

# ON THE LARGE DEFLECTION OF AN ANNULAR PLATE OF THE MATERIAL HAVING A NON-LINEAR STRESS-STRAIN RELATION

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## Summary

In the paper, a large deflection of a simply supported annular plate of an aluminium alloy, which has a non-linear continuous stress-strain curve, subjected to a uniform lateral load along its inner edge is analysed by taking account of the effects of compressibility of the material and membrane force induced in the plate according to its deflection. As the result of analysis, distributions of bending moment, membrane force, components of surface strain, radial displacement and deflection were obtained. The analytical values of deflection and strain components on both surfaces agreed well with the results of the corresponding experiment.

Previously, a remarkable difference between the analytical and the corresponding experimental results of the surface strain, especially on radial component, appeared on the bending of annular plate made of mild steel for more than a certain value of load. The difference was attributed to the delay of yielding of the mild steel having upper and lower yield points according to the restraint of the region being still in elastic state under large gradient of stress distribution which is especially remarkable in annular plate.

According to such an assumption, the difference between the corresponding two results cannot appear for the material having non-linear continuous stress-strain relation such as aluminium alloy. Therefore, to confirm the assumption in analysing the bending problem of annular plate made of an aluminium alloy is another essential intension of this paper. As mentioned above, the analytical values of surface strain components agreed well with the corresponding experimental ones. Accordingly, the above mentioned assumption was confirmed well by comparing the results of the two investigations concerning the bending of annular plates made of mild steel and aluminium alloy.

## 1. Introduction

In the plastic analysis of the bending problem of annular plate, the limit analysis has been used by several authors<sup>1)~5)</sup>. This method, however, would be convenient to estimate the limit load of elements of the complex structures made of the material which may be approximated as a rigid-perfectly plastic body rather than to analyse precisely the detailed process of successive elasto-plastic deformation of the plate under external loads.

Concerning the detailed analysis of the elasto-plastic bending of annular plate, Sokolovsky<sup>6)</sup>, Ohashi and Murakami<sup>10)</sup> analysed some problems in the case of elastic-perfectly plastic material for small deflection. Ohashi *et al.*<sup>11)</sup> extended

the analysis to large deflection and compared the analytical result with that of the corresponding experiment performed with a mild steel specimen.

On the other hand, for the material having a non-linear stress-strain relation, since there is not any well-defined yield point and the elastic and plastic ranges cannot be distinguished from each other, the limit analysis cannot be used as a proper approximation. In the analysis of small deflection of a circular plate of the material with non-linear stress-strain relation performed by Sokolovsky<sup>9)</sup>, the stress-strain relation was expressed with a single power function and the incompressibility of material was assumed through the whole plate. However, in such a material, as the elastic and plastic parts of strain are kept of the same order over a wide range, the assumption of incompressibility over the whole plate cannot be a suitable approximation. Moreover, the single power function is not a precise approximation of the mechanical property of such materials especially within a small range of strain less than  $10^{-2}$ .

Accordingly, taking account of the effect of compressibility of material and expressing the stress-strain relation with a kind of exponential function, the present authors<sup>12)</sup> analysed a small deflection of a circular plate made of pure aluminium under uniform lateral load and obtained a result which agreed well with the corresponding experimental one. They also extended their method to the corresponding problem with large deflection<sup>13)</sup> in the case of an aluminium alloy, for which a third order polynomial form is a good approximation of its stress-strain relation, and obtained a result having an excellent agreement with that of the corresponding experiment.

In this paper, a large deflection of a simply supported annular plate of the aluminium alloy subjected to a uniform lateral load along its inner edge is analysed in the analogous method mentioned above. Another intension of this paper consists in giving an explicit solution to the following problem.

In the previous analysis<sup>11)</sup> of large deflection of an annular plate of mild steel, the analytical results obtained by assuming the transition from elastic state to plastic at its lower yield point showed a remarkable difference in comparing with those of the corresponding experiment. In that paper<sup>11)</sup>, therefore, the difference was explained by considering a delay of yielding of the material having an upper yield point according to the restraint of the region still remaining in elastic state under the large gradient of stress distribution which is especially remarkable in annular plate, because there was no such a remarkable difference between the analytical result obtained by the same assumption and the corresponding experimental one in the corresponding case of circular plate. According to the above mentioned assumption, the difference between the corresponding two results cannot appear in the case of material having a non-linear continuous stress-strain relation such as aluminium alloy. Therefore, to confirm the assumption in analysing the corresponding problem using the aluminium alloy has an important meaning to examine the propriety of the explanation in the previous paper<sup>11)</sup>.

As the result of investigation using the aluminium alloy, the analytical and experimental results agreed well with each other and the above mentioned problem was given an explicit solution.

## 2. Fundamental Equations

A simply supported annular plate subjected to a uniform lateral load along

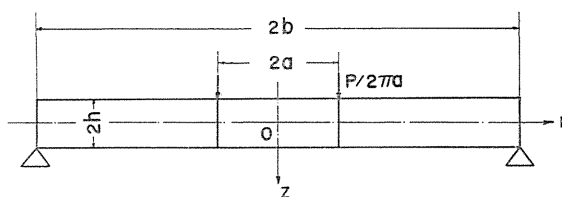


FIG. 1. Annular plate and coordinate system.

its free inner edge is shown in Fig. 1 with a cylindrical coordinate system  $r$ ,  $\theta$  and  $z$ . Since the fundamental equations for a large deflection of the annular plate of aluminium alloy may be derived by a trivial modification from those obtained for a circular plate<sup>13)</sup>, their derivation is outlined in the following.

According to Kirchhoff's hypothesis, the radial and circumferential components of strain in an element of the plate are expressed as follows:

$$\varepsilon_r = e_r + z\alpha_r, \quad \varepsilon_\theta = e_\theta + z\alpha_\theta, \quad (1)$$

where  $e_r$ ,  $e_\theta$ ,  $\alpha_r$  and  $\alpha_\theta$  are the radial and circumferential components of unit elongation and curvature and are related with the radial displacement  $u$  and deflection  $w$  of the plate as in the next relation, respectively.

$$e_r = \frac{du}{dr} + \frac{1}{r} \left( \frac{dw}{dr} \right)^2, \quad e_\theta = \frac{u}{r}, \quad \alpha_r = -\frac{d^2w}{dr^2}, \quad \alpha_\theta = -\frac{1}{r} \frac{dw}{dr}. \quad (2)$$

From these components, the compatibility conditions of strain are obtained as follows:

$$\frac{de_\theta}{dr} + \frac{1}{r} (e_\theta - e_r) + \frac{1}{2} r \alpha_\theta^2 = 0, \quad \frac{d\alpha_\theta}{dr} + \frac{1}{r} (\alpha_\theta - \alpha_r) = 0. \quad (3)$$

The equilibrium conditions of the plate element are considered as in the next forms.

$$\frac{dT_r}{dr} + \frac{1}{r} (T_r - T_\theta) = 0, \quad \frac{dM_r}{dr} + \frac{1}{r} (M_r - M_\theta) + T_r \frac{dw}{dr} + \frac{P}{2\pi r} = 0, \quad (4)$$

where  $T_r$ ,  $T_\theta$ ,  $M_r$  and  $M_\theta$  are components of membrane force and bending moment, respectively, and  $P$  denotes the total amount of the lateral load.

By taking account of the compressibility of material with the use of parameter  $c$  defined in relation to the state of deformation with the equivalent stress  $\bar{\sigma}$  and strain  $\bar{\epsilon}$  as well as the volumetric strain coefficient  $\alpha$  in the next form<sup>12)</sup>,

$$c = 2\bar{\sigma}\alpha/3\bar{\epsilon}, \quad (5)$$

the equivalent strain in the plane stress state is defined as follows:

$$\frac{3}{4} \bar{\epsilon}^2 = \frac{1+c+c^2}{(1+2c)^2} (\varepsilon_r + \varepsilon_\theta)^2 - \varepsilon_r \varepsilon_\theta. \quad (6)$$

Expressing  $e_r$ ,  $e_\theta$ ,  $\alpha_r$  and  $\alpha_\theta$  in the next trigonometric forms with the use of

new parameters  $E_0(r)$ ,  $E_1(r)$ ,  $\omega_0(r)$  and  $\omega_1(r)$ ,

$$\left. \begin{aligned} \frac{e_r}{e_\theta} &= \frac{2E_1}{\sqrt{3}} \left\{ \cos\left(\omega_1 \mp \frac{\pi}{3}\right) + c \cos \omega_1 \right\}, \\ \frac{\alpha_r}{\alpha_\theta} &= \frac{2E_0}{\sqrt{3}h} \left\{ \cos\left(\omega_0 \mp \frac{\pi}{3}\right) + c \cos \omega_0 \right\}, \end{aligned} \right\} \quad (7)$$

and using the Hencky's relation between the components of stress and strain together with the stress-strain relation in the form of a third order polynomial,

$$\bar{\sigma} = a_0 \bar{e} + a_1 \bar{e}^2 + a_2 \bar{e}^3, \quad (8)$$

the components of membrane force and bending moment acting on the unit width of the plate element are expressed as follows;

$$\left. \begin{aligned} \frac{T_r}{T_\theta} &= \pm \frac{4E_1 h}{3} \left[ \beta \sin\left(\omega_1 \pm \frac{\pi}{3}\right) I_1 + \sin\left(\omega_0 \pm \frac{\pi}{3}\right) I_2 \right], \\ \frac{M_r}{M_\theta} &= \pm \frac{4E_0 h^2}{3} \left[ \beta \sin\left(\omega_1 \pm \frac{\pi}{3}\right) I_2 + \sin\left(\omega_0 \pm \frac{\pi}{3}\right) I_3 \right], \end{aligned} \right\} \quad (9)$$

where  $a_0$ ,  $a_1$  and  $a_2$  are material constants, and  $\beta = E_1/E_0$ ,  $\zeta = z/h$ ,  $I_1 = \int_{-1}^1 \frac{\bar{\sigma}}{e} d\zeta$ ,  $I_2 = \int_{-1}^1 \frac{\bar{\sigma}}{e} \zeta d\zeta$  and  $I_3 = \int_{-1}^1 \frac{\bar{\sigma}}{e} \zeta^2 d\zeta$ .

Substituting (7) and (9) into (3) and (4), respectively, the conditions of equilibrium and compatibility are reduced to the following differential equations.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} dE_0/dr \\ d\beta/dr \\ d\omega_0/dr \\ d\omega_1/dr \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \quad (10)$$

Concrete expressions of  $I_1$ ,  $I_2$  and  $I_3$  as well as of the elements  $(a_{ij})$  and  $(b_i)$  ( $i, j=1, 2, 3, 4$ ) in the above equations are omitted here for conciseness. Analogous expressions for the circular plate under uniform lateral load are shown in detail in the previous paper<sup>13</sup>. The deflection of the plate can be found from the next differential equation obtained from (2).

$$\frac{dw}{dr} = -r\alpha_0. \quad (11)$$

The boundary conditions at inner and outer edges are expressed as

$$M_r = T_r = 0 \quad \text{at } r = a, \quad (12)$$

$$M_r = T_r = w = 0 \quad \text{at } r = b, \quad (13)$$

respectively.

### 3. Numerical Calculation and its Results

A numerical example of the above described fundamental equations was per-

formed for an annular plate with dimension ratios  $b/h=40$  and  $b/a=4$  made of an aluminium alloy. The chemical components of the material are shown in Table 1 and the result of calibration test are shown in Figs. 2 and 3 concerning the relations between  $\bar{\sigma}$  and  $\bar{\epsilon}$  as well as  $c$  and  $\bar{\epsilon}$ . The values of  $a_0$ ,  $a_1$  and  $a_2$  also entered in Fig. 2.

TABLE 1. Chemical Components of Aluminium Alloy

Cu	Si	Fe	Mn	Mg	Zn	Cr	Tr	Al
0.03%	0.09	0.20	0.63	4.52	0.02	0.19	0.02	R

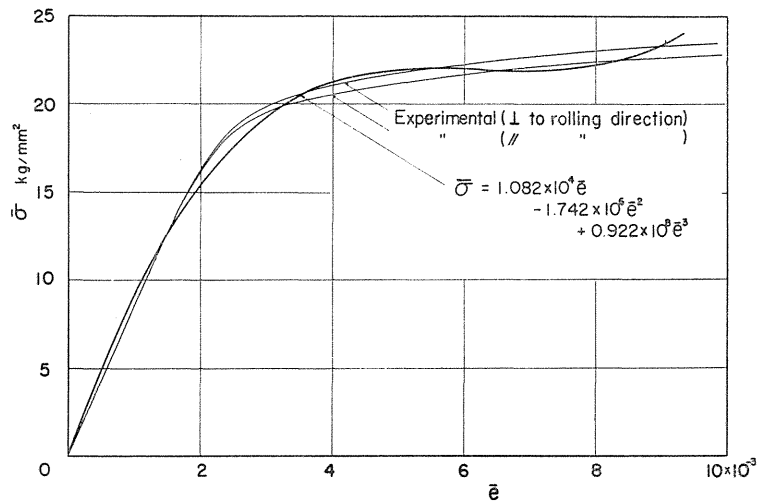
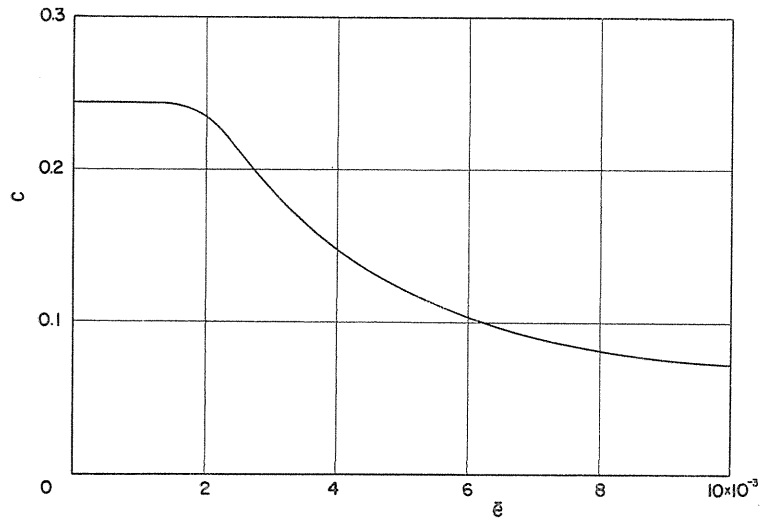


FIG. 2. Stress-strain relation of aluminium alloy.

FIG. 3. Relation between parameter  $c$  and equivalent strain.

Numerical integration of (10) and (11) was performed by an electronic computer with a trial method as in the following. Though the values of the parameters  $\omega_0$  and  $\omega_1$  at inner edge can be determined from the boundary conditions (12), the values of the other parameters  $E_0$  and  $\beta$  are difficult to know in advance. Accordingly, prescribing these values at the inner edge first approximately from the result of elastic analysis, the integration is carried out from inner edge to outer edge. Then, the correct values of these two parameters at inner edge are obtained modifying them so that the result of calculation may satisfy the outer edge condition (13).

The value of parameter  $c$  depends not only on the property of the material but on the state of deformation. However, the state of deformation cannot be known in advance. Therefore, as a first approximation for finding the value of  $c$ , the above mentioned equations are solved for each value of lateral load under the condition of incompressibility of the material ( $c=0$ ). As the result of this calculation, the distributions of strain components in the plate are obtained for each value of load. According to these values, by using (6) for  $c=0$ , a distribution of the equivalent strain  $\bar{e}$  in the plate can be found. A volumetric mean of these value in the plate can be easily estimated and is designated as  $\bar{e}_{eq}$  for each load. Assuming that the value of  $\bar{e}_{eq}$  represent on the average the state of deformation of the plate to the specified value of lateral load, the value of  $c$  corresponding to  $\bar{e}_{eq}$  may be found from Fig. 3.

Calculating again the fundamental equations with the value of  $c$  herein obtained, the state of deformation of the plate under the corresponding value of the lateral load can be analysed taking into account the effect of compressibility of the material. Some example of the results of calculation are shown in Figs. 4 to 7. In these figures, it is clearly observed that the radial components of bending moment and membrane force are quite small compared with the cor-

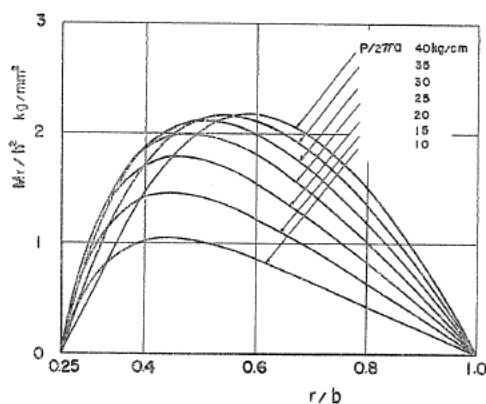


FIG. 4. Distribution of radial bending moment  $M_r/h^2$ .

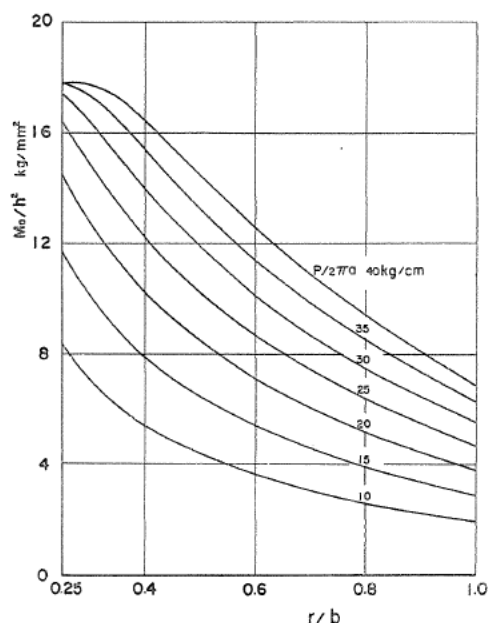


FIG. 5. Distribution of circumferential bending moment  $M_\theta/h^2$ .

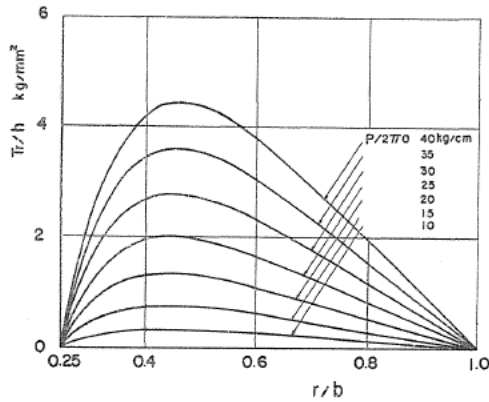


FIG. 6. Distribution of radial membrane force  $T_r/h$ .

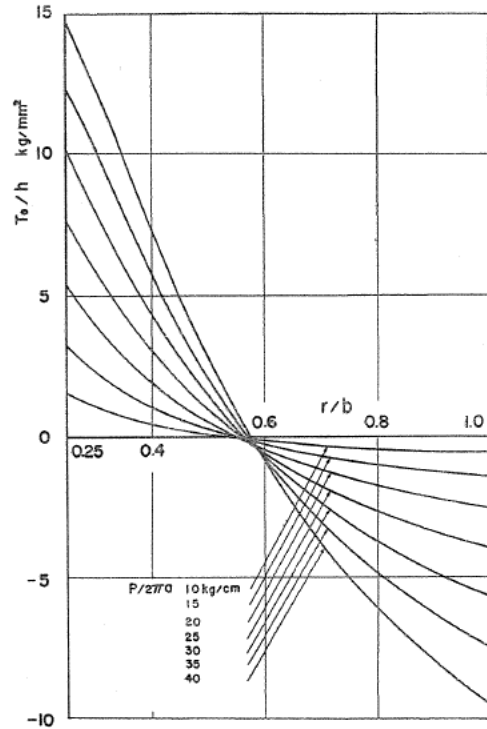


FIG. 7. Distribution of circumferential membrane force  $T_\theta/h$ .

responding circumferential ones and that the ratio of membrane force to bending moment both in radial and circumferential components are larger than that obtained for the circular plate<sup>13</sup>. The gradient of distributions of these values are far larger than those in the circular plate<sup>13</sup>.

#### 4. Experimental Examination

In order to discuss the validity of the above obtained analytical result in comparing with the corresponding actual behavior, an experiment was carried out by using aluminium alloy. As shown in Fig. 8, an annular plate of thickness  $2h=5$  mm, inner and outer diameters 49 mm and 202 mm, respectively, was supported on the supporting block of inner diameter  $2b=200$  mm ( $b/h=40$ ) and was sub-

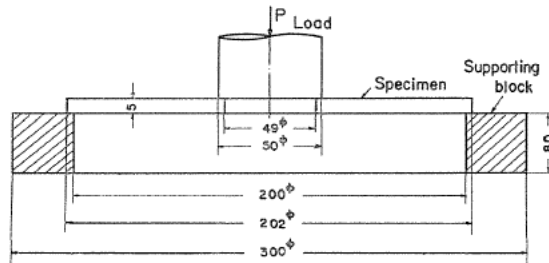


FIG. 8. Scheme of experimental apparatus.

jected to lateral load by means of a loading block of outer diameter  $2a=50$  mm ( $b/a=4$ ). The deflection and the strain components on both surfaces were measured by dial indicators and wire resistance strain gauges, respectively. The results of the experiment are shown in Figs. 9 to 13 in comparing with the corresponding analytical results.

### 5. Discussion and Conclusion

As shown in Fig. 9, though the experimental value of the deflection at each measured point is somewhat larger than that of the corresponding calculation over the whole range of total load, the latter represents a fairly good approximation to the former as a whole. With respect to the difference between these

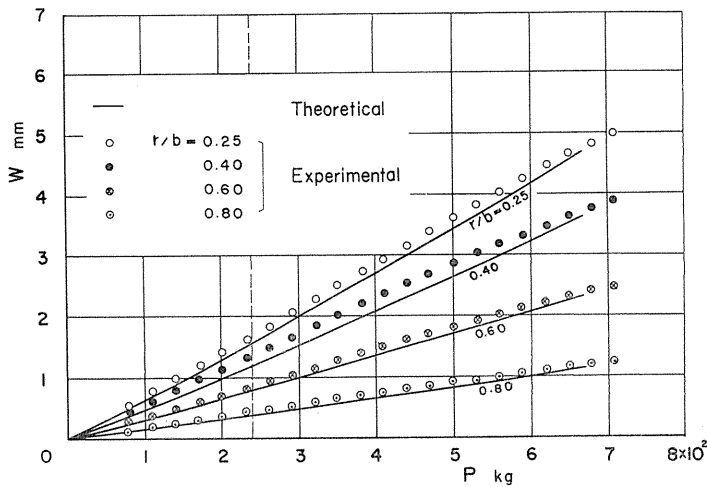


FIG. 9. Theoretical and experimental values of deflection.

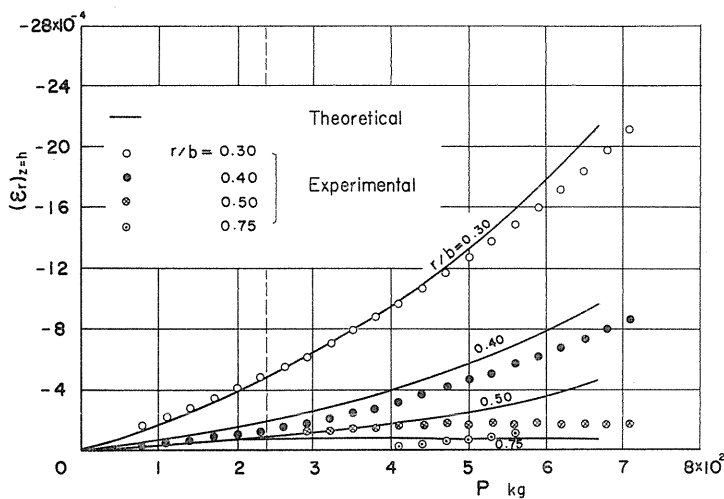
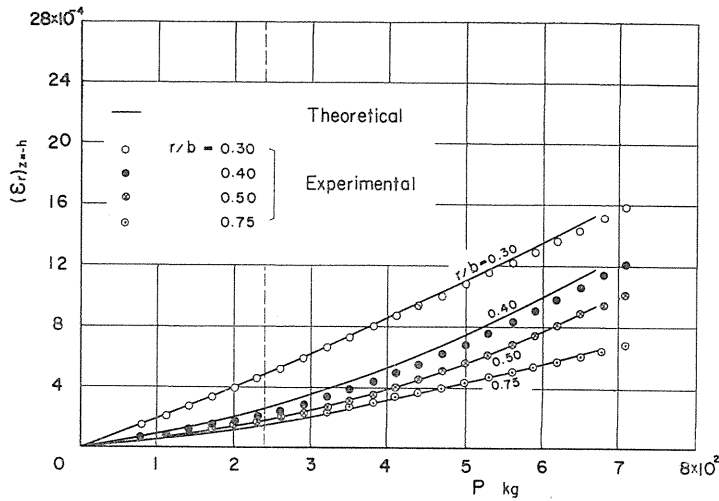
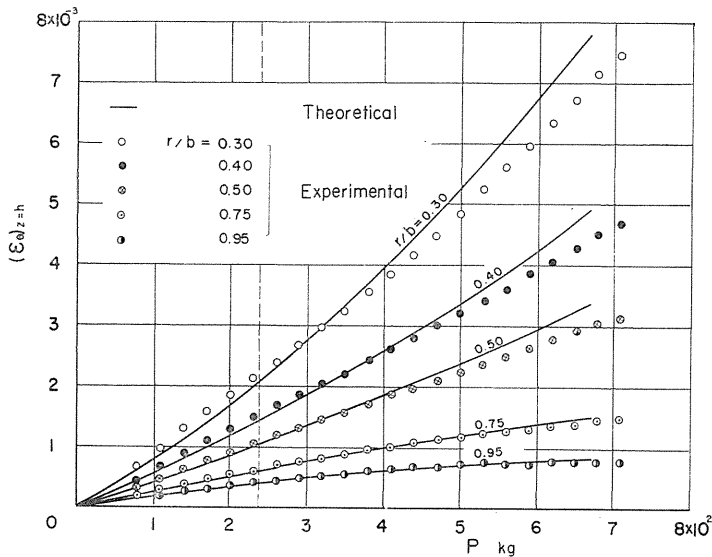
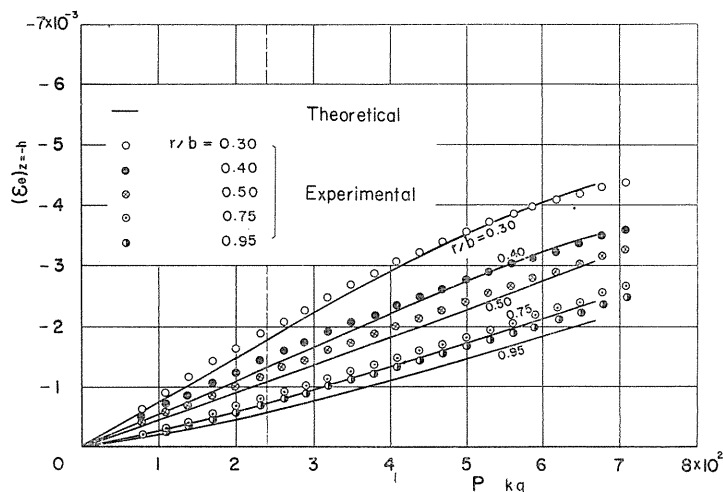


FIG. 10. Radial strain component on the surface  $z=h$ .

FIG. 11. Radial strain component on the surface  $z = -h$ .FIG. 12. Circumferential strain component on the surface  $z = h$ .

two results, the value herein obtained is somewhat larger than that obtained in the previous analysis<sup>13)</sup> for a large deflection of a circular plate of the same material using the analogous fundamental equations. The reason why the experimental value of deflection is larger than that of calculation may consist in the fact that the stress-strain relation (8) deviates a little from the result of calibration test to the side of smaller value of strain in the range of  $\bar{\epsilon} < 1500 \times 10^{-6}$  (Fig. 2).

The absolute values of radial strain components  $(\epsilon_r)_{z=h}$  shown in Fig. 10 and  $(\epsilon_r)_{z=-h}$  shown in Fig. 11 differ remarkably from each other. This means that

FIG. 13. Circumferential strain component on the surface  $z = -h$ .

the effect of membrane force induced in the annular plate is so distinguished that the strain distribution becomes asymmetric with respect to the middle plane of the plate. The differences between the analytical and experimental values of these strain components are nearly equal order to that of the deflection. As shown in Figs. 12 and 13, the values of the circumferential strain components are larger than the corresponding radial ones. The absolute value of  $(\varepsilon_\theta)_{z=h}$  is the largest of all strain components and its analytical value approximates the experimental one in good accuracy. The difference between the components on both surfaces is remarkable also in this respect. In considering the fact that the strain distributions on both surfaces of the plate are affected by the local inhomogeneity of the material, it may be said that the analytical results herein obtained agree accurately with the corresponding experimental ones.

Fig. 14 shows a comparison between the experimental and analytical results of the deflection at  $r/b=0.25$  (at the loading circle) for several assumptions. The solid curve shows the analytical result obtained taking account of the membrane force and the compressibility of material by means of the parameter  $c$  and is the best approximation to the experimental result. The analytical result obtained under the condition of incompressibility of the material corresponds to the chain curve lying under the solid one. The fine solid curve and the dashed curve show the results obtained by the analysis in which the membrane force is neglected and the compressibility of the material is and is not considered, respectively.

As shown in Fig. 2, the stress-strain relation of the aluminium alloy obtained by calibration test also may be approximated roughly with two straight lines representing the elastic and the strain hardening parts and the corresponding yield point may be regarded as the intersection of these lines. Therefore, incipience of plastic region in the plate may be assumed to appear at the equivalent strain corresponding to the roughly approximated yield point, the value of which may be obtained as  $\bar{\varepsilon}=2400 \times 10^{-6}$  from Fig. 2. As the largest value of the equivalent strain in the simply supported annular plate subjected to a uniform lateral load

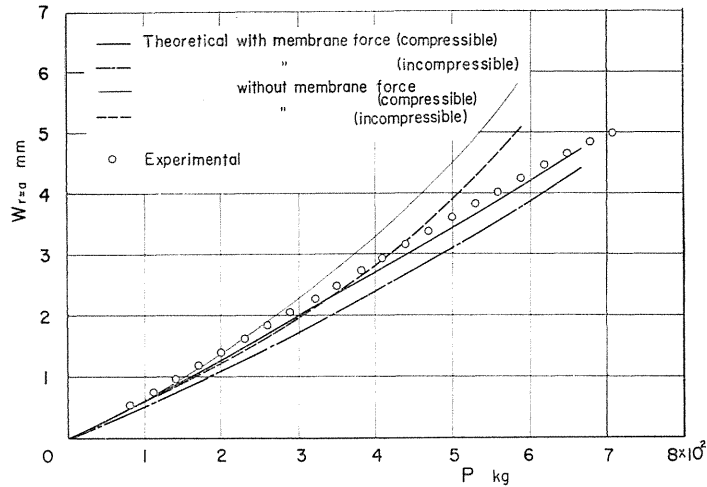


FIG. 14. Deflection at  $r/b=0.25$  obtained analytically for several assumptions.

along free inner edge appears at the inner edge on the surface  $z=h$ , from the analytical result, the corresponding external load to the incipience of plastic region is obtained as  $P=237$  kg. The value is entered in Figs. 9 to 13 with dashed lines.

Figs. 15 to 19 show the corresponding results obtained for mild steel<sup>11)</sup> having elastic-perfectly plastic stress-strain relation with well-defined yield point. In these figures, the small circles show the experimental results at measured points and the solid curves show the corresponding analytical results obtained by assuming that the transition from elastic state to plastic occurs at the stress state corresponding to the lower yield point of the material. The dashed line shows the incipience of plastic region according to the analytical result. In Figs. 16 and

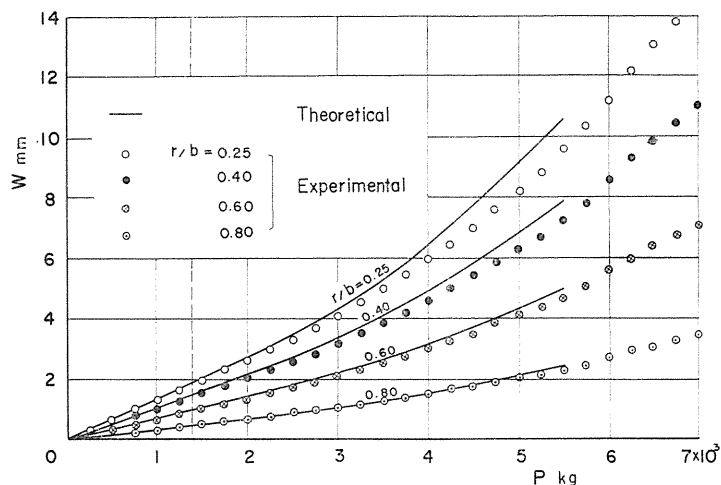
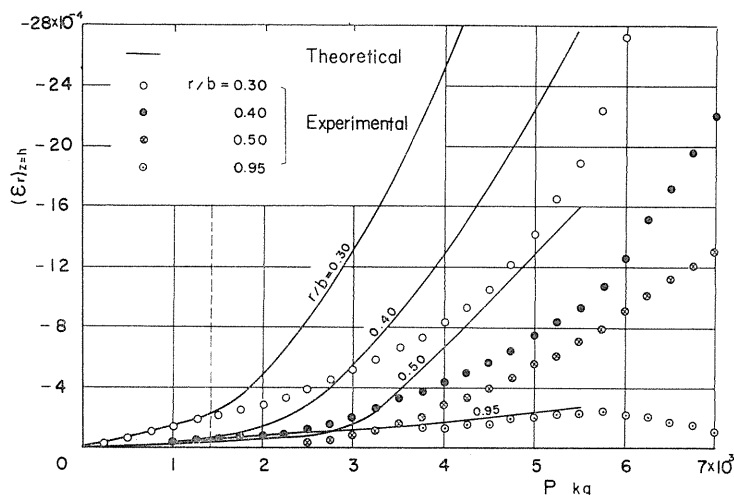
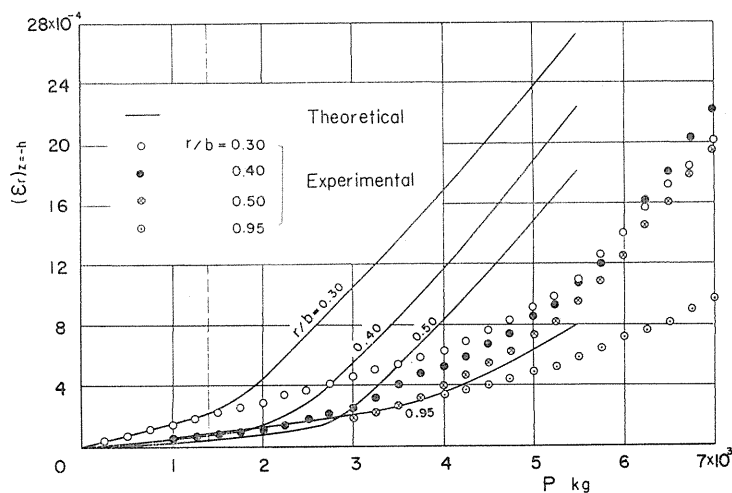
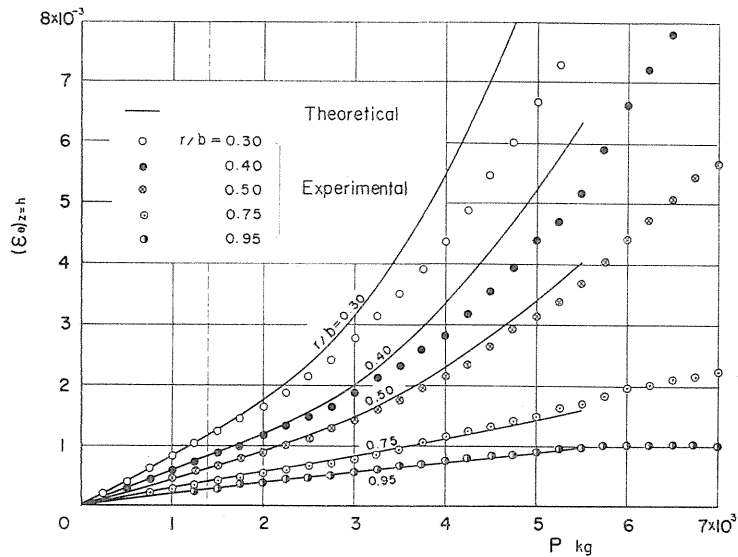
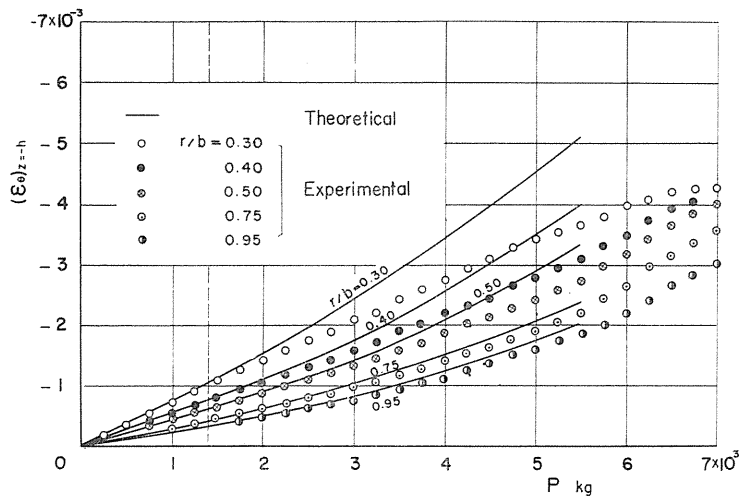


FIG. 15. Analytical and experimental values of deflection (mild steel) ( $2h=10$  mm).

FIG. 16. Radial strain components on the surface  $z=h$  (mild steel).FIG. 17. Radial strain component on the surface  $z=-h$  (mild steel).

17, the analytical values of the radial strain components increase rapidly after the incipience of plastic region while the corresponding experimental values increase as if they follow to the extension of linear elastic state for a while and after that they run parallel to the analytical curves. In the previous paper<sup>11)</sup>, such a remarkable difference between the two values was attributed to the fact that the transition from elastic state to plastic in mild steel having the upper and lower yield points was not realized at the lower one but delayed to the upper one. That is, when the stress gradient is remarkable as in the bending of annular plate, it is considered that the delay is increased by the restraint of the part kept in elastic state of lower stress level to the part which otherwise might be changed into plastic state. If the above mentioned assumption is correct, such a

FIG. 18. Circumferential strain component on the surface  $z=h$  (mild steel).FIG. 19. Circumferential strain component on the surface  $z=-h$  (mild steel).

delay of yielding cannot appear for the material having non-linear continuous stress-strain relation as in the aluminium alloy used here.

Actually, in Figs. 10 and 11 corresponding to Figs. 16 and 17, the experimental and the corresponding analytical values are in good agreement and there is not any remarkable difference having a marked trend as in the case of mild steel after the incipience of plastic region in the plate. Therefore, it may be said that the incipience of plastic state in a element of material having upper and lower yield points may be delayed by the restraint of the part of low stress level in its elastic state when the gradient of stress distribution is distinguished. Such a conclusion explains well the reason why the remarkable difference between the analytical

and the experimental results appears in the bending problem of annular plate of mild steel while the difference are not so distinguished in the bending problems of circular plate of the same material and those of circular and annular plates of aluminium alloy.

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