ON A SIMILAR SOLUTION OF LAMINAR HALF JET ALONG A CURVED STREAMLINE

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An analysis on the mixing of laminar half jet along a curved streamline is presented. Equations of motion referred to streamline coordinates are simplified by boundary layer approximations and integrated under the assumptions of similarity. A curved potential flow is dissipated by viscosity in the mixing region, resulting in a maximum value in the velocity distribution across the stream for convex flows, while the distribution is monotonous for concave flows. Pressure distributions across the stream are also presented.

1. Introduction

The curved half jet of laminar mixing has first solved by Uchida and Watanabe[1] by the use of streamline coordinates developed by Yen and Toba[2]. In the present paper a little improvement concerning the reference velocity along zero-streamline is applied to the previous paper and tables of numerical calculation are added.

2. Fundamental Equations

A steady two-dimensional laminar mixing flow of incompressible fluid is considered. All physical variables are non-dimensionalized by reference quantities, where $(1/2)\rho_S u_S^2$ is used for the reference pressure.

To describe curved flow field, streamline coordinates α and β, and their extension parameter $h_\alpha = \partial s/\partial\alpha$ and $h_\beta = \partial n/\partial\beta$ are introduced as shown in Fig. 1. These variables should satisfy Gauss' equation of orthogonality:

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{h_\alpha}\frac{\partial h_\beta}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{1}{h_\beta}\frac{\partial h_\alpha}{\partial\beta}\right)=0 \qquad (1)$$

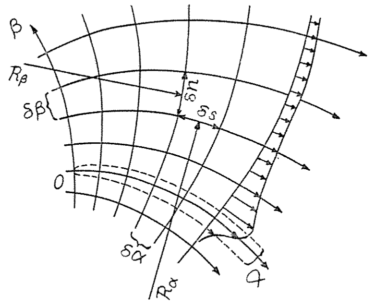

FIG. 1. Streamline coordinates.

The equation of continuity is identically satisfied by the stream function ψ, which is identified by β in streamline coordinates.

$$u \equiv \frac{1}{h_\beta}\frac{\partial\psi}{\partial\beta} = \frac{1}{h_\beta}, \qquad v \equiv \frac{1}{h_\alpha}\frac{\partial\psi}{\partial\alpha} = 0 \qquad (2)$$

Navier-Stokes' equations of motion are given by

$$\frac{u}{h_\alpha}\frac{\partial u}{\partial \alpha} = -\frac{1}{2h_\alpha}\frac{\partial p}{\partial \alpha} + \frac{1}{R_e}\frac{1}{h_\beta}\frac{\partial}{\partial \beta}\left[\frac{1}{h_\alpha h_\beta}\frac{\partial (h_\alpha u)}{\partial \beta}\right] \tag{3}$$

$$\frac{u}{h_\beta}\frac{\partial u}{\partial \beta} - \frac{u}{h_\alpha h_\beta}\frac{\partial (h_\alpha u)}{\partial \beta} = -\frac{1}{2h_\beta}\frac{\partial p}{\partial \beta} - \frac{1}{R_e}\frac{1}{h_\alpha}\frac{\partial}{\partial \alpha}\left[\frac{1}{h_\alpha h_\beta}\frac{\partial (h_\alpha u)}{\partial \beta}\right] \tag{4}$$

where 1/2 in pressure term originates in the non-dimensionalization of p by the reference dynamic pressure, and Reynolds' number R_e is defined by $R_e = u_s l_s / \nu$.

3 Boundary Layer Approximation

The boundary layer approximation is now introduced. The orders of magnitude are assumed to be,

$$\text{Order}[\beta] = R_e^{-1/2}, \quad \text{Order}[h_\alpha, h_\beta, u, p, \alpha] = 1$$

Gauss' equation and equations of motion are simplified into the following forms:

$$\frac{\partial}{\partial \beta}\left(\frac{1}{h_\beta}\frac{\partial h_\alpha}{\partial \beta}\right) = 0 \tag{5}$$

$$\frac{u}{h_\alpha}\frac{\partial u}{\partial \alpha} = -\frac{1}{2h_\alpha}\frac{\partial p}{\partial \alpha} + \frac{1}{R_e}\frac{1}{h_\beta}\frac{\partial}{\partial \beta}\left[\frac{1}{h_\alpha h_\beta}\frac{\partial (h_\alpha u)}{\partial \beta}\right] \tag{6}$$

$$-\frac{u^2}{h_\alpha h_\beta}\frac{\partial h_\alpha}{\partial \beta} = -\frac{1}{2h_\beta}\frac{\partial p}{\partial \beta}, \quad \text{or} \quad -\frac{u^2}{R_\alpha} = -\frac{1}{2h_\beta}\frac{\partial p}{\partial \beta} \tag{7}$$

where $1/R_\alpha = (1/h_\alpha h_\beta)\cdot\partial h_\alpha/\partial\beta$ represents curvature of streamline.

Elimination of pressure from the equations of motion yields vorticity equation:

$$\frac{\partial}{\partial \alpha}(h_\beta u \zeta) = \frac{1}{R_e}\frac{\partial}{\partial \beta}\left[\frac{h_\alpha}{h_\beta}\frac{\partial \zeta}{\partial \beta}\right] \tag{8}$$

where vorticity $\zeta = -(1/h_\alpha h_\beta)\cdot\partial(h_\alpha u)/\partial\beta$.

As the dependent variables, h_α and $\lambda \equiv h_\alpha/h_\beta = u h_\alpha$ are used in places of h_α and h_β. With these variables, Gauss' and vorticity equations are given by

$$\frac{\partial}{\partial \beta}\left(\frac{\lambda}{h_\alpha}\frac{\partial h_\alpha}{\partial \beta}\right) = 0 \tag{9}$$

$$\frac{\partial}{\partial \alpha}\left(\frac{\lambda}{h_\alpha^2}\frac{\partial \lambda}{\partial \beta}\right) = \frac{1}{R_e}\frac{\partial}{\partial \beta}\left[\lambda\frac{\partial}{\partial \beta}\left(\frac{\lambda}{h_\alpha^2}\frac{\partial \lambda}{\partial \beta}\right)\right] \tag{10}$$

Since $h_\alpha = h_\beta$ in outer potential flow, λ should tend to unity at $\beta = +\infty$. Boundary conditions to connect flow smoothly to the outer flow and to the inner fluid at rest are expressed by

$$\begin{aligned} &\lambda = 1, \quad \partial\lambda/\partial\beta = 0 \quad &\text{at} \quad \beta = +\infty \\ &\lambda = 0 &\text{at} \quad \beta = -\infty \end{aligned} \tag{11}$$

4. Similar Solutions

A similar solution can be obtained by introducing the single independent variable η and its functions:

$$\eta = R_e^{1/2}\alpha^{-1/2}\beta,\ \lambda = \Lambda(\eta),\ h_\alpha = \alpha^m H(\eta) \qquad u = 1/h_\beta = \alpha^{-m}\Lambda/H,\ p = \alpha^{-2m}P(\eta) \tag{12}$$

where m represents the effect of streamwise pressure gradient. The width of mixing region develops in proportion to $\sqrt{\alpha}$.

Gauss' equation is deformed and integrated as follows:

$$[(\Lambda/H)H']' = 0 \qquad \therefore \quad (\Lambda/H)H' = C \tag{13}$$

where $'$ means $d/d\eta$ and C represents a curvature parameter as shown by $1/R_\alpha = C\sqrt{R_e}\alpha^{-(1/2)-m}H^{-1}$.

The vorticity equation is given by

$$2[\Lambda(H^{-2}\Lambda\Lambda')']' + \eta(H^{-2}\Lambda\Lambda') + (1+4m)H^{-2}\Lambda\Lambda' = 0 \tag{14}$$

Substituting Eq. (13) into Eq. (14), H is eliminated and differential equation for Λ can easily be obtained.

5. Similar Solutions with Zero Pressure Gradient along the Zero-streamline

The solution of zero streamwise pressure gradient will be the most significant case of similar solutions. Similarity parameters for this case are given by $m=0$.

$$\eta = R_e^{1/2}\alpha^{-1/2}\beta,\ \lambda = \Lambda(\eta),\ h_\alpha = H(\eta),\ u = 1/h_\beta = \Lambda/H,\ p = P(\eta) \tag{15}$$

Eliminating H from Eqs. (13) and (14), an ordinary differential equation for Λ is obtained.

$$[2\Lambda(\Lambda^2)'' + (\eta - 4C)(\Lambda^2)']' - 2C\Lambda^{-1}[2\Lambda(\Lambda^2)'' + (\eta - 4C)(\Lambda^2)'] = 0 \tag{16}$$

which can be integrated once to give:

$$2\Lambda(\Lambda^2)'' + (\eta - 4C)(\Lambda^2)' = B\exp\left(2C\int_0^\eta \Lambda^{-1}d\eta\right) \tag{17}$$

When $B \neq 0$, the right hand side of Eq. (17) tends to $+\infty$ at $\eta = +\infty$ for $C>0$, and to $-\infty$ at $\eta = -\infty$ for $C<0$. To avoid this singularity B is set to zero. Then Eq. (17) is deformed to

$$\Lambda\Lambda'' + (\Lambda')^2 + (1/2)(\eta - 4C)\Lambda' = 0 \tag{18}$$

Boundary conditions are

$$\begin{aligned} &\Lambda = 1,\ \Lambda' = 0 \quad &&\text{at} \quad \eta = +\infty \\ &\Lambda = 0 &&\text{at} \quad \eta = -\infty \end{aligned} \tag{19}$$

in which $\Lambda' = 0$ is automatically satisfied by putting $B=0$.

Eq. (18) is integrated numerically by connecting to the asymptotic outer solution:

$$\Lambda_{\eta\to\infty} = 1 - \sqrt{\pi}A\,\mathrm{erfc}[(1/2)(\eta - 4C)] \tag{20}$$

at $\eta=7$ for $C=0$, which leads to other solutions of Λ for $C\neq0$ by simple shift of coordinate η. The value of A is chosen to give infinitesimal of $\Lambda(-\infty)=\text{constant}=K$. $A=0.16468$ is used for the present calculation.

The function H is calculated by integrating Eq. (13).

$$H/H_0=\exp\left[C\int_0^{\eta}\Lambda^{-1}d\eta\right] \tag{21}$$

where H_0 is H at $\eta=0$.

The results of Λ and H_0/H for several values of C are shown in Fig. 2, positive value of C corresponding to the flow along a convex streamline.

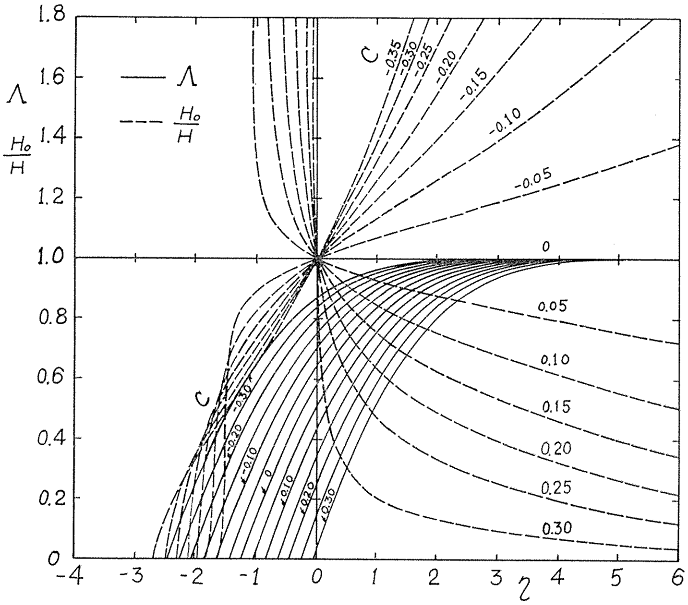

FIG. 2. Similar solution, Λ and H_0/H.

In order to know the form of zero-streamline, its curvature is now calculated.

$$\left(\frac{1}{R_\alpha}\right)_{\beta=0}=\left(\frac{1}{h_\alpha h_\beta}\frac{\partial h_\alpha}{\partial\beta}\right)_{\beta=0}=\sqrt{\frac{R_e}{\alpha}}\left(\frac{\Lambda}{H^2}H'\right)_{\eta=0}=C\sqrt{R_e}\frac{1}{\sqrt{\alpha}}\frac{1}{H_0} \tag{22}$$

The inclination angle of zero-streamline θ is reduced from the geometrical relation.

$$\frac{d\theta}{d\alpha}=-\frac{h_\alpha}{R_\alpha}=-C\frac{\sqrt{R_e}}{\sqrt{\alpha}}\quad\text{or}\quad\theta-\theta_0=-2C\sqrt{R_e}\sqrt{\alpha} \tag{23}$$

Taking $\theta_0=0$, it leads to $(d\alpha/d\theta)_{\beta=0}=1/(2C^2R_e)$. With $(\partial s/\partial\alpha)_{\beta=0}=H_0$, $(dx/ds)_{\beta=0}=\cos\theta$ and $(dy/ds)_{\beta=0}=\sin\theta$ are deformed to give

$$\frac{dx}{d\theta}=\frac{H_0}{2C^2R_e}\cos\theta,\quad\frac{dy}{d\theta}=\frac{H_0}{2C^2R_e}\sin\theta \tag{24}$$

The x, y coordinates of zero-streamline are obtained by integrating Eq. (24) with the origin $x=y=0$ at $\theta=0$.

$$\begin{aligned} x &= (H_0/2\,C^2R_e)(\cos\theta+\theta\sin\theta-1) \\ y &= (H_0/2\,C^2R_e)(\sin\theta-\cos\theta) \end{aligned} \qquad (25)$$

The forms of zero-streamline are shown in Fig. 3.

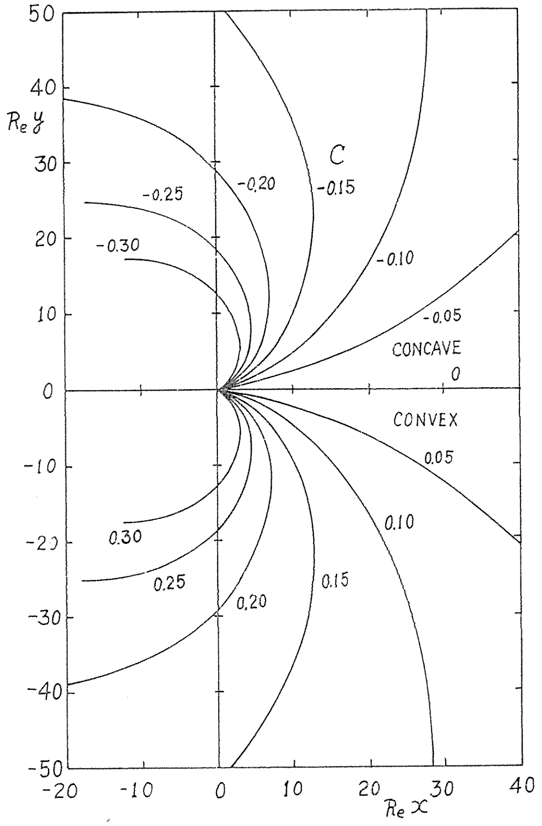

FIG. 3. Zero-streamlines.

The non-dimensional arc length along zero-streamline and normal to the streamline are given by the following integrations, choosing $s=0$ at $\alpha=0$ and $n=0$ at $\beta=0$.

$$s_{\beta=0} = \int_0^\alpha (h_\alpha)_{\beta=0}\,d\alpha = H_0\alpha$$

$$n = \int_0^\beta h_\beta\,d\beta = \sqrt{\frac{\alpha}{R_e}}\int_0^\eta \frac{H}{A}\,d\eta = \sqrt{\frac{\alpha}{R_e}}\int_{H_0}^H \frac{dH}{C} = \sqrt{\frac{\alpha}{R_e}}\,\frac{H-H_0}{C} \qquad (26)$$

A combined parameter corresponding to η is introduced, defining

$$(R_e/H_0)^{1/2}\cdot n/\sqrt{s} = C^{-1}[(H/H_0-1] \qquad (27)$$

The velocity of hypothetical basic flow is calculated from Eq. (27).

$$H_0/H = [1 + C(R_e/H_0)^{1/2}\cdot n/\sqrt{s}]^{-1} \tag{28}$$

The distribution of velocity along the normal to streamline is calculated by $u/H_0^{-1} = A(H_0/H)$ and some examples are shown in Fig. 4 as functions of $\sqrt{R_e/H_0}(n/\sqrt{s})$. The origin of abscissa indicates the position of zero-streamline. The volocity distribution for $C=0$ is confirmed to be identical with that of usual straight mixing flow3).

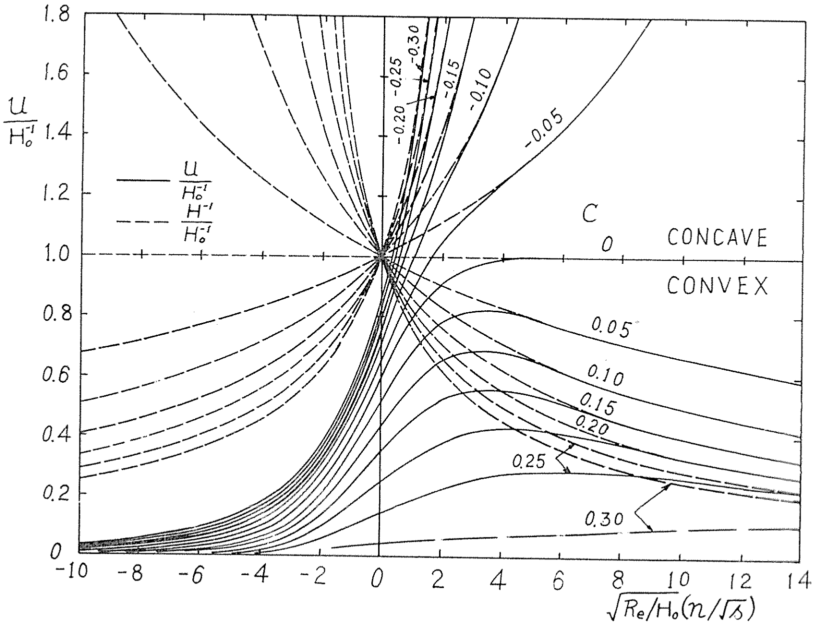

FIG. 4 .Velocity distributions.

It is noticed that in convex flow the velocity of hypothetical inviscid flow H_0/H decreases as n increases, and that the effect of viscosity decelerate flows in the inner region, thus resulting in the occurrance of a maximum velocity in between. For concave flow the velocity decreasing monotonously from outer to inner region.

The velocity ratio distribution referred to the velocity on the zero-streamline is shown in Fig. 5. The effect of increasing C from zero is somewhat complicated, at first decreasing and then increasing.

The pressure gradient across the flow is reduced from Eq. (7).

$$P' = 2A^2H^{-3}H' = 2CAH^{-2} \tag{29}$$

The pressure distribution caused by curvature effect is given by integration of Eq. (29).

$$(p - p_0)/H_0^{-2} = (P - P_0)/H_0^{-2} = 2C\int_0^{\eta} A(H/H_0)^{-2}d\eta \tag{30}$$

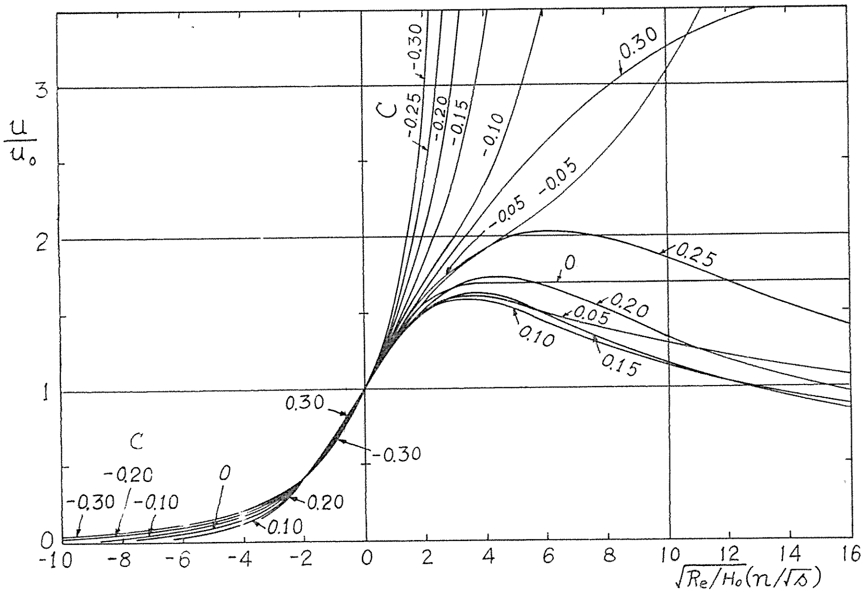

FIG. 5. Normalized velocity distributions.

The pressure expressed in terms of the dynamic pressure on the zero-streamline is given by

$$\frac{p-p_0}{u_0^2}=\frac{P-P_0}{u_0^2}=\frac{H_0^{-2}}{u_0^2}\left[2C\int_0^{\eta}\frac{\Lambda}{(H/H_0)^2}d\eta\right] \tag{31}$$

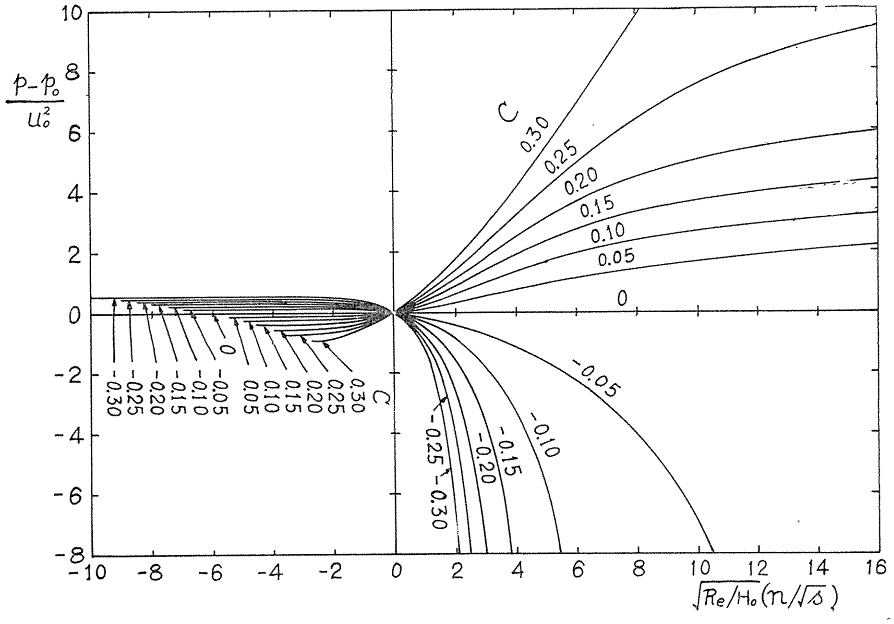

FIG. 6. Normalized pressure distributions.

This normalized pressure distribution is shown in Fig. 6. It is found that for convex flows the pressure increases from a lower constant value at inner part to a higher value at outer boundary. The difference of pressure increases as C increases.

The shearing stress along the zero-streamline, which is non-dimensionalized by the reference dynamic pressure, is expressed by

$$(\tau_{\alpha\beta})_{\beta=0}=\left(\left[\frac{2}{R_e}\frac{h_\alpha}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{u}{h_\alpha}\right)\right]\right)_{\beta=0}=\frac{2H_0^{-2}}{\sqrt{R_e\alpha}}[\Lambda(\Lambda'-2C)]_{\eta=0} \tag{32}$$

The shearing stress and its normalized form referred to dynamic pressure on the zero-streamline are shown in Fig. 7. With the increase of C the shearing stress decreases until it reaches zero at $C=0.313$.

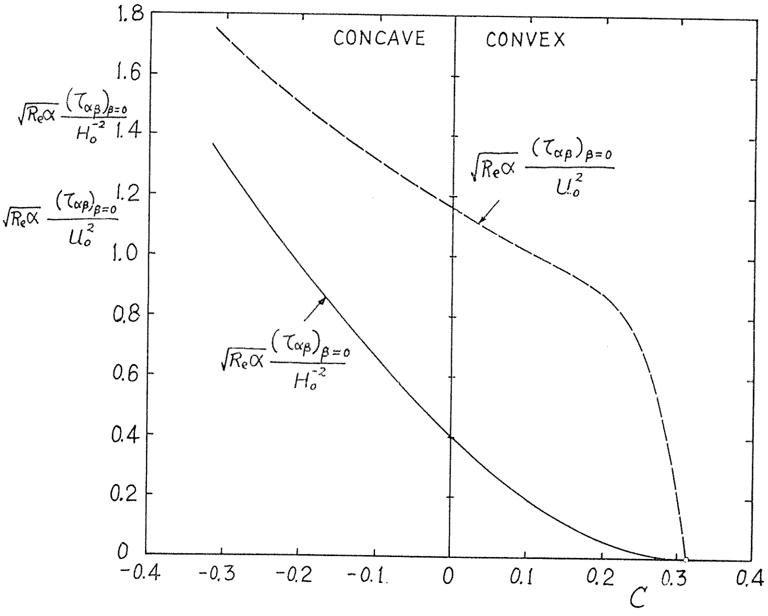

FIG. 7. Shearing stress along zero-streamline.

Calculated numerical values concerning with the present analysis are given in Tables 1 and 2.

6. Conclusion

Curvature effects on the incompressible laminar mixing in free jet boundary are analysed theoretically. Improving the previous paper the influence of curvature and of viscous mixing on the velocity and pressure distribution is investigated by the similar solution referred to the streamline coordinates. Tables of calculated values are also added.

References

1) Uchida, S. and Watanabe, K., On a Similar Solution of Curved Half Jet, Proc. 11th Int. Cong. Appl. Mech. Munich (1964), ed. H. Goertler, Berlin, Springer, pp. 677-683, 1966.
2) Yen, K. T. and Toba, K., A Theory of the Two-Dimensional Laminar Boundary Layer Over a Curved Surface, Jour. Aero. Sci., Vol. 28, No. 11, pp, 877-884, 1961.
3) Chapman, D. R., Laminar Mixing of a Compressible Fluid, NACA TN 1800, 1949.

TABLE 1. Pressure difference and Shearing stress

C	$\dfrac{u_0}{H_0^{-1}}$	$\dfrac{p_\infty - p_0}{H_0^{-2}}$	$\dfrac{p_\infty - p_0}{u_0^2}$	$\sqrt{R_e\alpha}\dfrac{(\tau_{\alpha\beta})_{\beta=0}}{H_0^{-2}}$	$\sqrt{R_e\alpha}\dfrac{(\tau_{\alpha\beta})_{\beta=0}}{u_0^2}$
−0.30	0.8701	$-\infty$	$-\infty$	1.2970	1.7131
−0.25	0.8385	$-\infty$	$-\infty$	1.1260	1.6017
−0.20	0.8013	$-\infty$	$-\infty$	0.9620	1.4980
−0.15	0.7582	$-\infty$	$-\infty$	0.8059	1.4018
−0.10	0.7086	$-\infty$	$-\infty$	0.6591	1.3127
−0.05	0.6520	$-\infty$	$-\infty$	0.5229	1.2300
0	0.5881	0	0	0.3988	1.1532
0.05	0.5163	0.8945	3.3559	0.2882	1.0814
0.10	0.4363	0.7626	4.0062	0.1928	1.0130
0.15	0.3478	0.6047	4.9995	0.1143	0.9448
0.20	0.2505	0.4240	6.7563	0.0543	0.8652
0.25	0.1445	0.2287	10.947	0.0147	0.7015
0.30	0.0320	0.0402	39.290		

TABLE 2. Velocity and Pressure distributions

C	η	$\sqrt{\dfrac{R_e}{H_0}}\dfrac{n}{\sqrt{s}}$	Λ	$\dfrac{H^{-1}}{H_0^{-1}}$	$\dfrac{u}{H_0^{-1}}$	$\dfrac{u}{u_0}$	$\dfrac{p-p_0}{H_0^{-2}}$	$\dfrac{p-p_0}{u_0^2}$
−0.30	−2.4	−26.508	0.0320	0.1117	0.0036	0.0041	0.4209	0.5560
	−2.2	−10.0149	0.1445	0.2497	0.0361	0.0415	0.4205	0.5554
	−2.0	−6.4534	0.2505	0.3406	0.0853	0.0981	0.4183	0.5525
	−1.6	−3.5268	0.4363	0.4859	0.2120	0.2436	0.4034	0.5328
	−1.2	−2.0913	0.5881	0.6145	0.3614	0.4153	0.3653	0.4825
	−0.8	−1.1754	0.7086	0.7393	0.5239	0.6021	0.2931	0.3871
	−0.4	−0.5138	0.8013	0.8664	0.6943	0.7979	0.1754	0.2317
	0	0	0.8701	1.0000	0.8701	1.0000	0	0
	0.4	0.4179	0.9190	1.1433	1.0507	1.2076	−0.2472	−0.3265
	0.8	0.7685	0.9520	1.2996	1.2373	1.4220	−0.5827	−0.7696
	1.2	1.0689	0.9732	1.4720	1.4325	1.6464	−1.0266	−1.3560
	1.6	1.3298	0.9859	1.6638	1.6403	1.8852	−1.6044	−2.1192
	2.0	1.5586	0.9930	1.8782	1.8652	2.1436	−2.3488	−3.1025
	2.4	1.7602	0.9968	2.1189	2.1122	2.4275	−3.3018	−4.3612
	2.8	1.9385	0.9986	2.3897	2.3864	2.7427	−4.5174	−5.9669
	3.2	2.0963	0.9995	2.6947	2.6932	3.0953	−6.0653	−8.0114
	3.6	2.2363	0.9998	3.0384	3.0378	3.4913	−8.0343	−10.6123
	4.0	2.3603	0.9999	3.4258	3.4256	3.9370	−10.5382	−13.9195
	4.4	2.4704	1.0000	3.8626	3.8625	4.4392	−13.7215	−18.1242
	4.8	2.5679	1.0000	4.3551	4.3550	5.0052	−17.7684	−23.4696

TABLE 2. (*Continued*)

C	η	$\sqrt{\frac{R_e}{H_0}}\frac{n}{\sqrt{s}}$	Λ	$\frac{H^{-1}}{H_0^{-1}}$	$\frac{u}{H_0^{-1}}$	$\frac{u}{u_0}$	$\frac{p-p_0}{H_0^{-2}}$	$\frac{p-p_0}{u_0^2}$
−0.25	−2.2	−19.439	0.0320	0.1707	0.0055	0.0065	0.3383	0.4812
	−2.0	−7.9881	0.1445	0.3337	0.0482	0.0575	0.3375	0.4801
	−1.6	−3.8232	0.3478	0.5113	0.1778	0.2121	0.3277	0.4662
	−1.2	−2.1976	0.5163	0.6454	0.3332	0.3974	0.2979	0.4238
	−0.8	−1.2225	0.6520	0.7659	0.4994	0.5956	0.2388	0.3397
	−0.4	−0.5328	0.7582	0.8825	0.6691	0.7980	0.1422	0.2023
	0	0	0.8385	1.0000	0.8385	1.0000	0	0
	0.4	0.4348	0.8968	1.1220	1.0061	1.2000	−0.1960	−0.2788
	0.8	0.8027	0.9372	1.2511	1.1725	1.3984	−0.4547	−0.6468
	1.2	1.1217	0.9638	1.3897	1.3395	1.5975	−0.7865	−1.1188
	1.6	1.4029	0.9804	1.5402	1.5100	1.8009	−1.2039	−1.7125
	2.0	1.6535	0.9900	1.7046	1.6876	2.0127	−1.7225	−2.4502
	2.4	1.8783	0.9952	1.8853	1.8763	2.2378	−2.3619	−3.3596
	2.8	2.0808	0.9979	2.0842	2.0798	2.4805	−3.1466	−4.4758
	3.2	2.2637	0.9991	2.3037	2.3017	2.7452	−4.1072	−5.8421
	3.6	2.4290	0.9997	2.5462	2.5453	3.0357	−5.2817	−7.5127
	4.0	2.5785	0.9999	2.8140	2.8137	3.3557	−6.7168	−9.5540
	4.4	2.7138	1.0000	3.1100	3.1099	3.7090	−8.4699	−12.0477
	4.8	2.8362	1.0000	3.4371	3.4370	4.0992	−10.6113	−15.0936
	5.0	2.8930	1.0000	3.6133	3.6133	4.3094	−11.8537	−16.8609
−0.20	−2.0	−14.593	0.0320	0.2552	0.0082	0.0102	0.2565	0.3995
	−1.8	−6.4589	0.1445	0.4363	0.0631	0.0787	0.2555	0.3979
	−1.6	−4.3176	0.2505	0.5366	0.1344	0.1678	0.2516	0.3918
	−1.2	−2.3524	0.4363	0.6800	0.2967	0.3702	0.2303	0.3587
	−0.8	−1.2872	0.5881	0.7953	0.4677	0.5836	0.1849	0.2880
	−0.4	−0.5580	0.7086	0.8996	0.6375	0.7955	0.1097	0.1708
	0	0	0.8013	1.0000	0.8013	1.0000	0	0
	0.4	0.4557	0.8701	1.1003	0.9574	1.1947	−0.1481	−0.2306
	0.8	0.8440	0.9190	1.2031	1.1056	1.3797	−0.3384	−0.5270
	1.2	1.1842	0.9520	1.3104	1.2475	1.5567	−0.5752	−0.8957
	1.6	1.1488	0.9732	1.4238	1.3856	1.7291	−0.8633	−1.3443
	2.0	1.7636	0.9859	1.5449	1.5231	1.9007	−1.2087	−1.8823
	2.4	2.0148	0.9930	1.6750	1.6633	2.0757	−1.6190	−2.5213
	2.8	2.2454	0.9968	1.8152	1.8094	2.2580	−2.1037	−3.2760
	3.2	2.4577	0.9986	1.9667	1.9640	2.4509	−2.6743	−4.1646
	3.6	2.6533	0.9995	2.1306	2.1295	2.6574	−3.3449	−5.2089
	4.0	2.8338	0.9998	2.3082	2.3077	2.8798	−4.1323	−6.4351
	4.4	3.0003	0.9999	2.5004	2.5003	3.1201	−5.0566	−7.8745
	4.8	3.1541	1.0000	2.7087	2.7086	3.3801	−6.1414	−9.5638
	5.2	3.2960	1.0000	2.9343	2.9343	3.6617	−7.4144	−11.5463
−0.15	−1.8	−11.200	0.0320	0.3731	0.0119	0.0157	0.1787	0.3109
	−1.6	−5.2825	0.1445	0.5579	0.0806	0.1063	0.1774	0.3086
	−1.2	−2.5829	0.3478	0.7208	0.2507	0.3306	0.1644	0.2859
	−0.8	−1.3764	0.5163	0.8289	0.4279	0.5644	0.1325	0.2305
	−0.4	−0.5913	0.6520	0.9185	0.5989	0.7899	0.0784	0.1364
	0	0	0.7582	1.0000	0.7582	1.0000	0	0
	0.4	0.4819	0.8385	1.0779	0.9038	1.1920	−0.1039	−0.1807
	0.8	0.8945	0.8968	1.1550	1.0357	1.3660	−0.2341	−0.4071
	1.2	1.2597	0.9372	1.2330	1.1556	1.5240	−0.3912	−0.6806
	1.6	1.5901	0.9638	1.3132	1.2657	1.6694	−0.5764	−1.0025
	2.0	1.8938	0.9804	1.3968	1.3694	1.8061	−0.7907	−1.3754
	2.4	2.1757	0.9900	1.4844	1.4696	1.9383	−1.0361	−1.8023
	2.8	2.4390	0.9952	1.5769	1.5694	2.0699	−1.3152	−2.2878
	3.2	2.6860	0.9979	1.6748	1.6712	2.2042	−1.6313	−2.8376
	3.6	2.9181	0.9991	1.7785	1.7769	2.3436	−1.9885	−3.4589
	4.0	3.1365	0.9997	1.8885	1.8879	2.4899	−2.3915	−4.1600
	4.4	3.3421	0.9999	2.0053	2.0051	2.6445	−2.8462	−4.9508
	4.8	3.5358	1.0000	2.1293	2.1292	2.8083	−3.3588	−5.8425
	5.2	3.7181	1.0000	2.2610	2.2610	2.9820	−3.9369	−6.8481
	5.4	3.8053	1.0000	2.3299	2.3298	3.0728	−4.2530	−7.3979

TABLE 2. (*Continued*)

C	η	$\sqrt{\frac{R_e}{H_0}}\frac{n}{\sqrt{s}}$	Λ	$\frac{H^{-1}}{H_0^{-1}}$	$\frac{u}{H_0^{-1}}$	$\frac{u}{u_0}$	$\frac{p-p_0}{H_0^{-2}}$	$\frac{p-p_0}{u_0^2}$
−0.10	−1.6	−8.7753	0.0320	0.5326	0.0170	0.0240	0.1080	0.2591
	−1.4	−4.3587	0.1445	0.6964	0.1007	0.1420	0.1065	0.2122
	−1.2	−2.9477	0.2505	0.7723	0.1935	0.2731	0.1022	0.2035
	−0.8	−1.5016	0.4363	0.8694	0.3793	0.5353	0.0831	0.1656
	−0.4	−0.6358	0.5881	0.9402	0.5530	0.7803	0.0492	0.0979
	0	0	0.7086	1.0000	0.7086	1.0000	0	0
	0.4	0.5152	0.8013	1.0543	0.8449	1.1923	−0.0640	−0.1275
	0.8	0.9578	0.8701	1.1059	0.9623	1.3580	−0.1423	−0.2834
	1.2	1.3527	0.9190	1.1564	1.0627	1.4997	−0.2341	−0.4662
	1.6	1.7142	0.9520	1.2069	1.1490	1.6215	−0.3387	−0.6746
	2.0	2.0512	0.9732	1.2581	1.2243	1.7277	−0.4558	−0.9078
	2.4	2.3691	0.9859	1.3105	1.2920	1.8233	−0.5851	−1.1654
	2.8	2.6713	0.9930	1.3645	1.3550	1.9123	−0.7268	−1.4474
	3.2	2.9601	0.9968	1.4205	1.4159	1.9982	−0.8811	−1.7549
	3.6	3.2367	0.9986	1.4786	1.4765	2.0837	−1.0489	−2.0884
	4.0	3.5021	0.9995	1.5390	1.5381	2.1707	−1.2308	−2.4512
	4.4	3.7570	0.9998	1.6018	1.6015	2.2601	−1.4280	−2.8439
	4.8	4.0018	0.9999	1.6672	1.6671	2.3526	−1.6416	−3.2694
	5.2	4.2370	1.0000	1.7352	1.7352	2.4487	−1.8731	−3.7304
	5.6	4.4630	1.0000	1.8060	1.8060	2.5487	−2.1239	−4.2299
−0.05	−1.4	−7.0048	0.0320	0.7406	0.0237	0.0363	0.0475	0.1117
	−1.2	−3.6159	0.1445	0.8469	0.1224	0.1877	0.0463	0.1090
	−0.8	−1.6836	0.3478	0.9224	0.3208	0.4920	0.0383	0.0902
	−0.4	−0.6967	0.5163	0.9663	0.4989	0.7651	0.0227	0.0535
	0	0	0.6520	1.0000	0.6520	1.0000	0	0
	0.4	0.5585	0.7582	1.0287	0.7800	1.1963	−0.0292	−0.0686
	0.8	1.0388	0.8385	1.0548	0.8844	1.3564	−0.0639	−0.1504
	1.2	1.4702	0.8968	1.0793	0.9679	1.4845	−0.1035	−0.2435
	1.6	1.8695	0.9372	1.1031	1.0338	1.5856	−0.1473	−0.3465
	2.0	2.2466	0.9638	1.1265	1.0858	1.6653	−0.1946	−0.4577
	2.4	2.6079	0.9804	1.1499	1.1274	1.7290	−0.2450	−0.5763
	2.8	2.9572	0.9900	1.1735	1.1618	1.7818	−0.2982	−0.7015
	3.2	3.2970	0.9952	1.1974	1.1917	1.8277	−0.3540	−0.8328
	3.6	3.6289	0.9979	1.2217	1.2191	1.8697	−0.4124	−0.9700
	4.0	3.9535	0.9991	1.2464	1.2453	1.9099	−0.4732	−1.1130
	4.4	4.2714	0.9997	1.2716	1.2711	1.9495	−0.5366	−1.2621
	4.8	4.5829	0.9999	1.2973	1.2971	1.9893	−0.6025	−1.4173
	5.2	4.8882	1.0000	1.3235	1.3234	2.0297	−0.6712	−1.5788
	5.6	5.1875	1.0000	1.3502	1.3502	2.0708	−0.7427	−1.7469
	5.8	5.3349	1.0000	1.3638	1.3638	2.0916	−0.7795	−1.8335
0	−1.2	−5.6832	0.0320	1.0000	0.0320	0.0320	0	0
	−1.0	−3.0013	0.1445	〃	0.1445	0.1445	〃	〃
	−0.8	−1.9669	0.2505	〃	0.2505	0.2505	〃	〃
	−0.6	−1.2940	0.3478	〃	0.3478	0.3478	〃	〃
	−0.4	−0.7826	0.4363	〃	0.4363	0.4363	〃	〃
	−0.2	−0.3623	0.5163	〃	0.5163	0.5163	〃	〃
	0	0	0.5881	〃	0.5881	0.5881	〃	〃
	0.2	0.3225	0.6520	〃	0.6520	0.6520	〃	〃
	0.4	0.6164	0.7086	〃	0.7086	0.7086	〃	〃
	0.6	0.8890	0.7582	〃	0.7582	0.7582	〃	〃
	0.8	1.1454	0.8013	〃	0.8013	0.8013	〃	〃
	1.0	1.3892	0.8385	〃	0.8385	0.8385	〃	〃
	1.2	1.6232	0.8701	〃	0.8701	0.8701	〃	〃
	1.4	1.8495	0.8968	〃	0.8968	0.8968	〃	〃
	1.6	2.0698	0.9190	〃	0.9190	0.9190	〃	〃
	1.8	2.2852	0.9372	〃	0.9372	0.9372	〃	〃
	2.0	2.4969	0.9520	〃	0.9520	0.9520	〃	〃
	2.2	2.7056	0.9638	〃	0.9638	0.9638	〃	〃
	2.4	2.9121	0.9732	〃	0.9732	0.9732	〃	〃

TABLE 2. (*Continued*)

C	η	$\sqrt{\frac{R_e}{H_0}}\frac{n}{\sqrt{s}}$	Λ	$\frac{H^{-1}}{H_0^{-1}}$	$\frac{u}{H_0^{-1}}$	$\frac{u}{u_0}$	$\frac{p-p_0}{H_0^{-2}}$	$\frac{p-p_0}{u_0^2}$
0	2.6	3.1168	0.9804	1.0000	0.9804	0.9804	0	0
	2.8	3.3202	0.9859	〃	0.9859	0.9859	〃	〃
	3.0	3.5226	0.9900	〃	0.9900	0.9900	〃	〃
	3.2	3.7243	0.9930	〃	0.9930	0.9930	〃	〃
	3.4	3.9255	0.9952	〃	0.9952	0.9952	〃	〃
	3.6	4.1263	0.9968	〃	0.9968	0.9968	〃	〃
	3.8	4.3268	0.9979	〃	0.9979	0.9979	〃	〃
	4.0	4.5271	0.9986	〃	0.9986	0.9986	〃	〃
	4.2	4.7274	0.9991	〃	0.9991	0.9991	〃	〃
	4.4	4.9275	0.9995	〃	0.9995	0.9995	〃	〃
	4.6	5.1276	0.9997	〃	0.9997	0.9997	〃	〃
	4.8	5.3276	0.9998	〃	0.9998	0.9998	〃	〃
	5.0	5.5277	0.9999	〃	0.9999	0.9999	〃	〃
	5.2	5.7277	0.9999	〃	0.9999	0.9999	〃	〃
	5.4	5.9277	1.0000	〃	1.0000	1.0000	〃	〃
	5.6	6.1277	1.0000	〃	1.0000	1.0000	〃	〃
	5.8	6.3277	1.0000	〃	1.0000	1.0000	〃	〃
	6.0	6.5277	1.0000	〃	1.0000	1.0000	〃	〃
0.05	−1.0	−4.6719	0.0320	1.3048	0.0417	0.0808	−0.0323	−0.1210
	−0.8	−2.4723	0.1445	1.1411	0.1649	0.3194	−0.0298	−0.1117
	−0.4	−0.9103	0.3478	1.0477	0.3644	0.7058	−0.0181	−0.0680
	0	0	0.5163	1.0000	0.5163	1.0000	0	0
	0.4	0.6967	0.6520	0.9663	0.6301	1.2205	0.0226	0.0849
	0.8	1.2913	0.7582	0.9394	0.7122	1.3796	0.0483	0.1811
	1.2	1.8305	0.8385	0.9162	0.7682	1.4879	0.0758	0.2844
	1.6	2.3388	0.8968	0.8953	0.8029	1.5551	0.1043	0.3914
	2.0	2.8307	0.9372	0.8760	0.8210	1.5903	0.1331	0.4995
	2.4	3.3157	0.9638	0.8578	0.8268	1.6014	0.1617	0.6067
	2.8	3.8000	0.9804	0.8403	0.8238	1.5958	0.1898	0.7120
	3.2	4.2879	0.9900	0.8235	0.8152	1.5791	0.2105	0.8144
	3.6	4.7821	0.9952	0.8070	0.8032	1.5558	0.2434	0.9134
	4.0	5.2844	0.9979	0.7910	0.7893	1.5298	0.2689	1.0089
	4.4	5.7959	0.9991	0.7753	0.7746	1.5005	0.2934	1.1008
	4.8	6.3173	0.9997	0.7600	0.7597	1.4715	0.3169	1.1892
	5.2	6.8490	0.9999	0.7449	0.7448	1.4427	0.3396	1.2741
	5.6	7.3914	1.0000	0.7302	0.7301	1.4142	0.3613	1.3557
	6.0	7.9448	1.0000	0.7157	0.7157	1.3863	0.3823	1.4342
	6.2	8.2257	1.0000	0.7086	0.7086	1.3725	0.3924	1.4722
0.10	−0.8	−3.8741	0.0320	1.6224	0.0522	0.1196	−0.0480	−0.2522
	−0.6	−1.9898	0 1445	1.2484	0.1804	0.4136	−0.0416	−0.2183
	−0.4	−1.1169	0.2505	1.1257	0.2820	0.6464	−0.0306	−0.1608
	0	0	0.4363	1.0000	0.4363	1.0000	0	0
	0.4	0.8140	0.5881	0.9247	0.5438	1.2464	0.0378	0.1988
	0.8	1.5016	0.7086	0.8694	0.6161	1.4121	0.0796	0.4180
	1.2	2.1264	0.8013	0.8246	0.6608	1.5146	0.1229	0.6458
	1.6	2.7199	0.8701	0.7862	0.6840	1.5679	0.1663	0.8738
	2.0	3.3008	0.9190	0.7518	0.6909	1.5836	0.2087	1.0963
	2.4	3.8811	0.9520	0.7204	0.6858	1.5720	0.2492	1.3095
	2.8	4.4696	0.9732	0.6911	0.6726	1.5415	0.2876	1.5110
	3.2	5.0724	0.9859	0.6635	0.6541	1.4992	0.3236	1.7000
	3.6	5.6940	0.9930	0.6372	0.6328	1.4503	0.3571	1.8959
	4.0	6.3376	0.9968	0.6121	0.6101	1.3985	0.3881	2.0390
	4.4	7.0059	0.9986	0.5880	0.5872	1.3117	0.4168	2.1900
	4.8	7.7005	0.9995	0.5650	0.5646	1.2942	0.4434	2.3295
	5.2	8.4231	0.9998	0.5428	0.5427	1.2439	0.4680	2.4584
	5.6	9.1751	0.9999	0.5215	0.5215	1.1952	0.4906	2.5774
	6.0	9.9568	1.0000	0.5011	0.5010	1.1484	0.5115	2.6872
	6.4	10.7722	1.0000	0.4814	0.4814	1.1034	0.5308	2.7886

TALBE 2. (*Continued*)

C	η	$\sqrt{\frac{R_e}{H_0}}\frac{n}{\sqrt{s}}$	Λ	$\frac{H^{-1}}{H_0^{-1}}$	$\frac{u}{H_0^{-1}}$	$\frac{u}{u_0}$	$\frac{p-p_0}{H_0^{-2}}$	$\frac{p-p_0}{u_0^2}$
0.15	−0.6	−3.2154	0.0320	1.9316	0.0618	0.1776	−0.0476	−0.3936
	−0.4	−1.5062	0.1445	1.2919	0.1867	0.5369	−0.0363	−0.3002
	0	0	0.3478	1.0000	0.3478	1.0000	0	0
	0.4	0.9997	0.5163	0.8696	0.4489	1.2908	0.0447	0.3696
	0.8	1.8294	0.6520	0.7847	0.5116	1.4712	0.0924	0.7644
	1.2	2.5829	0.7582	0.7208	0.5465	1.5714	0.1403	1.1601
	1.6	3.3036	0.8385	0.6687	0.5606	1.6121	0.1865	1.5422
	2.0	4.0163	0.8968	0.6240	0.5596	1.6091	0.2300	1.9018
	2.4	4.7378	0.9372	0.5846	0.5479	1.5753	0.2702	2.2341
	2.8	5.4801	0.9638	0.5488	0.5290	1.5211	0.3068	2.5370
	3.2	6.2529	0.9804	0.5160	0.5059	1.4546	0.3399	2.8104
	3.6	7.0638	0.9900	0.4855	0.4807	1.3822	0.3696	3.0555
	4.0	7.9191	0.9952	0.4571	0.4549	1.3080	0.3960	3.2743
	4.4	8.8240	0.9979	0.4304	0.4295	1.2349	0.4196	3.4689
	4.8	9.7833	0.9991	0.4053	0.4049	1.1643	0.4405	3.6418
	5.2	10.8011	0.9997	0.3817	0.3815	1.0970	0.4590	3.7952
	5.6	11.8815	0.9999	0.3594	0.3594	1.0334	0.4755	3.9314
	6.0	13.0285	1.0000	0.3385	0.3385	0.9733	0.4901	4.0522
	6.4	14.2464	1.0000	0.3188	0.3188	0.9166	0.5031	4.1592
	6.6	14.8833	1.0000	0.3094	0.3094	0.8895	0.5090	4.2082
0.20	−0.4	−2.6222	0.0320	2.1028	0.0672	0.2684	−0.0339	−0.5395
	−0.2	−0.9344	0.1445	1.2298	0.1777	0.7095	−0.0189	−0.3015
	0	0	0.2505	1.0000	0.2505	1.0000	0	0
	0.4	1.3364	0.4363	0.7891	0.3443	1.3743	0.0423	0.6735
	0.8	2.4100	0.5881	0.6748	0.3968	1.5840	0.0856	1.3646
	1.2	3.3822	0.7086	0.5965	0.4227	1.6873	0.1273	2.0290
	1.6	4.3176	0.8013	0.5366	0.4300	1.7166	0.1660	2.6453
	2.0	5.2520	0.8701	0.4877	0.4244	1.6940	0.2011	3.2035
	2.4	6.2097	0.9190	0.4460	0.4099	1.6362	0.2322	3.7003
	2.8	7.2093	0.9520	0.4095	0.3899	1.5563	0.2596	4.1366
	3.2	8.2665	0.9732	0.3769	0.3668	1.4641	0.2834	4.5158
	3.6	9.3948	0.9859	0.3473	0.3424	1.3670	0.3040	4.8432
	4.0	10.6065	0.9930	0.3204	0.3182	1.2700	0.3216	5.1243
	4.4	11.9129	0.9968	0.2956	0.2947	1.1764	0.3367	5.3647
	4.8	13.3247	0.9986	0.2729	0.2725	1.0877	0.3496	5.5701
	5.2	14.8524	0.9995	0.2519	0.2517	1.0048	0.3606	5.7454
	5.6	16.5064	0.9998	0.2325	0.2324	0.9279	0.3700	5.8948
	6.0	18.2978	0.9999	0.2146	0.2146	0.8567	0.3779	6.0222
	6.4	20.2383	1.0000	0.1981	0.1981	0.7908	0.3848	6.1307
	6.8	22.3403	1.0000	0.1829	0.1829	0.7300	0.3906	6.2233
0.25	−0.2	−1.9541	0.0320	1.9552	0.0625	0.4325	−0.0136	−0.6506
	0	0	0.1445	1.0000	0.1445	1.0000	0	0
	0.4	2.1295	0.3478	0.6526	0.2270	1.5703	0.0297	1.4203
	0.8	3.7372	0.5163	0.5170	0.2669	1.8467	0.0584	2.7956
	1.2	5.1820	0.6520	0.4364	0.2840	1.9653	0.0846	4.0498
	1.6	6.5789	0.7582	0.3781	0.2867	1.9836	0.1078	5.1613
	2.0	7.9881	0.8385	0.3337	0.2798	1.9357	0.1280	6.1264
	2.4	9.4502	0.8968	0.2974	0.2667	1.8452	0.1452	6.9521
	2.8	10.9979	0.9372	0.2667	0.2500	1.7295	0.1598	7.6496
	3.2	12.6600	0.9638	0.2401	0.2314	1.6011	0.1720	8.2336
	3.6	14.4638	0.9804	0.2166	0.2124	1.4695	0.1821	8.7185
	4.0	16.4353	0.9900	0.1957	0.1938	1.3408	0.1905	9.1192
	4.4	18.6007	0.9952	0.1770	0.1761	1.2187	0.1974	9.4490
	4.8	20.9858	0.9979	0.1601	0.1598	1.1053	0.2030	9.7200
	5.2	23.6175	0.9991	0.1448	0.1447	1.0013	0.2077	9.9421
	5.6	26.5237	0.9997	0.1310	0.1310	0.9064	0.2115	10.1240
	6.0	29.7346	0.9999	0.1186	0.1186	0.8203	0.2146	10.2729
	6.4	33.2828	1.0000	0.1073	0.1073	0.7423	0.2171	10.3949
	6.8	37.2039	1.0000	0.0971	0.0971	0.6717	0.2192	10.4945
	7.0	39.3165	1.0000	0.0923	0.0923	0.6389	0.2201	10.5387