SEQUENTIAL VECTOR SPACE DESCRIPTION OF DISCRETE CONTROL SYSTEM

Kahei NAKAMURA

Automatic Control Laboratory

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1. Introduction

In this paper, is proposed the *S.V.S.* (*Sequential Vector Space*) theory which is a humble theory for the discrete control system. In S.V.S. theory, the time functions such as input, output, and impulse response of a system, are expressed as sequential vectors composed of discrete functions which are, for example, sectional pieces of the function defined only in the sampling period. And the output S.V. (Sequential Vector) is related to input S.V. in simple product form with S.T.M. (Sequential Transfer Matrix), each column of which is derived from S.V. of impulse response of the system.

Although the sequential vector description is available to linear discrete system, the features especially appears in analysis and synthesis of non-linear discrete control system.

In this paper are treated at first the definition of S.V. and S.T.M., at second the sequential vector description of simple descrete system subjected to discrete-value input and discrete-function input, at last analysis and synthesis of a representative non-linear control system.

2. Definition of Sequential Vector and Sequential Transfer Matrix

A deterministic function can be expressed as a vector which is composed of discrete functions. We refer to that vector as *Sequential Vector* or *S.V.*.

The sequential vector of the input or output function of a general linear or non-linear system is referred to as input or output sequential vector respectively. The relation between the input and output sequential vectors of a system can be expressed by means of a matrix which is named as *Sequential Transfer Matrix* or *S.T.M.*, which is composed of the sequential vector of the impulse response of the system.

Now, we will use next notations:

$$\left.\begin{aligned}&\text{Input S.V.}: \ \boldsymbol{f}(t)=\{f_{t_j}(t-t_j)\}'=(f_{t_0}(t-t_0),\, f_{t_1}(t-t_1),\ \ldots)'\\ &\text{Output S.V.}: \ \boldsymbol{c}(t)=\{c_{t_j}(t-t_j)\}'=(c_{t_0}(t-t_0),\, c_{t_1}(t-t_1),\ \ldots)'\\ &\qquad\qquad\qquad\qquad\qquad (j=0,1,2,\ \ldots)\end{aligned}\right\} \qquad (2\text{-}1)^{*}$$

and

* The symbol $'$ means the transpose of a vector.

$$\text{S.T.M.}: \ \boldsymbol{H}(t) = \boldsymbol{H}(\boldsymbol{g}(t)) \tag{2-2}$$

where $f_{t_j}(t-t_j)$ or $c_{t_j}(t-t_j)$, $j=0, 1, 2, \ldots$ is a time function starting at the sampling time t_j, relative to input $f(t)$ or output $c(t)$, (*c.f.* Figure 2-1). $\boldsymbol{g}(t)$ is the sequential vector of impluse response of the system. The dimension of S.V. can be definite and very large. Then the system response (input-output relation) in the sequential vector space is generally expressed by a vector equation of Eq. (2-3).

$$\boldsymbol{c}(t) = \boldsymbol{H}(t) * \boldsymbol{f}(t) \qquad (2\text{-}3)^{**}$$

When $f_{t_j}(t)$ and $c_{t_j}(t)$ are defined only in a period of $T_j \leq t < T_{j+1}$, the input and output vectors will be expressed by functions of τ which is a time variable defined only in that period. That is

$$\begin{aligned} \boldsymbol{f}(t) &= \boldsymbol{f}(\tau) = \{f_{t_j}(\tau)\}' = (f_{t_0}(\tau), f_{t_1}(\tau), \ldots)' \\ \boldsymbol{c}(t) &= \boldsymbol{c}(\tau) = \{c_{t_j}(\tau)\}' = (c_{t_0}(\tau), c_{t_1}(\tau), \ldots)' \end{aligned} \qquad (2\text{-}4)^{*}$$

And the input-output relation can be written as

$$\boldsymbol{c}(\tau) = \boldsymbol{H}(\tau) * \boldsymbol{f}(\tau). \qquad (2\text{-}5)^{**}$$

As a special case, all the components of a sequential vector or a sequential transfer matrix can take the sampled values, as we will treat in Sec. 3. 1. Then, thereafter, we will refer to a sequential vector which is composed of only the sampled values as (ordinary) Sequential Vector, and the sequential vector composed of the sampled functions as the modified Sequential Vector or modified S.V. In the same way, we will name a S.T.M. composed of the sample values as the (ordinary) S.T.M. and a S.T.M. composed of the sampled functions as the modified S.T.M..

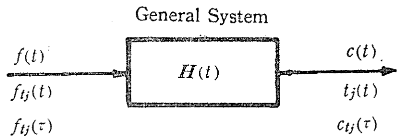

FIG. 2-1. System response of a general system.

3. System Response of Linear System Described in Sequential Vector Space

3. 1. System Response of A Single Discrete System in Sequential Vector Space

First of all, we will consider a most primitive sampled-data system as shown in Figure 3. 1-1, and assume that for simplicity the sampling action in the input and output sides are both taking place at the same time. The input sequential

* The symbol ′ means the transpose of a matrix.

** The symbol * means the convolution which will be explained in Sec. 3. 2.

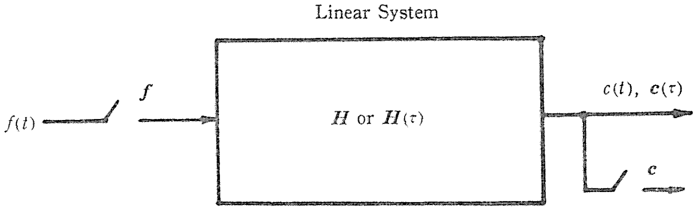

FIG. 3.1-1. System response of a single discrete system.

vector is composed of the sampled values of $f(t)$, and the output sequential vector is composed of the sampled values or the sampled functions of $c(t)$. They can be expressed by Eq. (3.1-1).

$$\begin{aligned} f &= \{f(jT)\}' = \{f_j\}' = (f_0, f_1, \ldots, f_n)' \\ c &= \{c(jT)\}' = \{c_j\}' = (c_0, c_1, \ldots, c_n)' \\ c(\tau) &= \{c(jT+\tau)\}' = \{c_j(\tau)\}' = (c_0(\tau), c_1(\tau), \ldots, c_n(\tau))' \end{aligned} \tag{3.1-1}$$

where $j = 0, 1, 2, \ldots, n$.

Let the sequence vector of impulse response of the system be

$$g = (g_0, g_1, \ldots, g_n)'. \tag{3.1-2}$$

By the fundamental theory of linear system, the relation of Eq. (3.1-3) can be directly derived.

$$c = \begin{pmatrix} c_0 \\ c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{pmatrix} = \begin{pmatrix} g_0 & 0 & \cdots & 0 \\ g_1 & g_0 & & \cdot \\ \cdot & \cdot\,\cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot\,\cdot \\ g_n & g_{n-1} & \cdots & g_0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{pmatrix} \tag{3.1-3}$$

Comparing Eq. (2-1) and Eq. (3.1-3), the S.T.M. of a linear system can be generally expressed by Eq. (3.1-4).

$$H = \begin{pmatrix} g_0 & 0 & \cdots & 0 \\ g_1 & g_0 & & \cdot \\ \cdot & \cdot\,\cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot\,\cdot \\ g_n & g_{n-1} & \cdots & g_0 \end{pmatrix} \tag{3.1-4}$$

Now, we will introduce a matrix S defined by Eq. (3.1-5).

$$S = \begin{pmatrix} 0 & 0 & \cdots\cdots & 0 \\ 1 & 0 & & \cdot \\ 0 & 1 & \cdot & \cdot \\ \cdot & \cdot\,\cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot\;\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\,\cdot \\ 0 & 0 & \cdots 1 & 0 \end{pmatrix} \tag{3.1-5}$$

S is an operator which shifts the raw of a vector or matrix downward by one raw, operating to it from the left side. Then the relations of Eq. (3.1-6) and Eq. (3.1-7) can be derived.

$$\begin{pmatrix} 0 \\ g_0 \\ g_1 \\ \cdot \\ \cdot \\ \cdot \\ g_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & & 0 \\ 1 & 0 & & & \cdot \\ 0 & 1 & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \cdot \\ \cdot \\ \cdot \\ g_n \end{pmatrix} = Sg \tag{3.1-6}$$

$$\begin{pmatrix} 0 \\ 0 \\ g_0 \\ g_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ g_{n-2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & & & 0 \\ 0 & 0 & \cdot & & & \cdot \\ 1 & 0 & \cdot & & & \cdot \\ 0 & 1 & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ g_n \end{pmatrix} = S^2 g \tag{3.1-7}$$

Therefore,

$$H = (g,\ Sg,\ S^2 g,\ \ldots,\ S^n g) \tag{3.1-8}$$

and

$$\begin{aligned} c &= H \cdot f \\ &= (g,\ Sg,\ S^2 g,\ \ldots,\ S^n g) f. \end{aligned} \tag{3.1-9}$$

Eq. (3.1-9) can be further modified as follows:

$$\begin{aligned} c &= f_0 \cdot g + f_1 S g + f_2 S^2 g + \cdots + f_n S^n g \\ &= (f_0 + f_1 S + f_2 S^2 + \cdots + f_n S^n) g \\ &= \left(\sum_{i=0}^{n} f_i S^i\right) g \end{aligned} \tag{3.1-10}$$

Eq. (3.1-9) can be also written in the other form. That is,

$$\begin{aligned} c &= (f,\ Sf,\ S^2 f \cdots S^n f) g \\ &= (g_0 + g_1 S + g_2 S^2 + \cdots + g_n S^n) f \\ &= \left(\sum_{i=0}^{n} g_i S^i\right) f. \end{aligned} \tag{3.1-11}$$

Since $S^i = 0$ for $i > n$,
then

$$\sum_{i=0}^{n} f_i S^i = \sum_{i=0}^{\infty} f_i S^i = [F(z)]_{\substack{z^{-1} \to S \\ 1 \to I}} = F(S) \tag{3.1-12}$$

and

$$\sum_{i=0}^{n} g_i S^i = [G(z)]_{\substack{z^{-1} \to S \\ 1 \to I}} = G(S). \tag{3.1-13}$$

We will refer to $F(S)$ and $G(S)$ in Eq. (3.1–12) and Eq. (3.1–13) as S-Transform of the input $f(t)$ and the impulse response $g(t)$ respectively, and now use the symbol of Eq. (3.1–14).

$$\begin{aligned} &G(S)=\mathscr{S}[g(t)] \\ \text{or}\quad &G(S)=\mathscr{S}[G(s)] \end{aligned} \tag{3.1-14}$$

The symbol of $\mathscr{S}$ means the taking of S-Transform of the function in the bracket. $G(S)$ or $F(S)$ is a matrix which is obtained by the substitution of z^{-1} and 1 in $G(z)$ or $F(z)$ (Z-Transform of $f(t)$ or $g(t)$) by S and I respectively. Thus, Eq. (3.1–11) can be written as Eq. (3.1–15) and Eq. (3.1–16).

$$c=F(S)g \tag{3.1-15}$$

$$c=G(S)f=Hf \tag{3.1-16}$$

where

$$H=\mathscr{S}[g(t)]=G(S). \tag{3.1-17}$$

Eq. (3.1–16) is the vector equation expressing the relation between the input and output sequential vectors, and Eq. (3.1–17) shows that the sequential transfer matrix of H is the S-Transform of the impulse response of the system and can be directly derived from its pulse transfer function by substituting z^{-1} and 1 by S and I respectively.

The modified output sequential vector $c(\tau)$ can be also described by the equation analogous to Eq. (3.1–16). Let the modified sequential vector of impulse response $g(t)$ be as Eq. (3.1–18).

$$g(t)=(g_0(\tau),\ g_1(\tau),\ \ldots,\ g_n(\tau))' \tag{3.1-18}$$

Referring to Eq. (3.1–11), the modified sequential vector of output can be expressed as

$$\begin{aligned} c(t)&=(f,\ Sf,\ S^2f,\ \ldots,\ S^nf)g(\tau) \\ &=(g_0(\tau),\ g_1(\tau)S,\ \ldots,\ g_n(\tau)S^n)f \\ &=(\sum_{i=0}^{n}g_i(\tau)S^i)f \\ &=G(S,\ \tau)f, \end{aligned} \tag{3.1-19}$$

where

$$G(S,\ \tau)=\sum_{i=0}^{\infty}g_i(\tau)S^i=[G(z,\ \tau)]_{\substack{z^{-1}\to S\\ 1\to I}} \tag{3.1-20}$$

$$G(z\tau)=[zG(z,\ m)]_{m=\tau/T}. \tag{3.1-21*}$$

Corresponding to the definition of Eq. (3.1–14), we will use the symbol of Eq. (3.1–22) for the operation of Eq. (3.1–20).

$$\left.\begin{aligned} &G(S,\ \tau)=\mathscr{S}_\tau[g(t)] \\ \text{or}\quad &G(S,\ \tau)=\mathscr{S}_\tau[G(s)] \end{aligned}\right\} \tag{3.1-22}$$

* $G(z,\ m)$ is the conventional modified Z-Transform of $g(t)$.

The symbol $\mathscr{S}_\tau$ means the operation of modified S-Transformation. Then if we put as

$$H(\tau) = \mathscr{S}_\tau[g(t)] = \boldsymbol{G}(S, \tau), \tag{3.1-23}$$

the modified input-output sequential vector relation of the system can be followed as Eq. (3.1-24).

$$\boldsymbol{c}(\tau) = \boldsymbol{H}(\tau)\boldsymbol{f} \tag{3.1-24}$$

$\boldsymbol{H}(\tau)$ refers to as modified S.T.M. and can be derived from the modified Pulse Transfer Function of the system by substituting z^{-1} and 1 by S and I respectively and is expressed by Eq. (3.1-25).

$$\boldsymbol{H}(\tau) = \begin{bmatrix} g_0(\tau) & 0 & 0 \cdots\cdots & 0 \\ g_1(\tau) & g_0(\tau) & 0 & \cdot \\ g_2(\tau) & g_1(\tau) & g_0(\tau) & \cdot \\ \cdot & \cdot & \cdot \quad \ddots & \cdot \\ \cdot & \cdot & \cdot \quad\quad \ddots & \cdot \\ \cdot & \cdot & \cdot \quad\quad\quad \ddots & \cdot \\ g_n(\tau) & g_{n-1}(\tau) & g_{n-2}(\tau) \cdots & g_0(\tau) \end{bmatrix} \tag{3.1-25}$$

3. 2. System Response of A Discrete System Subjected to Discrete Function Input

In this section, we will investigate the computing method of the output S.V. of a plant subjected to the discrete function input. The discrete function is referred to as a discontinuous function which has the different sectional function in each sampling interval. For example, the input of a plant preceded by sampler and hold circuit or sampler and interporator as shown in Figure 3. 2-1 is generally a discrete function. The sequential vector of discrete function is the modified sequential vector.

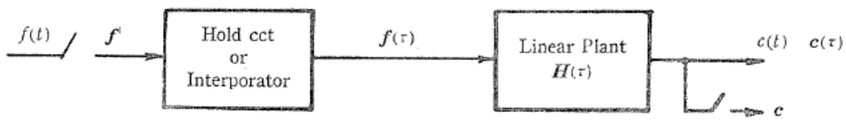

FIG. 3. 2-1. Linear plant subjected to the discrete function input.

Let the holding or interporating function of the hold circuit or interporator be $h(\tau)$. The output of the interporator can be expressed by Eq. (3.2-1).

$$\boldsymbol{f}(\tau) = \boldsymbol{h}(\tau)\boldsymbol{f} \tag{3.2-1}$$ *

Now the output of plant is a summation of the sectional convolutional integral. That is,

$$c_j(\tau) = \sum_{i=0}^{j-1}\int_0^T g_{(j-i)}(\tau-\nu) f_i(\nu)\,d\nu + \int_0^\tau g_0(\tau-\nu) f_j(\nu)\,d\nu, \quad j = 0, 1, 2, \ldots \tag{3.2-2}$$

* Note that $\boldsymbol{f}(\tau)$ which expressed by Eq. 3.2-1 is different from the sequential vector of the original input function $f(t)$.

where $f_i(\tau)$, $c_j(\tau)$ and $g_k(\tau)$ is respectively a component of the modified sequential vectors of the input, output and impulse response of the system. If we introduce a new expression of Eq. (3.2-3),

$$\left.\begin{aligned}\int_0^{\tau} g_k(\tau-\nu)f_i(\nu)d\nu &= [g_k(\tau)*f_i(\tau)]_{\tau} = (g_k*f_i)_{\tau} \\ \int_0^{T} g_k(\tau-\nu)f_i(\nu)d\nu &= [g_k(\tau)*f_i(\tau)]_{T} = (g_k*f_i)_{T}\end{aligned}\right\} \tag{3.2-3}$$

the following expressions can be obtained.

$$\left.\begin{aligned}c_0(\tau) &= (g_0*f_0)_{\tau} \\ c_1(\tau) &= (g_1*f_0)_T + (g_0*f_1)_{\tau} \\ c_2(\tau) &= (g_2*f_0)_T + (g_1*f_1)_T + (g_0*f_2)_{\tau} \\ &\vdots \\ c_n(\tau) &= (g_n*f_0)_T + (g_{n-1}*f_1)_T + \cdots + (g_0*f_n)_{\tau}\end{aligned}\right\} \tag{3.2-4}$$

A set of equations of Eq. (3.2-4) can be rewritten by the vector equation of Eq. (3.2-5) and Eq. (3.2-6)

$$c(\tau) = \begin{Bmatrix} c_0(\tau) \\ c_1(\tau) \\ c_2(\tau) \\ \cdot \\ \cdot \\ \cdot \\ c(\tau) \end{Bmatrix} = \begin{bmatrix} g_0(\tau) & 0 & 0 \cdots\cdots & 0 \\ g_1(\tau) & g_0(\tau) & 0 & 0 \\ g_2(\tau) & g_1(\tau) & g_0(\tau) & 0 \\ \cdot & \cdot & \cdot \quad \cdot & \cdot \\ \cdot & \cdot & \cdot \quad \cdot & \cdot \\ \cdot & \cdot & \cdot \quad \cdot & \cdot \\ g_n(\tau) & g_{n-1}(\tau) & g_{n-2}(\tau) & g_0(\tau) \end{bmatrix} * \begin{Bmatrix} f_0(\tau) \\ f_1(\tau) \\ f_2(\tau) \\ \cdot \\ \cdot \\ \cdot \\ f_n(\tau) \end{Bmatrix} \tag{3.2-5}$$

$$c(\tau) = [H(\tau)*f(\tau)] \tag{3.2-6}$$

The symbol of the thick brackets appeared in Eq. (3.2-4) ~ Eq. (3.2-6) shows that the time shifting of ν in the convolutional integrand is to be subjected to the definition of Eq. (3.2-3). It must be noted that the upper limit of the convolutional integral involving $g_0(\tau)$ is τ, and those of the other convolutions are T. Here it should be noted that Eq. (3.2-6) is an extention of Eq. (3.1-15) which is for the case of a discrete value input, to the case of a discrete function input.

Ex. 3.2-1

Now, we will learn the practical calculating procedure through a very simple example.

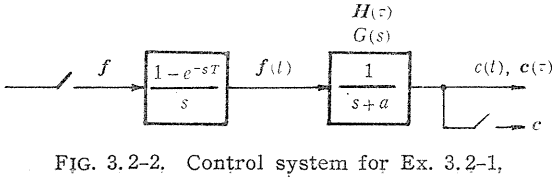

FIG. 3.2-2. Control system for Ex. 3.2-1,

By applying Eq. (3.1–23),

$$H(\tau)=\mathscr{S}_\tau[G(s)]=\left[\frac{e^{-a\tau}}{1-e^{-aT}z^{-1}}\right]_{\substack{z^{-1}\to S\\ 1\to I}}=\frac{e^{-a\tau}I}{I-e^{-aT}S}$$

$$=e^{-a\tau}I+e^{-a(T+\tau)}S+e^{-a(2T+\tau)}S^2+\cdots \tag{i}$$

On the other hand, the discrete function input $f(\tau)$ is expressed by Eq. (ii).

$$f(\tau)=[1(\tau)\cdot I][f]=\begin{Bmatrix} f_0\cdot 1(\tau)\\ f_1\cdot 1(\tau)\\ f_2\cdot 1(\tau)\\ \cdot\\ \cdot\\ \cdot\\ f_n\cdot 1(\tau)\end{Bmatrix} \tag{ii}$$

where $1(\tau)$: unit function

Then the modified output sequential vector $c(\tau)$ is obtained by using Eq. (3.2–5) and Eq. (3.2–6).

$$c(\tau)=[H(\tau)*f(\tau)]=\left[\frac{e^{-a\tau}I}{I-e^{-aT}S}*f(\tau)\right] \tag{iii}$$

$$c(\tau)=\begin{bmatrix} e^{-a\tau} & 0 & 0 & \cdots\cdot 0\\ e^{-a(T+\tau)} & e^{-a\tau} & 0 & 0\\ e^{-a(2T+\tau)} & e^{-a(T+\tau)} & e^{-a\tau} & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ e^{-a(nT+\tau)} & e^{-a(\overline{n-1}T+\tau)} & \cdots & \cdot\, e^{-a\tau}\end{bmatrix}*\begin{Bmatrix} f_0\cdot 1(\tau)\\ f_1\cdot 1(\tau)\\ f_2\cdot 1(\tau)\\ \cdot\\ \cdot\\ \cdot\\ f_n\cdot 1(\tau)\end{Bmatrix}$$

$$c(\tau)=\begin{Bmatrix} (e^{-a\tau}*1(\tau))_\tau f_0\\ (e^{-a(T+\tau)}*1(\tau))_T f_0+(e^{-a\tau}*1(\tau))_\tau f_1\\ (e^{-a(2T+\tau)}*1(\tau))_T f_0+(e^{-a(T+\tau)}*1(\tau))_T f_1+(e^{-a\tau}*1(\tau))_\tau f_2\\ \cdot \qquad \cdot\\ \cdot \qquad \cdot\\ \cdot \qquad \cdot\\ (e^{-a(nT+\tau)}*1(\tau))_T f_0+(e^{-a(\overline{n-1}T+\tau)}*1(\tau))_T f_1+\cdots+(e^{-a\tau}*1(\tau))_\tau f_n\end{Bmatrix} \tag{iv}$$

Referring to the convolution Table in APPENDIX,

$$\left.\begin{aligned} c_0(\tau)&=\frac{1}{a}(1-e^{-a\tau})f_0\\ c_1(\tau)&=\frac{1}{a}e^{-a\tau}(1-e^{-aT})f_0+\frac{1}{a}(1-e^{-a\tau})f_1\\ c_2(\tau)&=\frac{1}{a}e^{-a\tau}e^{-aT}(1-e^{-aT})f_0+\frac{1}{a}e^{-a\tau}(1-e^{-aT})f_1+\frac{1}{a}(1-e^{-a\tau})f_2\\ &\cdot\qquad\cdot\qquad\cdot\\ &\cdot\qquad\cdot\qquad\cdot\\ &\cdot\qquad\cdot\qquad\cdot\\ c_n(\tau)&=\frac{1}{a}e^{-a\tau}e^{-a(n-1)T}(1-e^{-aT})f_0+\frac{1}{a}e^{-a\tau}e^{-a(n-2)T}(1-e^{-aT})f_1+\cdots+\frac{1}{a}(1-e^{-a\tau})f_n \end{aligned}\right\} \tag{v}$$

c is obtained by putting $\tau=0$ in $c(\tau)$.

$$\left.\begin{array}{l} c_0 = 0 \\ c_1 = \frac{1}{a}(1-e^{-aT})f_0 \\ c_2 = \frac{1}{a}(1-e^{-aT})(e^{-aT}f_0+f_1) \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ c_n = \frac{1}{a}(1-e^{-aT})(e^{-a(n-1)T}f_0+e^{-a(n-2)T}f_1+\cdots+e^{-aT}f_{n-2}+f_{n-1}) \end{array}\right\} \quad \text{(vi)}$$

3. 3. System Response of Continuous System Described in Sequential Vector Space

In this section we will study how to treat the continuous system in sequential vector space. The input-output relation of continuous system, of course, can be described very simply and clearly by Laplace Transformation method. While the object of treatment in this section about the continuous system very lies in showing that the sequential vector method is also available for the continuous system.

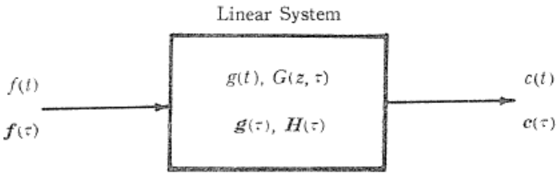

FIG. 3. 3-1. System response of general continuous system.

Now, let us consider a simple continuous system shown in Figure 3. 3-1. The output of the system $c(t)$ is clearly given by the next convolutional integral:

$$c(t) = \int_0^t f(t-\lambda)g(\lambda)\,d\lambda \qquad (3.3\text{-}1)$$

By putting as

$$t = jT+\tau\ ;\ j=0,1,2,\ldots,$$
$$\lambda = iT+\nu\ ;\ i=0,1,\ldots,j,$$

and making further modification, we can derive the expression of Eq. (3. 3-2).

$$\left.\begin{array}{l} c_0(\tau) = (g_0 * f_0)_\tau \\ c_1(\tau) = (g_1 * f_0)_\tau + (g_0 * f_1)_T \\ c_2(\tau) = (g_2 * f_0)_\tau + (g_1 * f_1)_T + (g_0 * f_2)_T \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ c_n(\tau) = (g_n * f_0)_\tau + (g_{n-1} * f_1)_T + \cdots + (g_0 * f_n)_T \end{array}\right\} \quad (3.3\text{-}2)$$

Where

$$\left.\begin{aligned}\int_0^{\tau} g_i(\nu) f_k(\tau-\nu)\, d\nu &= [g_i(\tau) * f_k(\tau)]\!]_{\tau} = (g_i * f_k)\!)_{\tau} \\ \int_0^{T} g_i(\nu) f_k(\tau-\nu)\, d\nu &= [g_i(\tau) * f_k(\tau)]\!]_{T} = (g_i * f_k)\!)_{T}\end{aligned}\right\} \qquad (3.3\text{-}3)$$

The expression of Eq. (3.3-2) causes the vector expression of Eq. (3.3-4) or Eq. (3.3-5).

$$c(\tau) = \begin{Bmatrix} c_0(\tau) \\ c_1(\tau) \\ c_2(\tau) \\ \cdot \\ \cdot \\ \cdot \\ c_n(\tau) \end{Bmatrix} = \begin{Bmatrix} g_0(\tau) & 0 & 0 \cdots\cdots 0 & \\ g_1(\tau) & g_0(\tau) & 0 \cdots\cdots 0 & \\ g_2(\tau) & g_1(\tau) & g_0(\tau) \cdots 0 & \\ \cdot & \cdot & \cdot\ \cdot & \\ \cdot & \cdot & \cdot \quad \cdot & \\ \cdot & \cdot & \cdot \qquad \cdot & \\ g_n(\tau) & g_{n-1}(\tau) & g_{n-2}(\tau) & g_0(\tau) \end{Bmatrix} * \begin{bmatrix} f_0(\tau) \\ f_1(\tau) \\ f_2(\tau) \\ \cdot \\ \cdot \\ \cdot \\ f_n(\tau) \end{bmatrix} \qquad (3.3\text{-}4)$$

$$c(\tau) = [H(\tau) * f(\tau)]\!] \qquad (3.3\text{-}5)$$

where, the meaning of the thick brackets in Eq. (3.3-2) to Eq. (3.3-5) is clarified in the similar way to that in Sec. 3. 2. It is noticeable that the upper limit of the convolutional integrals involving $f_0(\tau)$ are τ, and those of the other convolutions are T.

Now, we must take note that, as being exchangeable of f and g in convolution integral of Eq. (3.3-1), the output $c(\tau)$ can be also expressed as Eq. (3.3-6).

$$c(\tau) = [\![H(\tau) * f(\tau)] \qquad (3.3\text{-}6)$$

The relation of Eq. (3.3-6) is the same form as Eq. (3.2-6). The difference between Eq. (3.3-5) and Eq. (3.3-6) presents in the execution of matrix operation, and can be conclusively said as follows: "The shifting of the denominator of $H(\tau)$ to the left side of the vector oquation is possible in the execution of matrix operation by Eq. (3.3-5), but is impossible by Eq. (3.3-6)". This is caused by the fact that

(a) The denominator of $H(\tau)$ performs the weighting operation on elements of each column,

(b) On the integration limits of the convolution integrals made by $f_0(\tau)$ and the first column of $H(\tau)$, all of them are τ for the case of Eq. (3.3-5) but not for Eq. (3.3-6).

In accordance with the above two facts, the weighting operation by the denominator can be put off after the execution of the convolution integral for the case of Eq. (3.3-5), but not for Eq. (3.3-6). This is the very reason for whether or not the shifting of the denominator of $H(\tau)$ is possible.

Next, we will consider briefly the computation method of $c_j(T)$, $j=0, 1, 2, \ldots$. When $\tau=T$, all the integration limits in Eq. (3.3-5) and in Eq. (3.2-4) are T. Then, the shifting of the denominator of $H(\tau)$ is always possible.

3. 4. Introduction of Initial Condition to Sequential Vector Equation

It is very simple how to introduce the initial conditions to the sequential vector equation. Now, for convenience, we will consider the second order system.

Let the system differential equation be

$$\ddot{x} + a\dot{x} + bx = m \tag{3.4-1}$$

$$X(s) = G(s)M(s) + G(s)\dot{x}(0) + (s+a)G(s)x(0)$$

$$X(z, \tau) = \mathscr{Z}_\tau[G(s)M(s)] + G(z, \tau)\dot{x}(0) + [aG(z, \tau) + \dot{G}(y, \tau)]\times(0)$$

Then $$x(\tau) = [H(\tau)*m(\tau)] + \dot{x}(0)H(\tau)e + x(0)[aH(\tau) + \dot{H}(\tau)]e \tag{3.4-2}$$

or $$x(\tau) = [H(\tau)*m(\tau)] + \dot{x}(0)\,g(\tau) + x(0)[ag(\tau) + \dot{g}(\tau)] \tag{3.4-3}$$

where

$$\begin{aligned}
G(s) &= \frac{1}{s^2+as+b} \\
G(z,\ \tau) &= \mathscr{Z}_\tau[G(s)] \\
\dot{G}(z,\ \tau) &= \mathscr{Z}_\tau[sG(s)] \\
H(\tau) &= \mathscr{S}_\tau[G(s)] = [G(z,\ \tau)]_{\substack{z^{-1}\to S \\ 1\to I}} \\
\dot{H}(\tau) &= \mathscr{S}_\tau[sG(s)] = [\dot{G}(z,\ \tau)]_{\substack{z^{-1}\to S \\ 1\to I}} \\
g(\tau) &= (g_0(\tau),\ g_1(\tau)\cdots g_n(\tau))' \\
\dot{g}(\tau) &= (\dot{g}_0(\tau),\ \dot{g}_1(\tau)\cdots \dot{g}_n(\tau))' \\
e &= (1, 0, 0, \ldots, 0)'
\end{aligned} \tag{3.4-4}$$

Eq. (3.4-2) or Eq. (3.4-3) is the system sequential vector equation involving the initial condition. It is quite possible to extend the above discussion to the general higher order system.

4. System Response of Non-linear System Described in Sequential Vector Space

4.1. General Aspects of Sequential Vector Method for Non-linear System

In general, any non-linear system can be described by combination of linear systems and zero memory non-linear systems [7]. For example, a non-linear system which is described by the following non-linear equation

$$\ddot{x} + x\dot{x} + x^2 = f \tag{4.1-1}$$

can be expressed by the block diagram of Figure (4.1-1).

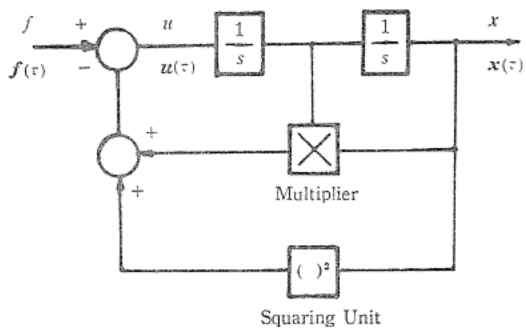

FIG. 4.1-1. General non-linear system.

In general, it is very difficult to obtain the dynamical behavior of this system by the conventional procedure. The author now proposes a method to secure the transient response of such complex non-linear system by applying the sequential vector space procedure. The very merit of the sequential vector space method is based on the matter that the output sequential vector of zero memory non-linear element can be easily obtained by executing the corresponding non-linear operation on each component of the input sequential vector.

4. 2. System Response of Non-Linear Open Loop System

4. 2. 1. Open Loop System with Squaring Unit

Now we consider a discrete system with a squaring unit shown in Figure 4. 2–1.

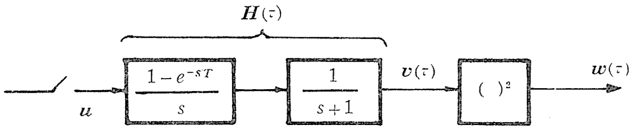

FIG. 4. 2–1. Open loop non-linear system with squaring unit.

The system relation can be expressed by Equation (4. 2–1) and Equation (4. 2–2).

$$v(\tau) = H(\tau)u \tag{4.2-1}$$

$$w(\tau) = v(\tau) \times v(\tau) = v^2(\tau) = (v_0^2(\tau),\ v_1^2(\tau), \ldots, v_n^2(\tau)) \tag{4.2-2}$$

The symbol $\times$ in the vector equation of Equation (4. 2–2) means the non-linear scalar operation of $v_j(\tau) \times v_j(\tau)$ relative to components of the vector $v(\tau)$.

By applying the computed result of Ex. 3. 2–1, $v(\tau)$ is straightly given by Equation (4. 2–3).

$$\left.\begin{aligned}
v_0(\tau) &= u_0(1-e^{-\tau}) \\
v_1(\tau) &= u_0(1-e^{-T})e^{-\tau} + u_1(1-e^{-\tau}) \\
v_2(\tau) &= (u_0e^{-T}+u_1)(1-e^{-T})e^{-\tau} + u_2(1-e^{-\tau}) \\
v_3(\tau) &= (u_0e^{-2T}+u_1e^{-T}+u_2)(1-e^{-T})e^{-\tau} + u_3(1-e^{-\tau}) \\
&\ \ \vdots \\
v_n(\tau) &= (u_0e^{-(n-1)T}+u_1e^{-(n-2)T}+\cdots+u_{n-1})(1-e^{-T})e^{-\tau} + u_n(1-e^{-\tau})
\end{aligned}\right\} \tag{4.2-3}$$

The squaring operation in Equation (4. 2–2) are performed by the way shown by Equation (4. 2–4).

$$\begin{aligned}
w_0(\tau) &= v_0^2(\tau) = u_0^2(1-e^{-\tau})^2 \\
w_1(\tau) &= v_1^2(\tau) = u_0^2(1-e^{-T})^2e^{-2\tau} + u_1^2(1-e^{-\tau})^2 + 2u_0u_1(1-e^{-T})(1-e^{-\tau})e^{-\tau} \\
w_2(\tau) &= v_2^2(\tau) = (u_0^2e^{-2T}+u_1^2)(1-e^{-T})^2e^{-2\tau} + u_2^2(1-e^{-\tau})^2 \\
&\quad + 2u_0u_1(1-e^{-T})^2e^{-(T+2\tau)} + 2(u_0u_2e^{-T}+u_1u_2)(1-e^{-T})(1-e^{-\tau})e^{-\tau}
\end{aligned}$$

$$
\left.\begin{aligned}
w_3(\tau) = v_3^2(\tau) &= (u_0^2 e^{-4T} + u_1^2 e^{-2T} + u_2^2)(1-e^{-T})^2 e^{-2\tau} + u_3^2(1-e^{-\tau})^2 \\
&\quad + 2(u_0u_1e^{-3T} + u_0u_2e^{-2T} + u_1u_2e^{-T})(1-e^{-T})^2 e^{-2\tau} \\
&\quad + 2(u_0u_3e^{-2T} + n_1u_3e^{-T} + u_2u_3)(1-e^{-T})(1-e^{-\tau})e^{-\tau} \\
&\quad \vdots \\
w_n(\tau) &= v_n^2(\tau) \\
&= (u_0^2 e^{-2(n-1)T} + u_1^2 e^{-2(n-2)T} + \cdots + u_{n-1}^2)(1-e^{-T})^2 e^{-2\tau} \\
&\quad + u_n^2(1-e^{-\tau})^2 \\
&\quad + 2\times\left\{\begin{aligned} & u_0u_1e^{-(2n-3)T} + u_0u_2e^{-(2n-4)T} + \cdots \\ & + u_0u_{n-1}e^{-(n-1)T} + u_1u_2e^{-(2n-5)T} \\ & + u_1u_3e^{-(2n-6)T} + \cdots + u_1u_{n-1}e^{-(n-2)T} \\ & \qquad\qquad + \cdots + u_{n-2}u_{n-1}e^{-T} \end{aligned}\right\}(1-e^{-T})^2 e^{-2\tau} \\
&\quad + 2(u_0u_ne^{-(n-1)T} + u_1u_ne^{-(n-2)T} + \cdots + u_{n-1}u_n) \\
&\qquad\qquad \times (1-e^{-T})(1-e^{-\tau})e^{-\tau}
\end{aligned}\right\} \quad (4.2\text{-}4)
$$

4. 3. System Response of Non-linear Discrete Feedback System

4. 3. 1. Servo System with Multiplier in the Loop

In this section, we will discuss how to secure the transient response of the non-linear discrete feedback control system. Now, we will show the calculating procedure for the system shown in Figure 4. 3-1.

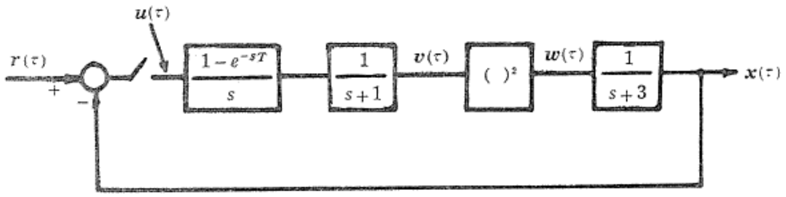

FIG. 4. 3-1. A discrete feedback control system containing squaring unit.

The executing procedure to calculate the system response is as follows:

1. Calculation of $w(\tau)$ by Equation (4. 2-4).
2. Computation of $x(\tau)$.
3. Computation of u.

The output sequential vector $x(\tau)$ is given by Equation (4. 3-1).

$$
\begin{aligned}
x(\tau) &= \mathscr{S}\left[\frac{1}{s+3}\right] * w(\tau) = \frac{e^{-3\tau}I}{I-e^{-3T}S} * w(\tau) \\
&= [e^{-3\tau}I + e^{-3(T+\tau)}S + e^{-3(2T+\tau)}S^2 + \cdots] * w(\tau) \qquad (4.3\text{-}1)
\end{aligned}
$$

The sequential vector $w(\tau)$ can be calculated directly by using Equation (4. 2-4). As $w(\tau)$ is a discrete function, the final output $x(\tau)$ should be computed by the procedure which is described in Section 3. 2.

$$
\begin{pmatrix} x_0(\tau) \\ x_1(\tau) \\ x_2(\tau) \\ x_3(\tau) \\ \cdot \\ \cdot \\ \cdot \\ x_n(\tau) \end{pmatrix} = \begin{bmatrix} e^{-3\tau} & 0 & 0 & 0 & 0 \\ e^{-3(T+\tau)} & e^{-3\tau} & 0 & 0 & \cdot \\ e^{-3(2T+\tau)} & e^{-3(T+\tau)} & e^{-3\tau} & 0 & \cdot \\ e^{-3(3T+\tau)} & e^{-3(2T+\tau)} & e^{-3(T+\tau)} & e^{-3\tau} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ e^{-3(nT+\tau)} & e^{-3(\overline{n-1}T+\tau)} & e^{-3(\overline{n-2}T+\tau)} & \cdots\cdots & e^{-3\tau} \end{bmatrix} * \begin{pmatrix} w_0(\tau) \\ w_1(\tau) \\ w_2(\tau) \\ w_3(\tau) \\ \cdot \\ \cdot \\ \cdot \\ w_n(\tau) \end{pmatrix}
$$

$$
= \begin{pmatrix} (e^{-3\tau} * w_0(\tau))_\tau \\ e^{-3T}(e^{-3\tau} * w_0(\tau))_T + (e^{-3\tau} * w_1(\tau))_\tau \\ e^{-6T}(e^{-3\tau} * w_0(\tau))_T + e^{-3T}(e^{-3\tau} * w_1(\tau))_T + (e^{-3\tau} * w_2(\tau))_\tau \\ e^{-9T}(e^{-3\tau} * w_0(\tau))_T + e^{-6T}(e^{-3\tau} * w_1(\tau))_T + e^{-3T}(e^{-3\tau} * w_2(\tau))_T + (e^{-3\tau} * w_3(\tau))_T \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ e^{-3nT}(e^{-3\tau} * w_0(\tau))_T + e^{-3(n-1)T}(e^{-3\tau} * w_1(\tau))_T + \cdots + (e^{-3\tau} * w_n(\tau))_T \end{pmatrix} \tag{4.3-2}
$$

$$
\left.
\begin{aligned}
x_0(\tau) &= \frac{1}{3}u_0^2(1 - 3e^{-\tau} + 3e^{-2\tau} - e^{-3\tau}) \\
x_1(\tau) &= \frac{1}{3}u_1^2 + u_1[(1 - e^{-T})u_0 - u_1]e^{-\tau} + [(1 - e^{-T})u_0 - u_1]^2 e^{-2\tau} \\
&\qquad + \left[-(1 - e^{-T})^2(2 + e^{-T})\frac{1}{3}u_0^2 - \frac{1}{3}u_1^2 + (1 - e^{-T})u_0u_1 \right] e^{-3\tau} \\
x_2(\tau) &= \frac{1}{3}u_2^2 + u_2[(1 - e^{-T})(u_0e^{-T} + u_1) - u_2]e^{-\tau} + [(1 - e^{-T})(u_0e^{-T} + u_1) - u_2]^2 e^{-2\tau} \\
&\qquad + \left[-(1 - e^{-T})^2(2 + e^{-T})\frac{1}{3}(u_0^2e^{-3T} + u_1^2) - \frac{1}{3}u_2^2 - (1 - e^{-T})^2(1 + e^{-T})u_0u_1e^{-T} \right. \\
&\qquad \left. + (1 - e^{-T})(u_0u_2e^{-T} + u_1u_2) \right] e^{-3\tau} \\
x_3(\tau) &= \frac{1}{3}u_3^2 + u_3[(1 - e^{-T})(u_0e^{-2T} + u_1e^{-T} + u_2) - u_3]e^{-\tau} \\
&\qquad + [(1 - e^{-T})(u_0e^{-2T} + u_1e^{-T} + u_2) - u_3]^2 e^{-2\tau} \\
&\qquad + \left[-(1 - e^{-T})^2(2 + e^{-T})\frac{1}{3}(u_0^2e^{-6T} + u_1^2e^{-3T} + u_2^2) - \frac{1}{3}u_3^2 \right. \\
&\qquad - (1 - e^{-T})^2(1 + e^{-T})(u_0u_1e^{-4T} + u_0u_2e^{-2T} + u_1u_2e^{-T}) \\
&\qquad \left. + (1 - e^{-T})(u_0e^{-2T} + u_1e^{-T} + u_2)u_3 \right] e^{-3T} \\
x_n(\tau) &= \frac{1}{3}u_n^2 + u_n[(1 - e^{-T})(u_0e^{-(n-1)T} + u_1e^{-(n-2)T} + \cdots + u_{n-1}) - u_n]e^{-\tau} \\
&\qquad + [(1 - e^{-T})(u_0e^{-(n-1)T} + u_1e^{-(n-2)T} + \cdots + u_{n-1}) - u_n]^2 e^{-2\tau} \\
&\qquad + \left[-(1 - e^{-T})^2(2 + e^{-T})\frac{1}{3}(u_0^2e^{-3(n-1)T} + u_1^2e^{-3(n-2)T} + \cdots + u_{n-1}^2) \right. \\
&\qquad - \frac{1}{3}u_n^2 - (1 - e^{-T})^2(1 + e^{-T})(u_0u_1e^{-(3n-5)T} + u_0u_2e^{-(3n-7)T} + \cdots \\
&\qquad + u_0u_{n-1}e^{-(n-1)T} + u_1u_2e^{-(3n-8)T} + \cdots + u_1u_{n-1}e^{-(n-2)T} + \cdots \\
&\qquad \left. + u_{n-2}u_{n-1}e^{-T}) + (1 - e^{-T})(u_0e^{-(n-1)T} + u_1e^{-(n-2)T} + \cdots + u_{n-2})u_n \right] e^{-3\tau}
\end{aligned}
\right\} \tag{4.3-3}
$$

Thus, the deviation or control u can be secured as follows:

The relation of Equation (4.3-4) or Equation (4.3-5) is satisfied on the system of Figure 4.3-1, because the output $x(\tau)$ does not have discontinuous component at sampling time.

$$u = r(0) - x(0) \tag{4.3-4}$$

or

$$\left.\begin{array}{l} u_0 = r_0 - x_0 \\ u_1 = r_1 - x_1 \\ \cdot \qquad\quad \cdot \\ \cdot \qquad\quad \cdot \\ \cdot \qquad\quad \cdot \\ u_n = r_n - x_n \end{array}\right\} \tag{4.3-5}$$

while, x_j, $j=0, 1, 2, \ldots, n$ is obtained as Equation (4.3-6) by putting $\tau=0$ in Equation (4.3-3).

$$\begin{aligned}
x_0(0) &= 0 \\
x_1(0) &= (1-e^{-T})^2\left(1-\frac{2+e^{-T}}{3}\right)u_0^2 = \frac{1}{3}(1-e^{-T})^3u_0^2 \\
x_2(0) &= (1-e^{-T})^2\left[u_0^2\left(1-\frac{2+e^{-T}}{3}e^{-T}\right)e^{-2T} + u_1^2\left(1-\frac{2+e^{-T}}{3}\right)\right. \\
&\quad \left. + u_0u_1(2-(1+e^{-T}))\,e^{-T}\right] \\
&= (1-e^{-T})^3\left[u_0^2\left(1+\frac{1}{3}e^{-T}\right)e^{-2T} + \frac{1}{3}u_1^2 + u_0u_1e^{-T}\right] \\
x_3(0) &= (1-e^{-T})^2\left[u_0^2\left(1-\frac{2+e^{-2}}{3}e^{-2T}\right)e^{-4T} + u_1^2\left(1-\frac{2+e^{-T}}{3}e^{-T}\right)e^{-2T}\right. \\
&\quad + u_2^2\left(1-\frac{2+e^{-T}}{3}\right) + u_0u_1(2-(1+e^{-T})e^{-T})e^{-3T} + u_0u_2(2-(1+e^{-T}))e^{-2T} \\
&\quad \left. + u_1u_2(2-(1+e^{-T}))\,e^{-T}\right] \\
&= (1-e^{-T})^3\left[u_0^2\left(1+e^{-T}+\frac{1}{3}e^{-2T}\right)e^{-4T} + u_1^2\left(1+\frac{1}{3}e^{-T}\right)e^{-2T} + u_2^2\frac{1}{3}\right. \\
&\quad \left. + u_0u_1(2+e^{-T})\,e^{-3T} + u_0u_2e^{-2T} + u_1u_2e^{-T}\right] \\
&\cdot \qquad\qquad\qquad \cdot \\
&\cdot \qquad\qquad\qquad \cdot \\
&\cdot \qquad\qquad\qquad \cdot \\
x_n(0) &= (1-e^{-T})^2 \\
&\quad \times\left[u_0^2\left(1-\frac{2+e^{-T}}{3}e^{-(n-1)T}\right)e^{-2(n-1)T} + u_1^2\left(1-\frac{2+e^{-T}}{3}e^{-(n-2)T}\right)e^{-2(n-2)T}\right. \\
&\quad + \cdots + u_{n-1}^2\left(1-\frac{2+e^{-T}}{3}\right) + u_0u_1(2-(1+e^{-T})e^{-(n-2)T})e^{-(2n-3)T} \\
&\quad + u_0u_2(2-(1+e^{-T})e^{-(n-3)T})e^{-(2n-4)T} + \cdots + u_0u_{n-1}(2-(1+e^{-T}))e^{-(n-1)T} \\
&\quad + u_1u_2(2-(1+e^{-T})e^{-(n-3)T})e^{-(2n-5)} + \cdots + u_1u_{n-1}(2-(1+e^{-T}))e^{-(n-2)T} \\
&\quad \cdots\cdots\cdots\cdots\cdots \\
&\quad \left. + u_{n-2}u_{n-1}(2-(1+e^{-T}))\,e^{-T}\right]
\end{aligned} \tag{4.3-6}$$

Since r_j is given, by substituting $x_j(0)$ of Equation (4.3-6) in to Equation (4.3-5), u_j to be secured is obtained successively by solving the recurrent equation of Equation (4.3-7)

$$\left.\begin{aligned}
u_0 &= r_0 \\
u_1 &= r_1 - \frac{1}{3}(1-e^{-T})^3 u_0^2 \\
u_2 &= r_2 - (1-e^{-T})^3\left[u_0^2\left(1+\frac{1}{3}e^{-T}\right)e^{-2T} + \frac{1}{3}u_1^2 + u_0 u_1 e^{-T}\right] \\
u_3 &= r_3 - (1-e^{-T})^3\left[u_0^2\left(1+e^{-T}+\frac{1}{3}e^{-2T}\right)e^{-4T} + u_1^2\left(1+\frac{1}{3}e^{-T}\right)e^{-2T}\right. \\
&\qquad \left. + u_2^2\frac{1}{3} + u_0 u_1(2+e^{-T})e^{-3T} + u_0 u_2 e^{-2T} + u_1 u_2 e^{-T}\right] \\
&\ \vdots \qquad \vdots
\end{aligned}\right\} \quad (4.3\text{-}7)$$

4. 3. 2. Second Order Non-Linear Servo System

By the way similar to the procedure developed for the system of Figure 4.3-1, we can also secure the system response of the non-linear system shown in Figure 4.3-2. The important process is summarized as follows:

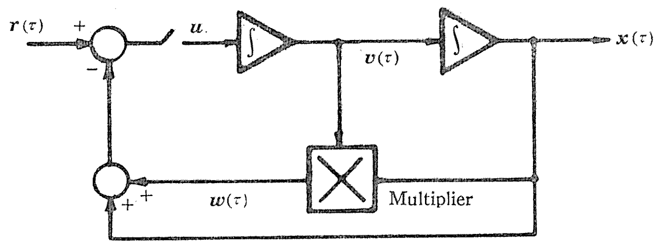

FIG. 4.3-2. A discrete feedback control system involving a multiplier.

1. Compute $v(\tau)$ by $v(\tau) = \mathscr{S}_\tau\left[\frac{1}{s}\right]u$
2. Compute $x(\tau)$ by $x(\tau) = \mathscr{S}_\tau\left[\frac{1}{s}\right] * v(\tau)$
3. Compute $w(\tau)$ by $w(\tau) = v(\tau) \times x(\tau)$
4. Compute $u(0)$ by $u = r(0) - w(0) - x(0)$

The full explanation of the computation procedure is omitted.

4. 3. 3. Consideration of Saturation Phenomena

In the calculating procedure of the system response in sequential vector space, it is very simple to take into consideration the effect of saturation.

Now, we will assume that the squaring unit in the system of Figure 4.3-1 is followed by a saturation unit. The effect of saturation should be appreciated in the calculation process of $w(\tau)$. After the evaluation of a u_j by Equation (4.3-7), the value of $w_j(\tau)$ corresponding to that u_j is checked whether it goes over the saturation limit S or not. When any w_j does not exceed the value of S, the evaluation process can be progress: However, if some v_j goes over S, the value of x_j to be used in Equation (4.3-7) must be specified to the value of x_j calculated by Equation (4.3-1) under the consideration of v_j's saturation. That is, the value u_j can be successively determined under the checking of w_j.

5. System Response of Time Variable System

The system response of time-variable systems can be also secured by the same way as the procedure which has been described in the preceding chapter.

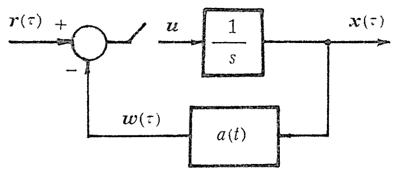

FIG. 5-1. A simple time-variable system.

Now, we will consider the simple time-variable system of Figure 5-1. The system equation can be expressed by

$$x(\tau) = \mathscr{S}_\tau\left[\frac{1}{s}\right]u = \frac{1(\tau)I}{I-S}u \tag{5-1}$$

$$w(\tau) = A(\tau)x(\tau) \tag{5-2}$$

where

$$A(\tau) = \begin{bmatrix} a_0(\tau) & 0 & 0\cdots\cdots 0 \\ 0 & a_1(\tau) & & \cdot \\ 0 & 0 & a_2(\tau) & \cdot \\ \cdot & \cdot & \cdot\ \ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \ \ \ \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot \\ 0 & 0 & 0\cdots\cdots & a_n(\tau) \end{bmatrix} \tag{5-3}$$

$$a_j(\tau) = a(t_j+\tau),\ j=0, 1, 2, \ldots, n.$$

From Eq. (5-1) and Eq. (5-2), Eq. (5-4) and Eq. (5-5) can be derived.

$$\left.\begin{aligned} x_0(\tau) &= u_0 1(\tau) \\ x_1(\tau) &= u_0 1(T+\tau) + u_1 1(\tau) \\ x_2(\tau) &= u_0 1(2T+\tau) + u_1 1(T+\tau) + u_2 1(\tau) \\ &\ \ \vdots \qquad\qquad\qquad \vdots \\ x_n(\tau) &= u_0 1(nT+\tau) + u_1 1(\overline{n-1}T+\tau) + \cdots + u_n 1(\tau) \end{aligned}\right\} \tag{5-4}$$

$$\left.\begin{aligned} w_0(\tau) &= a_0(\tau)\,x_0(\tau) \\ w_1(\tau) &= a_1(\tau)\,x_1(\tau) \\ w_2(\tau) &= a_2(\tau)\,x_2(\tau) \\ &\vdots \\ w_n(\tau) &= a_n(\tau)\,x_n(\tau) \end{aligned}\right\} \tag{5-5}$$

Then, the components of the error sequential vector u are calculated by the next relation.

$$\begin{aligned} u_j &= r_j(0) - w_j(0) \\ &= r_j(0) - a_j(0)\,x_j(0) \\ &= r_j(0) - a_j(0)\sum_{l=0}^{j} u_l \end{aligned} \tag{5-6}$$

Then

$$u_j = \frac{1}{1+a_j(0)}\left[r_j(0) - a_j(0)\sum_{l=0}^{j-1} u_l\right]. \tag{5-7}$$

Applying Eq. (5-7) to Eq. (5-4), the components of $x(\tau)$ can be secured.

$$x_j(\tau) = \sum_{j=0}^{n}\left(\frac{1}{1+a_j(0)}\right)\left[r_j(0) - a_j(0)\sum_{l=0}^{j-1} u_l\right]1(jT+\tau) \tag{5-8}$$

The above procedure can be straightly extended to the general time-variable system such as the higher order time-variable system or the non-linear multi-loop time-variable discrete system.

6. System Synthesis Through S.T.M.

The system synthesis through S.T.M. is applicable to the various system such as linear, non-linear, and time-variable systems which operate in the discrete or discrete-continuous combined mode. The fundamental features of this synthesis method are:

(1) The method is a unique one available to almost all kinds of systems.

(2) Being possible to process in quite mechanical manner, the method is very suitable to the computer processing, even if it will usually require the operation and computation of matrixes and determinants.

We have two methods of synthesis in sequential vector space. One is a direct method and the other is an asymptotic method. The former is suitable for the design of control of the well-known plant and the latter is for the extremely complex well-known or the poorly-known plant. And, the former is very convenient for the general off-line design, while the latter may be a powerful approach for the on-line or real-time designing of the sophiscated adaptive or learning control.

In this paper, only the direct synthesis method will be explained on the linear and non-linear simple systems.

6. 1. Synthesis of Discrete Linear Control System

The design of discrete linear control system can be, of course, proceeded by the conventional Z-Transform method. Here, we will briefly state on the synthesis in sequential vector space by a simple example.

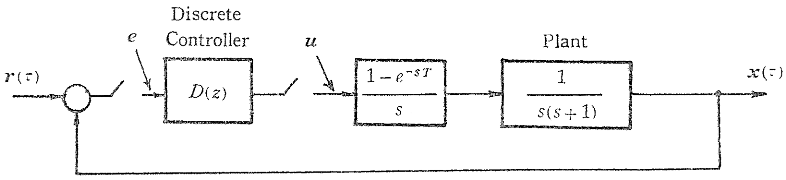

FIG. 6.1-1. A simple descrete linear control system.

We will design the system of Figure 6.1-1 so that it has the dead-beat response.

The modified S.T.M. of the plant with hold is given by Equation (6.1-1).

$$H(\tau)=\mathscr{S}_{\tau}\Big[\frac{1-e^{-sT}}{s}\cdot\frac{1}{s(s+1)}\Big]$$
$$=\frac{(-1+\tau+e^{-\tau})I+\{1+T+e^{-T}-(1+e^{-T})\tau-2e^{-\tau}\}S+\{-(1+T)e^{-T}+e^{-T}\tau+e^{-\tau}\}S^2}{I-(1+e^{-T})S+e^{-T}S^2} \tag{6.1-1}$$

Then, the modified output S.V. is obtained as follows:

$$x(\tau)=H(\tau)u$$
$$[I-(1+e^{-T})S+e^{-T}S^2]x(\tau)=[(-1+e^{-\tau})I+\{1+T+e^{-T}-(1+e^{-T})\tau-2e^{-\tau}\}S$$
$$+\{-(1+T)e^{-T}+e^{-T}\tau+e^{-\tau}\}S^2]u \tag{6.1-2}$$

$$\left.\begin{aligned}
x_0(\tau)&=u_0(-1+\tau+e^{-\tau})\\
x_1(\tau)&=u_0[T-(1-e^{-T})e^{-\tau}]+u_1(-1+\tau+e^{-\tau})\\
x_2(\tau)&=u_0[T-(1-e^{-T})e^{-(T+\tau)}]+u_1[T-(1-e^{-T})e^{-\tau}]+u_2(-1+\tau+e^{-\tau})\\
x_3(\tau)&=u_0[T-(1-e^{-T})e^{-(2T+\tau)}]+u_1[T-(1-e^{-T})e^{-(T+\tau)}]+u_2[T-(1-e^{-T})e^{-\tau}]\\
&\qquad+u_3(-1+\tau+e^{-\tau})\\
&\vdots\\
x_n(\tau)&=u_0[T-(1-e^{-T})e^{-(\overline{n-1}\,T+\tau)}]+u_1[T-(1-e^{-T})e^{-(\overline{n-2}\,T+\tau)}]+\cdots\\
&\qquad+u_n(-1+\tau+e^{-\tau})
\end{aligned}\right\} \tag{6.1-3}$$

or

$$\left.\begin{aligned}
x_0(\tau)&=-u_0+u_0\tau+u_0e^{-\tau}\\
x_1(\tau)&=u_0T-u_1+u_1\tau+[-u_0(1-e^{-T})+u_1]e^{-\tau}\\
x_2(\tau)&=(u_0+u_1)T-u_2+u_2\tau+[-(1-e^{-T})(u_0e^{-T}+u_1)+u_2]e^{-\tau}\\
x_3(\tau)&=(u_0+u_1+u_2)T-u_3+u_3\tau+[-(1-e^{-T})(u_0e^{-2T}+u_1e^{-T}+u_2)+u_3]e^{-\tau}\\
&\vdots\\
x_n(\tau)&=(u_0+u_1+\cdots+u_n)T-u_n+u_n\tau+[-(1-e^{-T})(u_0e^{-(n-1)T}+\cdots\\
&\qquad+u_{n-1})+u_n]e^{-\tau}
\end{aligned}\right\} \tag{6.1-4}$$

The $x_j(\tau)$ in Equation (6.1-4) has the three terms with $1(\tau)$, τ and $e^{-\tau}$. Then, in order that the system presents the dead-beat response for the unit step input, the three control of u_0, u_1, and u_2 must be determined so that the coefficients of each term in the equation of $x_2(\tau)$ in Equation (6.1-4) satisfy the relation of Equation (6.1-5).

$$\left.\begin{aligned} (u_0+u_1)T-u_2&=1\\ u_2&=0\\ (u_0e^{-T}+u_1)(1-e^{-T})-u_2&=0 \end{aligned}\right\} \tag{6.1-5}$$

Solving Equation (6.1-5), we obtain

$$\left.\begin{aligned} u_0&=\frac{1}{T(1-e^{-T})}\\ u_1&=\frac{-e^{-T}}{T(1-e^{-T})}\\ u_2&=0 \end{aligned}\right\} \tag{6.1-6}$$

Applying Equation (6.1-6) to the quation of $x_j(\tau)$, $j\geq 3$ in Equation (6.1-4) and considering $x_j(0)=0, j=3, 4, \ldots$, controls of u_3, u_4, ... are determined as Equation (6.1-7).

$$u_3=u_4=\cdots=0 \tag{6.1-7}$$

Thus

$$\boldsymbol{u}=\begin{bmatrix} \dfrac{1}{T(1-e^{-T})}\\ \dfrac{-e^{-T}}{T(1-e^{-T})}\\ 0\\ 0\\ \cdot\\ \cdot\\ \cdot \end{bmatrix} \tag{6.1-8}$$

Output vector components corresponding to $\boldsymbol{u}$ of Equation (6.1-8) are given by Equation (6.1-9).

$$\left.\begin{aligned} x_0(\tau)&=\frac{1}{T(1-e^{-T})}(-1+\tau+e^{-\tau})\\ x_1(\tau)&=\frac{1}{T(1-e^{-T})}(T+e^{-T}-e^{-T}\tau-e^{-\tau})\\ x_2(\tau)&=1(\tau)\\ x_3(\tau)&=x_4(\tau)=\cdots=1(\tau) \end{aligned}\right\} \tag{6.1-9}$$

The components of $\boldsymbol{x}(0)$ are obtained by putting $\tau=0$ in Equation (6.1-9).

$$\left.\begin{aligned} x_0(0) &= 0 \\ x_1(0) &= -1 + T + e^{-T} \\ x_2(0) &= x_3(0) = \cdots = 1 \end{aligned}\right\} \qquad (6.1\text{-}10)$$

Then, the S.V. of deviation $e(0)$ comes to

$$e(0) = \begin{bmatrix} r_0(0) - x_0(0) \\ r_1(0) - x_1(0) \\ r_2(0) - x_2(0) \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 2 - T - e^{-T} \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}.$$

Thus, the P.T.F. of the sampling controller is given by Equation (6.1-11).

$$D(z) = \frac{U(z)}{E(z)} = \frac{\dfrac{1}{T(1-e^{-T})}(1 - e^{-T}z^{-1})}{1 + (2 - T - e^{-T})z^{-1}} \qquad (6.1\text{-}11)$$

6. 2. Synthesis of Discrete Non-Linear Control System

In this section, we will investigate the synthesis procedure of the discrete non-linear system through an example.

Now, consider a non-linear servo system shown by Figure (6.2-1) and design the system to present a dead-beat response for the step function input.

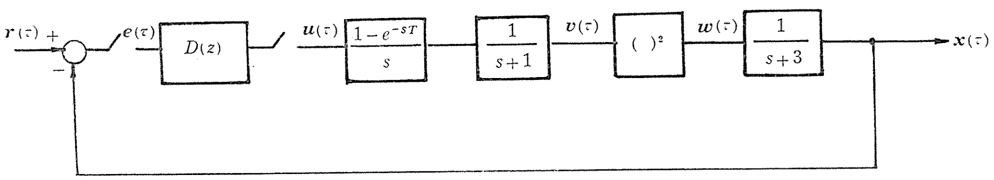

FIG. 6.2-1. A discrete servo system with squaring unit.

The expression of $x(\tau)$ is the same as the $x(\tau)$ given by Equation (4.3-3). All the components of $x(\tau)$ have four terms with $1(\tau)$, $e^{-\tau}$, $e^{-2\tau}$, and $e^{-3\tau}$, and the dead-beat condition for the unit step input can be secured by putting as Equation (6.2-1) on the $x_3(\tau)$ in Equation (4.3-3).

$$\left.\begin{aligned} &\text{constant term} = r_3 = 1 \\ &\text{term with } e^{-\tau} = 0 \\ &\text{term with } e^{-2\tau} = 0 \\ &\text{term with } e^{-3\tau} = 0 \end{aligned}\right\} \text{ for } x_3(\tau) \qquad (6.2\text{-}1)$$

However, as being clarified by the inspection of Equation (4.3-3), the second and third condition in Equation (6.2-1) are quite the same. Then the condition of Equation (6.2-1) should be substituted by Equation (6.2-2).

$$\left.\begin{array}{l}\text{constant term} = 1\\ \text{term with } e^{-\tau} = 0\\ \text{term with } e^{-3\tau} = 0\end{array}\right\} \text{ for } x_2(\tau) \tag{6.2-2}$$

Thus, we get the following simultaneous equations.

$$\left.\begin{array}{l}u_2^2 = 3 \quad \text{or} \quad u_2 = \sqrt{3}\\ L(u_0e^{-T} + u_1) = u_2\\ -M(u_0^2e^{-3T} + u_1^2) - \frac{1}{3}u_2^2 - Nu_0u_1e^{-T} + L(u_0e^{-T} + u_1)u_2 = 0\end{array}\right\} \tag{6.2-3}$$

where

$$\left.\begin{array}{l}L = 1 - e^{-T}\\ M = \frac{1}{3}(1 - e^{-T})^2(2 + e^{-T}) = \frac{2}{3} - e^{-T} + \frac{1}{3}e^{-3T}\\ N = (1 - e^{-T})^2(1 + e^{-T}) = 1 - e^{-T} - e^{-2T} + e^{-3T}\end{array}\right\} \tag{6.2-4}$$

Equation (6.2-3) reduces to Equation (6.2-5).

$$\left.\begin{array}{l}L(u_0e^{-T} + u_1) = \sqrt{3}\\ M(u_0^2e^{-3T} + u_1^2) + Ne^{-T}u_0u_1 - 2 = 0\end{array}\right\} \tag{6.2-5}$$

Solving Equation (6.2-5), u_0 and u_1 can be secured.

$$\left.\begin{array}{l}u_0 = -\dfrac{\sqrt{3}}{2e^{-T}(1-e^{-2T})}\left[1 \pm \sqrt{1 + 4e^{-T}\left(\dfrac{1+e^{-T}}{1-e^{-T}}\right)}\right]\\ u_1 = \dfrac{\sqrt{3}}{1-e^{-T}}\left[1 + \dfrac{1}{2(1+e^{-T})}\left\{1 \pm \sqrt{1 + 4e^{-T}\left(\dfrac{1+e^{-T}}{1-e^{-T}}\right)}\right\}\right]\end{array}\right\} \tag{6.2-6}$$

Here, it is noticeable that we have acquired the two kinds of control modes shown by Equation (6.2-6).

The succeeding controls u_j, $j=3, 4, \ldots$ are obtained as Equation (6.2-7), by solving the general condition of complete settling of Equation (6.2-8) using the above secured u_0 and u_1.

$$u_3 = u_4 = \cdots = \sqrt{3} \tag{6.2-7}$$

$$\left.\begin{array}{l}u_j^2 = 3, \qquad j \geq 3\\ L(u_0e^{-(n-1)T} + u_1e^{-(n-2)T} + \cdots + u_{n-1}) = u_j\\ M(u_0^2e^{-3(n-1)T} + u_1^2e^{-3(n-2)T} + \cdots + u_n^2)\\ + N(u_0u_1e^{-(3n-5)T} + u_0u_2e^{-(3n-7)T} + \cdots + u_0u_{n-1}e^{-(n-1)T}\\ \qquad + u_1u_2e^{-(3n-8)T} + \cdots + u_1u_{n-1}e^{-(n-2)T}\\ \qquad \cdots\cdots\cdots\cdots\\ \qquad + u_{n-2}u_{n-1}e^{-T}) - 2 = 0\end{array}\right\} \tag{6.2-8}$$

The results of Equation (6.2-7) can also be derived by the mathematical reduction method on Equation (6.2-8).

The output $x_j(0)$ is calculated by applying the u_j of Equation (6.2-6) and Equation (6.2-7) to Equation (4.3-6). That is,

$$\left.\begin{aligned} x_0(0) &= 0 \\ x_1(0) &= \frac{1}{3}(1-e^{-T})^3 u_0^2 \\ x_2(0) &= x_3(0) = \cdots = 1 \end{aligned}\right\} \qquad (6.2\text{-}9)$$

and the $e_j(0)$ are derived as follows:

$$\left.\begin{aligned} e_0(0) &= r_0(0) - x_0(0) = 1 \\ e_1(0) &= 1 - \frac{1}{3}(1-e^{-T})^3 u_0^2 \\ e_2(0) &= e_3(0) = \cdots = 0 \end{aligned}\right\}. \qquad (6.2\text{-}10)$$

Consequently, the sampling controller can be designed as Equation (6.2-11).

$$\left.\begin{aligned} D(z) &= \frac{u_0 + u_1 z^{-1} + \sqrt{3}^{-2}(1-z^{-1})^{-1}}{1+\left[1-\frac{1}{3}(1-e^{-T})^3 u_0^2\right]z^{-1}} \\ &= \frac{u_0 - (u_0 - u_2)z^{-1} + (\sqrt{3} - u_1)z^{-2}}{1 + \frac{1}{3}(1-e^{-3T})u_0^2 z^{-1} - \left\{1 - \frac{1}{3}(1-e^{-3T})u_0^2\right\}z^{-2}} \end{aligned}\right\} \qquad (6.2\text{-}11)$$

Where the u_0 and u_1 in Equation (6.2-11) are evaluated by Equation (6.2-6).

Here, we must pay some attention on the following subjects.

(1) In this example, the complete settling condition of Equation (6.2-1) is simplified to the form of Equation (6.2-2). However, for the general non-linear servo system this deduction will be impossible.

(2) The dead-beat condition of Equation (6.2-3) has been solved very simply, even though it is a quadratic simultaneous equation. However, for the general case, the condition comes to the simultaneous non-linear equation, and its solution will not be performed in a so simple way. The asymptotic method which will be investigated in the other papers will be an available approach to that case but has been omitted in this paper.

7. Conclusion

In this paper, the author has tried to make a new system description from the view point of sequential vector space and to apply it to the calculation of system responce of linear and nonlinear discrete system. Several interesting principles and techniques have been developed. It has been clarified that regardless of variety and complxity of system construction, the system responce can be always secured and the system synthesis can be also performed by an unique method with S.T.M. Especially it is a valuable result that the computation of transient responce and synthesis of optimal control (dead-beat responce) for the time-variable and/or nonlinear system can be performed in a quite mechanical way. The proposed design procedure of discrete controller for nonlinear discrete control system is quite a unique one and supply a very useful argorithm for the design of computer control.

This work is a minor part of research which was developed at Battelle Memorial Institute, Columbus Ohio, 1966~1967. And the author wishes to devote a deep acknowledgement to Dr. J. T. Tou, Dr. J. D. Hill and other staff members for their suggestions and continuous encouragement.

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Appendix

Computation Table of Convolution Integral (I)

$$(g_i * f_0)_\tau=\int_{-0}^{\tau-0} g(iT+\nu)f(\tau-\nu)d\nu$$

$$(g_i * f_k)_T=\int_{-0}^{T-0} g(iT+\nu)f(kT+\tau-\nu)d\nu$$

$f(t)\rightarrow$ / $g(t)\downarrow$		$1(t)$		t		e^{-bt}		$\sin\omega t$	
		$f_0(\tau)$	$f_k(\tau)$	$f_0(\tau)$	$f_k(\tau)$	$f_0(\tau)$	$f_k(\tau)$	$f_0(\tau)$	$f_k(\tau)$
$1(t)$	$g_i(\tau)$	τ *	T	$\frac{\tau^2}{2}$ *	$(k-1/2)T^2+T\tau$	$\frac{1}{b}(1-e^{-b\tau})$ *	$\frac{1}{b}e^{-b(\overline{k-1}T+\tau)}\times(1-e^{-bT})$	$-\frac{1}{\omega}(1-\cos\omega\tau)$	$\frac{1}{\omega}[\cos\omega(\overline{k-1}T+\tau)-\cos\omega(kT+\tau)]$
t	$g_i(\tau)$	$iT\tau+\frac{\tau^2}{2}$ *	$iT^2+\frac{T^2}{2}$	$\frac{iT}{2}\tau^2+\frac{\tau^3}{6}$ *	$\left(ik-\frac{i}{2}+\frac{k}{2}-\frac{1}{3}\right)T^3+\left(i+\frac{1}{2}\right)T^2\tau$	$\frac{1}{b}\left[\left(iT-\frac{1}{b}\right)(1-e^{-b\tau})+\tau\right]$	$\frac{1}{b}e^{-b(\overline{k-1}T+\tau)}\times\left[\left(iT-\frac{1}{b}\right)(1-e^{-bT})+T\right]$	$\frac{1}{\omega}\left[iT(1-\cos\omega\tau)-\frac{1}{\omega}\sin\omega\tau+\tau\right]$	$\frac{1}{\omega}\left[(i+1)T\cos\omega(\overline{k-1}T+\tau)-iT\cos\omega(kT+\tau)+\frac{1}{\omega}\sin(\overline{k-1}T+\tau)-\frac{1}{\omega}\sin\omega(kT+\tau)\right]$
t^2	$g_i(\tau)$	$i^2T^2\tau+iT\tau^2+\frac{\tau^3}{3}$ *	$\left(i^2+i+\frac{1}{3}\right)T^3$	$\frac{1}{2}i^2T^2\tau^2+\frac{1}{3}iT\tau^3+\frac{1}{12}\tau^4$ *	$i^2kT^3\tau+\left(\frac{i^2}{2}+ik\right)T^2\tau^2+\frac{1}{3}(i+k)T\tau^3+\frac{1}{12}\tau^4$	$\frac{1}{b}\left[\left(i^2T^2-\frac{2iT}{b}+\frac{1}{b^2}\right)\times(1-e^{-b\tau})+2\left(iT-\frac{1}{b}\right)\tau+\tau^2\right]$	$\frac{1}{b}e^{-(\overline{k-1}T+\tau)}\times\left[\left(i^2T^2-\frac{2iT}{b}+\frac{1}{b^2}\right)\times(1-e^{-bT})+2\left(iT-\frac{1}{b}\right)T+T^2\right]$	$\frac{1}{\omega}\left[\left(i^2T^2-\frac{2}{\omega^2}\right)\times(1-\cos\omega\tau)-\frac{1}{\omega}2iT\sin\omega\tau+2iT\tau+\tau^2\right]$	$\frac{1}{\omega}\left\{(i^2+2i)T^2-\frac{2}{\omega^2}\right\}\times\cos\omega(\overline{k-1}T+\tau)+\frac{1}{\omega}\left(\frac{2}{\omega^2}-i^2T^2\right)\cos(kT+\tau)+\frac{1}{\omega^2}(i+1)2T\sin(\overline{k-1}T+\tau)-\frac{2iT}{\omega^2}\sin\omega(kT+\tau)$
e^{-at}	$g_i(\tau)$	$\frac{1}{a}e^{-iaT}\times(1-e^{-a\tau})$	$\frac{1}{a}e^{-iaT}\times(1-e^{-aT})$	$\frac{1}{a}e^{-iaT}\times\left(\tau-\frac{1-e^{-a\tau}}{a}\right)$	$\frac{1}{a}e^{-iaT}\times\left[\left(\overline{k-1}T+\tau-\frac{1}{a}\right)\times(1-e^{-aT})+T\right]$	$\left(\frac{1}{b-a}\right)e^{-aiT}(e^{-a\tau}-e^{-b\tau})$	$\left(\frac{1}{b-a}\right)e^{-aiT}\times e^{-b(\overline{k-1}T+\tau)}\times(e^{-aT}-e^{-bT})$	$\frac{e^{-aiT}}{a^2+\omega^2}\times[\omega e^{-a\tau}-\omega\cos\omega\tau-a\sin\omega\tau]$	$\frac{e^{-aiT}}{a^2+\omega^2}\times[e^{-aT}\{\omega\cos\omega(\overline{k-1}T+\tau)-a\sin(\overline{k-1}T+\tau)\}-\omega\cos\omega(kT+\tau)+a\sin\omega(kT+\tau)]$

The columns marked by * show the computed values of $(g_i * f_0)_\tau$, and the other columns show those of $(g_i * f_k)_T$, $i=0,1,2,\ldots$, $k=1,2,\ldots$.

Computation Table of Convolution Integral (II)

$$(g_0 * f_i)_\tau = \int_{-0}^{T-0} g(\tau-\nu) f(iT+\nu)\, d\nu$$

$$(g_k * f_i)_T = \int_{-0}^{T-0} g(kT+\tau-\nu) f(iT+\nu)\, d\nu$$

$g(t) \downarrow$ \ $f(t) \rightarrow$		$1(t)$	t	e^{-bt}	$\sin \omega t$
		$f_i(\tau)$	$f_i(\tau)$	$f_i(\tau)$	$f_i(\tau)$
$1(t)$	$g_0(\tau)$	τ *	$iT\tau+\frac{\tau^2}{2}$ *	$\frac{1}{b}e^{-biT}(1-e^{-b\tau})$ *	$\frac{1}{\omega}[-\cos\omega(iT+\tau)+\cos\omega iT]$
	$g_k(\tau)$	T	$\left(i+\frac{1}{2}\right)T^2$	$\frac{1}{b}e^{-biT}(1-e^{-bT})$	$\frac{1}{\omega}[-\cos\omega(iT+\tau)+\cos\omega iT]$
t	$g_0(\tau)$	$\frac{\tau^2}{2}$ *	$\frac{1}{2}iT\tau^2+\frac{1}{6}\tau^3$ *	$\frac{1}{b}e^{-biT}\left(\tau-\frac{1-e^{-b\tau}}{b}\right)$ *	$\frac{1}{\omega}\left[-\frac{1}{\omega}\sin\omega(iT+\tau)+\tau\cos\omega iT + \frac{1}{\omega}\sin\omega iT\right]$
	$g_k(\tau)$	$\left(k-\frac{1}{2}\right)T^2+T\tau$	$\left(ik+\frac{k-i}{2}-\frac{1}{3}\right)T^2+\left(i+\frac{1}{2}\right)T^2\tau$	$\frac{1}{b}e^{-biT}\left[\left(\overline{k-1}\,T+\tau-\frac{1}{b}\right)(1-e^{-bT}) + T\right]$	$\frac{1}{\omega}\left[-(\overline{k-1}\,T+\tau)\cos\omega(i+1)T + (kT+\tau)\cos\omega iT-\frac{1}{\omega}\sin\omega(i+1)T + \frac{1}{\omega}\sin\omega iT\right]$
t^2	$g_0(\tau)$	$\frac{\tau^3}{3}$ *	$\frac{1}{3}iT\tau^3+\frac{1}{12}\tau^4$ *	$\frac{1}{b}e^{-biT}\left[\frac{2(1-e^{-b\tau})}{b^2}-\frac{2\tau}{b}+\tau^2\right]$	$\frac{1}{\omega}\left[\frac{2}{\omega^2}\cos\omega(iT+\tau)-\left(\tau^2+\frac{2}{\omega^2}\right)\cos\omega iT + \frac{2\tau}{\omega}\sin\omega iT\right]$
	$g_k(\tau)$	$\left(k^2-k+\frac{1}{3}\right)T^3 +(2k-1)T^2\tau+T\tau^2$	$\left(ik^2-ik+\frac{i-2k}{3}+\frac{k^2}{2}+\frac{1}{4}\right)T^4 +\left(2ik-i-k-\frac{2}{3}\right)T^3\tau+\left(i+\frac{1}{2}\right)T^2\tau^2$	$\frac{1}{b}e^{-biT}\left[\left\{\left(kT+\tau-\frac{1}{b}\right)^2+\left(T+\frac{1}{b}\right)^2 -2(kT+\tau)T\right\}(1-e^{-bT})+(2k-1)T^2 +2\left(\tau-\frac{1}{b}\right)T\right]$	$-\frac{1}{\omega}\left[(\overline{k-1}\,T+\tau)^2\cos\omega(i+1)T+\left\{(kT +\tau)^2-\frac{2}{\omega^2}\right\}\cos\omega iT-\frac{2}{\omega}(\overline{k-1}\,T+\tau) \sin\omega(i+1)T+\frac{2(kT+\tau)}{\omega}\sin\omega iT\right]$
e^{-at}	$g_0(\tau)$	$\frac{1}{a}(1-e^{-a\tau})$ *	$\frac{1}{a}\left[\left(iT-\frac{1}{a}\right)(1-e^{-a\tau})+\tau\right]$ *	$\left(\frac{1}{b-a}\right)e^{-biT}(e^{-a\tau}-e^{-b\tau})$	$\frac{1}{a^2+\omega^2}[-\omega\cos\omega(iT+\tau)+a\sin\omega(iT+\tau) +e^{-a\tau}(\omega\cos\omega iT-a\sin\omega iT)]$
	$g_k(\tau)$	$\frac{1}{a}e^{-(\overline{k-1}\,T+\tau)} \times(1-e^{-a\tau})$	$\frac{1}{a}e^{-(\overline{k-1}\,T+\tau)}\left[\left(iT-\frac{1}{a}\right)(1-e^{-aT})+T\right]$	$\left(\frac{1}{b-a}\right)e^{-(a(\overline{k-1}\,T+\tau)+biT)}(e^{-aT}-e^{-bT})$	$\frac{1}{a^2+\omega^2}e^{-a(\overline{k-1}\,T+\tau)}[-\omega\cos\omega(i+1)T +a\sin(i+1)T +e^{-aT}(\omega\cos\omega iT-a\sin\omega iT)$

The columns marked by * show the computed values of $(g_0 * f_i)_\tau$ and the other columns show those of $(g_k * f_i)_T$, $i=0, 1, 2, \ldots, k=1, 2, \ldots$.