STUDIES IN NON UNI-DIMENSIONAL FILTRATION

FILTRATION ON CYLINDRICAL, SPHERICAL AND SQUARE SURFACES

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1. Introduction

The previous works in filtration have been almost exclusively restricted to uni-dimensional phenomena, because of the practical importance in actual industries. Strictly speaking, uni-dimensional phenomena are to be encountered only when there is some sort of a retaining wall which compels the flow into uni-dimension. If a cake is deposited either internally or externally on either a cylindrical element or a spherical surface, or either on a circular or a rectangular leaf, non uni-dimensional theories will play an important role in the analysis for filtration. While some work has been done to apply a filtration theory to scale-up problems, only an ideal case of a three-dimensional filtration on a circular leaf has been considered by Brenner[1,2]. Applying Brenner's work, a term entitled "effective filtration area factor" has been developed theoretically and empirically for the cases of both a three-dimensional filtration on a circular leaf and a two-dimensional filtration on a rectangular leaf[3], while Yoshioka[5] has attempted to analyze the flow variation through non uni-dimensional cakes.

Starting from basic differential equations for flow through porous media, non uni-dimensional filtration theories are developed in view of the effective filtration area factor j_N[3]. In this paper, theoretical and experimental methods are presented for obtaining values of j_N for two-dimensional filtration on cylindrical (tubular) surfaces and three-dimensional filtration both on square leaves and spherical surfaces.

2. Basic Equations for Non Uni-dimensional Filtration

The conventional Ruth's equation, the basic equation for uni-dimensional filtration, is written as

$$\left(\frac{dv}{d\theta}\right)_1 = \frac{g_c \cdot \Delta p}{\mu\alpha \dfrac{W}{A}} = \frac{g_c \cdot \Delta p}{\mu\alpha w} \tag{1}$$

In Eq. (1), the medium resistance R_m is neglected, and v is the volume of filtrate per unit area, θ the time, Δp the filtration pressure, μ the viscosity of filtrate, α the average cake resistance, A the filtration area, and $w(=W/A)$ is the total mass of dry cake solids per unit area.

In accordance with the concept of the "effective filtration area factor j_N", Ruth's equation is modified[3] for non uni-dimensional filtration as

$$\left(\frac{dv}{d\theta}\right)_N=\frac{g_c\cdot\Delta p}{\mu\alpha\dfrac{W}{A_e}}=\frac{g_c\cdot\Delta p}{\mu\alpha\dfrac{W}{A}\cdot\dfrac{A}{A_e}}=\frac{g_c\cdot\Delta p}{\mu\alpha\dfrac{w}{j_N}} \tag{2}$$

where the subscript N is employed to emphasize that the equation can be used for general problems of uni-dimensional and non uni-dimensional filtrations, and A_e denotes the effective filtration area defined by Eq. (2).

i) Basic Flow Equation

Using vector notations, a basic flow equation through a non uni-dimensional filter cake can be generally written as

$$\mathbf{u}=\frac{\mathbf{q}}{\varepsilon}-\frac{\mathbf{r}}{1-\varepsilon}\quad\text{or}\quad\varepsilon\mathbf{u}=\mathbf{q}-\frac{\varepsilon\mathbf{r}}{1-\varepsilon}=\mathbf{q}-e\mathbf{r}$$

where $\mathbf{u}$ is a local value of the relative velocity of liquid to solids, $\mathbf{q}$ a local apparent velocity of liquid, $\mathbf{r}$ a local apparent velocity of solids, ε a local porosity, e is a local void ratio of filter cake defined by $e=\varepsilon/(1-\varepsilon)$. Actually, $\mathbf{u}$, $\mathbf{q}$, $\mathbf{r}$, ε and e are functions of position, time, and operational conditions. Since $\varepsilon\mathbf{u}$ is the apparent relative velocity of liquid to solids, the above equation can be presented in view of a local value α of Ruth's filtration resistance as

$$\varepsilon\mathbf{u}=\mathbf{q}-e\mathbf{r}=-\frac{g_c}{\alpha(1-\varepsilon)\rho_s}\cdot\nabla p \tag{3}$$

where ρ_s is the true density of cake solids, and p denotes a local instantaneous pressure in the cake.

ii) Continuity Equation

The total mass of liquid M at a time θ in a filter chamber of the volume V_{ch} can be written in the form

$$M(\theta)=\iiint_{V_{ch}}(\rho\varepsilon)\cdot dV_{ch} \tag{4}$$

and the accumulation rate of liquid mass in chamber can be written as

$$\frac{dM}{d\theta}=-\iint_S(\rho\mathbf{q})\cdot\mathbf{n}dS \tag{5}$$

where $\mathbf{n}$ represents the normal unit vector of a surface S. Differentiating Eq. (4) with respect to θ, one gets

$$\frac{dM}{d\theta}=\iiint_{V_{ch}}\frac{\partial(\rho\varepsilon)}{\partial\theta}\cdot dV_{ch} \tag{6}$$

while the right-hand side of Eq. (5) can be rewritten by Gauss' divergence theorem in the form

$$-\frac{dM}{d\theta}=\iint_{S}(\rho\mathbf{q})\cdot\mathbf{n}dS=\iiint_{V_{ch}}\nabla\cdot(\rho\mathbf{q})\cdot dV_{ch} \tag{7}$$

Substituting Eq. (7) into Eq. (6) one obtains

$$\iiint_{V_{ch}}\left[\frac{\partial(\rho\varepsilon)}{\partial\theta}+\nabla\cdot(\rho\mathbf{q})\right]dV_{ch}=0$$

or

$$\frac{\partial(\rho\varepsilon)}{\partial\theta}+\nabla\cdot(\rho\mathbf{q})=0 \tag{8}$$

Filtrate being incompressible, Eq. (8) becomes

$$\frac{\partial\varepsilon}{\partial\theta}+\nabla\cdot\mathbf{q}=0 \tag{9}$$

In accordance with the same procedure as mentioned above, the continuity equation of solids may be represented in the following form

$$\frac{\partial\{\rho_s(1-\varepsilon)\}}{\partial\theta}+\nabla\cdot(\rho_s\mathbf{r})=0 \quad \text{or} \quad -\frac{\partial\varepsilon}{\partial\theta}+\nabla\cdot\mathbf{r}=0 \tag{10}$$

Substitution of Eq. (10) into Eq. (9) gives

$$\nabla\cdot\mathbf{q}+\nabla\cdot\mathbf{r}=0 \tag{11}$$

Eqs. (9), (10) and (11) can be viewed as the basic continuity equations for flow through filter cakes.

3. Non Uni-dimensional Filtration on Cylindrical and Spherical Surfaces

i) Simplified Equations for Incompressible Cakes

In this paper emphasis is placed on obtaining useful equations for practical design purposes without making the analysis unduly complex. Current development of the mathematical art of filtration depends on a number of assumptions which have been indicated by Tiller.[4] In addition to these postulates, the following items are assumed.

1. An incompressible, homogeneous and isotropic filter-cake, that is, the constant values of the specific resistance α and the porosity ε.
2. Negligibly small resistance of the filter medium.
3. A constant applied filtration pressure.

Based upon the assumptions listed above, Eqs. (3) and (9) may be rewritten as

$$\mathbf{q}=-\frac{g_c}{\mu\alpha(1-\varepsilon)\rho_s}\cdot\nabla p=-\frac{kg_c}{\mu}\cdot\nabla p \tag{12}$$

$$\nabla\cdot\mathbf{q}=0 \tag{13}$$

respectively, where k is the permeability coefficient, represented by

$$k=\frac{1}{\alpha(1-\varepsilon)\rho_s}=\text{const.}$$

Substitution of Eq. (12) in Eq. (13) leads to

$$\mathrm{div}\left[-\frac{kg_c}{\mu}\cdot\nabla p\right]=-\frac{kg_c}{\mu}\,\mathrm{div}\,(\nabla p)=0$$

$$\text{or}\quad \nabla^2 p=0 \tag{14}$$

In order to obtain the solution for a non uni-dimensional problem in a specified coordinate system, the pressure variation is to be calculated from Eq. (14), and then the flow rate may be determined from Eq. (12). It should be noted that the cake profiles of non uni-dimensional filtration coincide with the equi-pressure surfaces, the pattern of filtrate flow following to potential flow.

ii) Two-dimensional Filtration on Cylindrical Surfaces

The problem of determining the filter cake deposition on a cylindrical surface as a function of time is best discussed in a system of the cylindrical coordinates (r, ϕ, z), as shown in Fig. 1. According to the coordinates, the following equations hold.

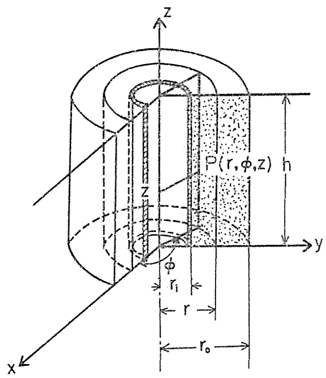

FIG. 1. Cake on Cylindrical Coordinates.

$$\nabla p=\frac{\partial p}{\partial r}\,\mathrm{e}_r+\frac{1}{r}\cdot\frac{\partial p}{\partial \phi}\,\mathrm{e}_\phi+\frac{\partial p}{\partial z}\,\mathrm{e}_z \tag{15}$$

$$\nabla^2 p=\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)+\frac{\partial}{\partial \phi}\left(\frac{1}{r}\cdot\frac{\partial p}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial p}{\partial z}\right)\right] \tag{16}$$

where e_r, e_ϕ and $\mathbf{e}_z$ are the unit vectors for the cylindrical system.

The equi-pressure surface within the filter cake being identical with the cylindrical surface of a constant radius r, that is $p=p(r, \theta)$, Eqs. (14) and (16) yield

$$\nabla^2 p=\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)=0$$

Integrating the above equation and substituting both the boundary conditions ($p=p_i$ at the medium surface $r=r_i$, $p=p_0$ at the cake surface $r=r_0$) and the initial condition ($r=r_0=r_i$ at $\theta=0$), one gets

$$p = p_0 + (p_0 - p_i)\frac{\ln(r/r_0)}{\ln(r_0/r_i)}$$

Differentiating the above equation partially with respect to r leads to

$$\frac{\partial p}{\partial r} = \frac{1}{r}\cdot\frac{\Delta p}{\ln(r_0/r_i)}$$

where the filtration pressure $\Delta p = p_0 - p_i = \text{const}$. Substituting in Eq. (12) yields

$$\mathbf{q} = -\frac{kg_c}{\mu}\cdot\nabla p = -\frac{kg_c}{\mu}\cdot\frac{1}{r}\cdot\frac{\Delta p}{\ln(r_0/r_i)}\cdot \mathbf{e}_r$$

$$\text{or}\quad q = |\mathbf{q}| = \frac{kg_c}{\mu}\cdot\frac{1}{r}\cdot\frac{\Delta p}{\ln(r_0/r_i)}$$

Therefore, the flow rate of filtrate at the medium is given by

$$\left(\frac{dv}{d\theta}\right)_{II,cy} = q|_{r=r_i} = \frac{kg_c}{\mu}\cdot\frac{1}{r_i}\cdot\frac{\Delta p}{\ln(r_0/r_i)}$$

$$= \frac{1}{\alpha(1-\varepsilon)\rho_s}\cdot\frac{g_c\cdot\Delta p}{\mu r_i\ln(r_0/r_i)} \qquad (17)$$

The total volume V_c of the filter cake is given by

$$V_c = \pi h(r_0^2 - r_i^2) \qquad (18)$$

where h is the length of the cylinder. The total mass of dry cake per unit filter-medium area is given by

$$w = \frac{V_c}{A}\rho_s(1-\varepsilon) = \frac{r_i}{2}\left[\left(\frac{r_0}{r_i}\right)^2 - 1\right]\rho_s(1-\varepsilon)$$

Substituting in Eq. (17), one obtains

$$\left(\frac{dv}{d\theta}\right)_{II,cy} = \frac{1}{2}\left[\left(\frac{r_0}{r_i}\right)^2 - 1\right]\frac{1}{\ln(r_0/r_i)}\cdot\frac{g_c\cdot\Delta p}{\mu\alpha w} \qquad (19)$$

Eq. (19) represents the rate equation for the two-dimensional filtration on a cylindrical surface at a constant pressure and the subscripted $(dv/d\theta)_{II,cy}$ is employed.

iii) Three-dimensional filtration on Spherical Surfaces

The three-dimensional filtration on a spherical surface may be best discussed in a system of the spherical coordinates (r, λ, ϕ), as shown in Fig. 2, where the following equations hold.

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\cdot\frac{\partial p}{\partial\lambda}\mathbf{e}_\lambda + \frac{1}{r\sin\lambda}\cdot\frac{\partial p}{\partial\phi}\mathbf{e}_\phi$$

$$\nabla^2 p = \frac{1}{r^2\sin\lambda}\left[\frac{\partial}{\partial r}\left(r^2\sin\lambda\frac{\partial p}{\partial r}\right) + \frac{\partial}{\partial\lambda}\left(\sin\lambda\frac{\partial p}{\partial\lambda}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\lambda}\cdot\frac{\partial p}{\partial\phi}\right)\right]$$

The pressure distribution within a spherical cake being independent of λ and ϕ,

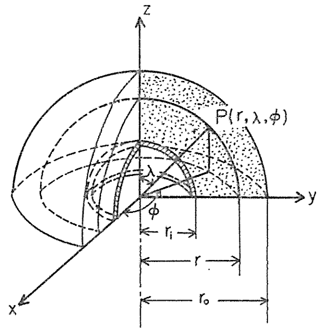

FIG. 2. Cake on Spherical Coordinates.

the equi-pressure surfaces coincide with the spherical surfaces of constant radii r and $p=p\ (r,\ \theta)$.

According to the same mathematical procedure mentioned in (ii), one obtains the rate equation for the three-dimensional filtration on a spherical surface at a constant pressure.

$$\left(\frac{dv}{d\theta}\right)_{III,\,sp}=\frac{1}{3}\left(\frac{r_0}{r_i}\right)\left[\left(\frac{r_0}{r_i}\right)^2+\left(\frac{r_0}{r_i}\right)+1\right]\frac{g_c\cdot\Delta p}{\alpha\mu w} \tag{20}$$

4. Effective Filtration Area Factor

In constant pressure filtration, the flow rate equations on a cylindrical and a spherical surface have been represented in the forms of Eqs. (19) and (20). It is apparent that the flow rate for non uni-dimensional filtration is a function of (r_0/r_i). For practical purposes of numerical calculations, it may be more convenient to replace r_0 by the volume of cake v_c per unit medium area. For two-dimensional filtration on a cylindrical surface of radius r_i:

Cake forming outside the surface;

$$\frac{V_c}{r_i}=\frac{V_c}{A}\cdot\frac{1}{r_i}=\frac{\pi h(r_0^2-r_i^2)}{2\pi h r_i^2}=\frac{1}{2}\left[\left(\frac{r_0}{r_i}\right)^2-1\right]$$

$$\frac{r_0}{r_i}=\sqrt{1+2\frac{v_c}{r_i}} \tag{21}$$

Cake forming inside the surface;

$$\frac{r_0}{r_i}=\sqrt{1-2\frac{v_c}{r_i}} \tag{21'}$$

For three-dimensional filtration on a spherical surface of radius r_i:

Cake forming outside the surface;

$$\frac{r_0}{r_i}=\sqrt[3]{1+3\frac{v_c}{r_i}} \tag{22}$$

Cake forming inside the surface

$$\frac{r_0}{r_i} = \sqrt[3]{1 - 3\frac{v_c}{r_i}} \tag{22'}$$

With these changes of (r_0/r_i) in the right hand side of Eqs. (19) and (20), the reference of Eqs. (2), (19) and (20) will serve to define the filtration area factors j_N as follow:

$$\begin{aligned} j_{II,cy} &= \frac{1}{2}\left[\left(\frac{r_0}{r_i}\right)^2 - 1\right]\frac{1}{\ln(r_0/r_i)} \\ &= \frac{\pm 2\frac{v_c}{r_i}}{\ln\left(1 \pm 2\frac{v_c}{r_i}\right)} \qquad \begin{pmatrix} + : & r_0 \geqq r_i \\ - : & r_0 \leqq r_i \end{pmatrix} \end{aligned} \tag{23}$$

$$\begin{aligned} j_{III,sp} &= \frac{1}{3}\left(\frac{r_0}{r_i}\right)\left[\left(\frac{r_0}{r_i}\right)^2 + \left(\frac{r_0}{r_i}\right) + 1\right] \\ &= \frac{1}{3}\left[\left(1 \pm 3\frac{v_c}{r_i}\right) + \left(1 \pm 3\frac{v_c}{r_i}\right)^{2/3} + \left(1 \pm 3\frac{v_c}{r_i}\right)^{1/3}\right] \\ & \qquad \begin{pmatrix} + : & r_0 \geqq r_i \\ - : & r_0 \leqq r_i \end{pmatrix} \end{aligned} \tag{24}$$

In Fig. 3, the theoretical values of j_N are illustrated. j_N is a unique function of (v_c/r_i) or (r_0/r_i) for each non uni-dimensional filtration.

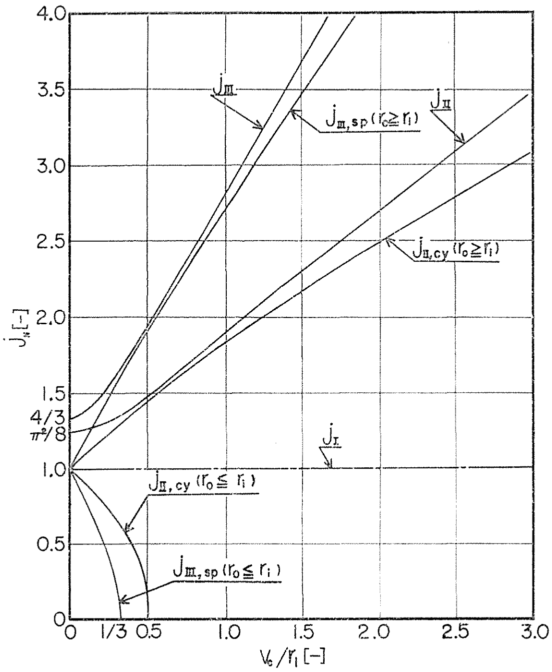

FIG. 3. Theoretical values of j_{II}, $j_{II,cy}$, j_{III} and $j_{III,sp}$.

5. Experimental Results

i) Experimental Equipment

For studying the problems of non uni-dimensional filtration, square test surfaces of 3×3, 5×5 and 8×8 cm^2 (Fig. 4-1 a) and cylindrical surfaces of $r_i = 1.25$, 2.50, 3.75 and 5.00 cm (Fig. 4-1 b) are used. The schematic picture of the apparatus used is shown in Fig. 4-2. Filter-cel slurry (compressibility coefficient

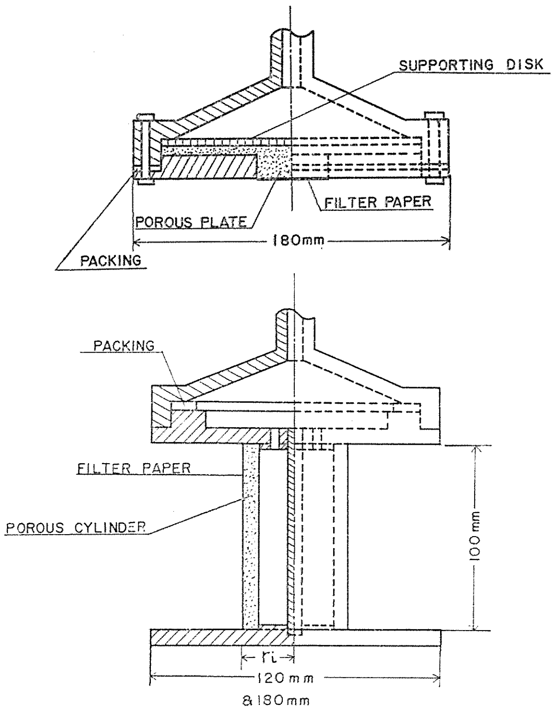

FIG. 4-1 a. Square Test Leaf.
4-1 b. Cylindrical Test Surface.

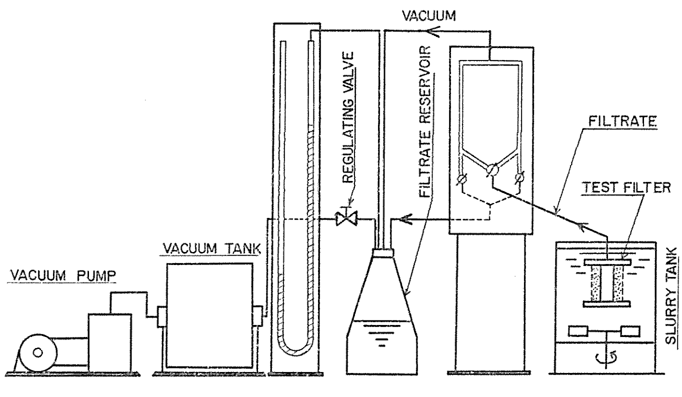

FIG. 4-2. Schematic Picture of Experimental Apparatus.

of cake $n=0.03$; slurry concentration $s=0.059$) is filtered at a constant pressure of $\Delta p=0.835$ Kg/cm² vacuum. For numerical treatment of three-dimensional filtration on the square leaves, the equivalent radius $r_{eq}=a/2$ is used as a representative size of a leaf, where a is the side length of a square leaf.

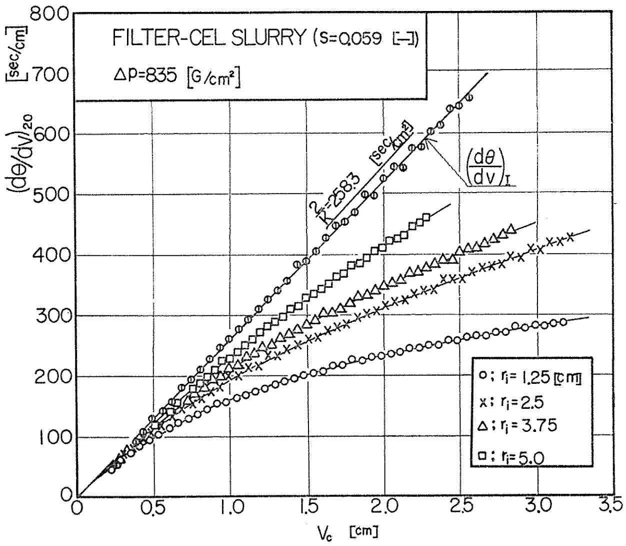

FIG. 5. $(d\theta/dv)_{II,cy}$ vs. v_c.

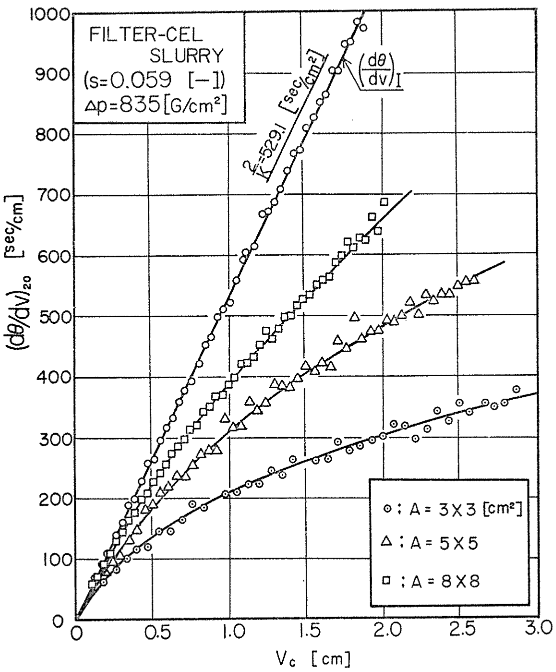

FIG. 6. $(d\theta/dv)_{III,sq}$ vs. v_c.

ii) *Experimental Results and Procedures*

Figs. 5 and 6 show experimental data of $(d\theta/dv)_N$ vs. v_c, together with unidimensional filtration data. It is apparent that $[(1/r_i)(d\theta/dv)]_N$ vs. (v_c/r_i) represents a unique relation as may be seen from Fig. 7 or Eq. (17).

The experimental values of j_N can be well determined from experimental values of $(dv/d\theta)_I$ and $(dv/d\theta)_N$ in view of the defining equation of j_N. It is apparrent from Eqs. (1) and (2) that the ratio of $(dv/d\theta)_N$ to $(dv/d\theta)_I$ at an equal value of w or v_c equals j_N.

$$j_N \equiv \frac{A_e}{A} = \frac{(dv/d\theta)_N}{(dv/d\theta)_I} = \frac{(d\theta/dv)_I}{(d\theta/dv)_N} \tag{25}$$

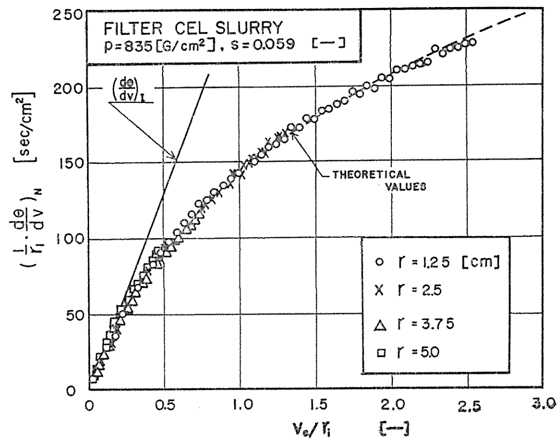

FIG. 7. $[(1/r_i)(d\theta/dv)]_{II,cy}$ vs. (v_c/r_i).

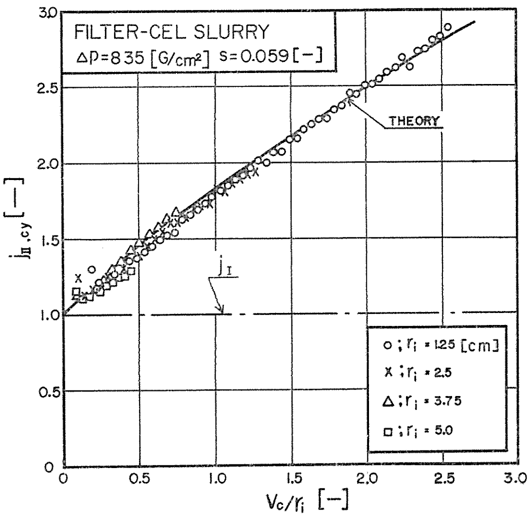

FIG. 8. $j_{II,cy}$ vs. (v_c/r_i).

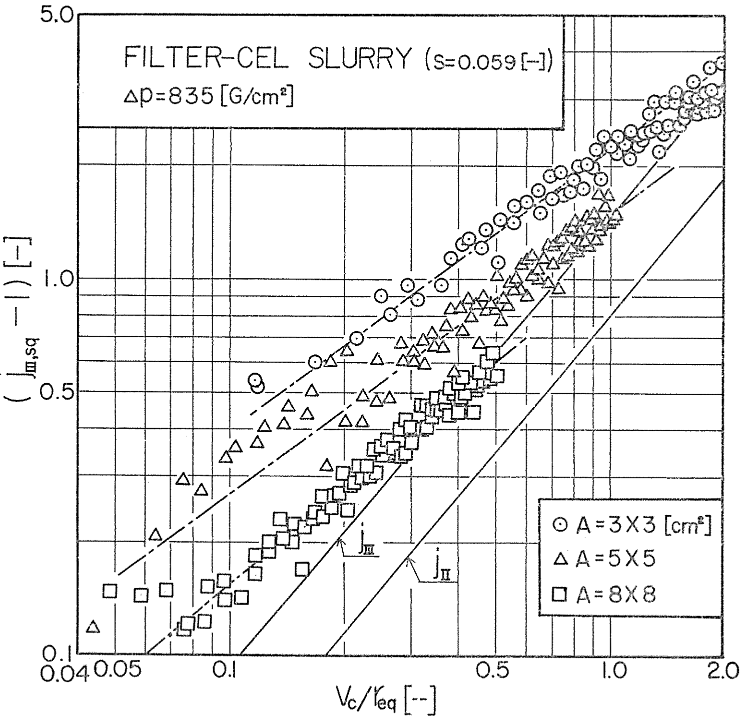

FIG. 9. $j_{III,sq}$ vs. (v_c/req).

Experimental j_N-values thus obtained are compared favorably with the theoretical j_N-values as show in Fig. 8.

Experimental j_N-values for three-dimensional filtration on square leaves are illustrated in Fig. 9, together with j_{III} on circular leaves and j_{II} on rectangular leaves[3]. It can be safely said that $j_{III,sq}$ approaches j_{III} as both the cake volume v_c and the side length a increase.

6. Conclusions

It has been demonstrated that the non uni-dimensional filtration problems on a cylindrical surface, a spherical surface and a square leaf can be solved in view of the effective filtration area factor j_N as previously defined[3].

Basic equations for non uni-dimensional filtration are presented. $j_{II,cy}$ on a cylindrical surface and $j_{III,sp}$ on a spherical surface are evaluated theoretically. An analytical method of determining j_N from experimental data are also discussed.

Reasonable coincidence between theories and experiments are reported. To increase still further the accuracy for non uni-dimensional calculations, both the medium resistance and the cake compressibility should be considered.

Acknowledgement

The authors acknowledge with gratitude the gifts of porous filter tubes and plates by Sango Toki Co., Ltd.

Nomenclatures

A	: filter-cloth area	[cm^2]
A_e	: effective filtration area	[cm^2]
a	: side length of a square medium	[cm]
e	: local void ratio, $e=\varepsilon/(1-\varepsilon)$	[−]
$\mathbf{e}_r$, $\mathbf{e}_\phi$, $\mathbf{e}_z$, $\mathbf{e}_\lambda$	: unit vectors	
g_c	: coversion factor	[dyne/G]
h	: length of a cylindrical filter surface	[cm]
j_N	: effective filtration area factor, defined by Eq. (2)	[−]
k	: permeability coefficient	[cm^2]
M	: liquid mass in cake	[g]
n	: compressibility coefficient of cake	[−]
p	: local pressure in cake at time θ	[G/cm^2]
p_i	: pressure at filter medium	[G/cm^2]
p_0	: pressure at cake surface	[G/cm^2]
Δp	: applied filtration pressure	[G/cm^2]
∇p	: gradient of p	[G/(cm^2·cm)]
$\mathbf{q}$	: velocity vector of filtrate	[cm/sec]
q	: superficial local velocity of filtrate at time θ	[cm/sec]
$\mathbf{r}$	: local apparent solid-migration velocity vector	[cm/sec]
r	: radius	[cm]
r_{eq}	: equivalent radius of square medium	[cm]
r_i	: radius of filter medium	[cm]
r_0	: radius of cake surface	[cm]
S	: hypothetical enclosure in space	[cm^2]
s	: mass fraction of solids in slurry	[−]
$\mathbf{u}$	: relative velocity vector of filtrate to solids	[cm/sec]
v	: volume of filtrate per unit medium area	[cm^3/cm^2]
V_c	: cake volume	[cm^3]
V_{ch}	: filter chamber volume	[cm^3]
v_c	: cake volume per unit medium area	[cm^3/cm^2]
W	: total mass of dry solids in cake	[g]
w	: mass of dry cake solids per unit medium area	[g/cm^2]
z	: coordinate	[cm]

Greek letters

α	: specific resistance of cake	[cm/g]
λ	: coordinate	[−]
ε	: porosity of filter cake	[−]
θ	: time	[sec]
μ	: viscosity of filrate	[g/(cm−sec)]
ρ	: density of filtrate	[g/cm^3]
ρ_s	: true density of cake solids	[g/cm^3]
ϕ	: coordinate	[−]

Suffix

I : uni-dimensional filtration

II : two-dimensional filtration on rectangular leaf
II, cy : two-dimensional filtration on cylindrical surface
III : three-dimensional filtration on circular leaf
III, sq: three-dimensional filtration on square leaf
III, sp: three-dimensional filtration on spherical surface
N : non uni-dimensional filtration

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