RESEARCH REPORTS

ALLOWABLE REGION OF THE PLANT PARAMETER VARIATION TO THE TERMINAL MARGIN IN OPTIMAL CONTROL SYSTEM

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1. Introduction

In the theory of control, the description of the controlled object is made by mathematical models, or this procedure may perhaps be only way that can be used by engineers and scientists. If we can describe the controlled object exactly, often the description becomes very complicated formulation, and then we cannot treat it moreover analyticaly. So the use of mathematical model is inevitable and the discrepancy between the model and the practical plant is not avoidable. The sensitivity analysis is devised and used in this point of view. However the discrepancy between the model and the practical plant is not given before hand, only what one can do in synthesizing stage of control is to take such care of high sensitive elements as not to make their deviation too large.

However here is one important imformation that should be used but is not used in optimal control design as yet. This is the margin for controlled results. These margins are always fully utilized in ordinary design work of the whole engineering field. The margin admits the inaccuracy of mathematical model.

This paper introduces one example of above utilization problems of the margin in energy minimum problems, that is to make $\int_0^t u^2 dt$ minimum, and in the case of fixed terminal conditions. The margin is assumed to be given to the terminal condition and we will determine allowable ambiguity of the mathematical model, that is the allowable range of model parameter changes. In the above problem, the performance index $\int_0^t u^2 dt$ is supposed as an auxiliary one which is used in place of avoiding the saturation of manipulating variables.

The control system configuration is assumed to have open loop type. Therefore if an optimal controller obtained by using given mathematical model—hereafter the mathematical model is called nominal system—can be realized ideally and be applied to the practical plant, the output trajectories become different from the theoretically expected ones, however the value of the performance index cannot differ from the theoretical one.

2. Variation equation and inverse sensitivity problem

We will assume the system state transition equation (nominal system) in the form,

$$\dot{x}(t) = f(x(t),\ m(t),\ w) \tag{1}$$

x : an n-state vector
m : a q-control signal vector
w : nominal system parameter, a p-vector.

Let the uncertainty of system parameter value be Δw. Then the state transition equation* of the practical plant changes as follows:

$$\dot{x}(t) = f(x(t),\ m(t),\ w + \Delta w). \tag{2}$$

The difference of solution trajectories between eqs. (1) and (2) is

$$\Delta x(t) = S\Delta w. \tag{3}$$

This is, of course, of the first order approximation, and here S is sensitivity matrix and takes the following form:

$$S = \begin{bmatrix} \frac{\partial x_1(t)}{\partial w_1} & \cdots & \frac{\partial x_1(t)}{\partial w_p} \\ \vdots & & \vdots \\ \frac{\partial x_n(t)}{\partial w_1} & \cdots & \frac{\partial x_n(t)}{\partial w_p} \end{bmatrix}, \tag{4}$$

where S is generally an $n \times p$ matrix. If only terminal state is considered, the eq. (3) becomes

$$\Delta x(T) = S(T)\Delta w, \tag{5}$$

where T is a terminal time and

$$S(T) = \left[\frac{\partial x_i(t)}{\partial w_j}\right]_{t=T}, \qquad \begin{matrix} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, p \end{matrix} \tag{6}$$

We want to solve the eq. (5) inversely with respect to given terminal margin. However since generally $S(T)$ is an $n \times p$ matrix and singular, we cannot solve it and must obtain something that is meaningful in the engineering point of view.

3. Terminal margin and meaningful solutions for eq. (5) in engineering

The terminal margin is assumed to be some region given around the terminal point and called Γ. The range of S will be denoted by $\mathscr{R}(S)$, that is

* In this article all the state variables are assumed to be directly observable.

$$\mathscr{R}(S) = \{\xi \mid S\Delta w = \xi,\ \Delta w \in R^p\}, \tag{7}$$

where R^p is a p dimentional Euclidean space of the system parameter. The necessary and sufficient condition for the existence of solution of eq. (5) is

$$\Delta x(T) \in \mathscr{R}(S). \tag{8}$$

Moreover $\Delta x(T)$ must be in Γ, so the condition (8) becomes

$$\Delta x(T) \in \mathscr{R}(S) \cap \Gamma = \Pi. \tag{9}$$

Here the condition (9) only means the existence of solution but does not mean its uniqueness.

Now we must consider how Γ is given practically. Γ is a quantity that is very much in field engineering sense and is not given by new sense of the control engineering. Γ is determined, for example, by plant operation conditions, or product quality in the then market, etc. Therefore Γ is not given so adequately that the inverse solution of eq. (5) can be easily obtainable. Thus ordinarily Γ contains an extra region out of the intersection between $\mathscr{R}(S)$ and Γ, that is Π. If in eq. (5), $\Delta x(T)$ is in this $\Gamma-\Pi$ region, there is no inverse solution of eq. (5) and we must find some solution which is meaningful in engineering. However every cases where the solution exists and is unique, or not unique, or the solution does not exist can be simultaneously treated by generalized inverse matrix method stated in the following section.

4. Application of generalized inverse matrix method[1)2)]

The generalized inverse matrix denoted by $S^\dagger$ with respect to matrix S is defined as follows:
$S^\dagger$ is a solution matrix for the following four matrix equations:

$$\left.\begin{aligned} &1,\ SS^\dagger S = S \\ &2,\ S^\dagger SS^\dagger = S^\dagger \\ &3,\ (SS^\dagger)^* = SS^\dagger \\ &4,\ (S^\dagger S)^* = S^\dagger S \end{aligned}\right\}, \tag{10}$$

where * designates the conjugate transpose of a matrix.

4.1. The best approximated solution

In the following equation

$$S\Delta w = \Delta x(T), \tag{11}$$

we shall define the best approximated solution of eq. (11) denoted by Δw_0 such that Δw_0 has the smallest norm among w which minimizes $\|S\Delta w - \Delta x(T)\|$. That is, next two conditions must be satisfied simultaneously:

$$\left.\begin{aligned} &1),\ \|S\Delta w - \Delta x(T)\| \geq \|S\Delta w_0 - \Delta x(T)\| \\ &2),\ \|\Delta w\| \geq \|\Delta w_0\| \end{aligned}\right\}, \tag{12}$$

where $\|\Delta x(T)\|^2=\langle\Delta x(T), \Delta x(T)\rangle$, $\|\Delta w\|^2=\langle\Delta w, \Delta w\rangle$. Here $\langle\ \rangle$ indecates an inner product and the norm of vector is Euclidean norm.

The introduction of the best approximated solution can make the solution of eq. (5) unique in every cases and the unified solution can be written in the following formula,

$$\Delta w_0 = S^\dagger \Delta x(T). \tag{13}$$

This situation will be explained in detail in what follows.

4.2. A case where $\Delta x(T)\in\Pi$

In this case the solution of eq. (5) exists and can be expressed in the form

$$\Delta w = S^\dagger \Delta x(T) + (I - S^\dagger S)Z, \tag{14}$$

where I is the identity matrix and Z is an arbitrary vector in R^p, that is, $(I-S^\dagger S)Z$ is the null space of S, $N(s)$. Now the first condition in eq. (12) becomes zero, that is, $\|S\Delta w-\Delta x(T)\|=\|S\Delta w_0-\Delta x(T)\|=0$ because eq. (14) is the solution of eq. (5) and the second condition in eq. (12) can remove the null space and thus the best approximated solution takes the form of eq. (13). When S is non-singular and the unique solution exists in eq. (5), $S^\dagger=S^{-1}$, $S^\dagger S=S^{-1}S=I$ and $\Delta w_0=S^\dagger\Delta x(T)$ $=S^{-1}\Delta x(T)$.

The null space is an intrinsic characteristic of the system—plant and optimal controller—and whether it exists or not is regardless of Γ. So we did not consider it in this report. However it is an important thing in design work to find whether it exists or not because it is an intrinsic allowable region of the plant parameter change.

4.3. A case where $\Delta x(T)\in\Gamma-\Pi$

This case has no exact solution of eq. (5). However we can obtain the above mentioned approximated or meaningful solution as follows. First we shall decompose $\Delta x(T)$ into its othogonal projection $\Delta x_p(T)$ on Π** and the perpendicular component to $\Delta x_p(T)$ and then the best approximated solution will be found for $\Delta x_p(T)$, that is,

$$\left.\begin{aligned} &\Delta x(T) = \Delta x_p(T) + \Delta x_p^{\perp}(T),\ \ \Delta x_p(T)\in\Pi \\ &\Delta w_0 = S^\dagger \Delta x_p(T) \end{aligned}\right\}. \tag{15}$$

However the second relation is equivalent to

$$\Delta w_0 = S^\dagger \Delta x(T), \tag{15$'$}$$

because $S^\dagger\Delta x_p^{\perp}(T)=0$ and we need not to find $\Delta x_p(T)$ and only need to calculate $S^\dagger$.

4.4. Calculation of $S^\dagger$ [3]

From the defining equation (10), we obtain

** Strictly on $\mathscr{R}(S)$. However in the examples shown in section 5, the form of Γ will be taken or narrowed such that the othogonal projection on $\mathscr{R}(S)$ of an arbitrary vector in Γ will just be on Π.

$$S^{\dagger} = (S^*S)^{\dagger}S^*. \tag{16}$$

Therefore we need to obtain the generalized inverse matrix of S^*S. However if each column vector in S is independent, S^*S is non-singular, thus $(S^*S)^{\dagger}=(S^*S)^{-1}$. If S were not so, $(S^*S)^{-1}$ does not exist and $(S^*S)^{\dagger}$ must be found.

According to the reference article (3), $S^{\dagger}$ can be expressed by the characteristic polynominal of S^*S, that is

$$S^*S = \sum_{\lambda\in\sigma(S^*S)} \lambda \left(\frac{\prod_{\lambda\neq\theta\in\sigma(S^*S)} (S^*S-\theta I)}{\prod_{\lambda\neq\theta\in\sigma(S^*S)} (\lambda-\theta)} \right), \tag{17}$$

$$S^{\dagger} = \sum_{\lambda\in\sigma(S^*S)} \lambda^{\dagger} \left(\frac{\prod_{\lambda\neq\theta\in\sigma(S^*S)} (S^*S-\theta I)}{\prod_{\lambda\neq\theta\in\sigma(S^*S)} (\lambda-\theta)} \right) S^*. \tag{18}$$

Here $\sigma(S^*S)$ is the spectrum (or eigenvalues) of S^*S and λ, θ are members of $\sigma(S^*S)$ and $\lambda^{\dagger}=1/\lambda\,(\lambda\neq 0)$, $\lambda^{\dagger}=0\ (\lambda=0)$.

5. Allowable region of plant parameter variation

Already we have shown that we can obtain the best approximated solution with respect to the corresponding vector in Π to an arbitrarily given vector in Γ. Now we must define an allowable region (range) of the plant parameter variation corresponding to the above obtained solution group.

We can find the maximum norm solution among the group because of continuity of norm operation. Then we will define the allowable region such that it has the maximum norm as the supremum.

The generalized inverse method of matrix is applicable only when the vector norm is defined by inner product. In this report, we used Euclidean norm. When Euclidean norm is used, n-dimentional sphere can be easily expressive. Thus first we shall assume that Γ is an n-dimentional sphere, that is

$$\Gamma = \{\Delta x(T) \mid \|\Delta x(T)\|_2 \leq a\}, \tag{19}$$

where a is the radius of the sphere and $\| \ \|_2$ designates Euclidean norm. Next we shall change the radius into unit length by using a vector transformation.

$$\Delta y(T) = F\Delta x(T), \qquad F\text{: transformation matrix} \tag{20}$$

$$\Gamma' = \{\Delta y(T) \mid \|\Delta y(T)\|_2 \leq 1\}. \tag{21}$$

Therefore, by using eq. (13)

$$\Delta w_0 = S^{\dagger}F^{-1}\Delta y(T). \tag{22}$$

Taking norm of each side of eq. (22), we have

$$\|\Delta w_0\|_2 = \|S^{\dagger}F^{-1}\Delta y(T)\|_2. \tag{23}$$

As $S^{\dagger}$ is a linear operator[4] on Γ and Γ is of finite dimention, $S^{\dagger}$ is a bounded (continous) linear operator. F^{-1} is also a bounded linear operator, therefore

$S^{\dagger}F^{-1}$ is also. Thus $\max \|\varDelta w_0\|_2$ can be obtained as follows:

$$\max \|\varDelta w_0\|_2 = \max_{\|\varDelta y(T)\|_2 \leq 1} \|S^{\dagger}F^{-1}\varDelta y(T)\|_2 = \|S^{\dagger}F^{-1}\|_2 = \sqrt{\lambda_{\max}}\,. \qquad (24)^{5)}$$

Here λ_{mxa} is the maximum eigenvalue among the eigenvaelus of $(S^{\dagger}F^{-1})^*S^{\dagger}F^{-1}$.

In this case, matrix F is $\frac{1}{a}I$, and

$$F^{-1} = aI. \qquad (25)$$

Thus according to eq. (24), if we can find the maximum eigenvalue of $(S^{\dagger}F^{-1})^*S^{\dagger}F^{-1}$ by using, for example, the characteristic equation of it, we can determine the allowable region $\varLambda$ of parameter variation as follows:

$$\varLambda = \{\varDelta w \mid \|\varDelta w\|_2 \leq \|S^{\dagger}F^{-1}\|_2 = \sqrt{\lambda_{\max}}\}. \qquad (26)$$

Next we shall assume that $\varGamma$ is a n-dimentional polyhedron:

$$\varGamma = \{\varDelta x(T) \mid |\varDelta x_i(T)| \leq a_i, \quad i = 1, \ldots, n\}. \qquad (27)$$

As the sphere case, we shall change the unit length of each component of $\varDelta x(T)$ in order to mke it a regular polyhedron. Thus

$$\varDelta y(T) = F\varDelta x(T), \qquad F\text{: transformation matrix,}$$

$$\varGamma' = \{\varDelta y(T) \mid |\varDelta y_i(T)| \leq 1, \quad i = 1, \ldots, n\}. \qquad (28)$$

F is in the form:

$$F = \begin{bmatrix} \frac{1}{a_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_n} \end{bmatrix}. \qquad (29)$$

The solution of eq. (5) is

$$\varDelta w_0 = S^{\dagger}F^{-1}\varDelta y(T). \qquad (30)$$

In seeking $\max \|\varDelta w_0\|$, we must consider that the whole region of $\varGamma$ is not alway in $\mathscr{R}(S)$. If this is the case, the othogonal projection of a certain vector in $\varGamma - \varPi$ on $\mathscr{R}(S)$ may occur to be out of $\varGamma$. For safety's sake, we will narrow $\varGamma$ slightly, that is, consider the inscribed sphere of $\varGamma'$. According to eq. (28) the radius of this sphere is equal to 1, so the following treatment is the same as the previous case.

6. Example

Let us specify an optimal control problem of the second order system having

one parameter as follows:
The state transition equation,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -w \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u,$$

the boundary condition,

$$x_1(0) = 1, \quad x_1(T) = 0, \quad x_2(0) = 0, \quad x_2(T) = 0,$$

the performance index,

$$J = \int_0^T \frac{1}{2} [u(t)]^2 dt, \qquad T = 1.$$

(Problem) To the nominal parameter value $w=3$ and the given sphere of radius 0.01 (terminal margin), that is,

$$\Gamma = \{\Delta x(T) \mid \|\Delta x(T)\|_2 \leq 0.01\},$$

find the allowable region of w to Γ.

(Solution) By using Maximum principle, the open loop type optimal control law and the optimal trajectory can be obtained. Then by solving the sensitivity differential equation[6], the dynamic sensitivity at $t=T=1$, $S(T)$ can be found. Thus,

$$S = \left(\frac{\partial x_1(T)}{\partial w}\Big|_{w=3}, \frac{\partial x_2(T)}{\partial w}\Big|_{w=3} \right)' = (0.718, \ 0.359)'$$

$$S^* = (0.718, \ 0.359)$$

$$S^*S = (0.718, \ 0.359)(0.718, \ 0.359)' = 0.645.$$

Therefore the eigenvalue of S^*S is only $\lambda=0.645$. The caluculation of $S^\dagger$ is,

$$S^\dagger = (S^*S)^\dagger S^* = (1/0.645)(0.718, \ 0.359) = (1.1136, \ 0.5567).$$

The transformation matrix is

$$F = \begin{pmatrix} \frac{1}{0.01} & 0 \\ 0 & \frac{1}{0.01} \end{pmatrix}, \qquad F^{-1} = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix}.$$

Thus,

$$\max_{\|\Delta x(T)\|_2 \leq 0.01} \|\Delta w\|_2 = \max_{\|\Delta y(T)\|_2 \leq 1} \|\Delta w\|_2 = \max_{\|\Delta y(T)\|_2 \leq 1} |\Delta w| = \|S^\dagger F^{-1}\| = \sqrt{\lambda_{\max}}$$

$\lambda_{\max}$ can be caluculated as follows:

$$(S^\dagger F^{-1})^* = (F^{-1})^* S^{\dagger *} = (0.011136, \ 0.005567)'$$

$$(S^\dagger F^{-1})^* S^\dagger F^{-1} = \begin{pmatrix} 0.000124 & 0.000062 \\ 0.000062 & 0.000031 \end{pmatrix}.$$

The eigenvalues of $(S^{\dagger}F^{-1})^{*}S^{\dagger}F^{-1}$ are $\lambda=0$ and $\lambda=0.000155$, therefore $\lambda_{\max}=0.000155$, that is $\max|\Delta w|=\sqrt{\lambda_{\max}}=0.01245$. Then $\left|\frac{\Delta w}{w}\right|\times 100=0.415\%$. Corresponding to the maximum eigenvalue, the eigenvector on $\mathscr{R}(S)$ is

$$\Delta \boldsymbol{x}(T)=(0.00894,0.00447)' \qquad \|\Delta \boldsymbol{x}(T)\|_2=0.01.$$

That is, $\Delta \boldsymbol{x}(T)$ shows the intersection point between $\mathscr{R}(S)$ and Γ which is the terminal point of Π. Investigating $N(S)$, we found that $N(S)=\{0\}$, because $S^{\dagger}S=I$, $I-S^{\dagger}S=0$.

7. Conclusion

In this report only energy minimum problem was treated. However in the cases of the other performance index, for example, in which the terms of quadratic form with respect to the state variables are containd too in the integrand of perfomance index, the above stated method is applicable also, if the performance index can be thought auxiliary and hitting the terminal point is the main object. The reason why we did not treat the optimal control problem as the socalled terminal value problem itself is that in the treatment of terminal control the control often results in a Bang-Bang type and the dynamic sensitivity coefficients become difficult to be caluculated[7].

And also the reason why we did not treat the given terminal margin as a kind of manifold around the terminal point is that even if the problem were treated in the above stated manner, only one phase point will be selected optimally in the manifold and since this point often takes a position on the boundary of the manifold and the terminal deviation happens around this selected point, the initial setting of a manifold loses its original meaning.

Lastly only the terminal margin was considered in this report, however as the sensitivity matrix $S(t)$ is a time function between the initial and final time, there is such a possibility that we can develop the consideration of margin along the whole trajectory. This is a future problem.

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