AN OPTIMAL FEED BACK COMPOSITION FOR THE SECOND ORDER SYSTEM WITH THE PERFORMANCE INDEX CONSTRAINED BY THE DYNAMIC SENSITIVITY

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1. Introduction

The optimal control problems with performance index constrained by the dynamic sensitivity can be treated by adding new state variables to the original ones. That is, the dynamic sensitivity function $\dfrac{dx}{dw} = \left(\dfrac{\partial x_i}{\partial w_j}\right)\left(\begin{matrix} i=1, \dots, n \\ j=1, \dots, p \end{matrix}\right)$ can be treated as new state variables. If notation ${}_iy(t) \triangleq \dfrac{\partial x}{\partial w_i}$ $(i=1, \dots, p)$ is used, the expanded system is as follows: the state transition equations are

$$\begin{cases} \dfrac{dx}{dt} = f(t, x, w, u) \\ \dfrac{d_iy}{dt} = \dfrac{\partial f}{\partial x}\,{}_iy + \dfrac{\partial f}{\partial w_i} \qquad i=1, 2, \dots, p \end{cases} \tag{1}$$

where x is an n-dimentional state vector, u is an m-dimentional manipulating signal vector, w is a p-dimentional plant parameter vector and f is an n-dimentional function, the performance index functional is

$$J = g(x, u, {}_1y, {}_2y, \dots, {}_py), \tag{2}$$

the boundary conditions are, for example,

$$\begin{cases} x(0) = x_0, \ {}_iy(0) = 0 \qquad (i=1, \dots, p) \\ x(T) \text{ is free or given } {}_iy(T) \quad (i=1, \dots, p) \text{ is free.} \end{cases} \tag{3}$$

As one could see in the above statement, the systems order—which means number of the state variables—increases rapidly. However any algorithm that can give the optimal solution automatically without human heuristic ability has not as yet been found for higher order systems excepting few special problems.

The above consideration guided us to use the classical optimization technique, that is, Wiener-Hopf integral equation method[1)2)], in order to investigate what practically significant results are produced by the sensitivity consideration added to the ordinary performance index. However since this method is only applicable to the systems having one input and one output, we cannot utilize the information of the inner state variables whose introduction may be said to be one of the most important development in modern control engineering (science).

2. Closed loop system and performance index

Fig. 1 shows the closed loop system block-diagram where $G(s, p)$ is the plant transfer function in which $p=(p_1, p_2, \dots, p_n)$ is the parameter vector as in the previous section, $G_c(s)$ is the controller transfer function which must be determined optimally, $C(s)$, $u(s)$ and $e(s)$ are controlled signal, manipulating signal and error (actuating) signal, respectively.

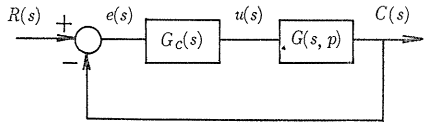

FIG. 1. Closed loop system.

Let us denote the dynamic sensitivity with respect to one parameter p by $S_p(t)$, then the ordinary quadratic form integral type performance index constrained by $S_p(t)$ is as folloews:

$$J=\int_0^{\infty}[e^2(t)+k^2u^2(t)+\sigma^2\{S_p(t)\}^2]dt \tag{4}$$

where k and σ are constant weighting factors. By using Parseval theorem, transforming eq. (4) into frequency domain, we shall obtain

$$J=\frac{1}{2\pi j}\int_{-j\infty}^{+j\infty}[e(s)e(-s)+k^2u(s)u(-s)+\sigma^2\{S_p(s)\,S_p(-s)\}]ds \tag{5}$$

where $s=jw$.

When we will treat the case of many parameters variation, if we denote their dynamic sensitivities by $S_{p_j}(t)$, we can expand eq. (5) to the case of many parameters variation as follows:

$$J=\frac{1}{2\pi j}\int_{-j\infty}^{+j\infty}[e(s)e(-s)+k^2u(s)u(-s)+\sum_{j=1}^{n}\sigma_j^2\{S_{p_j}(s)S_{p_j}(-s)\}]ds \tag{6}$$

In what follows, since we can always expand one parameter case as shown above, we shall show the case of one parameter variation alone for simplicity.

3. Sensitivity function

Many types of sensitivity function can be defined. However it is a problem studied in future to find what kind of sensitivity function should be used corresponding to the problem given. However since it is expected that we shall obtain qualitatively similar results in spite of the form of the sensitivity function to be used, we only used Bode type sensitivity function which results in an easy treatment in what follows.

In Fig. 1, let the closed loop transfer functions be $T(s)$, then the Bode Type sensitivity function can be written in the Laplace transformed form as follows:

$$S_p^B(s)=\frac{\partial \ln T(s)}{\partial \ln p}. \tag{7}$$

Here

$$T(s)=\frac{G_c(s)G(s)}{1+G_c(s)G(s)}=\frac{C(s)}{R(s)}. \tag{8}$$

From eqs. (7) and (8), we obtain

$$S_p^B(s)=\frac{\partial \ln T}{\partial T}\frac{\partial T}{\partial p}\frac{\partial p}{\partial \ln p}=\frac{p}{T(s)}\frac{\partial T(s)}{\partial p}=p\Big[\frac{1}{G(s)}-\frac{u(s)}{R(s)}\Big]\frac{\partial G(s)}{\partial p}$$
$$=p\Big[\frac{1}{G(s)}-\frac{u(s)}{R(s)}\Big]G_p(s),\quad G_p(s)=\frac{\partial G(s)}{\partial p}. \tag{9}$$

Eq. (9) shows that the Bode sensit.vity function contains $u(s)$ linearly. This results in convenient treatment of eq. (5) because its integrand becomes quadratic with respect to $u(s)$.

4. Optimal control signal

Introducing eq. (9) into eq. (5), we obtain the performance index in the form,

$$J=\frac{1}{2\pi j}\int_{-j\infty}^{+j\infty}\Big[(R-Gu)(\overline{R}-\overline{Gu})+k^2u\overline{u}+\sigma^2p^2\Big(\frac{1}{G}-\frac{u}{R}\Big)\Big(\frac{1}{\overline{G}}-\frac{\overline{u}}{\overline{R}}\Big)G_p\overline{G}_p\Big]ds. \tag{10}$$

Hereafter all functions of s will be written in the above form in which the argument s is dropped, and the upper bar on each function indicates its complex conjugate.

We shall denote the optimal manipulating signal as u^*, its arbitrary variation as u_1, and an arbitrary small real number as ε, that is, $u=u^*+\varepsilon u_1$, $(\overline{u}^*+\varepsilon\overline{u}_1)$ and intoduce them into eq. (10). By means of the usual development of the classical calculus of variation, let the first variation be equal to zero, then we shall obtain the next relation,

$$\int_{-j\infty}^{+j\infty}\Big[\Big(G\overline{G}+k^2+\sigma^2p^2\frac{G_p\overline{G}_p}{R\overline{R}}\Big)u^*-\Big(R\overline{G}+\sigma^2p^2\frac{G_p\overline{G}_p}{G\overline{R}}\Big)\Big]u_1ds=0. \tag{11}$$

As u^* and u are stable manipulating signals, their poles must exist in the left half plane (LHP) of the complex plane. Thus also u_1 has its poles in LHP. Therefore in order to establish eq. (11), the integrand function in [ ] of eq. (11) must be regular in the right half plane (RHP). So let a certain regular function in RHP be $\overline{X}$, we shall obtain the next relation from eq. (11) and call it Wiener-Hopf equation:

$$\Big(k^2+G\overline{G}+\sigma^2p^2\frac{G_p\overline{G}_p}{R\overline{R}}\Big)u^*-\Big(R\overline{G}+\sigma^2p^2\frac{G_p\overline{G}_p}{G\overline{R}}\Big)=\overline{X}. \tag{12}$$

Eq. (12) usually can be solved by using so-called spectral factorization method and the partial fraction decomposition. We will define a function $Y(s)$ as follows:

$$\left\{\begin{array}{l}\Big\{k^2+G\overline{G}+\sigma^2p^2\dfrac{G_p\overline{G}_p}{R\overline{R}}\Big\}^+\triangleq Y(s)\\ k^2+G\overline{G}+\sigma^2p^2\dfrac{G_p\overline{G}_p}{R\overline{R}}=Y(s)Y(-s)\end{array}\right. \tag{13}$$

where $\{\cdot\}^+$ indicates all the factors of zeros and poles of $\cdot$ in LHP and $\{\cdot\}^-$ shows the factors in RHP.

By replacing eq. (12) by eq. (13), we have

$$Y\overline{Y}u^* - Z = \overline{X}, \tag{14}$$

where

$$Z(s) \triangleq R\overline{G} + \sigma^2 p^2 \frac{G_p\overline{G}_p}{G\overline{R}}. \tag{15}$$

From eq. (14), we obtain

$$Yu^* - \frac{Z}{\overline{Y}} = \frac{\overline{X}}{\overline{Y}}. \tag{16}$$

If $Z/\overline{Y}$ is decomposed into partial fractions in such manner that the terms having poles in LHP or RHP are collected separately, that is, from eq. (16) we have

$$Yu^* - \left[\frac{Z}{\overline{Y}}\right]_+ = \frac{\overline{X}}{\overline{Y}} + \left[\frac{Z}{\overline{Y}}\right]_-. \tag{17}$$

Here

$$\left[\frac{Z}{\overline{Y}}\right]_+ = \sum_i \frac{A_i}{S-\alpha_i}, \qquad \alpha_i = \text{a pole in LHP}, \tag{18}$$

$$\left[\frac{Z}{\overline{Y}}\right]_- = \sum_j \frac{B_j}{S-\beta_i}, \qquad \beta_j = \text{a pole in RHP}. \tag{19}$$

From eq. (17) we have the optimal manipulating signal as follows:

$$u^* = \frac{1}{Y}\left[\frac{Z}{\overline{Y}}\right]_+. \tag{20}$$

Introducing eqs. (13) and (15) into eq. (20), u^* can be expressed precisely as

$$u^* = \frac{1}{\left(k^2 + G\overline{G} + \sigma^2 p^2 \frac{G_p\overline{G}_p}{R\overline{R}}\right)^+} \left[\frac{R\overline{G} + \sigma^2 p^2 \frac{G_p\overline{G}_p}{G\overline{G}}}{\left(k^2 + G\overline{G} + \sigma^2 p^2 \frac{G_p\overline{G}_p}{R\overline{R}}\right)^-}\right]_+. \tag{21}$$

5. Application to the second order system

We shall assume the transfer function of the second order system as follows:

$$G(s) = \frac{1}{S^2+2\zeta S+1} \ (\zeta > 0). \tag{22}$$

Thus the dynamic sensitivity function with respect to ζ is

$$G_\zeta(s) = \frac{-2S}{(S^2+2\zeta S+1)^2}. \tag{23}$$

Here if we will make the weighting factor k^2 in the performance index to be equal to 1 and the reference input to be a step function, that is, $k^2=1$, $R(s)=1/s$, we have, corresponding to each term in eq. (21),

$$\left(k^2+G\bar{G}+\sigma^2\zeta^2\frac{G_\zeta\bar{G}_\zeta}{R\bar{R}}\right)^+$$

$$=\frac{\{(S^2+2\zeta S+1)^2(S^2-2\zeta S+1)^2+(S^2+2\zeta S+1)(S^2-2\zeta S+1)+4\sigma^2\zeta^2S^4\}^+}{(S^2+2\zeta S+1)^2}, \tag{24}$$

$$\left(k^2+G\bar{G}+\sigma^2\zeta^2\frac{G_\zeta\bar{G}_\zeta}{R\bar{R}}\right)^- = \frac{\{\cdot\}^-}{(S^2-2\zeta S+1)^2}=\bar{Y}, \tag{25}$$

where $\{\cdot\}$ is equal to $\{\cdot\}$ in the numerator of eq. (24).

And moreover,

$$\left[\frac{R\bar{G}+\sigma^2\zeta^2G_\zeta\bar{G}_\zeta/G\bar{R}}{\bar{Y}}\right]$$

$$=\frac{\{(S^2+2\zeta S+1)(S^2-2\zeta S+1)+4\sigma^2\zeta^2S^4\}}{S(S^2+2\zeta S+1)\{(S^2+2\zeta S+1)^2(S^2-2\zeta S+1)^2+(S^2+2\zeta S+1)(S^2-2\zeta S+1)+4\sigma^2\zeta^2S^4\}^-} \tag{26}$$

Thus

$$\left[\frac{R\bar{G}+\sigma^2\zeta^2G_\zeta\bar{G}_\zeta/G\bar{R}}{\bar{Y}}\right]_+ = \frac{r_1}{S}+\frac{r_2}{S+\alpha}+\frac{r_3}{S+\beta} \tag{27}$$

where $(S+\alpha)(S+\beta)=S^2+2\zeta S+1$ and r_1, r_2 and r_3 are the residues at $S=0$, $-\alpha$ and $-\beta$.

The optimal closed loop transfer function T^* can be expressed by the optimal manipulating signal u^* as

$$T^*=\frac{G}{R}u^*. \tag{28}$$

Thus from eqs. (21), (24), (26), (27) and (28), we shall obtain

$$T^*=\frac{(r_1+r_2+r_3)S^2+(2\zeta r_1+\beta r_2+\alpha r_3)S+r_1}{[(S^2+2\zeta S+1)^2(S^2-2\zeta S+1)^2+(S^2+2\zeta S+1)(S^2-2\zeta S+1)+4\sigma^2\zeta^2S^4]^+} \tag{29}$$

and as the optimal controller, we have

$$G_c^*(S)=$$

$$\frac{T^*}{G(1-T^*)}=\frac{(S^2+2\zeta S+1)\{(r_1+r_2+r_3)S^2+(2\zeta r_1+\beta r_2+\alpha r_3)S+r_1\}}{\{(S^2+2\zeta S+1)^2(S^2-2\zeta S+1)^2+(S^2+2\zeta S+1)(S^2-2\zeta S+1)+4\sigma^2\zeta^2S^4\}^+ - \{(r_1+r_2+r_3)S^2+(2\zeta r_1+\beta r_2+\alpha r_3)S+r_1\}}. \tag{30}$$

6. Zeros and poles of T^*

The zeros are the roots of the numerator in eq. (29):

$$\text{the numerator}=(r_1+r_2+r_3)S^2+(2\zeta r_1+\beta r_2+\alpha r_3)S+r_1. \tag{31}$$

The residues r_1, r_2 and r_3 are given respectively as follows:

$$\begin{cases} r_1 = 1/(\cdot)^-_{s=0} \\ r_2 = 4\,\sigma^2\zeta^2\alpha^3/(\alpha-\beta)(\cdot)^-_{s=-\alpha} \\ r_3 = -4\,\sigma^2\zeta^2\beta^3/(\alpha-\beta)(\cdot)^-_{s=-\beta} \end{cases} \tag{32}$$

where $(\cdot)^-$ shows the numerator in eq. (25).

Now we will investigate the case where $\sigma \to \infty$. In this case, if we take an arbitrary small positive number ε, we can let $(\cdot)^-$ be approximately as follows:

$$(\cdot)^- \approx (S-\varepsilon)^2 R_\infty. \tag{33}$$

Here R_∞ indicates the product of factors having infinite large roots. From eq. (32)

$$\begin{cases} \beta r_2 + \alpha r_3 = 0 \\ \dfrac{r_2 + r_3}{r_1} = 4\,\sigma^2\zeta^2\varepsilon^2. \end{cases} \tag{34}$$

While when $\sigma \to \infty$, $S=\varepsilon$ fulfils the eq. $(\cdot)=0$. This shows that as $\sigma \to \infty$, $\varepsilon^2 \to 1/2\,\sigma\zeta$ and therefore $r_2+r_3/r_1 \to \infty$, that is $r_1 \to 0$. Thus as $\sigma \to \infty$, both the constant term and the first degree term in eq. (31) vanish and the two zeros approch to the origin of the complex plane.

The poles are 4 roots in LHP of the denominator in eq. (29). The characteristic equation:

$$\frac{4\,\sigma^2\zeta^2 S^4}{(S^2+2\,\zeta S+1)^2(S^2-2\,\zeta S+1)^2+(S^2+2\,\zeta S+1)(S^2-2\,\zeta S+1)} = -1 \tag{35}$$

must hold when $\sigma \to \infty$. This means that the two poles approch to the origin of the complex plane, and the other two poles approch to infinity as $\sigma \to \infty$.

In eq. (35) the difference between the degree of the denominator and the numerator is 4, therefore the asymptotes of the infinite poles have angles of $\pm 45°$ and $\pm 135°$ between the abscissa.

7. Numerical evaluation of the poles

In order to obtain the 4 poles stated above, we must solve the next 8 degrees algebraic equation and pick up its 4 roots in LHP,

$$(S^2+2\,\zeta S+1)^2(S^2-2\,\zeta S+1)^2+(S^2+2\,\zeta S+1)(S^2-2\,\zeta S+1)+4\,\sigma^2\zeta^2 S^4 = 0. \tag{36}$$

Since eq. (36) is a 4 degrees equation with respect to S^2, we can obtain its roots numerically by using Newton-Raphson iteration method and Ferrari method.

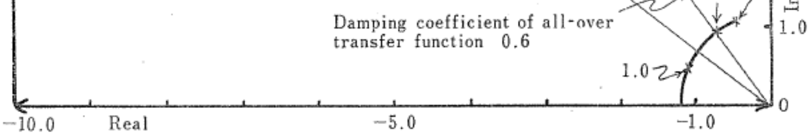

FIG. 2. $\sigma=0$ and the plant damping coefficient ζ changes.

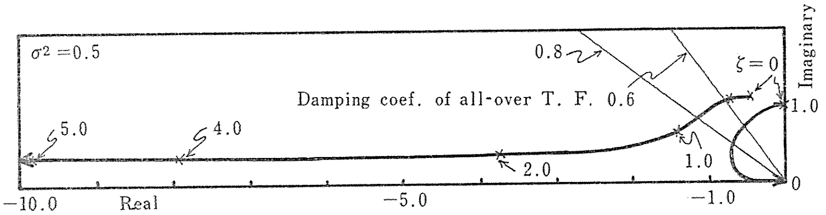

FIG. 3. $\sigma^2 = 0.5$ and ζ changes.

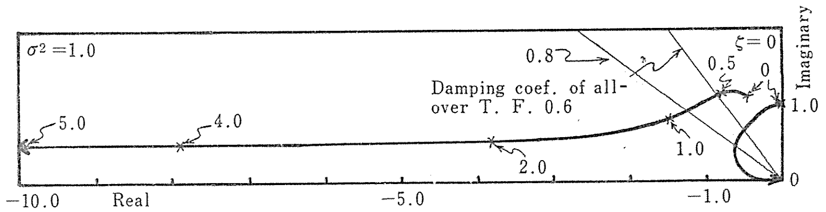

FIG. 4. $\sigma^2 = 1.0$ and ζ changes.

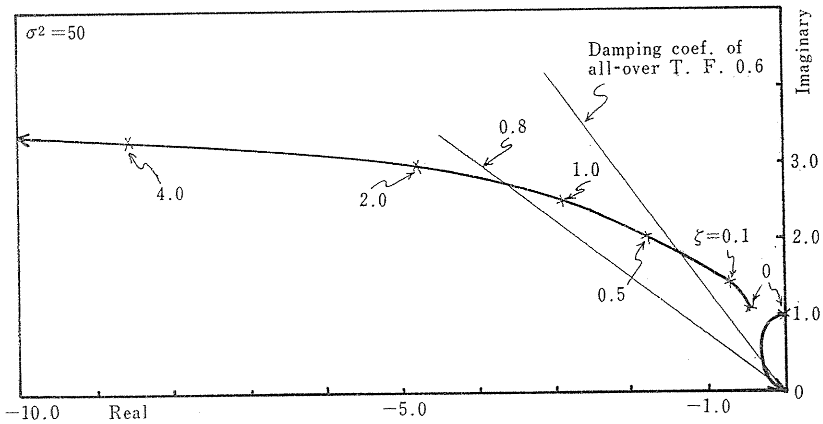

FIG. 5. $\sigma^2 = 50$ and ζ changes.

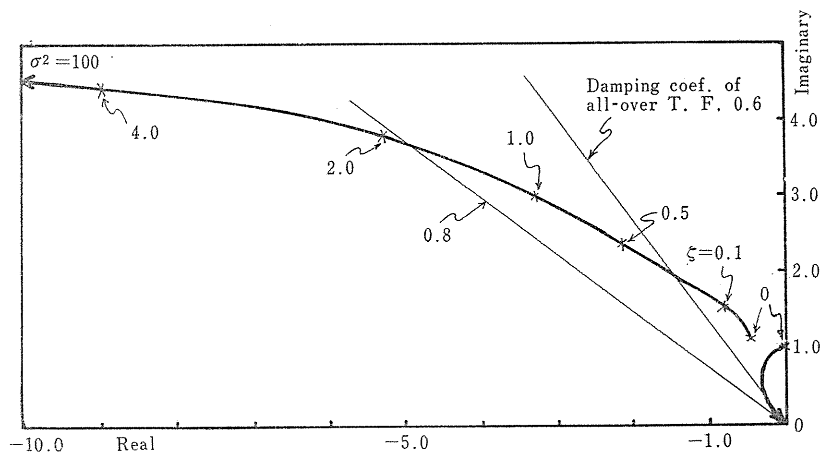

FIG. 6. $\sigma^2 = 100$ and ζ changes.

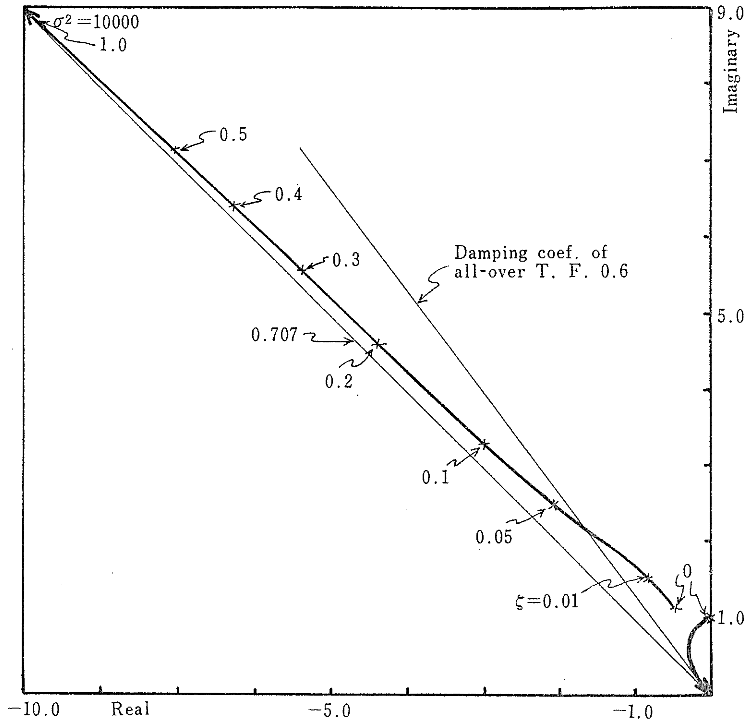

FIG. 7. $\sigma^2=10000$ and ζ changes.

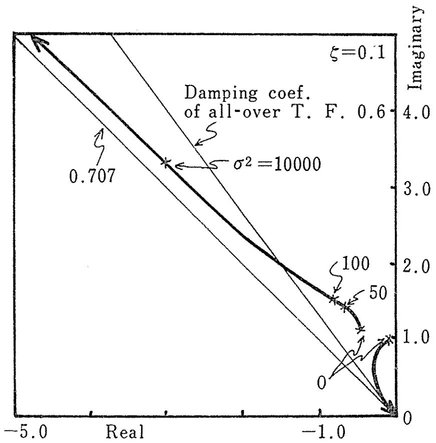

FIG. 8. The plant damping coefficient $\zeta=0.1$ and σ^2 changes.

In from Fig. 2 to Fig. 10 the obtained root loci of several cases are shown. Each figure only shows the behavior of the locus in the second quadrant of the complex plane because of the existence of symmetric nature to the others. Fig. 2 ~ 7 are the cases where ζ changes under constant σ. On the contrary, Fig. 8 ~ 10 are the cases of constant ζ and variable σ. We can see that each figure has two branches of the root locus, that is, one branch tends to the origin of the complex plane and the other goes to infinity as ζ increases infinitely. In all the figures, the lines marked as 0.6 and 0.8 shows such pole positions of all-over transfer function that its damping coefficients are equal to 0.6 and 0.8 and the good stability can be expected in the servomechanism between these two lines.

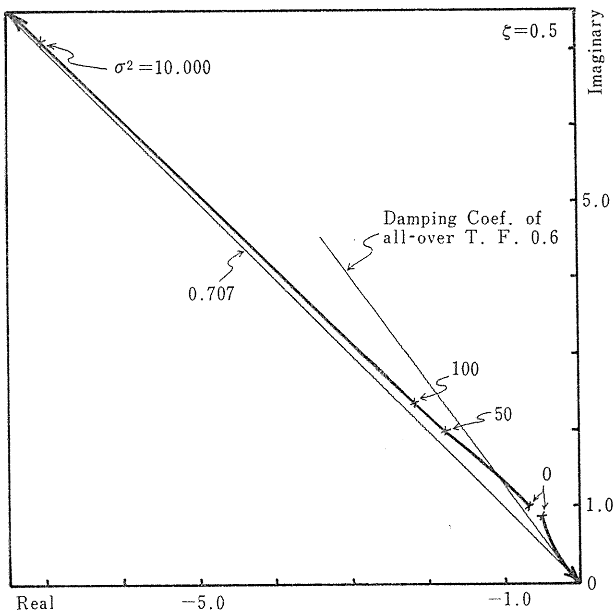

FIG. 9. $\zeta=0.5$ and σ^2 changes.

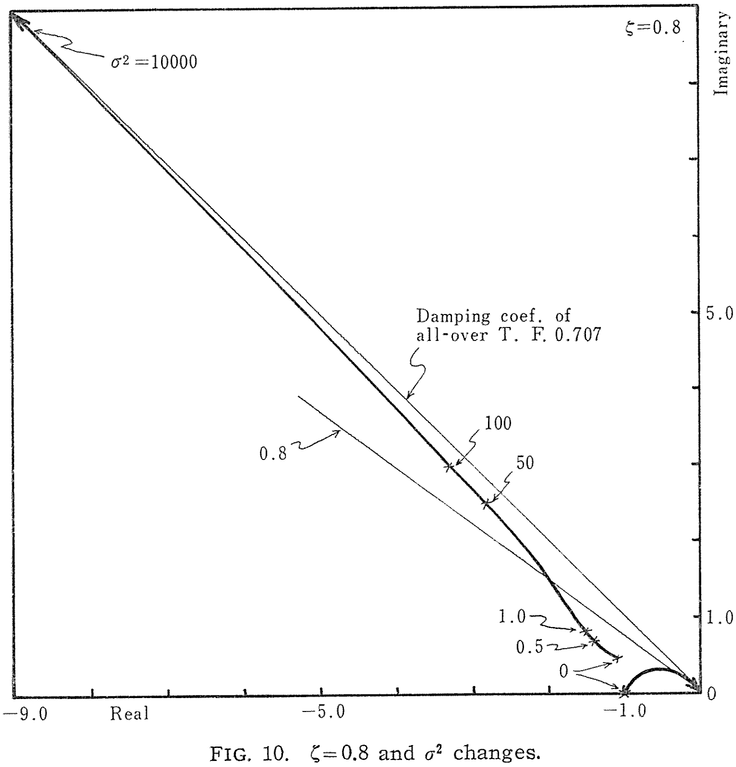

FIG. 10. $\zeta=0.8$ and σ^2 changes.

8. Conclusion

If we take a certain large value of σ, we may not consider the branch of the root locus tending to the origin because the poles with respect to this branch can be cancelled by the zeros. Fig. 2 shows that in the case of $\sigma=0$, the all-over transfer characteristics change very remarkably as ζ changes.

However as σ gradually increases from Fig. 3 to Fig. 7, the infinity branch gradually comes near the asymptote line of 135° that expresses $\zeta_c=\sin 135°=0.707$. This shows that if we give some large value to the weighting coefficient σ of the dynamic sensitivity in the performance index, we can realize a very insensible and good ($\zeta_c \fallingdotseq 0.7$) system with respect to the all-over damping characteristics.

The next table shows the variable range of the original plant damping coefficient ζ under the condition that the change of the all-over damping coefficient ζ_c can remain between 0.6 and 0.8 depending upon the values of σ.

σ^2	1	50	10.000
ζ	0.6 ~ 0.8	0.4 ~ 1.3	0.04 ~ 10

However in this report ζ was only considered as a variable parameter, so if ζ changes from a small value to large one, that is, increases, then the characteristic angular frequency increases. This results in a rapid response and is desirable, but if the variation of ζ happens in the reverse direction, that is, decreases, then the response speed also decreases and this is an undesirable performance. However if the charachteristic frequency is also considered as a variable parameter, we can expect that the above stated defect will also be removed.

Reference

1) S. L. Chang: Synthesis of optimum control system, McGraw-Hill Co., 1961, pp. 11/35.
2) W. J. Budurka: Sensitivity constrained optimal control synthesis, IBM Journal, July 1957, pp. 427/435.