AN INVESTIGATION ON THE INTERACTION BETWEEN A SUPERSONIC BOUNDARY LAYER AND A SHOCK WAVE PRODUCED BY THE WEDGE SLOPE

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Summary

A supersonic boundary layer is interacted by the pressure rise produced by a wedge slope. This interaction is found to be similar with the shock wave reflection from the boundary layer. In the present paper the line of displacement thickness is calculated by the momentum integral equation and the outer supersonic flow along this line is obtained by the theory of linearized supersonic flow. Matching the outer and the inner flows, we can solve the entire flow field. The principle of calculation and some numerical results are presented.

1. Introduction

In a supersonic boundary layer the effect of abrupt change of surface slope is extended in some upstream distance, resulting in the change of external flow, which will react as a change of flow in the boundary layer. This is one of the interaction phenomena between supersonic external flow and that of boundary layer.

When a shock wave is impinged on the boundary layer of a flat plate, the pressure rise behind shock wave leaks out to the forward region through the subsonic part of boundary layer. Ritter and Kuo[1], Kuo[2] and others have investigated such form of interaction problems.

In the present paper a quite similar method of analysis developed by Ritter and Kuo is found to be useful to solve the supersonic boundary layer near the corner as shown in Fig. 1, and the general descriptions of the process are presented.

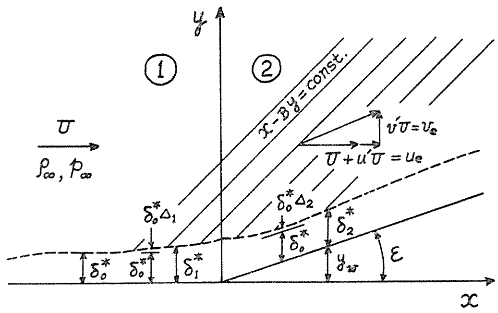

FIG. 1. Flow pattern of the boundary layer around a wedge slope.

2. Fundamental Equations in the Compressible Boundary Layer

The steady two-dimensional boundary layer flow of a compressible fluid is governed by equations of state, continuity, momentum in x and y directions and of energy as follows:

$$p = R\rho T \tag{1}$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{3}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{4}$$

$$\rho\left[u\frac{\partial(c_p T)}{\partial x} + v\frac{\partial(c_p T)}{\partial y}\right] = u\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{5}$$

where ρ denotes density, p pressure, T absolute temperature, R gas constant, and u, v are x, y components of velocity, respectively. k is heat conductivity and μ is coefficient of viscosity.

Assuming the unity of Prandtl number, *i.e.* $P_r \equiv \mu c_p/k = 1$, we can integrate the energy equation (5). For the adiabatic wall it is given

$$(1/2)u^2 + c_p T = \text{constant} = (1/2)u_e^2 + c_p T_e = c_p T_0 \tag{6}$$

where suffix e and 0 express the quantities of external flow and of stagnation point.

The stream function is introduced by eq. (2) defining

$$\rho u = \rho_0(\partial\Psi/\partial y), \ \rho v = -\rho_0(\partial\Psi/\partial x) \tag{7}$$

Assuming $\mu \propto T$, Howarth transformation

$$x_i = x, \ y_i = \left(\frac{p}{p_0}\right)^{1/2}\int_0^y \frac{T_0}{T}dy, \ \Psi = \left(\frac{p}{p_0}\right)^{1/2}\chi(x_i, y_i) \tag{8}$$

is applied to the equation of motion. Substituting eqs. (6) and (8) into eq. (3) we have

$$\begin{aligned}\frac{\partial^2\chi}{\partial x_i\partial y_i}\frac{\partial\chi}{\partial y_i} - \frac{\partial^2\chi}{\partial y_i^2}\frac{\partial\chi}{\partial x_i} &= u_e\frac{du_e}{dx_i}\left[\frac{T}{T_e} - \frac{\gamma}{2a_e^2}\chi\frac{\partial^2\chi}{\partial y_i^2}\right] + \nu_0\frac{\partial^3\chi}{\partial y_i^3} \\ &= \nu_0\frac{\partial^3\chi}{\partial y_i^3} + u_e\frac{du_e}{dx_i}\left\{1 + \frac{\gamma-1}{2a_e^2}\left[u_e^2 - \left(\frac{\partial\chi}{\partial y_i}\right)^2\right] - \frac{\gamma}{2a_e^2}\chi\frac{\partial^2\chi}{\partial y_i^2}\right\}\end{aligned} \tag{9}$$

Integration of eq. (9) across the boundary layer from $y_i = 0$ to $y_i = \delta_i$ gives the momentum integral equation.

$$\frac{d\theta_i}{dx_i} + \left[\left(2 - \frac{u_e^2}{2a_e^2}\right)\theta_i + \left(1 + \frac{\gamma-1}{2}\frac{u_e^2}{a_e^2}\right)\delta_i^*\right]\frac{1}{u_e}\frac{du_e}{dx_i} = \frac{\nu_0}{u_e^2}\left(\frac{\partial u}{\partial y_i}\right)_w \tag{10}$$

where δ_i^* and θ_i are displacement and momentum thickness, respectively, defined by

$$\delta_i^* = \int_0^{\delta_i}\left(1-\frac{u}{u_e}\right)dy_i, \quad \theta_i = \int_0^{\delta_i}\left(1-\frac{u}{u_e}\right)\frac{u}{u_e}dy_i \tag{11}$$

3. A Solution of Momentum Integral Equation

In order to solve eq. (10) the fourth power expression of velocity profile is introduced, denoting $\eta = y_i/\delta_i$

$$u/u_e = F(\eta) + \Lambda G(\eta)\ ; \quad F = 2\eta - 2\eta^3 + \eta^4, \ G = (1/6)\eta(1-\eta)^3 \tag{12}$$

where

$$\Lambda = \frac{\delta_i^2}{\nu_0}\left(1+\frac{\gamma-1}{2}M_e^2\right)\frac{du_e}{dx} \tag{13}$$

The boundary layer thicknsses are

$$\delta_i^* = \left(\frac{3}{10}-\frac{\Lambda}{120}\right)\delta_i, \quad \theta_i = \left(\frac{37}{315}-\frac{\Lambda}{945}-\frac{\Lambda^2}{9072}\right)\delta_i; \ \left(\frac{\partial u}{\partial y_i}\right)_w = \left(2+\frac{\Lambda}{6}\right)\frac{u_e}{\delta_i} \tag{14}$$

Transformation of δ_i^* is reduced by substituting eqs. (6) and (12) as follows:

$$\begin{aligned}\delta_i^* &\equiv \int_0^{\delta_i}\left(1-\frac{u}{u_e}\right)dy_i = \left(\frac{p}{p_0}\right)^{1/2}\int_0^{\delta}\frac{T_0}{T}\left(1-\frac{u}{u_e}\right)dy \\ &= \left(\frac{p}{p_0}\right)^{1/2}\frac{T_0}{T_e}\int_0^{\delta}\left[\left(\frac{T_e}{T}-1\right)+\left(1-\frac{\rho u}{\rho_e u_e}\right)\right]dy \\ &= \left(\frac{p}{p_0}\right)^{1/2}\frac{T_0}{T_e}\delta^* + \frac{\gamma-1}{2}M_e^2\left[-\frac{263}{630}+\frac{71\Lambda}{7560}+\frac{\Lambda^2}{9072}\right]\delta_i\end{aligned} \tag{15}$$

where

$$\delta^* \equiv \int_0^{\delta}\left(1-\frac{\rho u}{\rho_e u_e}\right)dy$$

Eliminating δ_i^* from eqs. (14) and (15), we have

$$\begin{aligned}&\frac{1}{9072}\frac{\gamma-1}{2}M_e^2\left[\frac{1}{\nu_0}\frac{du_e}{dx}\left(1+\frac{\gamma-1}{2}M_e^2\right)\right]^2\delta_i^5 \\ &+\frac{1}{\nu_0}\frac{du_e}{dx}\left(1+\frac{\gamma-1}{2}M_e^2\right)\left(\frac{71}{7560}\frac{\gamma-1}{2}M_e^2+\frac{1}{120}\right)\delta_i^3 - \left(\frac{263}{630}\frac{\gamma-1}{2}M_e^2+\frac{3}{10}\right)\delta_i \\ &\qquad = -\left(\frac{p}{p_0}\right)^{1/2}\frac{T_0}{T_e}\delta^*\end{aligned} \tag{16}$$

When the change of wall slope is very small, the external flow does not much deviate from the initial uniform flow. In such a case u_e can be expressed by

$$u_e = U(1+u'+\cdots), \ u' \ll 1 \tag{17}$$

Subsitiuting eq. (17) into eq. (16) and neglecting the higher order of u', we have

$$\frac{U}{\nu_0}\left(\frac{71}{7560\sigma^2}-\frac{1}{945\sigma}\right)\frac{du'}{dx}\delta_i^3 + \left(\frac{37}{315}-\frac{263}{630\sigma}-\frac{263}{630}\frac{\gamma-1}{\sigma^2}M_0^2u'\right)\delta_i$$

$$= \left(\sigma^{\frac{2-\gamma}{2\gamma-2}} - \frac{2-\gamma}{2} M_0^2 \sigma^{\frac{4-3\gamma}{2\gamma-2}} u'\right)\delta^* \tag{18}$$

where $\sigma = 1 - [(\gamma-1)/2]M_0^2$, and $M_0 = U/a_0$.

In the case of weak interaction δ_i should have a very close value with that of non-interaction flow, where $u'=0$. Therefore, the first approximation will be obtained by putting the zeroth order value of

$$\delta_i = \sigma^{\frac{2-\gamma}{2\gamma-2}}\left(\frac{263}{630\sigma} - \frac{37}{415}\right)^{-1}\delta^*$$

into the higher terms of eq. (18).

$$\begin{aligned}
\delta_i = {} & \sigma^{\frac{2-\gamma}{2\gamma-2}}\left(\frac{263}{630\sigma} - \frac{37}{315}\right)^{-1}\delta^* \\
& - \left[\frac{263}{630}\frac{\gamma-1}{\sigma^2}M_0^2\sigma^{\frac{2-\gamma}{2\gamma-2}}\left(\frac{263}{630\sigma} - \frac{37}{315}\right)^{-2} + \frac{2-\gamma}{2}M_0^2\sigma^{\frac{4-3\gamma}{2\gamma-2}}\left(\frac{263}{630\sigma} - \frac{37}{315}\right)^{-1}\right]u'\delta^* \\
& + \frac{U}{\nu_0}\left(\frac{71}{7560\sigma^2} - \frac{1}{945\sigma}\right)\left(\frac{263}{630\sigma} - \frac{37}{315}\right)^{-4}\sigma^{\frac{6-3\gamma}{2\gamma-2}}\frac{du'}{dx}\delta^{*3}
\end{aligned} \tag{19}$$

This is the form of the first approximation of δ_i expressed by δ^*.

When the change of quantities is not much deviated from undisturbed boundary layer, we can put

$$\delta^* = \delta_0^*(1+\varDelta), \quad \varDelta \ll 1 \tag{20}$$

where δ_0^* is the displacement thickness for the uniform external flow, which is expressed by $\delta_0^*(d\delta_0^*/dx) = \text{constant}$. Substituting eq. (20) into eq. (10) and retaining terms of the first power of $\varDelta$ and u', we have

$$g_1\varDelta + \frac{d\varDelta}{d\xi} = g_2 u' + g_3\frac{du'}{d\xi} + g_4\frac{d^2u'}{d\xi^2} - \frac{1}{g_1}\frac{d\varDelta}{d\xi}\left(g_2u' + 3g_3\frac{du'}{d\xi} + 5g_4\frac{d^2u'}{d\xi^2}\right) \tag{21}$$

where $\xi \equiv x/\delta_0^*$, $R_e \equiv U\delta_0^*/\nu_0$ and

$$\begin{aligned}
g_1 &= \frac{1}{R_e}\frac{(263)^2}{23310}\sigma^{-\frac{\gamma}{\gamma-1}}\left(1 - \frac{74}{263}\sigma\right)^2 \\
g_2 &= -g_1\left[1 - 2\frac{\gamma-1}{\sigma}M_0^2\left(\frac{1}{1-74\sigma/263} + \frac{2-\gamma}{2\gamma-2}\right)\right] \\
g_3 &= \frac{3-\gamma}{2\sigma}M_0^2 - 2 - \left(\frac{15934}{9731} + \frac{74588}{360047\sigma}\right)\left(1 - \frac{74}{263}\sigma\right)^{-1} \\
g_4 &= -R_e\sigma^{\frac{1}{\gamma-1}}\frac{5}{888}\left(\frac{630}{263}\right)^3\left(1 - \frac{74}{263}\sigma\right)^{-3}
\end{aligned} \tag{22}$$

4. Matching Condition with External Flow

The displacement effect of a boundary layer disturbs the external flow, which reflects to the flow in a boundary layer. By the matching condition the velocity vector of external flow should be tangential at the outer edge of the displacement thickness. Denoting the small height of the wall from a reference plane by y_w, the matching condition is expressed as follows:

$$\frac{v_e}{u_e}=\frac{d}{dx}(y_w+\delta^*) \tag{23}$$

Since disturbances caused by the interaction are much greater than the gradual change in undisturbed boundary layer, we can assume that $\delta_0^*=\text{constant}$. Introducing non-dimensional lengths

$$\xi \equiv x/\delta_0^*, \quad \eta \equiv y/\delta_0^*$$

the matching condition with the external flow of small perturbation is given by

$$\frac{v_e}{U}=\frac{d\eta_w}{d\xi}+\frac{d\Delta}{d\xi} \qquad \text{at } \eta=\eta_w \tag{24}$$

5. Solution of the Matched External Flow

The small perturbation theory is applied to the supersonic external flow. The perturbation potential ϕ which is defined by

$$u' \equiv (u_e-U)/U=\partial\phi/\partial\xi, \quad v' \equiv v_e/U=\partial\phi/\partial\eta \tag{25}$$

is governed by the linearized equation

$$B^2\frac{\partial^2\phi}{\partial\xi^2}-\frac{\partial^2\phi}{\partial\eta^2}=0, \quad B=\sqrt{M_\infty^2-1} \tag{26}$$

The general solution is given by

$$\phi=f(\xi-B\eta)+g(\xi+B\eta) \tag{27}$$

Since no disturbances are contained in the impinging wave, the second family should be vanished, *i.e.* $g=0$.

The matching condition (24) is deformed

$$\frac{\partial\phi}{\partial\eta}=\frac{d\eta_w}{d\xi}+\frac{d\Delta}{d\xi} \qquad \text{at } \eta=+0 \tag{28}$$

(i) Solution in the Upstream Region ($x\leqq 0$).

The solution of ϕ is given by

$$\phi_1=f_1(\xi-B\eta) \tag{29}$$

In this region, $\eta_w=\text{constant}=0$. From the matching condition we have

$$\left(\frac{\partial\phi}{\partial\eta}\right)_{+0}=-Bf_1'(\xi)=\frac{d\Delta}{d\xi} \tag{30}$$

Differentiating eq. (21) terms of the first order are retained.

$$g_1\frac{d\Delta}{d\xi}+\frac{d^2\Delta}{d\xi^2}=g_2\frac{du'}{d\xi}+g_3\frac{d^2u'}{d\xi^2}+g_4\frac{d^3u'}{d\xi^3} \tag{31}$$

Substituting eq. (30) into eq. (31) we have

$$g_4 f_1^{IV}(\xi) + g_3 f_1'''(\xi) + (g_2 + B) f_1''(\xi) + g_1 B f_1'(\xi) = 0 \tag{32}$$

The solution is given by

$$f_1(\xi) = A_0 + A_1 e^{\lambda_1 \xi} + A_2 e^{\lambda_2 \xi} + A_3 e^{\lambda_3 \xi} \tag{33}$$

where λ is the root of

$$g_4 \lambda^3 + g_3 \lambda^2 + (g_2 + B)\lambda + g_1 B = 0 \tag{34}$$

Numerical values of λ_1, λ_2 and λ_3 are shown in Figs. 2, 3 and 4, where $\lambda_1 < 0$, $\lambda_2 > 0$ and $\lambda_3 < 0$.

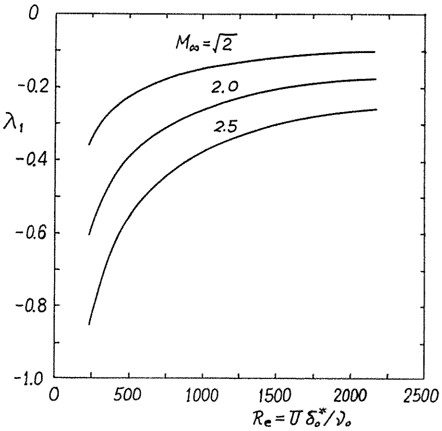

FIG. 2. Value of λ_1.

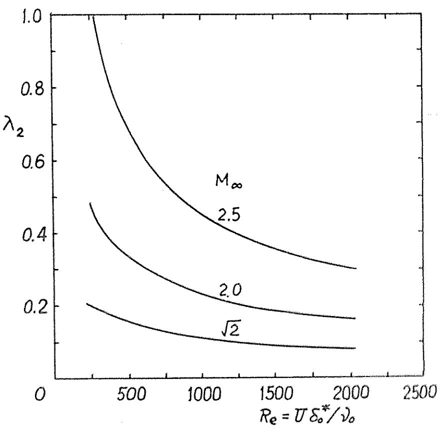

FIG. 3. Value of λ_2.

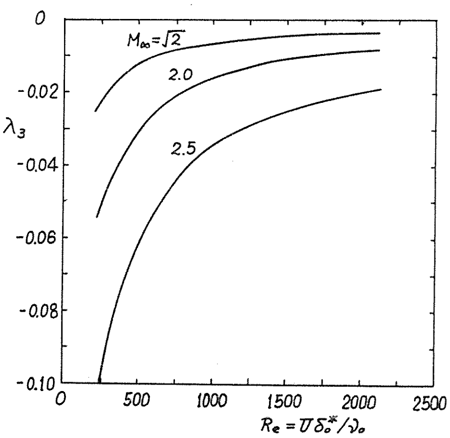

FIG. 4. Value of λ_3.

From the condition

$$f_1 = 0 \qquad \text{at } \xi = -\infty \tag{35}$$

$A_1=0$, $A_3=0$ and $A_0=0$. Therefore,

$$f_1(\xi - B\eta) = A_2 e^{\lambda_2(\xi - B\eta)} \tag{36}$$

The solution in upstream region is given by

$$\phi_1 = A e^{\lambda_2 \cdot (x - By)/\delta_0^*} \qquad \text{in } x - By \leqq 0 \tag{37}$$

Eq. (30) is integrated by the use of eq. (37), giving

$$\Delta_1 = -A \cdot B e^{\lambda_2 \xi} + \text{constant}$$

Since $\Delta_1(-\infty) = 0$, the constant is vanished. We have

$$\delta_1^* = \delta_0^* [1 - A \cdot B e^{\lambda_2 (x/\delta^*_0)}] \tag{38}$$

(ii) Solution in the Downstream Region ($x \geqq 0$)

The general form of solution is expressed by

$$\phi_2 = f_2(\xi - B\eta) \tag{39}$$

In this region, $\eta_w = \varepsilon\xi$, and therefore, the matching condition is reduced to

$$\left(\frac{\partial \phi}{\partial \eta}\right)_{+0} = -Bf_2'(\xi) = \varepsilon + \frac{d\Delta}{d\xi} \tag{40}$$

Substituting this equation into eq. (31) we have

$$g_4 f_2^{IV} + g_3 f_2''' + (g_2 + B) f_2'' + g_1 B f_2' = -g_1 \varepsilon \tag{41}$$

The complementary solution is found to be identical with eq. (33). The complete solution is

$$f_2(\xi) = C_0 + C_1 e^{\lambda_1 \xi} + C_2 e^{\lambda_2 \xi} + C_3 e^{\lambda_3 \xi} - (\varepsilon/B)\xi \tag{42}$$

Since $f_2(\xi)$ should have a finite value at infinite downstream, $C_2=0$. The constant value of velocity potential can be nullified without any loss of generality. Therefore

$$f_2(\xi - B\eta) = C_1 e^{\lambda_1(\xi - B\eta)} + C_3 e^{\lambda_3(\xi - B\eta)} - (\varepsilon/B)(\xi - B\eta) \tag{43}$$

The solution in downstream region is given by

$$\phi_2 = C e^{\lambda_1 \frac{x - By}{\delta_0^*}} + D e^{\lambda_3 \frac{x - By}{\delta_0^*}} - \frac{\varepsilon}{B} \frac{x - By}{\delta_0^*} \tag{44}$$

Integrating eq. (40) with eq. (44) we have

$$\Delta_2 = -C \cdot B e^{\lambda_1 \xi} - D \cdot B e^{\lambda_3 \xi} + E$$

where $E = \Delta_2(+\infty)$, since $\lambda_1 < 0$ and $\lambda_3 < 0$. The displacement thickness in downstream region is given by

$$\delta_2^* = \delta_0^*[1 - C\cdot Be^{\lambda_1(x/\delta_0^*)} - D\cdot Be^{\lambda_3(x/\delta_0^*)} + E] \tag{45}$$

The thickness at the infinite downstream is

$$\delta_2^*(+\infty) = \delta_0^*[1+E] \tag{46}$$

(iii) Determination of Integral Constants

Integral constants can be determined by some boundary conditions and matching conditions at $x=0$. Since the present calculation is based on a linearized theory, quantities can not be connected to all higher order derivatives at the origin. In this connection four conditions are introduced as follows:

(a) δ^* is continuous at $x=0$.

(b) p is continuous at $x=0$, $y=0$.

(c) Integrated deviation of total momentum should be vanished in the region of origin. It is shown later that this condition is equivalent to the continuity of pressure gradient at $x=0$.

(d) Since perturbations caused by interaction should be vanished at $x=+\infty$, the displacement thickness at infinite downstream can be assumed to be the same as that of undisturbed boundary layer. Individual conditions are now formulated.

(a) The continuity of δ^* at $x=0$.

This condition is given by

$$\delta_1^* = \delta_2^* \qquad \text{at } x=0 \tag{47}$$

Substituting eqs. (38) and (45) into eq. (47) we have

$$A = C + D - E/B \tag{48}$$

(b) The continuity of pressure at $x=0$, $y=0$.

Denoting the perturbation of pressure by p' we have

$$p = p_\infty + p'\,;\;\; p' = -\rho_\infty U^2 u' = -\rho_\infty U^2(\partial\phi/\partial\xi) \tag{49}$$

From eqs. (37) and (44) we have

$$p_1' = -\gamma M_\infty^2 p_\infty[A\lambda_2 e^{\lambda_2\frac{x-By}{\delta_0^*}}] \tag{50}$$

$$p_2' = -\gamma M_\infty^2 p_\infty[C\lambda_1 e^{\lambda_1\frac{x-By}{\delta_0^*}} + D\lambda_3 e^{\lambda_3\frac{x-By}{\delta_0^*}} - \frac{\varepsilon}{B}] \tag{51}$$

It is found that $p_2' = (\gamma M_\infty^2/B)p_\infty\varepsilon$ at $x=+\infty$ is just the same value of pressure growth in inviscid flow caused by the wedge with deflection angle of ε.

Condition (b) is described by

$$p_1' = p_2' \qquad \text{at } x=0,\ y=0 \tag{52}$$

Equating eqs. (50) and (51) we have

$$A\lambda_2 = C\lambda_1 + D\lambda_3 - (\varepsilon/B) \tag{53}$$

(c) Continuity of pressure gradient at $x=0$.

The momentum integral equation can be expressed by

$$K \equiv \frac{d}{dx}(\rho_e u_e^2 \theta) - \delta^* \frac{dp}{dx} - \tau_w$$
$$= \frac{d}{dx}(\rho_e u_e^2 \theta - \delta^* p) + p\frac{d\delta^*}{dx} - \tau_w = 0 \tag{54}$$

The present condition is expressed initially by

$$\lim_{x_0 \to 0} \int_{-x_0}^{x_0} K dx = 0$$

or

$$\lim_{x_0 \to 0} \left[\left| \rho_e u_e^2 \theta - \delta^* p \right|_{-x_0}^{x_0} + \int_{-x_0}^{x_0} p \frac{d\delta^*}{dx} dx - \int_{-x_0}^{x_0} \tau_w dx \right] = 0 \tag{55}$$

Since $p(d\delta^*/dx)$ is discontinuous but is finite and τ_w is also finite, the limits of the last two terms are vanished. Eq. (55) is reduced to

$$(\rho_e u_e^2 \theta)_2 - (\rho_e u_e^2 \theta)_1 = (p_2 - p_1)\delta^*(0) \tag{56}$$

From the condition (b), *i.e.* $p_1 = p_2$ at $x = y = 0$, we have

$$(\rho_e u_e^2 \theta)_2 = (\rho_e u_e^2 \theta)_1 \tag{57}$$

Linearizing by $u_e = U(1 + u' + \cdots)$, the differentiated form of eq. (57) is reduced to[1)]

$$\alpha_1 \delta_0^* \left(\frac{du_2'}{dx} - \frac{du_1'}{dx} \right) = \alpha_2 (u_2' - u_1') \tag{58}$$

where

$$\alpha_1 = -R_e \left(\frac{37}{315} - \frac{263}{630\sigma} \right)^{-3} \sigma^{\frac{6-3\gamma}{2\gamma-2}} \left[\frac{1}{111\sigma} + \left(\frac{71}{7560\sigma^2} - \frac{1}{945\sigma} \right) \left(\frac{37}{315} - \frac{263}{630\sigma} \right)^{-1} \right]$$
$$\alpha_2 = -\sigma^{\frac{2-\gamma}{2\gamma-1}} \left(\frac{37}{315} - \frac{263}{630\sigma} \right)^{-1} \left[2 + M_\infty^2 \left(\frac{263}{630} \frac{\gamma - 1}{\sigma} - 1 \right) \right] \tag{59}$$

The continuity of pressure p at the origin and $p' = -\rho_\infty U^2 u'$ lead the continuity of u'. At the origin of external flow it is written

$$u_1' = u_2' \tag{60}$$

Eq. (58) is reduced to

$$du_1'/dx = du_2'/dx \tag{61}$$

which expresses the continuity of pressure gradient at the origin. Using eqs. (37) and (44) we have

$$\left(\frac{du_1'}{dx} \right)_{x=y=0} = \frac{1}{\delta_0^*} A\lambda_2^2, \quad \left(\frac{du_2'}{dx} \right)_{x=y=0} = \frac{1}{\delta_0^*} (C\lambda_1^2 + D\lambda_3^2) \tag{62}$$

The present condition gives

$$A\lambda_2^2 = C\lambda_1^2 + D\lambda_3^2 \tag{63}$$

So far the conditions (a) to (c), which are expressed by eqs. (48), (53) and (63), give A, C, D by E as follows:

$$\begin{aligned} A &= \frac{-(\lambda_3+\lambda_1)}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)}\left[-\frac{\varepsilon}{B}\right] + \frac{\lambda_1\lambda_3}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)}\left[-\frac{E}{B}\right] \\ C &= \frac{\lambda_3+\lambda_2}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)}\left[-\frac{\varepsilon}{B}\right] - \frac{\lambda_2\lambda_3}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)}\left[-\frac{E}{B}\right] \\ D &= \frac{-(\lambda_2+\lambda_1)}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_3)}\left[-\frac{\varepsilon}{B}\right] + \frac{\lambda_1\lambda_2}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_3)}\left[-\frac{E}{B}\right] \end{aligned} \tag{64}$$

(d) The displacement thickness at the infinite downstream.
The momentum integral equation is expressed by

$$\frac{d\theta}{dx} + \theta\left[\frac{1}{\rho_e}\frac{d\rho_e}{dx} + \left(2+\frac{\delta^*}{\theta}\right)\frac{1}{u_e}\frac{du_e}{dx}\right] = \frac{\tau_w}{\rho_e u_e^2} \tag{65}$$

Considering the flow condition close to the separation region, it is assumed that τ_w is negligible small and that form parameter $H \equiv \delta^*/\theta$ is constant as a mean through concerned region. Eq. (65) is reduced to

$$\frac{1}{\theta}\frac{d\theta}{dx} + \frac{1}{\rho_e}\frac{d\rho_e}{dx} + (2+H)\frac{1}{u_e}\frac{du_e}{dx} = 0$$

It is integrated to give

$$\theta\rho_e u_e^{2+H} = \text{constant}$$

or removing suffix e

$$\theta_1\rho_1 u_1^{2+H} = \theta_2\rho_2 u_2^{2+H} \tag{66}$$

We have

$$\delta_2^*/\delta_1^* = \theta_2/\theta_1 = (\rho_1/\rho_2)(u_1/u_2)^{2+H} \tag{67}$$

When $\mu \propto T$, $P_r = 1$ and $(\partial T/\partial y)_w = 0$, the form parameter for the laminar compressible boundary layer is given by Lees as

$$H = 2.50 + 1.75(\gamma-1)M_\infty^2 \tag{68}$$

For the linearized flow we have

$$\rho_1/\rho_2 = 1 - M_\infty^2(\varepsilon/B), \quad u_1/u_2 = 1 + (\varepsilon/B) \tag{69)(70}$$

$$\delta_2^*/\delta_1^* = 1 + (2+H-M_\infty^2)(\varepsilon/B) = 1 + E$$

$$\therefore \quad E = (2+H-M_\infty^2)(\varepsilon/B) = [4.50 - (1.75\gamma - 2.75)M_\infty^2](\varepsilon/B) \tag{71}$$

Substituting this value of E into eq. (64), A, C, and D are obtained. We can, then, calculate displacement thickness and pressure distribution as shown in the following sections.

6. Pressure Distribution

As shown in eq. (49) pressure is calculated by

$$p = p_\infty + p' = p_\infty - \rho_\infty U^2(\partial\phi/\partial\xi) \tag{49}$$

The pressure in upstream and downstream region are given respectively by

$$p_1 = p_\infty[1 - \gamma M_\infty^2 \cdot A\lambda_2 e^{\lambda_2 \frac{x-By}{\delta_0^*}}] \tag{72}$$

$$p_2 = p_\infty\left[1 - \gamma M_\infty^2\left(C\lambda_1 e^{\lambda_1 \frac{x-By}{\delta_0^*}} + D\lambda_3 e^{\lambda_3 \frac{x-By}{\delta_0^*}} - \frac{\varepsilon}{B}\right)\right] \tag{73}$$

At the infinite distance they tend to limit values.

$$p_1(-\infty) = p_\infty, \ \ p_2(+\infty) = p_\infty[1 + \gamma M_\infty^2(\varepsilon/B)] \tag{74}$$

Two examples of pressure distribution are shown in Fig. 5. They have the overshoot and arrive the maximum value. The situation of this maximum is calculated by

$$(x/\delta_0^*)_{p\max} = (\lambda_3 - \lambda_1)^{-1}\ln(-C\lambda_1^2/D\lambda_3^2) \tag{75}$$

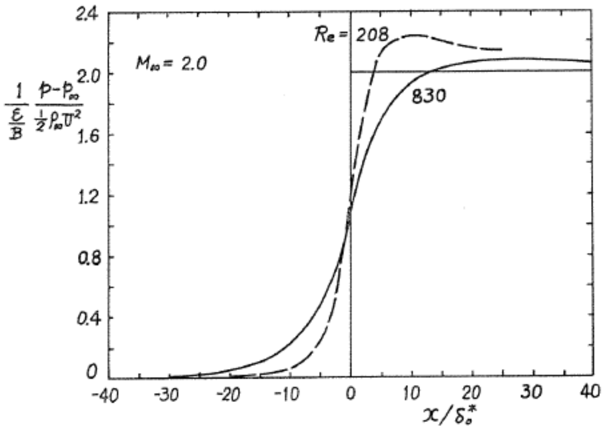

FIG. 5. Pressure distribution along the wall.

7. Displacement Thickness and Upstream Influence

Displacement thickness in upstream and downstream region are given, respectively, by

$$\delta_1^* = \delta_0^*[1 - A \cdot Be^{\lambda_2(x/\delta_0^*)}] \tag{38}$$

$$\delta_2^* = \delta_0^*[1 - C \cdot Be^{\lambda_1(x/\delta_0^*)} - D \cdot Be^{\lambda_3(x/\delta_0^*)} + E] \tag{45}$$

Fig. 6 shows a numerical example of perturbed displacement thickness along wedge slope which are both proportionate to ε.

The perturbed displacement thickness in upstream region is given by

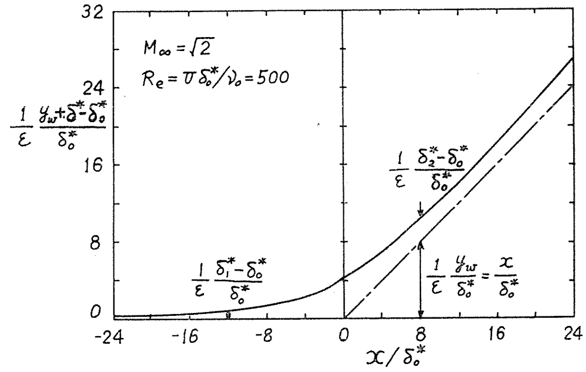

FIG. 6. Distribution of displacement thickness.

$$\delta_1^* - \delta_0^* = \delta_0^* A \cdot B e^{\lambda_2 (x/\delta_0^*)} \tag{76}$$

Defining the influence distance x_d by the situation where $\delta_1^* - \delta_0^*$ is α times of $\delta_1^* - \delta_0^*$ at the origin, it is given

$$x_d/\delta_0^* = -(1/\lambda_2)\ln(1/\alpha) \tag{77}$$

Some numerical examples are shown in Figs. 7, 8 and 9. Taking $\alpha=0.05$ we have $x_d/\delta_0^* \fallingdotseq 30$ at $M \fallingdotseq 2$, which meet with experimental result fairly well.

8. Separation Point

When the angle of wedge slope is increased, the perturbation becomes so strong that the flow is separated. The separation of flow occurs at $(\partial u/\partial y)_w=0$ or at $(\partial u/\partial y_i)_w=0$. It is given

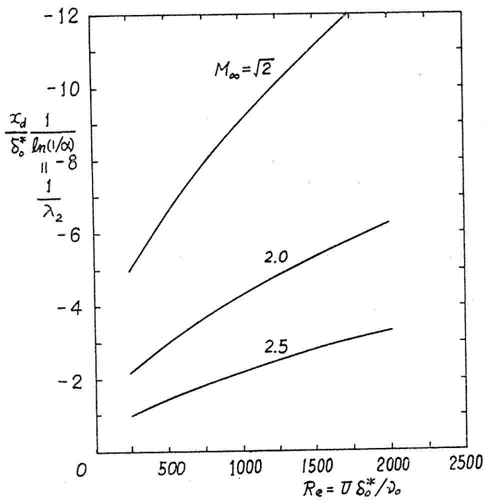

FIG. 7. Influence distance (1).

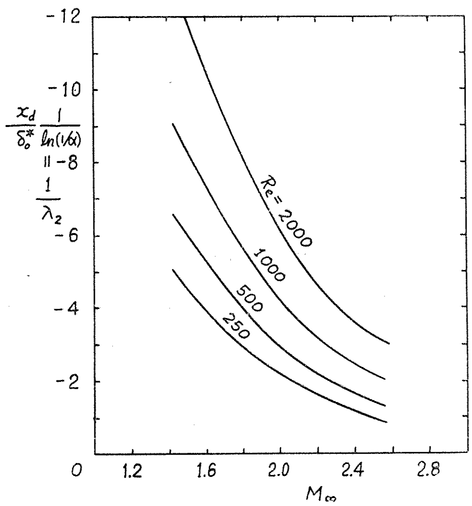

FIG. 8. Influence distance (2).

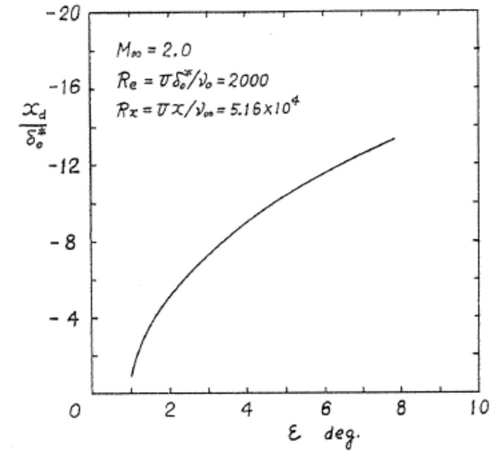

FIG. 9. Effect of wedge angle on the influence distance.

$$(\partial u/\partial y_i)_w = (u_e/\delta_i)(2+\Lambda/6)=0 \tag{78}$$

Therefore, separation point is given by $\Lambda=-12$ as in incompressible flow.

In the present linearized external flow Λ is given by eqs. (13) and (17).

$$\Lambda = \frac{\delta_i^2}{\nu_0}\frac{U}{\sigma}\frac{du'}{dx} = R_e\delta_0^*\left(\frac{37}{315}-\frac{263}{630\sigma}\right)^{-2}\sigma^{\frac{3-2\gamma}{\gamma-1}}\frac{du'}{dx} \tag{79}$$

where

$$\frac{du'}{dx} = A\frac{\lambda_2^2}{\delta_0^*}e^{\lambda_2(x/\delta_0^*)} \tag{80}$$

At the separation point x_s

$$\Lambda_s = AR_e\lambda_2^2\left(\frac{37}{315}-\frac{263}{630\sigma}\right)^{-2}\sigma^{\frac{3-2\gamma}{\gamma-1}}e^{\lambda_2(x_s/\delta_0^*)} = -12$$

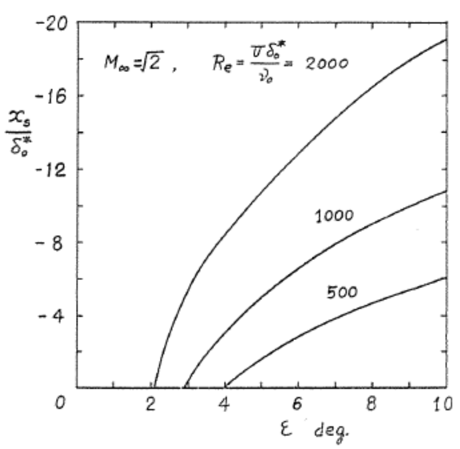

FIG. 10. Situation of separation point (1).

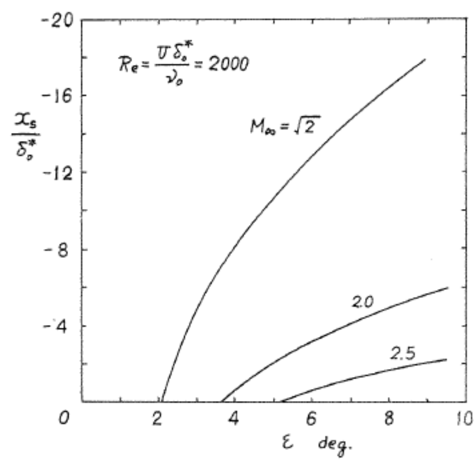

FIG. 11. Situation of separation point (2).

and, therefore, we have

$$\frac{x_s}{\delta_0^*} = -\frac{1}{\lambda_2}\ln\left[\frac{-AR_e\lambda_2^2}{12}\left(\frac{37}{315}-\frac{263}{630\sigma}\right)^{-2}\sigma^{\frac{3-2\gamma}{\gamma-1}}\right] \tag{81}$$

The value of x_s is affected by R_e through the effect of pressure gradient which suffers interaction between external flow and boundary layer thickness. Examples of numerical calculation are shown in Figs. 10 and 11. When the angle of wedge slope ε is increased, $(-A)$ and, therefore, $(-x_s)$ is increased. It is found that separation point is getting forward by increasing ε and R_e and by decreasing M_∞.

9. Conclusion

The effect of wedge slope in a supersonic boundary layer is investigated. Changing the compressible boundary layer into the incompressible one by Howarth transformation momentum integral equation is solved. It is connected with the linearized external flow and the interaction is matched on the outer edge of boundary layer. The growth of displacement thickness, the pressure distribution, the influence distance and the separation point are calculated.

References

1) Ritter, A. and Kuo Y-H., Reflection of a weak shock wave from a boundary layer along a flat plate, Pt. I, NACA TN 2868, 1953.
2) Kuo Y-H., Reflection of a weak shock wave from a boundary layer along a flat plate, Pt. II, NACA TN 2869, 1953.