RESEARCH REPORTS

ERROR ANALYSIS ON VARIOUS ELECTRICAL ANALOGUES SOLUTIONS FOR ONE, TWO AND THREE DIMENSIONAL HEAT-CONDUCTION PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction

Partial differential equations for heat conduction systems are easily transformed into their corresponding difference equations, and various electrical simulation techniques for solving the former have most commonly their analogy basis on the latter.

In these cases, two kinds of main causes given under may give rise to solution errors.

(1) difference of mathematical solutions between the two (often called "truncation error").

(2) allowances of constants set in particular simulation devices (in this text, called "coefficient error").

These errors depend in general upon number of divisions in space coordinates, principle of simulation techniques adopted and boundary conditions given to original equations, either.

In this paper, commonly known three kinds of simulation techniques that are Beuken Model (CR network simulator) and two well known methods by means of general purpose analogue computers (electronic differential analyzers) are considered for three kinds of boundary conditions respectively.

Thereupon emphasis is given to quantitative comparisons of errors in eigen values for each case.

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2. General Descriptions for One Dimensional Cases

2.1. Partial differential equation

One dimensional heat conduction equation, when normalized, is given as follows.

$$\partial\theta(t, x)/\partial t = \partial^2\theta(t, x)/\partial x^2, \text{ where } 0 \leq x \leq 1 \tag{1}$$

Boundary conditions given at both ends of heat conducting medium may generally be represented by three cases given under.

(1) Fixed temperatures are given at both ends.

(2) Fixed temperature at one end, while constant heat flow at the other end.

(3) Constant heat flows at both ends.

It may be worth saying that though only cases for particular boundary values of 0 and 1 (for either temperature or heat flow respectively) with 0 initial condition throughout the space coordinate are considered, conclusions given about eigen values in the following text still retain its generality.

2.2. Difference equation

Refering to Fig. 1 with θ_i denoting $\theta(t, i/N)$, Equation (1) may be transformed approximately into difference ordinary differential Equation (2) or (3).

$$\begin{aligned} \frac{d\theta_i}{dt} &= \frac{1}{(1/N)^2}(\theta_{i-1} - 2\,\theta_i + \theta_{i+1}) \\ &= N^2(\theta_{i-1} - 2\,\theta_i + \theta_{i+1}) \qquad (2) \\ &= N^2\{(\theta_{i-1} - \theta_i) - (\theta_i - \theta_{i+1})\} \qquad (3) \end{aligned}$$

Hereupon we define a column vector Y as following.

$$Y = \begin{pmatrix} \theta_0 \\ \theta_1 \\ \cdot \\ \cdot \\ \theta_N \end{pmatrix} \tag{4}$$

and ensues the vector equation,

$$dY/dt = N^2AY + F \tag{5}$$

where N^2A is a rectangular coefficient matrix, and F is a constant column vector of which elements are determined by given boundary conditions.

Thus comes from Equation (5) a problem of finding out all of eigen values as well as corresponding eigen vectors.

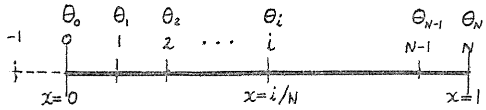

FIG. 1. One dimensional heat conduction.

2.3. Descriptions of simulation methods

Considered three kinds of simulation methods are shown in Figs. 2, 3 and 4 respectively. Both Figs. 2 and 3 refer to analogue computer methods and Fig. 4 to Beuken Model. It is evident that Fig. 2 expresses faithfully Equation (2), while Figs. 3 and 4 Equation (3). So, Fig. 2 is called 2nd difference mechanization, and Fig. 3 1st difference mechanization, Beuken Model also corresponding to the latter. Hereupon, difference of the two 1st differences appearing in Equation (3) is realized in Figs. 3 and 4.

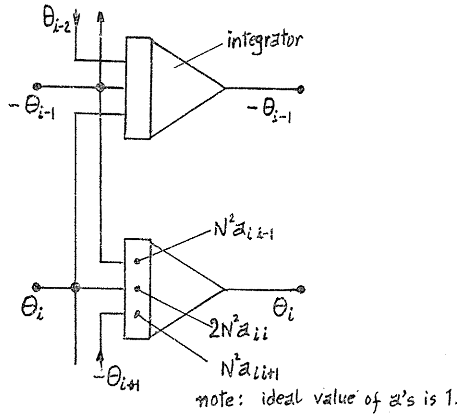

FIG. 2. 2nd difference mechanization by means of analogue computer.

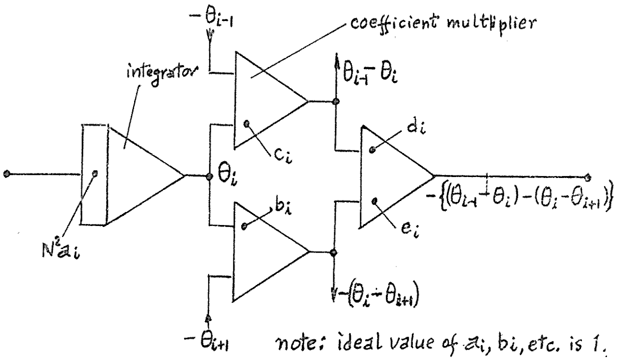

FIG. 3. 1st difference mechanization by means of analogue computer.

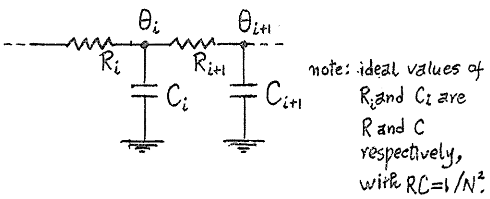

FIG. 4. Beuken Model.

In Figs. 2 and 3, signal voltages representing respective temperatures, which are indicated close to connecting lines, are showing, for simplicity, values which are to be so when all coefficients, a, b, etc. assume their own ideal value, 1 respectively.

Time scale factor which is to be taken into consideration when simulating Equation (2) or (3) by any one of Figs. 2, 3 and 4 has practically nothing to do with relative values of both truncation errors and coefficient errors, therefore the factor is always assumed to be 1 throughout the text.

2.4. Detailed aspects of causes for solution errors

(1) Approximation by difference equations.

(2) Coefficient allowances for analogue computers, and allowances of set values of condensers (C_i) and resistors (R_i) for Beuken Model (the latter is also called "coefficient" hereafter). In addition to the above two causes, the third one is to be mentioned somewhat.

(3) Imperfections of computing elements, such as (i) leakage resistance or absorption phenomenon in integrating condensers, (ii) stray time constant of operational resistors, (iii) amplification factor of operational amplifiers being less than infinity and also (iv) its dependency upon frequency, etc. are to be taken into consideration for use of analogue computers. For use of Beuke Model, the same is true also except that the above mentioned amplification factor is allowed to assume infinity.

It is to be recalled that solution of Equation (1) is inherently sum of exponentially decaying terms and comprises none of oscillating ones. So, frequency characteristics of operational amplifiers as well as stray time constant of operational resistors given in (3) may hardly affect solution errors. Accordingly we are enough to consider (1), (2) and (i) and (iii) in (3) for this study.

Let λ_0 denote eigen value of Equation (1), λ denote characteristic root of Equation (2) or (3), λ^* denote characteristic root of the similar Equation to (2) or (3) which exactly represents a particular simulator mathematically, and $\Delta\lambda$ denote $\lambda^*-\lambda_0$, then neglecting 2nd order infinitesimal terms, we get

$$\begin{aligned} \varepsilon &= (\lambda^* - \lambda_0)/\lambda_0 = \Delta\lambda/\lambda_0 = \Delta\lambda_{tr}/\lambda_0 + \Delta\lambda_{in}/\lambda_0 + \Delta\lambda_{co}/\lambda_0 \\ &= \varepsilon_{tr} + \varepsilon_{in} + \varepsilon_{co}, \\ &\text{and } \lambda = \lambda_0 + \Delta\lambda_{tr} \end{aligned} \tag{6}$$

where ε's mean relative eigen value errors, and suffixes, "*tr*", "*in*" and "*co*" abbrebiate "truncation", "integrator" and "coefficient" respectively, and of course each corresponds to the mentioned causes (1), (i) and (iii) in (3), and (2) in order.

2.5. Eigen value error due to imperfection of integrators, $\Delta\lambda_{in}$

Many authors report analyses of eigen value errors due to the mentioned inperfection of integrators, so on referring to them we can easily write down $\Delta\lambda_{in}$ as follows.

$$\Delta\lambda_{in} = -\frac{S+\lambda}{A} - \frac{1}{CR_l} - \frac{C'\lambda^{\alpha}}{C\tau^{1-\alpha}} \tag{7}$$

where λ means a value of characteristic root when integrators being assumed ideal, A amplification factor of operational amplifiers used for integrators, S sum of coefficient values set for each integrator, CR_l leakage time constant of integrating condenser C, and the third term in right members of Equation (7) is a term due to absorption phenomenon of C.

In deriving Formula (7), it is assumed that each constant appearing in the formula takes the same value throughout all integrators used.

Formula (7) is applicable to both 1st and 2nd mechanizations of analogue computers, and is also good for Beuken Model only if A is taken to be infinity.

In case when a time scale factor is chosen some appropriate value, as is usually the case, $\Delta\lambda_{in}$ shall become unimportant compared to the other errors. However, this may not be true for an unfavourable case such as given under. For instance, the more N, number of coordinate division, become, the more N^2, namely a coefficient of the right member in Equation (2) or (3), will become. This results inevitably in a larger time scale factor, because input coefficients of integrators are not practically allowed to take so large values. This ensues in general the less magnitude in eigen value, whereas magnitude of $1/CR_l$ keeps constant. Therefore the error $\Delta\lambda_{in}$ might hardly become negligible for such unfavouable cases.

2.6. Eigen value error due to coefficient allowances, $\Delta\lambda_{co}$

Variation of coefficients from their respective ideal values means a small change of coefficient matrix N^2A in Equation (5).

So, solutions obtained by the mentioned analogue simulations depend on the perturbed matrix, $N^2A^*\{=N^2(A+\Delta A)\}$, and naturally corresponding eigen values and eigen vectors should be found thereof. Hereupon ΔA is of course a small vaviation from the ideal A.

Perturbation theory of mathematics tells us that: neglecting 2nd order infinitesimals, k_{th} order eigen value deviation, $\Delta\lambda^{(k)}$ is generally given as follows.

$$\Delta\lambda^{(k)} = x_k^T(N^2\Delta A)X_k \tag{8}$$

where N^2A is in general a nth order rectangular matrix, and its eigen values $\lambda^{(k)}$ ($k=1, 2, \ldots, n$) are assumed to differ between one another, and X_k is corresponding eigen vectors. Again, X_k and x_k^T are defined as under.

$$X = (X_1,\ X_2,\ \ldots,\ X_k,\ \ldots,\ X_n), \tag{9}$$

a row vector, of which each element consists of a column vector.

$$X^{-1} = \begin{Bmatrix} x_1^T \\ \cdot \\ \cdot \\ \cdot \\ x_1^T \end{Bmatrix}, \tag{10}$$

a column vector, of which each element consists of a row vector.

Applying the above theory to the case of $\Delta\lambda_{co}^{(k)}$, we get

$$\lambda^{*(k)} = \lambda^{(k)} + \Delta\lambda_{co}^{(k)}.$$

Owing to Formula (8), $\Delta\lambda_{co}^{(k)}$ is able to be calculated for each boundary condition and again for each of the three simulation methods.

3. Eigen Value Errors in Case of Both Boundary Conditions given by Heat Flows

This case seems to be most important particularly for electro-heat systems. because it is considered usual for the systems to have *KW* rating for heat sources on one hand, and to have some heat insulation medium on the other hand. That is to say, both boundary conditions of the systems are given by something like heat flows not by temperatures. Therefore, some detailed descriptions for this case have been made under, while for the remaining cases (conf. 2.1) conclusions only have been summarized later.

3.1. Solutions of partial differential equation and of corresponding difference equation, and relative truncation error, ε_{tr}

Fig. 5 shows the case, and initial as well as boundary conditions are formulated as under.

$$\theta(0,\ x)=0\ ;\quad -\partial\theta(t,\ 0)/\partial x=1,\quad -\partial\theta(t,\ 1)/\partial x=0 \tag{11}$$

Taking advantage of Laplace transformation we get a solution for Equation (1) with (11).

$\theta(t,\ x)=\mathscr{L}^{-1}\left[\dfrac{\cosh\{\sqrt{s}\,(1-x)\}}{s\sqrt{s}\cdot\sinh\sqrt{s}}\right]$, where s is an independent variable for Laplace transformation,

$$=t+\frac{1}{2}x^2-x+\frac{1}{3}-\sum_{k=1}^{\infty}\frac{2}{k^2\pi^2}\cdot\varepsilon^{-k^2\pi^2 t}\cdot\cos k\pi x \tag{12}$$

$$\lambda_0^{(k)}=-k^2\pi^2,\ k=0,\ 1,\ 2,\ \ldots. \tag{13}$$

For difference equation, the boundary condition at $x=0$ can be approximated as follows.

$$-\partial\theta(t,\ 0)/\partial x=-(\theta_{-1}-\theta_1)/2(1/N)=1 \tag{14}$$

Thus referring to Equations (2) and (5) and using the boundary conditions, the corresponding difference equation reduces to

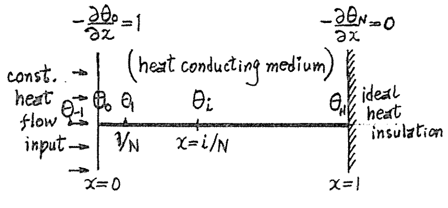

FIG. 5. Both boundary conditions are given by heat flows.

$$\frac{d}{dt}\begin{bmatrix}\theta_0\\ \theta_1\\ \cdot\\ \cdot\\ \theta_{N-1}\\ \theta_N\end{bmatrix} = N^2 \begin{array}{c} \begin{array}{cccccc}(0) & (1) & \cdot & \cdot & \cdot & (N)\end{array}\\ \begin{bmatrix}-2 & 2 & 0 & 0 & \cdot & 0\\ 1 & -2 & 1 & 0 & \cdot & 0\\ 0 & 1 & -2 & 1 & \cdot & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & \cdot & 0 & 1 & -2 & 1\\ 0 & \cdot & 0 & 0 & 2 & -2\end{bmatrix}\end{array} \begin{bmatrix}\theta_0\\ \theta_1\\ \cdot\\ \cdot\\ \theta_{N-1}\\ \theta_N\end{bmatrix} + 2N\begin{bmatrix}1\\0\\0\\ \cdot\\0\\0\end{bmatrix} \tag{15}$$

Through mathematical manipulation, we get

$$\lambda^{(k)} = -2N^2\{1-\cos(k\pi/N)\},\ k=0,1,\ \ldots,N \tag{16}$$

$$X_k = \begin{bmatrix}1\\ \cos(k\pi/N)\\ \cos(2k\pi/N)\\ \cdot\\ \cos(N-1)k\pi/N\\ \cos Nk\pi/N\end{bmatrix},\ \text{where } k=0,1,\ \ldots,N \tag{17}$$

$$x_k^T = \frac{2\alpha_k}{N}\left(\frac{1}{2},\ \cos\frac{k\pi}{N},\ \ldots,\ \cos\left\{(N-1)\frac{k\pi}{N}\right\},\ \frac{1}{2}\cos\left(N\frac{k\pi}{N}\right)\right)$$

$$\text{where}\quad \alpha_0=\alpha_N=1/2,\ \alpha_1=\alpha_2=\cdots=\alpha_{N-1}=1. \tag{18}$$

After finding out a particular integral of Equation (15), using (16), (17) and (18), we get the solution,

$$\theta_i = t + \frac{1}{2}\left(\frac{i}{N}\right)^2 - \frac{i}{N} + \frac{1}{3}\left(1-\frac{1}{4N^2}\right) - \frac{(-1)^i}{4N^2}\cdot\varepsilon^{\lambda^{(N)}t}$$
$$-\sum_{k=1}^{N-1}\frac{1}{2N^2}\cdot\mathrm{cosec}^2\left(\frac{k\pi}{2N}\right)^2\cdot\cos i\frac{k\pi}{N}\cdot\varepsilon^{\lambda^{(k)}t}. \tag{19}$$

The above solution naturally reduces to Solution (12) as N approaches to infinity.

From Formulae (6), (13) and (16), relative truncation error, $\varepsilon_{tr}^{(k)}$ is obtained.

$$\varepsilon_{tr}^{(k)} = \frac{2N^2}{k^2\pi^2}\left(1-\cos\frac{k\pi}{N}\right) - 1 \simeq -\frac{k^2\pi^2}{12N^2},\ \text{for } \frac{k\pi}{N} \ll 1 \tag{20}$$

As seen above, relative truncation error is approximately proportional to k^2, while inversely proportional to N^2, square of number of division.

The fact that "zero" eigen value exists for both partial differential and difference equations, namely $\lambda_0^{(0)} = \lambda^{(0)} = 0$, is to be emphasized. We are hereupon to understand that in regard to 0th order eigen value, of which value is zero, any number of division in a space coordinate gives rise to no truncation error.

3.2. Eigen value errors due to coefficient allowances, $\Delta\lambda_{co}$

Letting (i, j) element of ΔA be Δ_{ij}, we get $\Delta\lambda_{co}^{(k)}$ as a function of Δ_{ij} from Formulae (8), (17) and (18) as following.

$$\Delta\lambda_{co}^{(k)} = \alpha_k N\Big[(\Delta_{00}+\Delta_{NN}) + (\Delta_{01}+\Delta_{NN-1})\cdot\cos\frac{k\pi}{N}$$
$$+2\sum_{i=1}^{N-1}\cos\left(i\frac{k\pi}{N}\right)\cdot\left\{\Delta_{ii}\cos\left(i\frac{k\pi}{N}\right) + \Delta_{ii-1}\cos\left\{(i-1)\frac{k\pi}{N}\right\} + \Delta_{ii+1}\cos\left\{(i+1)\frac{k\pi}{N}\right\}\right\}\Big] \tag{21}$$

where α_k is the same as given in (18).

We are to pay attention to the fact that relation between Δ_{ij} and coefficient allowances differs depending on a simulation method adopted, and so these relations will be given under for each of the three simulations.

a) 2nd difference mechanization

Refering to Fig. 2 and letting relative coefficient variation from the ideal value of 1 be a', we get, for example,

$$N^2 \cdot a_{ij} = N^2 \cdot (1 + a'_{ij}),$$

and from Equation (2) and Fig. 2 the following equation ensues.

$$\begin{aligned} \frac{d\theta_i}{dt} &= N^2(a_{ii-1}\theta_{i-1} - 2\, a_{ii}\theta_i + a_{ii+1}\theta_{i+1}) \\ &= N^2\{(\theta_{i-1} - 2\,\theta_i + \theta_{i+1}) + (a'_{ii-1}\theta_{i-1} - 2\, a'_{ii}\theta_i + a'_{ii+1}\theta_{i+1})\} \end{aligned} \tag{22}$$

Therefore, $\Delta_{ii-1} = a'_{ii-1}, \quad \Delta_{ii} = -2\, a'_{ii}, \quad \Delta_{ii+1} = a'_{ii+1}.$ (23)

For particular simulation of θ_0, however, the corresponding simulation circuit differs somewhat from those of θ_i $(i \neq 0)$: it is to be as shown in Fig. 6, and Equation (24) is applied instead of (22).

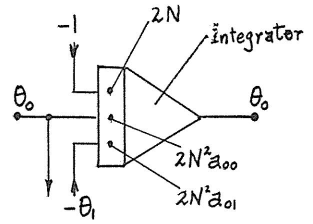

FIG. 6. Diagram for θ_0 in 2nd diff. mech.

$$\begin{aligned} \frac{d\theta_0}{dt} &= 2\, N^2(-a_{00}\theta_0 + a_{01}\theta_1) + 2\, N \\ &= 2\, N^2\{(-\theta_0 + \theta_1) + (-a'_{00}\theta_0 + a'_{01}\theta_1)\} + 2\, N \end{aligned} \tag{24}$$

Therefore, $\Delta_{00} = -2\, a'_{00}, \quad \Delta_{01} = 2\, a'_{01}.$ (25)

Thus substituting (23) and (25) for Δ's in Formula (21), we have

$$\Delta\lambda_{co}^{(k)} = 2\, N\Big[-(a'_{00} + a'_{NN}) - 2\sum_{i=1}^{N-1} a'_{ii}\cdot\cos^2\Big(i\frac{k\pi}{N}\Big) + \sum_{i=1}^{N}(a'_{ii-1} + a'_{i-1i})\cdot \cos\Big(i\frac{k\pi}{N}\Big)\cos\Big\{(i-1)\frac{k\pi}{N}\Big\}\Big]$$

$$\varepsilon_{co}^{(k)} = \Delta\lambda_{co}^{(k)}/\lambda_0^{(k)} = -\Delta\lambda_{co}^{(k)}/k^2\pi^2 \tag{26}$$

where $k = 1, 2, \ldots, N-1$, and for $k = N$, multiply 1/2 to the right member.

$$\Delta\lambda_{co}^{(0)} = N\Big\{\sum_{i=1}^{N}(a'_{ii-1} + a'_{i-1i}) - \Big(a'_{00} + a'_{NN} + 2\sum_{i=1}^{N-1} a'_{ii}\Big)\Big\} \tag{27}$$

Example 1) As an example, we take up the case that all a'_{ij}'s assume the same absolute value of ε', and their signs are such that all terms in the right member of (26) add to one another to produce the worst case.

Assuming N is an integer multiple of k, we have from (26)

$$\varepsilon_{co}^{(k)} = 2N^2\varepsilon' \cdot \{1+\cos(k\pi/N)\}/(k^2\pi^2) \simeq 4N^2\varepsilon'/(k^2\pi^2),\ k \neq 0,\ k\pi \ll N. \tag{28}$$

As seen above, relative eigen value error due to coefficient allowances of 2nd diff. mechanization is approximately proportional to N^2 and inversely proportional to k^2. The situation is, therefore, just oppsite to that of truncation error.

Example 2) As the 2nd example, we take up a case that a'_{ij}'s are considered to be independent probability variables of which each average value is zero, and each distribution assumes symmetry with the same standard deviation σ'.

From Formula (26), we easily have standard deviation of $\varepsilon_{co}^{(k)}$ as following.

$$\begin{aligned}\sigma_{co}^{(k)} &= N^{3/2}\cdot\{8+\cos(2k\pi/N) - 8/N\}^{1/2}\cdot\sigma'/(k^2\pi^2)\\ &\simeq 3N^{3/2}\sigma'/(k^2\pi^2),\ \text{where } N \geq 3,\ k \neq 0 \text{ and } k\pi \ll N.\end{aligned} \tag{29}$$

Therefore it is generally considered that for 2nd difference mechanization coefficient error increases as one and a half power of number of division.

b) 1st difference mechanization

Refering to Fig. 3 and letting relative coefficient variation from their ideal value of 1 be a', b', c', etc., we have the following equation like the same way.

$$\begin{aligned}\frac{d\theta_i}{dt} &= N^2 a_i\{d_i(b_{i-1}\theta_{i-1} - c_i\theta_i) - e_i(b_i\theta_i - c_{i+1}\theta_{i+1})\}\\ &= N^2\{(\theta_{i-1} - 2\theta_i + \theta_{i+1}) + (a'_i + b'_{i-1} + d'_i)\theta_{i-1}\\ &\qquad - (2a'_i + b'_i + c'_i + d'_i + e'_i)\theta_i + (a'_i + c'_{i+1} + e'_i)\theta_{i+1}\}\end{aligned} \tag{30}$$

Therefore, $\Delta_{ii-1} = a'_i + b'_{i-1} + d'_i,\quad \Delta_{ii+1} = a'_i + c'_{i+1} + e'_i,$

$$\Delta_{ii} = -(2a'_i + b'_i + c'_i + d'_i + e'_i). \tag{31}$$

For particular simulation of θ_0, however, Fig. 7 and the following Equation (32) are to be applied.

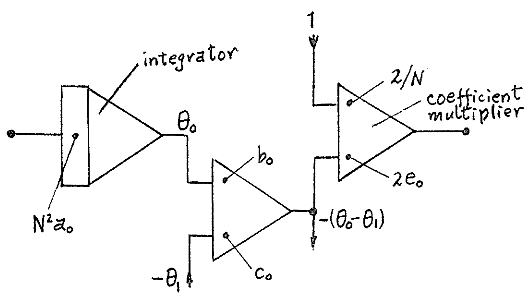

FIG. 7. Diagram for θ_0 in 1st diff. mech,

$$\frac{d\theta_0}{dt} = 2N^2 a_0 e_0(-b_0\theta_0 + c_1\theta_1) + 2Na_0$$
$$= 2N^2\{(-\theta_0+\theta_1) - (a_0'+b_0'+c_0')\theta_0 + (a_0'+c_1'+e_0')\theta_1\} + 2Na_0$$
$$\therefore \quad \Delta_{00} = -2(a_0'+b_0'+c_0'), \qquad \Delta_{01} = (a_0'+c_1'+e_0') \tag{32}$$

Thus substiting (31) and (32) for Δ's in Formula (21), we have

$$\Delta\lambda_{co}^{(k)} = 2N\Big[-(a_0'+a_N')\Big(1-\cos\frac{k\pi}{N}\Big) - 2\sum_{i=1}^{N-1} a_i'\Big(1-\cos\frac{k\pi}{N}\Big)\cdot\cos^2 i\frac{k\pi}{N}$$
$$+\sum_{i=1}^{N-1}(b_i'+e_i')\cos i\frac{k\pi}{N}\cdot\Big\{\cos(i+1)\frac{k\pi}{N} - \cos i\frac{k\pi}{N}\Big\}$$
$$+\sum_{i=1}^{N}(c_i'+d_i')\cos i\frac{k\pi}{N}\cdot\Big\{\cos(i-1)\frac{k\pi}{N} - \cos i\frac{k\pi}{N}\Big\}\Big]$$
$$\varepsilon_{co}^{(k)} = \Delta\lambda_{co}^{(k)}/\lambda_{co}^{(k)} = -\Delta\lambda_{co}^{(k)}/k^2\pi^2, \text{ where } k=1,2,\ldots,N-1, \tag{33}$$

and for $k=N$, multiply 1/2 to the right member.

$$\Delta\lambda_{co}^{(0)} = 0 \tag{34}$$

As comparing (33) to (26), we may understand that coefficient error for this case is in general smaller than that of the former 2nd diff. mechanization case, though this very case comprises more factors of the cause for giving rise to the error.

The situation may more easily be understood only if two special cases such as given before are considered like the same way.

Example 1)

$\varepsilon_{co}^{(k)} = 2N\varepsilon'\cdot[4k + N\{1-\cos(k\pi/N)\}]/(k^2\pi^2)$ for N being even multiple of k,

$= 2N\varepsilon'\cdot[2k\{3-\cos(k\pi/N)\} + N\{1-\cos(k\pi/N)\}]/(k^2\pi^2)$ for N being odd multiple of k. Therefore

$$\varepsilon_{co}^{(k)} \simeq 8N\varepsilon'/k\pi^2, \quad k \neq 0 \text{ and } k\pi \ll N \tag{35}$$

Example 2)

$$\sigma_{co}^{(k)} = 2N^{3/2}\sigma'\cdot[\{1-\cos(k\pi/N)\}\{7-5\cos(k\pi/N)\}$$
$$-4\{1-\cos(k\pi/N)\}^2/N]^{1/2}/(k^2\pi^2), \; N\geq 3$$
$$\simeq \sqrt{2N}\cdot\sigma'/k\pi, \quad k\neq 0 \text{ and } k\pi\ll N \tag{36}$$

Therefore, in case of 1st diff. mechanization, coefficient error increases proportionally to number of division N for the worst case, whereas for general cases does proportionally to half power of N.

c) Beuken Model

Refering to Fig. 4, let actual set values of resistors and condensers be R_i and C_i respectively, their idal value be R, C ($RC=1/N^2$), and respective relative errors be r_i', c_i', then we have

$$\frac{d\theta_i}{dt} = \frac{1}{C_i}\left\{\frac{1}{R_i}(\theta_{i-1}-\theta_i) - \frac{1}{R_{i+1}}(\theta_i - \theta_{i+1})\right\}$$
$$= N^2\{(\theta_{i-1} - 2\,\theta_i + \theta_{i+1}) - (c_i' + r_i')\theta_{i-1}$$
$$+ (2\,c_i' + r_i' + r_{i+1}')\,\theta_i - (c_i' + r_{i+1}')\,\theta_{i+1}\} \tag{37}$$

$$\therefore \quad \Delta_{ii-1} = -(c_i' + r_i'), \quad \Delta_{ii+1} = -(c_i' + r_{i+1}'),$$
$$\Delta_{ii} = 2\,c_i' + r_i' + r_{i+1}'. \tag{38}$$

For particular simulation of θ_0, however, Fig. 8 and the following Equation (39) are to be applied.

$$\frac{d\theta_0}{dt} = \frac{2}{C_0 R_1}(\theta_1 - \theta_0) + \frac{2\,I}{C_2}$$
$$= 2\,N^2\{(\theta_1 - \theta_0) + (c_i' + r_i')(\theta_0 - \theta_1)\} + \frac{2\,I}{C_0}$$
$$\therefore \quad \Delta_{00} = -\Delta_{01} = c_i' + r_i'. \tag{39}$$

From Formula (21), we have like the same way

$$\Delta\lambda_{co}^{(k)} = 2\,N\Big[(c_0' + c_N')\Big(1 - \cos\frac{k\pi}{N}\Big) + 2\sum_{i=1}^{N-1} c_i'\Big(1 - \cos\frac{k\pi}{N}\Big)\cdot\cos^2 i\,\frac{k\pi}{N}$$
$$+ \sum_{i=1}^{N} r_i'\Big\{\cos i\,\frac{k\pi}{N} - \cos\Big\{(i-1)\frac{k\pi}{N}\Big\}\Big\}^2\Big] \tag{40}$$

where $k = 1, 2, \ldots, N-1$, and for $k = N$, multiply $1/2$ to the right member.

$$\Delta\lambda_{co}^{(0)} = 0 \tag{41}$$

Considering the same two special cases, situations for Beuken Model case may be understood more.

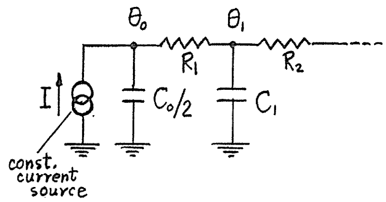

FIG. 8. Diagram for θ_0 in Beuken Model.

Example 1)

$$\varepsilon_{co}^{(k)} = 4\,N^2\varepsilon'\{1 - \cos(k\pi/N)\}/k^2\pi^2 \simeq 2\,\varepsilon' \tag{42}$$

Example 2)

$$\sigma_{co}^{(k)} = \sqrt{12\{1 - 2/(3\,N)\}}\cdot N^{3/2}\cdot\sigma'\{1 - \cos(k\pi/N)\}/k^2\pi^2$$
$$\simeq \sqrt{3/N}\cdot\sigma', \; N \geq 3, \; k \neq 0 \text{ and } k\pi \ll N. \tag{43}$$

Therefore, in case of Beuken Model, coefficient crror approximates a constant, about twice of ε', even for the worst case, and for general cases it does decrease nearly inversely proportionally to a half power of N. These remarkable features seem to be worthy of special attention.

Summarizing results of the above a), b) and c), Fig. 9 shows $\sigma_{co}^{(1)}$'s, standard deviations of relative errors in eigen values of 1st order due to coefficient allowances. Also shown in the figure is $\varepsilon_{tr}^{(1)}$, relative truncation error in eigen value of 1st order. We see how N, number of coordinate division, affects two kinds of eigen value errors just mentioned, and may read an optimum number of division for realizing the minimum error. It is to be noteworthy that when compared to 1st diff. mechanization, a less number of optimun division is obtained for 2nd diff. mechanization which inherenty needs less numbers of operational amplifiers, while in the latter case more error gives rise to.

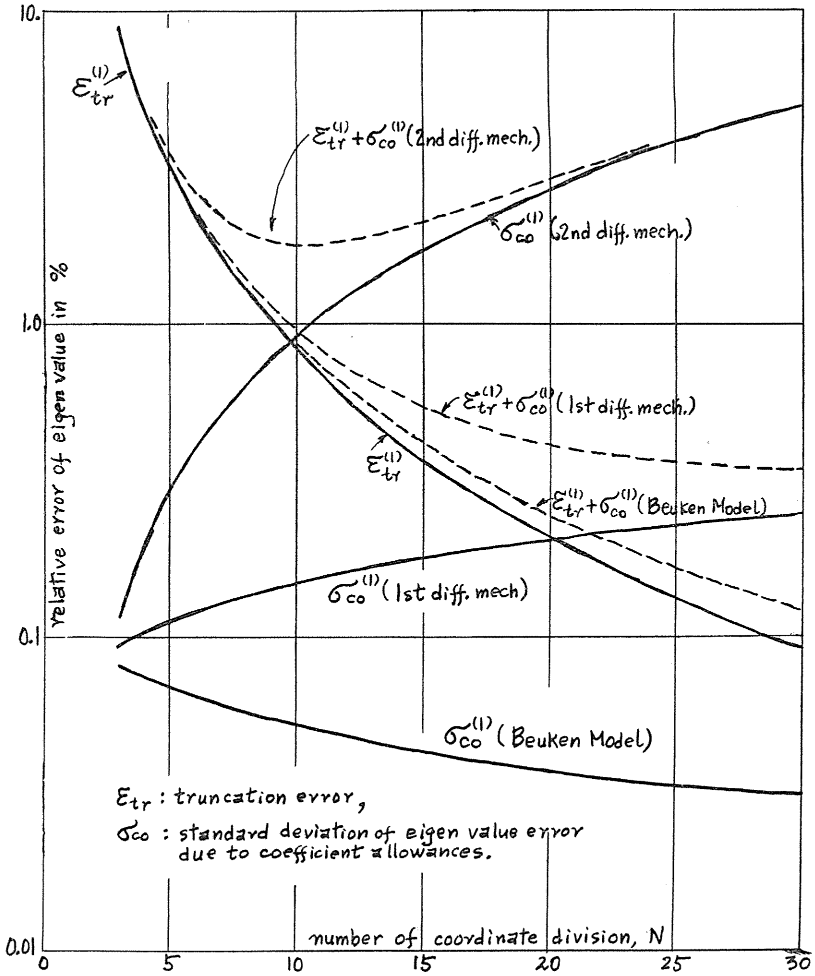

FIG. 9. Dependency of relative errors of 1st order eigen values upon number of division, in case $\sigma'=0.1\%$.

d) Special references to "0" eigen value, and simulator solutions

Special attention should be paid to the fact that "zero" eigen value must exist for the considering boundary conditions, and $\Delta\lambda_{co}^{(0)}$, namely error of this eigen value due to coefficient allowances, is given by Formula (27) for 2nd diff. mechanization, whereas it always reduces to zero for 1st diff. mechanization as well as for Beuken Model {cf. (34) and (41)}.

Accordingly, for the latter two simulation methods, each of the simulation solutions to be obtained should have a term of time "t" which represents a particular integral term, just in the same way as seen in the Solution (19) which represents the solution of Difference Equation (15), and again this equation corresponds to the ideal case with no coefficient allowances. (In case of the mentioned two methods with coefficient allowances, however, a coefficient of time t in the solution equation may have some error (see footnote*)).

On the other hand, in case of 2nd diff. mechanization, a small definit value of $\Delta\lambda_{co}^{(0)}$ exists, and accordingly the corresponding solution may well become exponentially diverged or saturated as time elapses depending on the sign of $\Delta\lambda_{co}^{(0)}$, thus the solution error might be so exaggerated as time increases, as is shown later.

Seeing back again to Solution (19), we will have a following formula which is good for large $t(-\lambda^{(1)}t \simeq \pi^2 t \gg 1)$ in place of (19).

$$\theta_i = t + K_i, \text{ where } K_i = (1/2)(i/N)^2 - i/N - (1/3)\{1 - 1/(4N^2)\} \tag{44}$$

Corresponding formula for 2nd diff. mechanization with coefficient allowances is given as follows through some methematical manipulation taking advantage of Laplace transformation.

$$\theta_i^* = (\varepsilon^{\gamma t} - 1)/\gamma + K_i \cdot \varepsilon^{\gamma t} \tag{45}$$

where γ denotes $\Delta\lambda_{co}^{(0)}$.

Formulae (45) and (27) have been checked and proved by experiments such as given under.

For example, Curve ① shown in Fig. 10 indicates plots of θ_0, ideal Solution (19) for $i=0$, namely temperature of the left boundary point in Fig. 5, when N is taken 10. For enough time-elapse, the curve reduces, needless to say, to that given by Formula (44).

Using an analog computer, partial differential equations in which N equals 10, or

$$\frac{d(\theta_i/5)}{d(20\,t)} = 5\left\{\frac{\theta_{i-1}}{5} - 2\frac{\theta_i}{5} + \frac{\theta_{i+1}}{5}\right\} \tag{46}$$

has been solved by means of 2nd diff. mechanization. The result for θ_0 is indicated as Curve ②, in which we can observe a tendency of saturation as time elapses.

Three marks of Δ close to the Curve ② show plots of values calculated from Formula (45) by putting $\gamma = -0.023$ and using K_i which has been found from

* The coefficient of t in the solution formula can be found by putting $\theta_i^* = \eta_i t + \zeta_i$ and substituting this for θ_i in Equation (30) or (37). Thus we obtain η_i which is not equal to 1,

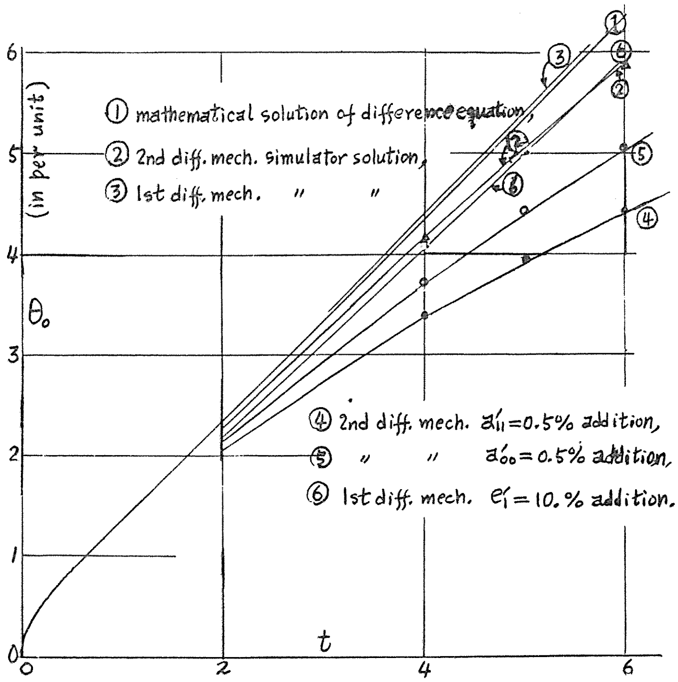

FIG. 10. Examples of solutions θ_0 obtained with analogue computer mechanization in case that boundary conditions are both given by constant heat flows.

Curve ① or from Formula (44). Mark Δ's are found to be well on Curve ②. The result may seem to prove Formula (45).

For another example, an allowance of $a'_{11} = 0.5\%$ has been intentionally added as shown in Fig. 11 to the very simulation circuit soon after getting Curve ②. The result is shown as Curve ④ from which $\gamma \simeq -0.123$ has been calculated. (three black circles indicate calculated values from Formula (45) by putting $\gamma = -0.123$). While, Equation (27) theoretically gives that $\Delta\lambda_{co}^{(0)} = -10 \times 2 \times 0.005 - 0.023$ (the latter corresponds to the value before addition of the 0.5%) $= -0.123$. Thus we see good agreement with the experimental value.

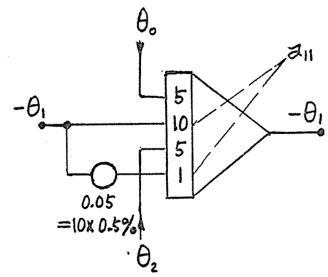

FIG. 11. Intentional allowance addition of $a'_{11}=0.5\%$.

Curve ⑤ indicates the computer result when $a'_{00} = 0.5\%$ has been added intentionally in the same way. From the curve we get $\gamma \simeq -0.073$ (three white circles show plots of calculated values from Equation (45) by putting $\gamma = -0.073$). While, Equation (27) gives theoretical value of $\Delta\lambda_{co}^{(0)}$ being -0.073 $(-10 \times 0.005 - 0.023)$. Good agreement of theory with experiment is thus obtained for 2nd diff. mechanization.

For 1st diff. mechanization, theory claims, as already indicated, that "0" eigen value should always exist irrespective of existence of coeff. allowances.

In order to prove above experimentally, $e_1' = 10\%$, a large allowance, has been intentionally added to the very simulation circuit just after getting Curve ③ which is the result without such an artificial addition of allowance. Curve ⑥ shows the result. Curves ③ and ⑥ thus obtained by means of 1st diff. mechanization definitely indicate linearity to larger values of time t, though a very little non-linearity might exist owing to zero eigen value error caused by imperfection of integrators adopted.

In order to prove existence of zero eigen value also for Beuken Model case, no experiment was tried, since now it seemed very natural and proper from view point of its physical construction consisting of a series of RC networks.

In the last place, though it does not seem to need add, zero eigen value does never exist for cases of the other boundary conditions which will be treated next ((1), (2) cases in paragraph 2.1).

4. Eigen Value Errors in Case of Both Boundary Conditions given by Temperatures

Fig. 12 shows the case. To find out eigen values, their errors, etc., almost the same procedures as before are followed. So, special mentions except conclusions are not given under.

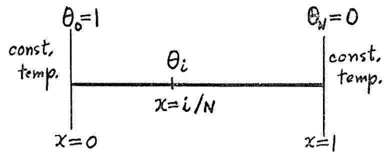

FIG. 12. Both boundary conditions are given by constant temperatures.

Numbers in parentheses to designate formulae or equations are shown with one dash in this chapter. Note that they are showing corresponding ones with the same number without dash in parentheses already given in the preceding chapter.

$$\theta(t, x) = \mathscr{L}^{-1}\left[\frac{\sinh\{\sqrt{s}(1-x)\}}{s\cdot\sinh\sqrt{s}}\right]$$

$$= 1 - x - \sum_{k=1}^{\infty}\frac{2}{k\pi}\cdot\varepsilon^{-k^2\pi^2 t}\cdot\sin k\pi x \qquad (12')$$

$\lambda_0^{(k)} = -k^2\pi^2$, $k = 1, 2, \ldots$, the same with (13) except $k \neq 0$. (13')

$$A = \begin{array}{c} \begin{array}{cccccc} (1) & (2) & \cdot & \cdot & \cdot & (N-1) \end{array} \\ \left|\begin{array}{cccccc} -2 & 1 & 0 & 0 & \cdot & 0 \\ 1 & -2 & 1 & 0 & \cdot & 0 \\ 0 & 1 & -2 & 1 & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & 0 & 1 & -2 & 1 \\ 0 & 0 & \cdot & 0 & 1 & -2 \end{array}\right| \end{array} \begin{array}{c} \\ (1) \\ (2) \\ \cdot \\ \cdot \\ \cdot \\ (N-1) \end{array} \quad \text{in} \quad (15')$$

Note that A for this case is of symmetry and $N-1$ order.

$$\lambda^{(k)} = -2N^2\{1-\cos(k\pi/N)\} \tag{16'}$$

This is the same with (16), except that k takes only1, 2, . . . , $N-1$, and in the following also the same k's are applied. (Note: Solutions (12′), (13′) and (16′) may often be found in reference books).

$$\theta_i = 1 - \frac{i}{N} - \sum_{k=1}^{N-1} \frac{1}{N} \cot \frac{k\pi}{2N} \cdot \varepsilon^{\lambda^{(k)}t} \cdot \sin i \frac{k\pi}{N} \tag{19'}$$

The above reduces to (12′) as N approches to infinity.

$$\varepsilon_{tr}^{(k)} = 2N^2\{1-\cos(k\pi/N)\}/(k^2\pi^2) - 1 \simeq -k^2\pi^2/12N^2 \tag{20'}$$

This is the same with (20).

a) 2nd diff. mechanization

$$\varepsilon_{co}^{(k)} = 2N^2\varepsilon'\{1+\cos(k\pi/N)\}/k^2\pi^2 \simeq 4N^2\varepsilon'/k^2\pi^2 \tag{28'}$$

This is the same with (28).

$$\sigma_{co}^{(k)} = N^{3/2}\sigma'\{8+\cos(2k\pi/N)\}^{1/2}/k^2\pi^2 \simeq 3N^{3/2}\sigma'/k^2\pi^2 \tag{29'}$$

Approximation formula only is the same with (29).

b) 1st diff. mechanization

$$\begin{aligned} \varepsilon_{co}^{(k)} &= 2N\varepsilon'[4k + N\{1-\cos(k\pi/N)\}]/k^2\pi^2, \\ \text{or} \quad & 2N\varepsilon'[2k\{1+\cos(k\pi/N)\} + N\{1-\cos(k\pi/N)\}]/k^2\pi^2 \\ &\simeq 8N\varepsilon'/k\pi^2 \end{aligned} \tag{35'}$$

$$\begin{aligned} \sigma_{co}^{(k)} &= \sqrt{2N}^{3/2}\sigma'[\{1-\cos(k\pi/N)\}\{7-5\cos(k\pi/N)\}]^{1/2}/k^2\pi^2 \\ &\simeq \sqrt{2N}\,\sigma'/k\pi \end{aligned} \tag{36'}$$

As seen above two approximation formulae are the same with corresponding ones in the preceding chapter.

c) Beuken Model

$$\varepsilon_{co}^{(k)} = 4N^2\varepsilon'\{1-\cos(k\pi/N)\}/k^2\pi^2 \simeq 2\varepsilon' \tag{42'}$$

This is the same with (42).

$$\sigma_{co}^{(k)} = \sqrt{12N}^{3/2}\sigma'\{1-\cos(k\pi/N)\}/k^2\pi^2 \simeq \sqrt{3/N}\,\sigma' \tag{43'}$$

Approximation formula only is the same with (43).

5. Eigen Value Errors in Case of Boundary Conditions given by Temperature and Heat Flow

Fig. 13 shows the case. Two dashed numbers in parentheses designating formulae, equations, etc., are showing corresponding ones in chapter 3.

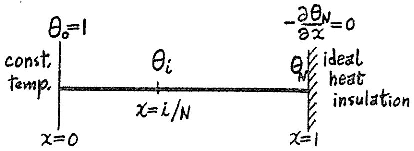

FIG. 13. Boundary conditions are given by const. temperature and heat flow.

$$\theta(t, x) = \mathscr{L}^{-1}\left[\frac{\cosh\{\sqrt{s}(1-x)\}}{s\cdot\cosh\sqrt{s}}\right]$$

$$= 1 - \sum_{k=1}^{\infty}\frac{2}{(k-1/2)\pi}\cdot\varepsilon^{-(k-1/2)^2\pi^2 t}\cdot\sin(k-1/2)\pi x \tag{12''}$$

$$\lambda_0^{(k)} = -(k-1/2)^2\pi^2, \quad k=1, 2, \ldots \tag{13''}$$

$$\mathbf{A} = \begin{array}{c} \begin{array}{cccccc} (1) & (2) & & & & (N) \end{array} \\ \left[\begin{array}{cccccc} -2 & 1 & 0 & 0 & \cdot & 0 \\ 1 & -2 & 1 & 0 & \cdot & 0 \\ 0 & 1 & -2 & 1 & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & 0 & 1 & -2 & 1 \\ 0 & 0 & \cdot & 0 & 2 & -2 \end{array}\right] \end{array} \begin{array}{c} (1) \\ (2) \\ \cdot \\ \cdot \\ \cdot \\ (N) \end{array} \tag{15''}$$

$$\lambda^{(k)} = -2N^2[1-\cos\{(k-1/2)\pi/N\}] \tag{16''}$$

where $k=1, 2, \ldots, N$, and in the following also the same k's are applied.

$$\theta_i = 1 - \sum_{k=1}^{N}\frac{1}{N}\cot\frac{(k-1/2)\pi}{2N}\cdot\varepsilon^{\lambda^{(k)}t}\cdot\sin\frac{i(k-1/2)\pi}{N} \tag{19''}$$

$$\varepsilon_{tr}^{(k)} = 2N^2[1-\cos\{(k-1/2)\pi/N\}]/(k-1/2)^2\pi^2 - 1$$

$$\simeq -(k-1/2)^2\pi^2/12N^2, \text{ for } (k-1/2)\pi/N \ll 1. \tag{20''}$$

a) 2nd diff. mechanization

$\varepsilon_{co}^{(k)} = 2N^2\varepsilon'[1+\cos\{(k-1/2)\pi/N\}]/(k-1/2)^2\pi^2$ for N being integer multiple of $2k-1$,

$$\simeq 4N^2\varepsilon'/(k-1/2)^2\pi^2 \tag{28''}$$

$$\sigma_{co}^{(k)} = N^{3/2}\sigma'[8+\cos\{2(k-1/2)\pi/N\} - 4/N]^{1/2}/(k-1/2)^2\pi^2$$

$$\simeq 3N^{3/2}\sigma'/(k-1/2)^2\pi^2 \tag{29''}$$

b) 1st diff. mechanization

$\varepsilon_{co}^{(k)} = 2N\varepsilon'[4(k-1/2) + N[1-\cos\{(k-1/2)\pi/N\}]]/(k-1/2)^2\pi^2$ for N being integer multiple of $2k-1$,

$$\simeq 8N\varepsilon'/(k-1/2)\pi^2 \tag{35''}$$

$$\sigma_{co}^{(k)} = \sqrt{2N}^{3/2}\sigma'[[1-\cos\{(k-1/2)\pi/N\}]\cdot[7-5\cos\{(k-1/2)\pi/N\}] \\ -2[1-\cos\{(k-1/2)\pi/N\}]^2/N]^{1/2}/(k-1/2)^2\pi^2 \\ \simeq \sqrt{2N}\sigma'/(k-1/2)\pi \qquad (36'')$$

c) *Beuken Model*

$$\varepsilon_{co}^{(k)} = 4N^2\varepsilon'[1-\cos\{(k-1/2)\pi/N\}]/(k-1/2)^2\pi^2 \\ \simeq 2\varepsilon' \qquad (42'')$$

$$\sigma_{co}^{(k)} = \sqrt{12(1-1/(3N))}\cdot N^{3/2}\sigma'[1-\cos\{(k-1/2)\pi/N\}]/(k-1/2)^2\pi^2 \\ \simeq \sqrt{3/N}\cdot\sigma'. \qquad (43'')$$

Through three chapters of 3, 4 and 5, we can well say that as regards corresponding approximation formulae, they are the same with one another except k is replaced by $k-1/2$ for chapter 5 case.

6. Analyses for Two and Three Dimensional Cases

We have seen hitherto that various approximation formulae derived for one dimensional cases have the same form throughout the three different kinds of boundary conditions, though eigen vector only differs somewhat to each other for respective boundary condition. (their detailed description were omitted for simplicity in the preceding chapters).

Therefore, we will take up here only the case for temperature boundary conditions, and analyses for other cases, if wanted, may be performed in the same way.

6.1. Two dimensional case

Refering to Fig. 14, normalized partial differential equation is

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2} + \alpha\frac{\partial^2\theta}{\partial y^2}, \quad \text{where } \alpha = a^2/b^2,\ 0\le x,\ y\le 1. \qquad (47)$$

For boundary condition of $\theta(0, y, t) = \theta(1, y, t) = 0$, $\theta(x, 0, t) = \theta(x, 1, t) = 0$, we have solution of Equation (47), as follows.

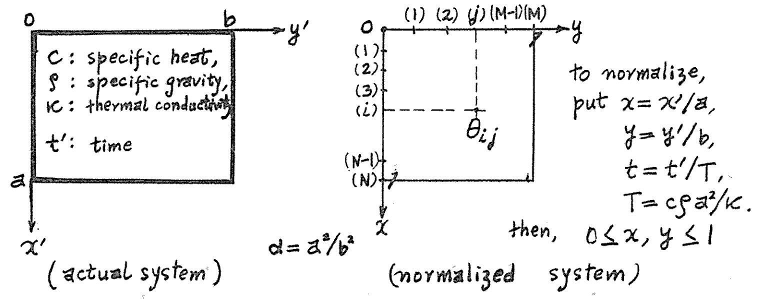

FIG. 14. Two dimensional heat conduction system.

$$\theta = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} C_{pq} \sin p\pi x \cdot \sin q\pi x \cdot \varepsilon^{-(p^2+\alpha q^2)\pi^2 t}, \tag{48}$$

where p and q correspond to k in one dimensional case, and C_{pq} will be defined by given intial conditions and eigen values given under.

$$\lambda_0^{(p,q)} = -(p^2 + \alpha q^2)\pi^2, \qquad p, q = 1, 2, \ldots . \tag{49}$$

If x-coordinate is divided into N sections and y into M sections, we will have the folling difference equation corresponding to Equation (2).

$$\frac{d\theta_{ij}}{dt} = N^2(\theta_{i-1j} - 2\theta_{ij} + \theta_{i+1j}) + \alpha M^2(\theta_{ij-1} - 2\theta_{ij} + \theta_{ij+1}) \tag{50}$$

Put $\beta = \alpha M^2/N^2$ and $\gamma = 1 + \beta$, then above equation reduces to

$$\frac{d\theta_{ij}}{dt} = N^2\{(\theta_{i-1j} - 2\gamma\theta_{ij} + \theta_{i+1j}) + \beta(\theta_{ij-1} + \theta_{ij+1})\} \tag{51}$$

Hereupon we define two vectors as following.

$$X_j = \begin{pmatrix} \theta_{1j} \\ \theta_{2j} \\ \cdot \\ \cdot \\ \cdot \\ \theta_{N-1j} \end{pmatrix}, \qquad Y = \begin{pmatrix} X_1 \\ X_2 \\ \cdot \\ X_j \\ \cdot \\ X_{M-1} \end{pmatrix} \tag{52}$$

We may herewith understand correspondency of the above to vector Y in Formula (4), of which element θ_i has now been replaced by vector X_j which represents temperature distributions along x-coordinate at $y=j$.

Using X_j's, the following left members are expressed respectively by vector products as are given in the following right side.

$$\begin{aligned}
&&& \quad\ \ (1) \qquad\qquad\quad (i) \\
\theta_{i-1j} - 2\gamma\theta_{ij} + \theta_{i+1j} &= (0, \ldots, 0, 1, -2\gamma, 1, 0, \ldots, 0)X_j \\
\theta_{ij-1} &= (0, \ldots, \quad 0, \quad 1, \ 0, \quad \ldots, 0)X_{j-1} \\
\theta_{ij+1} &= (0, \ldots, \quad 0, \quad 1, \ 0, \quad \ldots, 0)X_{j+1}
\end{aligned}$$

$$\therefore \frac{d\theta_{ij}}{dt} = N^2(0, .., \overset{(1)}{1}, \overset{(i)}{-2\gamma}, 1, .., 0)X_j + N^2\beta(0, .., \overset{(1)}{1}, \overset{(i)}{.}, .., 0)(X_{j-1} + X_{j+1}) \tag{53}$$

Therefore, from (52) and (53), we get

$$\frac{dX_j}{dt} = N^2 \begin{array}{c} \begin{array}{ccccc} (1) & & & & (N-1) \end{array} \\ \begin{pmatrix} -2\gamma, & 1, & 0, & \cdot & 0 \\ 1, & -2\gamma, & 1, & 0, & 0 \\ 0, & 1, & -2\gamma, & 1, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & \cdot & 0, & 1, & -2\gamma \end{pmatrix} \end{array} X_j + N^2\beta \begin{array}{c} \begin{array}{ccccc} (1) & & & & (N-1) \end{array} \\ \begin{pmatrix} 1 & 0 & \cdot & & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \cdot & \cdot & \cdot & \cdot & \\ 0 & 0 & \cdot & 0 & 1 \end{pmatrix} \end{array} (X_{j-1} + X_{j+1}) \tag{54}$$

Again, put
A^N = coefficient matrix of X_j in Equation (54), and
I^N = unit matrix appearing in (54).
Then we finally get

$$\frac{dY}{dt} = N^2 \begin{array}{c} \begin{array}{ccccc} (1) & & & & (M-1) \end{array} \\ \left\{ \begin{array}{ccccc} A^N & \beta I^N & 0 & \cdot & 0 \\ \beta I^N & A^N & \beta I^N & \cdot & 0 \\ 0 & \beta I^N & A^N & \beta I^N & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & \beta I^N & A^N \end{array} \right\} \end{array} Y = N^2 A^{NM} Y \tag{55}$$

where A^{NM} denotes coefficient rectangular matrix shown above and is of $(N-1)\cdot(M-1)$th order.

Solution of equation (55) is obtained by finding out eigen values of $N^2 A^{NM}$ and their corresponding eigen vectors just in the same way as in one dimensional case.

Descriptions of the procedures are omitted in this text for simplicity, but some of their conclusions are summarized in Table 1.

6.2. Three dimensional case

Referring to Fig. 15, we have in similar way three dimensional normalized partial differential and difference equations as given under.

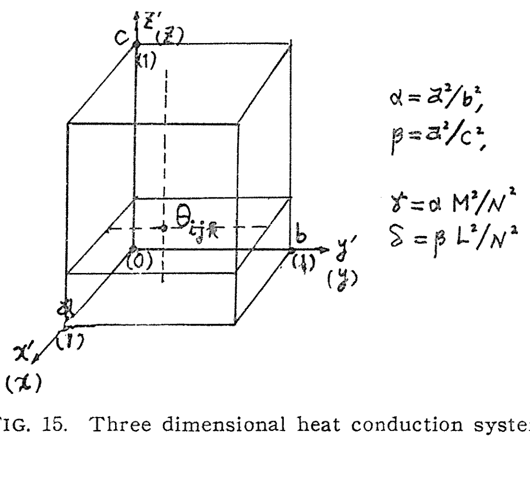

FIG. 15. Three dimensional heat conduction system.

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} + \alpha \frac{\partial^2 \theta}{\partial y^2} + \beta \frac{\partial^2 \theta}{\partial z^2} \tag{56}$$

where $\alpha = a^2/b^2$, $\beta = a^2/c^2$, $0 \leq x,\ y,\ z \leq 1$,

$$\frac{dZ}{dt} = N^2 A^{NML} Z \tag{57}$$

where

$$
Z=\begin{bmatrix} Y_1 \\ Y_2 \\ \cdot \\ Y_k \\ \cdot \\ Y_{L-1} \end{bmatrix}, \quad
Y_k=\begin{bmatrix} X_{1k} \\ X_{2k} \\ \cdot \\ X_{jk} \\ \cdot \\ X_{M-1k} \end{bmatrix}, \quad
X_{jk}=\begin{bmatrix} \theta_{1jk} \\ \theta_{2jk} \\ \\ \\ \cdot \\ \theta_{N-1jk} \end{bmatrix},
$$

$$
A^{NML}=\begin{bmatrix} A^{NM}, & \delta I^{NM}, & \cdot & 0 \\ \delta I^{NM}, & A^{NM}, & \delta I^{NM}, & \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \delta I^{NM}, & A^{NM}, \end{bmatrix}, \quad
A^{NM}=\begin{bmatrix} A^{N}, & \gamma I^{N}, & & 0 \\ \gamma I^{N}, & A^{N}, & \gamma I^{N}, & \\ & \cdot & \cdot & \\ 0, & & \gamma I^{N}, & A^{N} \end{bmatrix}
$$

$$
A^{N}=\begin{bmatrix} -2\xi, & 1, & & 0 \\ 1, & -2\xi, & 1, & \\ & \cdot & \cdot & \cdot \\ 0 & & 1, & -2\xi \end{bmatrix},
$$

I^{NM}=unit matrix of $(N-1)\cdot(M-1)$th order,

I^{N}=unit matrix of $(N-1)$th order,

$\xi=1+\gamma+\delta,\ \gamma=\alpha M^2/N^2,\ \delta=\beta L^2/N^2.$

In order to facilitate comparison among one, two and three dimensional cases, only some of the results which have been obtained for such simple case that $a=b=c$ and $N=M=L$ are shown in Table 1.

Description of the results for more general cases are omitted for favour of simplicity.

TABLE 1. Comparison of λ_0's, ε_{tr}'s and σ_{co}'s

Number of Dimension in Cartesian Coordinate	λ_0 $(\lambda_0^{(k)},\ \lambda_0^{(p,q)},\ \lambda_0^{(p,q,r)})$	ε_{tr} $(\varepsilon_{tr}^{(k)},\ \varepsilon_{tr}^{(p,q)},\ \varepsilon_{tr}^{(p,q,r)})$	σ_{co} $(\sigma_{co}^{(k)},\ \sigma_{co}^{(p,q)},\ \sigma_{co}^{(p,q,r)})$		
			2nd diff. mech.	1st diff. mech.	Beuken Model
One	$-k^2\pi^2$	$-k^2\cdot\dfrac{\pi^2}{12N^2}$	$3N^{3/2}\cdot\dfrac{\sigma'}{k^2\pi^2}$	$\sqrt{2}N^{1/2}\cdot\dfrac{\sigma'}{k\pi}$	$\dfrac{\sqrt{3}}{N^{1/2}}\cdot\sigma'$
Two	$-(p^2+q^2)\pi^2$	$-\dfrac{p^4+q^4}{p^2+q^2}\cdot\dfrac{\pi^2}{12N^2}$	$\sqrt{45}N\cdot\dfrac{\sigma'}{(p^2+q^2)\pi^2}$	$\sqrt{3}\cdot\dfrac{\sigma'}{\sqrt{p^2+q^2}\pi}$	$\dfrac{3}{\sqrt{2}N}\cdot\dfrac{\sqrt{p^4+p^2q^2+q^4}}{p^2q^2}\cdot\sigma'$
Three	$-(p^2+q^2+r^2)\pi^2$	$-\dfrac{p^4+q^4+r^4}{p^2+q^2+r^2}\cdot\dfrac{\pi^2}{12N^2}$	$\sqrt{141\frac{3}{4}}N^{1/2}\cdot\dfrac{\sigma'}{(p^2+q^2+r^2)\pi^2}$	$\dfrac{3}{\sqrt{2}N^{1/2}}\cdot\dfrac{\sigma'}{\sqrt{p^2+q^2+r^2}\,\pi}$	$\dfrac{3\sqrt{3}}{2N^{3/2}}\cdot\dfrac{\sqrt{p^4+q^4+r^4+p^2q^2+q^2r^2+r^2p^2}}{p^2+q^2+r^2}\cdot\sigma'$

Note: k; p, q; p, q, r; indicate order of eigen values for one, two and three dimensional cases respectively.

λ_0 : eigen value of partial differential equation.

ε_{tr} : relative truncation error.

σ_{co}: standard deviation of relative eigen value error due to coeff. allowances.

7. Conclusions

Featuring points of results are summarized as following:

(1) For using an analog computer, 2nd diff. mechanization needs less opera-

tional amplifiers, but gives rise to more error when compared to 1st diff. mechanization.

(2) Optimum number of coordinate division for attaining overall least eigen value error is less for 2nd diff. mechanization.

(3) Beuken Model is not only most economic for its construction, but also gives rise to the least error among all of the considered methods.

(4) Eigen value errors generally decrease with increasing number of coordinate division in Beuken Model, whereas in analog computer methods they usually do increase especially for simulation of one dimensional heat conduction.

(5) Effects of dimensional numbers of heat conduction space.

For two or three dimensional heat conduction simulation, a great number of simulation elements are needed, and naturally accumulation of errors due to such many inevitable coefficient allowances seems to appear. But this is not quite true, and on the contrary eigen value errors generally decrease as number of space dimensions increases.

Accordingly it may well be stated at least from the analytical point of view that reduction of dimensional number from three to two or from two to one in view to attain simplification of simulation seems to be avoided as much as possible, though of course more elements are needed.

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