) \leq \min_{1 \leq i \leq n} J(u_i) - J_n(u_n). \tag{4-1-47}$$

In the sequel we are concerned with the system (II), because the system (III) is identified with the system (II) by changing the operator $\boldsymbol{D}$ by the operator $\boldsymbol{B}$.

We approximate the system (II) by the following finite-dimensional system.

$$\begin{bmatrix} \underline{x}(t) \\ \xi_n(t) \end{bmatrix} = \begin{bmatrix} \phi(t)\,\underline{x}(0) + \int_0^t \phi(t-\tau)\, Gu(\tau)\, d\tau \\ \Phi_n(t)\,\xi_n(0) + \int_0^t \Phi_n(t-\tau)\, D_n H \phi(\tau)\, \underline{x}(0)\, d\tau + \int_0^t \Phi_n(t-\tau) D_n H \int_0^\tau \phi(\tau-\mu)\, Gu(\mu)\, dud\tau \end{bmatrix} \tag{4-1-48}$$

This the n-th approximate system of the system (II). Furthermore let $L_n(t)$, $\tilde{L}_n(t)$, $f_n(t)$ and $\tilde{f}_n(t)$ be defined as follows.

$$L_n(t)\,u = \begin{bmatrix} \int_0^t \phi(t-\tau)\, Gu(\tau)\, d\tau \\ \int_0^t \Phi_n(t-\tau)\, D_n H \int_0^\tau \phi(\tau-\mu)\, Gu(\mu)\, d\mu\, d\tau \end{bmatrix},$$

$$\tilde{L}(t) = \begin{bmatrix} L_n(t) \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix},$$

$$f_n(t) = \begin{bmatrix} \underline{x}_d(t) - \phi(t)\,\underline{x}(0) \\ \xi_{dn}(t) - \Phi_n(t)\,\xi_n(0) - \int_0^t \Phi_n(t-\tau)\, D_n H \phi(\tau)\, \underline{x}(0)\, d\tau \end{bmatrix},$$

$$\tilde{f}_n(t) = \begin{bmatrix} f_n(t) \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}.$$

The performance index $J_n(u)$ for this system (4-1-48) is

$$J_n(u) = \|u(\cdot)\|^2_{H^1} + \|L_n u - f_n\|^2_{L_2(T,\, \underline{X} \times \mathfrak{h}_n)}, \tag{4-1-49}$$

where $L_2(T, \underline{X} \times \mathfrak{h}_n)$ is the following inner product space,

$$\left.\begin{aligned} & e_n, f_n(\cdot) \in L_2(T, \underline{X} \times \mathfrak{h}_n), \\ & \langle e_n, f_n \rangle_{L_2(T,\, \underline{X} \times \mathfrak{h}_n)} = \int_0^T \langle e_n(t), f_n(t) \rangle_{\underline{X} \times \mathfrak{h}_n}\, dt, \\ & \|e_n\|^2_{L_2(T,\, \underline{X} \times \mathfrak{h}_n)} = \int_0^T \|e_n(t)\|^2_{\underline{X} \times \mathfrak{h}_n}\, dt. \end{aligned}\right\} \tag{4-1-50}$$

We define $H^1(T)\times L_2(T, \underline{X}\times \mathfrak{h}_n)$ to be the following product space.

$$\left.\begin{array}{l}(u_n, e_n),\ (v_n, f_n)\in \boldsymbol{H}^1(T)\times L_2(T, \underline{X}\times \mathfrak{h}_n),\\ \langle(u_n, e_n),\ (v_n, f_n)\rangle = \langle u_n, v_n\rangle_{\boldsymbol{H}^1}+\langle e_n, f_n\rangle_{L_2(T,\ \underline{X}\times \mathfrak{h}_n)},\\ \|(u_n, e_n)\|^2=\|u_n\|^2_{\boldsymbol{H}^1}+\|e_n\|^2_{L_2(T,\ X\times \mathfrak{h}_n)},\end{array}\right\} \qquad (4\text{-}1\text{-}51)$$

By Eq. (4-1-51), the performance index (4-1-49) can be rewritten as follows:

$$J_n(u)=\|(u, L_nu)-(0, f_n)\|^2. \qquad (4\text{-}1\text{-}52)$$

Then the n-th sub-optimal control u_n is obtained similarly as Eq. (4-1-41), so that

$$u_n=L_n^*[I+L_nL_n^*]^{-1}f_n=[I+L_n^*L_n]^{-1}L_n^*f_n,$$

and we can show in the same way as Eq. (4-1-51) that u_n tends to u_0 (Eq. (4-1-28)) uniformly in time when n tends to infinity.

It is clear that the results obtained in this section are easily extended to the systems which are described by hyperbolic equations.

4-2. Convergence of Sub-Optimal Regulators of Discrete-Time Systems

In the Sec. 3, are derived the recurrence relations which are satisfied by the optimal feedback controllers and the optimal state-estimators. But these recurrence relations are difficult to be solved, because they are infinite-dimensional equations. Then we have to calculate the sub-optimal feedback controllers and sub-optimal state-estimators by approximating the infinite-dimensional system by the finite-dimensional system which consists of finite-main modes of the orginal system. And we show that in the operator topology the sub-optimal feedback controllers and the sub-optimal state-estimators converge to the optimal ones respectively when the number of the modes used increases. From this result, we say that the estimated state of the finite-dimensional system converges to the one of the infinite-dimensional system, and the optimal control and optimal trajectory of the former converge to ones of the latter.

We construct the following finite-dimensional approximating system by using n-main modes.

$$\zeta(n, i+1)=\Phi(n)\zeta(n, i)+\boldsymbol{B}(n)u(i)+\boldsymbol{D}(n)d(i), \qquad (4\text{-}2\text{-}1)$$

where $\zeta(n, \cdot)\equiv(\zeta_1(\cdot), \zeta_2(\cdot), \ldots, \zeta_n(\cdot))'$,

$$\Phi(n)\equiv[\Phi_{ij}],\ 1\le i, j\le n,\ \boldsymbol{B}(n)\equiv[B_1', B_2', \ldots, B_n']',$$
$$\boldsymbol{D}(n)\equiv[D_1', D_2', \ldots, D_n']'.$$

This system is the n-th approximate system of the infinite-dimensional system (3-1-3). The operator $Q(n, j)$ is the projection of $Q(j)$ to the n-dimensional state space X_n, and the performance index is

$$E(\textstyle\sum_{j=1}^n[\langle\zeta(n, j),\ Q(n, j)\zeta(n, j)\rangle+\langle u(j-1),\ H(j-1)u(j-1)\rangle]). \qquad (4\text{-}2\text{-}2)$$

We embed this n-dimensional system into the state space X of the system

(3-1-3), and then obtain the following system.

$$\tilde{\zeta}(n, i+1) = \tilde{\Phi}(n)\tilde{\zeta}(n, i) + \tilde{B}(n)u(i) + \tilde{D}(n)d(i), \tag{4-2-3}$$

where $\tilde{\zeta}(n, i) = (\zeta(n, i)', 0, \ldots)'$,

$$\tilde{B}(n) \equiv [B'(n), 0, \ldots]', \ \tilde{D}(n) \equiv [D'(n), 0, 0, \ldots]',$$

$$\tilde{\Phi}(n) \equiv \left[\begin{array}{c|c} \Phi(n) & 0 \\ \hline 0 & 0 \end{array}\right].$$

4–2–1. Convergence of Sub-Optimal Feedback Controllers

The optimal feedback controller $F(n, \cdot)$ of the system (4–2–1) with the P.I. (4–2–2) is obtained from the following recurrence relations as done in Sec. 3-2-1.

$$\left.\begin{aligned} &F(n, N-\overline{k+1}) = -[B^*(n)S(n, k)B(n) + H(N-\overline{k+1})]^{-1}B^*(n)S(n, k)\Phi(n), \\ &S(n, k) = P(n, k) + Q(n, N-k), \ P(n, 0) = 0, \\ &P(n, k+1) = [\Phi(n) + B(n)F(n, N-\overline{k+1})]^*S(n, k)[\Phi(n) + B(n)F(n, N-\overline{k+1})] \\ &\qquad\qquad + F^*(n, N-\overline{k+1})H(N-\overline{k+1})F(n, N-\overline{k+1}). \end{aligned}\right\} \tag{4-2-4}$$

These equations can be easily solved because they contain only finite unknowns.

We set

$$\left.\begin{aligned} &\tilde{F}(n, j) \equiv [F(n, j), 0, 0, \ldots], \\ &\tilde{Q}(n, j) \equiv \left[\begin{array}{c|c} Q(n, j) & 0 \\ \hline 0 & 0 \end{array}\right], \quad \tilde{P}(n, j) \equiv \left[\begin{array}{c|c} P(n, j) & 0 \\ \hline 0 & 0 \end{array}\right], \\ &\tilde{S}(n, j) \equiv \left[\begin{array}{c|c} S(n, j) & 0 \\ \hline 0 & 0 \end{array}\right]. \end{aligned}\right\} \tag{4-2-5}$$

These are linear bounded operators from X to X.

Clearly

$$\lim_{n\to\infty} \|\tilde{Q}(n, j) - Q(j)\| = 0.$$

Next we show by the mathematical induction, that $F(n, j)$ converges to $F(j)$, *i.e.*,

$$\left.\begin{aligned} &\lim_{n\to\infty} \|\tilde{F}(n, j) - F(j)\| = 0, \ j = 0, 1, \ldots, N-1, \\ &\lim_{n\to\infty} \|\tilde{P}(n, j) - P(j)\| = 0, \ j = 1, 2, \ldots, N, \\ &\lim_{n\to\infty} \|\tilde{S}(n, j) - S(j)\| = 0, \ j = 0, 1, 2, \ldots, N. \end{aligned}\right\} \tag{4-2-6}$$

In the following proof the uniform boundedness of the operators $\tilde{F}(n, j)$, $\tilde{P}(n, j)$ and $\tilde{S}(n, j)$ in n plays an essential role.

Since $P(n, 0) = 0$, then $S(n, 0) = Q(n, N)$ and

$$\|S(n, 0)\| = \|\tilde{S}(n, 0)\| = \|\tilde{Q}(n, N)\| \leq \|Q(N)\|. \tag{4-2-7}$$

From Eq. (4–2–4),

$$F(n, N-1) = -[B^*(n)S(n, 0)B(n) + H(N-1)]^{-1}B^*(n)S(n, 0)\Phi(n),$$

and clearly

$$\|\boldsymbol{B}^*(n)\| \leq \|\boldsymbol{B}^*\|, \quad \|S(n, 0)\| \leq \|Q(N)\|, \quad \|\boldsymbol{\Phi}(n)\| \leq \|\boldsymbol{\Phi}\|. \tag{4-2-8}$$

Then

$$\begin{aligned}\|\tilde{\boldsymbol{F}}(n, N-1)\| &= \|\boldsymbol{F}(n, N-1)\| \\ &\leq \|\boldsymbol{B}^*\| \ \|Q(N)\| \ \|\boldsymbol{\Phi}\| \ \|[\boldsymbol{B}^*(n) S(n, 0) \boldsymbol{B}(n) + H(N-1)]^{-1}\|.\end{aligned} \tag{4-2-9}$$

Since the matrix $H(N-1)$ is a positive definite diagonal matrix, $H^{-1}(N-1)$ is also diagonal and positive definite. Hence

$$\begin{aligned}&[\boldsymbol{B}^*(n) S(n, 0) \boldsymbol{B}(n) + H(N-1)]^{-1} \\ &\quad = H^{-1}(N-1)[H^{-1}(N-1) \boldsymbol{B}^*(n) S(n, 0) \boldsymbol{B}(n) + I]^{-1}.\end{aligned} \tag{4-2-10}$$

We put

$$L(n, N) \equiv H^{-1}(N-1) \boldsymbol{B}^*(n) S(n, 0) \boldsymbol{B}(n). \tag{4-2-11}$$

Since $L(n, N)$ is a self-adjoint positive semi-definite matrix, there exists a self-adjoint positive semi-definite matrix $L_{1/2}(n, N)$ such that

$$L(n, N) = L_{1/2}(n, N) L_{1/2}(n, N).\text{[3]} \tag{4-2-12}$$

Therefore

$$\left.\begin{aligned}&I + L(n, N) = I + L_{1/2}(n, N) L^*_{1/2}(n, N), \\ &\|[I + L(n, N)]^{-1}\| \leq 1.\text{[4]}\end{aligned}\right\} \tag{4-2-13}$$

With these Eqs. (4-2-10) and (4-2-13), it can be shown

$$\|[\boldsymbol{B}^*(n) S(n, 0) \boldsymbol{B}(n) + H(N-1)]^{-1}\| \leq \|H^{-1}(N-1)\|. \tag{4-2-14}$$

From Eqs. (9) and (14), it follows

$$\|\tilde{\boldsymbol{F}}(n, N-1)\| \leq \|\boldsymbol{B}^*\| \ \|Q(N)\| \ \|\boldsymbol{\Phi}\| \ \|H^{-1}(N-1)\|. \tag{4-2-15}$$

Consequently $\tilde{\boldsymbol{F}}(n, N-1)$ is uniformly bounded in n.

From Eq. (4-2-4), it follows

$$[\tilde{\boldsymbol{B}}^*(n) \tilde{S}(n, 0) \tilde{\boldsymbol{B}}(n) + H(N-1)] \tilde{\boldsymbol{F}}(n, N-1) = -\tilde{\boldsymbol{B}}^*(n) S(n, 0) \tilde{\boldsymbol{\Phi}}(n).$$

On the other hand, from Eq. (3-2-2)

$$[\boldsymbol{B}^* S(0) \boldsymbol{B} + H(N-1)] \boldsymbol{F}(N-1) = -\boldsymbol{B}^* S(0) \boldsymbol{\Phi}.$$

The differences of the both sides of these two equations are

$$\begin{aligned}&[\tilde{\boldsymbol{B}}^*(n) \tilde{S}(n, 0) \tilde{\boldsymbol{B}}(n) + H(N-1)] \tilde{\boldsymbol{F}}(n, N-1) - [\boldsymbol{B}^* S(0) \boldsymbol{B} + H(N-1)] \boldsymbol{F}(N-1) \\ &\qquad = \boldsymbol{B}^* S(0) \boldsymbol{\Phi} - \tilde{\boldsymbol{B}}^*(n) \tilde{S}(n, 0) \tilde{\boldsymbol{\Phi}}(n).\end{aligned} \tag{4-2-16}$$

The right hand term of this equation is

$$\begin{aligned}&\boldsymbol{B}^* S(0) \boldsymbol{\Phi} - \tilde{\boldsymbol{B}}^*(n) \tilde{S}(n, 0) \boldsymbol{\Phi}(n) \\ &\quad = \boldsymbol{B}^*[S(0) \boldsymbol{\Phi} - \tilde{S}(n, 0) \tilde{\boldsymbol{\Phi}}(n)] + [\boldsymbol{B}^* - \boldsymbol{B}^*(n)] \tilde{S}(n, 0) \tilde{\boldsymbol{\Phi}}(n).\end{aligned}$$

And

$$\lim_n \tilde{S}(n, 0)\tilde{\Phi}(n) = S(0)\Phi,$$
$$\lim_n \tilde{B}^*(n) = B^*,$$
$$\|\tilde{S}(n, 0)\Phi(n)\| \le \|S(0)\|\|\Phi\|.$$

Then

$$\lim_n [B^*S(0)\Phi - \tilde{B}^*(n)\tilde{S}(n, 0)\tilde{\Phi}(n)] = 0. \tag{4-2-17}$$

The left hand term of Eq. (4–2–16) is

$$\begin{aligned}&[\tilde{B}^*(n)\tilde{S}(n, 0)\tilde{B}(n) + H(N-1)]\tilde{F}(n, N-1)\\ &-[B^*S(0)B + H(N-1)]F(N-1)\\ &=[\tilde{B}^*(n)\tilde{S}(n, 0)\tilde{B}(n) - B^*S(0)B]\tilde{F}(n, N-1)\\ &+[B^*S(0)B + H(N-1)][\tilde{F}(n, N-1) - F(N-1)].\end{aligned}$$

Since

$$\lim_n [\tilde{B}^*(n)\tilde{S}(n, 0)\tilde{B}(n) - B^*S(0)B] = 0,$$

and from Eq. (4–2–15), $\|F(n, N-1)\|$ is uniformly bounded in n, then the left hand term tends to $\lim_n [B^*S(0)B + H(N-1)][\tilde{F}(n, N-1) - F(N-1)]$. From Eq. (4–2–17), the above operator equals to 0. Then

$$\begin{aligned}&\lim_n [B^*S(0)B + H(N-1)][\tilde{F}(n, N-1) - F(N-1)]\\ &=[B^*S(0)B + H(N-1)][\lim_n ((\tilde{F}(n, N-1) - F(N-1))] = 0.\end{aligned}$$

Therefore for any ζ in X,

$$[B^*S(0)B + H(N-1)][\lim_n (\tilde{F}(n, N-1) - F(N-1))\zeta] = 0.$$

As $[B^*S(0)B + H(N-1)]$ is positive definite,

$$\lim_n [\tilde{F}(n, N-1) - F(N-1)]\zeta = 0.$$

This implies that

$$\|\lim_n \tilde{F}(n, N-1) - F(N-1)\| = \sup_{\|\zeta\|=1} \|[\lim_n \tilde{F}(n, N-1) - F(N-1)]\zeta\| = 0. \tag{4-2-18}$$

From Eq. (4–2–4)

$$\begin{aligned}\tilde{P}(n, 1) = [\tilde{\Phi}(n) + \tilde{B}(n)\tilde{F}(n, N-1)]^*\tilde{S}(n, 0)[\tilde{\Phi}(n) + \tilde{B}(n)\tilde{F}(n, N-1)]\\ + \tilde{F}^*(n, N-1)H(N-1)\tilde{F}(n, N-1).\end{aligned}$$

Since

$$\lim_n \tilde{B}^*(n) = B^*,\ \lim_n \tilde{F}^*(n, N-1) = F^*(N-1),\ \lim_n \tilde{\Phi}^*(n) = \Phi^*,$$

then from Eq. (3–2–3) it follows:

$$\lim_n \tilde{P}(n, 1) = [\Phi + BF(N-1)]^* S(0) [\Phi + BF(N-1)] + F^*(N-1) H(N-1) F(N-1) = P(1), \quad (4\text{-}2\text{-}19)$$

$$\|\tilde{P}(n, 1)\| \leq \|[\tilde{\Phi}(n) + \tilde{B}(n) \tilde{F}(n, N-1)]^* \tilde{S}(n, 0)[\tilde{\Phi}(n) + \tilde{B}(n)] \tilde{F}(n, N-1)\| + \|\tilde{F}^*(n, N-1) H(N-1) \tilde{F}(n, N-1)\|.$$

From Eq. (4–2–9)

$$\|\tilde{F}^*(n, N-1)\| = \|F^*(n, N-1)\| = \|\Phi^*(n) S(n, 0) B(n) [B^*(n) S(n, 0) B(n) + H(N-1)]^{-1}\| \leq \|\Phi^*\| \; \|S(0)\| \; \|B\| \; \|H^{-1}(N-1)\|.$$

Moreover $\|\tilde{P}(n, 1)\|$ is also uniformly bounded in n, because by Eq. (4–2–15), $\|\tilde{F}(n, N-1)\|$ is uniformly bounded in n.

Now we suppose that $\lim_n \tilde{P}(n, j) = P(j)$, $\lim_n \tilde{F}(n, N-j) = F(N-j)$ and $\|\tilde{P}(n, j)$ is uniformly bounded in n. As $\lim_n \tilde{Q}(n, j) = Q(j)$, it follows:

$$\lim_n \tilde{S}(n, j) = \lim_n [\tilde{P}(n, j) + \tilde{Q}(n, N-j)] = P(j) + Q(N-j) = S(j).$$

Since

$$\|S(n, j)\| = \|\tilde{S}(n, j)\| \leq \|\tilde{P}(n, j)\| + \|\tilde{Q}(n, N-j)\| \leq \|\tilde{P}(n, j)\| + \|Q(N-j)\|,$$

and we suppose that $\|\tilde{P}(n, j)\|$ is uniformly bounded in n, then $\|S(n, j)\|$ is uniformly bounded in n. By Eq. (4–2–4)

$$F(n, N-\overline{j+1}) = -[B^*(n) S(n, j) B(n) + H(j)]^{-1} B^*(n) S(n, j) \Phi(n).$$

Similarly as Eq. (4–2–14),

$$\|[B^*(n) S(n, j) B(n) + H(j)]^{-1}\| \leq \|H^{-1}(j)\|.$$

Hence

$$\|\tilde{F}(n, N-\overline{j+1})\| = \|F(n, N-\overline{j+1})\| \leq \|\Phi\| \; \|B^*\| \; \|H^{-1}(j)\| \; \|S(n, j)\|.$$

As $\|S(n, j)\|$ is uniformly bounded in n, $\|\tilde{F}(n, N-\overline{j+1})\|$ is uniformly bounded in n. From Eq. (4–2–4), it follows:

$$-[\tilde{B}^*(n) \tilde{S}(n, j) \tilde{B}(n) + H(N-\overline{j+1})] \tilde{F}(n, N-\overline{j+1}) = \tilde{B}^*(n) \tilde{S}(n, j) \tilde{\Phi}(n).$$

And from Eq. (3–2–4),

$$-[B^* S(j) B + H(N-\overline{j+1})] F(N-\overline{j+1}) = B^* S(j) \Phi.$$

The differences of both sides of these two equations are

$$-[\tilde{B}^*(n) \tilde{S}(n, j) \tilde{B}(n) + H(N-\overline{j+1})] \tilde{F}(n, N-\overline{j+1}) + [B^* S(j) B + H(N-\overline{j+1})] F(N-\overline{j+1}) = \tilde{B}^*(n) \tilde{S}(n, j) \tilde{\Phi}(n) - B^* S(j) \Phi.$$

When n tends to infinity, the right hand term to 0. On the other hand, the left hand term tends to

$$\|[B^*S(j)B+H(N-\overline{j+1})[\lim_n \tilde{F}(n, N-\overline{j+1})-F(N-\overline{j+1})]\|,$$

as $\|\tilde{F}(n, N-\overline{j+1})\|$ is uniformly bounded in n. Therefore

$$\|[B^*S(j)B+H(N-\overline{j+1})][\lim_n \tilde{F}(n, N-\overline{j+1})-F(N-\overline{j+1})]\|=0.$$

Since $B^*S(j)B+H(N-\overline{j+1})$ is positive definite, we can show that

$$\lim_n \tilde{F}(n, N-\overline{j+1})=F(N-\overline{j+1}), \tag{4-2-20}$$

as we proved Eq. (4–2–18).

By Eq. (4–2–4),

$$\begin{aligned}\tilde{P}(n, j+1)=&[\tilde{\Phi}(n)+\tilde{B}(n)\tilde{F}(n, N-\overline{j+1})]^*\tilde{S}(n, j)[\tilde{\Phi}(n)\\&+\tilde{B}(n)]\tilde{F}(n, N-\overline{j+1})]+\tilde{F}^*(n, N-\overline{j+1})H(N-\overline{j+1})\tilde{F}(n, N-\overline{j+1}).\end{aligned} \tag{4-2-21}$$

We have shown that $\|\tilde{S}(n, j)\|$ and $\|\tilde{F}(n, N-\overline{j+1})\|$ are uniformly bounded in n. Therefore $\|P(n, j+1)\|=\|\tilde{P}(n, j+1)\|$ is uniformly bounded in n. Since $\lim_n \tilde{B}(n)=B$ and $\lim_n \tilde{\Phi}(n)=\Phi$, it follows similary as Eq. (4–2–19)

$$\lim_n \tilde{P}(n, j+1)=P(j+1),$$

by Eqs. (3–2–5) and (4–2–21).

From above results, we obtain Eq. (4–2–6). Consequently $F(n, i)$ is the n-th approximate solution of $F(i)$, $i=0, 1, 2, \ldots, N-1$.

4–2–2. Convergence of Sub-Optimal State-Estimators

We investigate the sub-optimal state-estimator. We suppose that the output measurement operator $M(n)$ of the system (4–2–1) is the projection of M to X_n. That is

$$\left.\begin{aligned}m(i)&=M(n)\zeta(n, i)+n(i),\\ \tilde{M}(n)&=[M(n), 0, \ldots].\end{aligned}\right\} \tag{4-2-22}$$

$m(i)$ is the actual measurement output and the stochastic properties of the measurement noise is given. Then the estimated value $\hat{\zeta}(n, i|i)$ of $\zeta(n, i)$ which is calculated with Eq. (4–2–22) is generally not equal to the first n components of the estimated value $\hat{\zeta}(i|i)$ of $\zeta(i)$ calculated with Eq. (3–1–5). Then we must show that $\hat{\zeta}(n, i|i)$ converges to $\hat{\zeta}(i|i)$ in order to say that $\hat{\zeta}(n, i|i)$ is the n-th approximate solution of $\hat{\zeta}(i|i)$. In this section we prove that the estimator which estimates $\hat{\zeta}(n, i)$ converges in the operator topology to the one which estimates $\zeta(i)$. From this we can show that $\hat{\zeta}(n, i|i)$ converges to $\hat{\zeta}(i|i)$.

Let $\hat{\zeta}(n, i|k)$ be the optimal estimate of $\zeta(n, i)$ with measured outputs $m(0)$, $m(1), \ldots, m(k)$, and define $\check{\zeta}(n, i|k)$ by

$$\check{\zeta}(n, i|k) \equiv \zeta(n, i) - \hat{\zeta}(n, i|k). \tag{4-2-23}$$

The convariance matrix $C(n, k)$ is defined as

$$C(n, k) \equiv E([\check{\zeta}(n, k|k-1)][\check{\zeta}(n, k|k-1)]'). \tag{4-2-24}$$

This is a self-adjoint operator from X_n to X_n. We set

$$\Psi(n, k) \equiv \Phi(n) + B(n) F(n, N-k). \tag{4-2-25}$$

and define $K(n, k+1)$ by

$$\check{\zeta}(n, k+1|k+1) = K(n, k+1)\check{m}(k+1|k). \tag{4-2-26}$$

Then similarly as Eqs. (3-3-11), (3-3-18), (3-3-20), the following equations can be derived.

$$\left.\begin{aligned}
&\hat{\zeta}(n, k+1|k+1) = \Psi(n, k)\hat{\zeta}(n, k|k) + K(n, k+1)\check{m}(k+1|k)\\
&C(n, k+1) = \Psi(n, k)[I - K(n, k)M(n)]C(n, k)[I - K(n, k)M(n)]^*\Psi^*(n, k)\\
&\qquad + D(n)R(k)D^*(n) + \Psi(n, k)K(n, k)N(k)K^*(n, k)\Psi^*(n, k),\\
&K(n, k+1) = C(n, k+1)M^*(n)[M(n)C(n, k+1)M^*(n) + N(k+1)]^{-1}.
\end{aligned}\right\} \tag{4-2-27}$$

By Eq. (4-2-27), we can calculate the optimal estimate $\hat{\zeta}(n, i|i)$ of $\zeta(n, i)$ with the initial conditions $\hat{\zeta}(n, 0|-1)$ and $C(n, 0)$ when the measured output $m(i)$ is obtained.

The block diagram of this estimator can be drwan as the following Fig. 2.

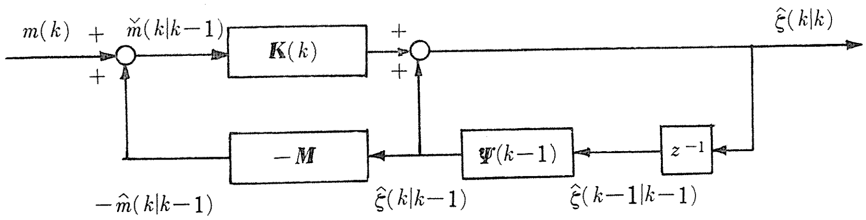

FIG. 2. Optimal Estimator for Finite-Dimensional System (4-2-1).

We define $\tilde{\Psi}(n, k)$, $\tilde{K}(n, k)$ and $\tilde{M}(n)$ as

$$\left.\begin{aligned}
&\tilde{\Psi}(n, k) \equiv \tilde{\Phi}(n) + \tilde{B}(n)\tilde{F}(n, N-k),\\
&\tilde{K}(n, k) \equiv [K'(n, k), 0, \ldots]',\\
&\tilde{M}(n) \equiv [M(n), 0, \ldots].
\end{aligned}\right\} \tag{4-2-28}$$

Clearly $\lim_n \tilde{\Psi}(n, k) = \Psi(k)$ and $\lim_n \tilde{M}(n) = M$.

Hence if we can show that

$$\lim_n \tilde{K}(n, k) = K(k), \tag{4-2-29}$$

then

$$\lim_{n} \hat{\zeta}(n, k|k)^{\sim} = \lim_{n} (\hat{\zeta}(n, k|k)', 0, \ldots)' = \hat{\zeta}(k|k).$$

In other wards, $\hat{\zeta}(n, k|k)$ is the n-th approximate solution of $\hat{\zeta}(k|k)$. Therefore we must prove Eq. (4–2–29), and the uniform boundedness of $\tilde{C}(n, k)$ and $\tilde{K}(n, k)$ in n plays an essential role in the following mathematical induction.

We set

$$\tilde{C}(n, k) \equiv [\check{\zeta}(n, k|k-1)^{\sim} \circ \check{\zeta}(n, k|k-1)^{\sim}]. \tag{4-2-30}$$

Then

$$\tilde{C}(n, k)\xi = \begin{bmatrix} C(n, k)\xi(n) \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \tag{4-2-31}$$

where $\xi = (\xi_1, \xi_2, \ldots)'$, and $\xi(n) = (\xi_1, \xi_2, \ldots, \xi_n)'$. By three equations (4–2–27), (4–2–28) and (4–2–29), it follows:

$$\left.\begin{aligned} \tilde{C}(n, k+1) &= \tilde{\Psi}(n, k)[I - \tilde{K}(n, k)\tilde{M}(n)]C(n, k)[I - \tilde{K}(n, k)\tilde{M}(n)]^{*}\tilde{\Psi}^{*}(n, k) \\ &\quad + \tilde{D}(n)R(k)\tilde{D}^{*}(n) + \tilde{\Psi}(n, k)\tilde{K}^{*}(n, k)N(k)\tilde{K}^{*}(n, k)\tilde{\Psi}^{*}(n, k), \\ \tilde{K}(n, k+1) &= \tilde{C}(n, k+1)\tilde{M}^{*}(n)[\tilde{M}(n)\tilde{C}(n, k+1)\tilde{M}^{*}(n) + N(k+1)]^{-1}. \end{aligned}\right\} \tag{4-2-32}$$

It is reasonable that we suppose

$$\lim_{n} \tilde{C}(n, 0) = C(0). \tag{4-2-33}$$

From Eq. (3–3–18)

$$C(0)M^{*} = K(0)[MC(0)M^{*} + N(0)],$$

and from Eq. (4–2–32) (4-2-34)

$$\tilde{C}(n, 0)\tilde{M}^{*}(n) = \tilde{K}(n, 0)[\tilde{M}(n)\tilde{C}(n, 0)\tilde{M}^{*}(n) + N(0)].$$

Since $N(0)$ is a positive definite diagonal matrix, the invese matrix $N^{-1}(0)$ exists. Furthermore since the following inequalities hold $\|\tilde{C}(n, 0)\| \leq \|C((0)\|$ and $\|\tilde{M}^{*}(n)\| \leq \|M^{*}\|$, then by Eq. (4–2–32) it follows:

$$\|K(n, 0)\| = \|\tilde{K}(n, 0)\| \leq \|C(0)\|\ \|M^{*}\|\ \|N^{-1}(0)\|\ \|[N^{-1}(0)\tilde{M}(n)\tilde{C}(n, 0)\tilde{M}^{*}(n) + I]^{-1}\|.$$

As $N^{-1}(0)\tilde{M}(n)\tilde{C}(n, 0)\tilde{M}^{*}(n)$ is self-adjoint and positive semi-definite, then similarly as Eq. (4–2–14),

$$\|[N^{-1}(0)\tilde{M}(n)\tilde{C}(n, 0)\tilde{M}^{*}(n) + I]^{-1}\| \leq 1.$$

Hence

$$\|K(n, 0)\| \leq \|C(0)\|\ \|M^{*}\|\ \|N^{-1}\|(0). \tag{4-2-35}$$

From this, $\|\tilde{K}(n, 0)\|$ is uniformly bounded in n.

The differences of both sides of two equations of Eq. (4–2–34) are

$$C(0)M^* - \tilde{C}(n,0)\tilde{M}^*(n)$$
$$= [K(0) - \tilde{K}(n,0)][MC(0)M^* + N(0)] + \tilde{K}(n,0)[MC(0)M^* - \tilde{M}(n)\tilde{C}(n,0)\tilde{M}^*(n)]. \quad (4\text{-}2\text{-}36)$$

Since $\|\tilde{K}(n,0)\|$ is uniformly bounded in n, and

$$\lim_n \tilde{M}^*(n) = M^*, \ \lim_n \tilde{C}(n,0) = C(0),$$

then the left hand term of Eq. (4–2–36) tends to 0 and the right hand term tends to

$$\lim_n [K(0) - \tilde{K}(n,0)][MC(0)M^* + N(0)],$$

when n tends to infinity. Then

$$\lim_n [MC(0)M^* + N(0)][K(0) - \tilde{K}(n,0)]^* = 0.$$

Therefore

$$[MC(0)M^* + N(0)][K(0) - \lim_n \tilde{K}(n,0)]^* = 0.$$

As $MC(0)M^* + N(0)$ is positive definite, $[\lim \tilde{K}(n,0) - K(0)]^* = 0$. Consequently,

$$\lim_n \tilde{K}(n,0) = K(0), \quad (4\text{-}2\text{-}37)$$

By Eq. (3–3–20),

$$C(1) = \Psi(0)[I - K(0)M]C(0)[I - K(0)]^*\Psi^*(0) + DR(0)D^*$$
$$+ \Psi(0)K(0)N(0)K^*(0)\Psi^*(0).$$

By the first equation of Eq. (4–2–32)

$$\tilde{C}(n,1) = \tilde{\Psi}(n,0)[I - \tilde{K}(n,0)\tilde{M}(n)]\tilde{C}(n,0)[I - \tilde{K}(n,0)\tilde{M}(n)]^*\tilde{\Psi}^*(n,0)$$
$$+ \tilde{D}(n)R(0)\tilde{D}^*(n) + \tilde{\Psi}(n,0)\tilde{K}(n,0)N(0)\tilde{K}^*(n,0)\tilde{\Psi}^*(n,0).$$

This equation implies that $\|\tilde{C}(n,1)\|$ is uniformly bounded in n.
Then it follows:

$$\lim_n \tilde{C}(n,1) = C(1), \quad (4\text{-}2\text{-}38)$$

by Eq. (4–2–37), because

$$\lim_n \tilde{C}(n,0) = C(0), \quad \lim_n \tilde{M}^*(n) = M^*$$
$$\lim_n \tilde{D}(n) = D, \quad \lim_n \tilde{\Psi}(n,0) = \Psi(0).$$

We suppose that $\|\tilde{C}(n,k)\|$ is uniformly bounded in n and

$$\lim_n \tilde{C}(n,k) = C(k), \quad (4\text{-}2\text{-}39)$$

By the second equation of Eq. (4–2–32), it follows:

$$\|\tilde{K}(n,k)\| \leq \|\tilde{C}(n,k)\| \, \|\tilde{M}^*(n)\| \, \|N^{-1}(k)\| \, \|[N^{-1}(k)\tilde{M}(n)\tilde{C}(n,k)\tilde{M}^*(n)+I]^{-1}\|.$$

Since $N^{-1}(k)\tilde{M}(n)\tilde{C}(n,k)\tilde{M}^*(n)$ is self-adjoint and posiitve semi-definite,

$$\|[N^{-1}(k)\tilde{M}(n)\tilde{C}(n,k)\tilde{M}^*(n)+I]^{-1}\| \leq 1,$$

similary as Eq. (4–2–13). Therefore $\|\tilde{K}(n,k)\|$ is uniformly bounded in n. By the second equation of Eq. (4–2–32), the following equations can be derived

$$\left.\begin{aligned} &\tilde{C}(n,k)\tilde{M}^*(n) = \tilde{K}(n,k)[\tilde{M}(n)\tilde{C}(n,k)\tilde{M}^*(n)+N(k)], \\ &\text{and by Eq. (3–3–17)} \\ &C(k)M^* = K(k)[MC(k)M^*+N(k)]. \end{aligned}\right\} \qquad (4\text{-}2\text{-}40)$$

Making use of the fact that $\|\tilde{K}(n,k\|$ is uniformly bounded in n, we can show from Eqs. (4–2–39) and (4–2–40) that

$$\lim_n \tilde{K}(n,k) = K(k), \qquad (4\text{-}2\text{-}41)$$

by the similar method which we used in order to obtain Eq. (4–2–37) from Eqs. (4–2–33) and (4–2–34).

By Eq. (3–3–20), it follows:

$$C(k+1) = \Psi(k)[I-K(k)M]C(k)[I-K(k)M]^*\Psi^*(k) + DR(k)D^* + \Psi(k)K(k)N(k)K^*(k)\Psi^*(k),$$

and by the first equation of Eq. (4–2–23),

$$\tilde{C}(n,k+1) = \tilde{\Psi}(n,k)[I-\tilde{K}(n,k)\tilde{M}(n)]\tilde{C}(n,k)[I-\tilde{K}(n,k)\tilde{M}(n)]^*\tilde{\Psi}^*(n,k) + \tilde{D}(n)R(k)\tilde{D}^*(n) + \tilde{\Psi}(n,k)\tilde{K}(n,k)N(k)\tilde{K}^*(n,k)\tilde{\Psi}^*(n,k).$$

Hence by Eq. (4–2–41), we can obtain the following equation

$$\lim_n \tilde{C}(n,k+1) = C(k+1), \qquad (4\text{-}2\text{-}42)$$

because $\|\tilde{C}(n,k)\|$ is uniformly bounded in n and

$$\lim_n \tilde{\Psi}(n,k) = \Psi(k).$$

And it can be easily shown that $\|\tilde{C}(n,k+1)\|$ is uniformly bounded in n.

Hence we have shown that

$$\lim_n \tilde{C}(n,k) = C(k), \quad k=0,1,2,\ldots.$$
$$\lim_n \tilde{K}(n,k) = K(k), \quad k=0,1,2,\ldots.$$

Then the state-estimator of Fig. 2 converges to the one of Fig. 1 in the operator topology and the former is the n-th approximate solution of the latter.

5. Concluding Remarks

An attempt has been made to present a general, unified discussion on the synthesis of the optimal controllers for distributed-parameter systems. Since "Functional Analysis" approach plays an essential role, portions of deriving procedures of optimal controls may seem to be somewhat abstract from the engineering viewpoint. However it is worthy to emphasize that the abstract approach with Functional Analysis can provide a better prospect of the basic properties of the problems and an extension of certain research results obtained on lumped-parameter systems.

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