ON FRICTIONAL VISCOSITY IN VIBRATING SYSTEMS

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Summary

When a constant force S is applied to a mass m which is under control of the Coulomb friction μmg between the horizontal base surface (See Fig. 1), the following wellknown rule is maintained as far as no other dynamical effects are concerned.

"When $S>\mu mg$, the velocity of the mass is ever accelerated, and when $S<\mu mg$, it is ever decelerated until the mass comes to standstill."

However, if the base surface is in vibration in the horizontal plane, and if some condition is fulfilled, the mean velocity of the mass can be kept constant as if there exists some viscous resistance. This phenomena may be called as "Frictional Viscosity." The mean velocity thus observed has, in general, components not only in the direction of S but also in the lateral direction to it.

§ 1. The mechanical configuration and notations

Let us consider a system shown in Fig. 1. The base surface AA is vibrating in some horizontal direction, and a free mass m weighing mg on it is acted by an external force S in x-direction.

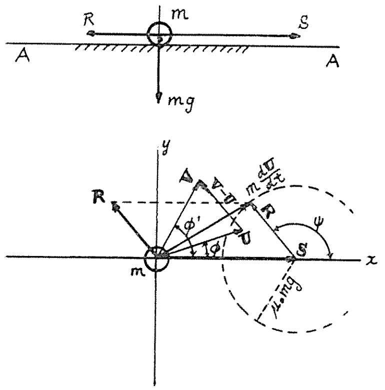

FIG. 1. Schematic drawing of the system in discussion.

m: mass

$\mathbf{U}$: velocity of m, $\mathbf{u}=\dfrac{\omega}{g}\mathbf{U}=ue^{i\phi}$

$\mathbf{V}$: velocity of the base, $\mathbf{v}=\dfrac{\omega}{g}\mathbf{V}=ve^{i\phi'}=a\cos\tau\, e^{i\phi'}$, $\mathbf{v}-\mathbf{u}=we^{i\psi}$

ϕ, ϕ', ψ: directional angles defined above, measured from x-axis.

μ: friction coefficient

$0\leq\mu\leq\mu_1$ when $\mathbf{v}-\mathbf{u}=0$,

$\mu=\mu_0$ when $\mathbf{v}-\mathbf{u}\neq 0$, $\mu_1>\mu_0$ in general.

S: external force, $\mathbf{s}=\mathbf{S}/mg$

R: frictional resistance

$\mathbf{R}=\mu_0 mge^{i\psi}$ when $\mathbf{v}-\mathbf{u}\neq 0$,

$|R|=\mu mg$ and $\mu<\mu_1$ when $\mathbf{v}-\mathbf{u}=0$.

ω: frequency of vibration of the base.

t: time, $\tau=\omega t$

$[\,\dot{}\,]=d/d\tau$

§2. Equation of motion

The frictional resistance to the mass acts in the direction of $\mathbf{v}-\mathbf{u}$, or $\mathbf{R}=\mu_0 mge^{i\psi}$, except the case when $\mathbf{v}=\mathbf{u}$. The angle ψ is given by the following equation

$$e^{i\psi}=\frac{\mathbf{v}-\mathbf{u}}{|\mathbf{v}-\mathbf{u}|}=\frac{(v_x-u_x)+i(v_y-u_y)}{\sqrt{(v_x-u_x)^2+(v_y-u_y)^2}}. \tag{1}$$

The equation of motion for the mass m

$$m\frac{d\mathbf{U}}{d\tau}=\mathbf{S}+\mathbf{R}$$

can be reduced to the non-dimensional one

$$\frac{d}{d\tau}(ue^{i\phi})=s+\mu_0 e^{i\psi}. \tag{2}$$

If, however, the condition $\mathbf{u}=\mathbf{v}$, or non-slip state, is once attained, it remains so till the time when $\left|\frac{d\mathbf{v}}{d\tau}-s\right|$ overcomes the value μ_1, inspite of the relation (2).

§3. A graphical method of solution

Eq. (2) can be solved approximately by a simple graphical method shown in Fig. 2. Let us select a small time interval $\Delta\tau=\pi/n$ where n is an integer of the order of 10 or so ($n=12$ is used in Fig. 2 and following examples), and assume the vectors $\mathbf{u}$ and $\mathbf{v}$ at a time $\tau=r\Delta\tau$ to be at the points r and r' in the hodograph plane respectively.

As the eq. (2) is equivalent to the difference equation

$$\Delta\mathbf{u}=\mathbf{s}\Delta\tau+\mu_0\Delta\tau e^{i\psi},$$

the point $r+1$ can be determined by the vector summation

$$\mathbf{u}_{r+1}=\mathbf{u}_r+\mathbf{s}\Delta\tau+\mu_0\Delta\tau e^{i\psi_{r\,1/2}},$$

where $\psi_{r+1/2}$ denotes the value of ψ at $\tau=\left(r+\frac{1}{2}\right)\Delta\tau$ and shows approximately the effective mean value during the time interval $\Delta\tau$.

The point s_r in Fig. 2 denotes the vector $\mathbf{u}+\mathbf{s}\Delta\tau$ and the point s_r' the vector

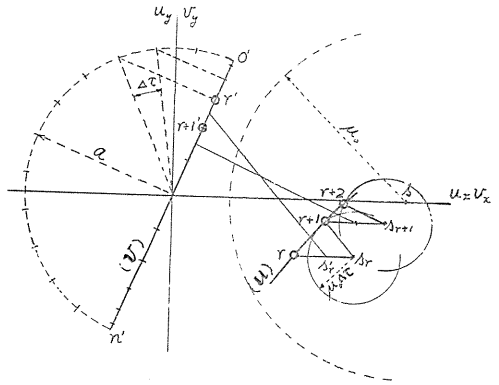

FIG. 2. Graphical method of solution.

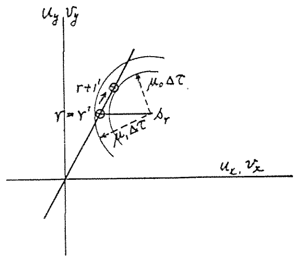

FIG. 3. Detection of non-slip range.

$\mathbf{u}+\mathbf{s}\Delta\tau/2$. The point $r+1$ can be found as the intersection of a straight line through s_r parallel to the line $s_r'\left(r+\frac{1}{2}\right)'$ and a small circle centered on s_r and with a radius $\mu_0\Delta\tau$, because in the triangle $\Delta r \cdot r+1 \cdot s_r$ the midpoint between r and $r+1$ lies just on the line $s_r'\left(r+\frac{2}{1}\right)'$ and is considered to denote approximately the vector $\mathbf{u}_{r+1/2}$. The curve connecting such points $0, \ldots . r, r+1, r+2, \ldots .$ gives a hodographic figure of $\mathbf{u}$. If the initial value of $\mathbf{u}$ is given, such a curve can always be plotted for any type of given $\mathbf{v}$ motion.

Sometime in the course of this process, one might find the condition that r just coincides with r'. In such cases a special treatment is needed. If the point $(r+1)'$ lies inside the circle mentioned above but with μ_1 instead of μ_0, non-slip motion shall be kept unchanged and $r+1$ again coincides with $(r+1)'$, (Fig. 3). In general, a much smaller $\Delta\tau$ shall be preferably used to find and discuss on such condition.

§ 4. Some examples, special features, and the Frictional Viscosity

(1) Case I; $s>\mu_0$.

Δu_x or $\frac{du_x}{d\tau}$ is always positive, so that the mass m is ever accelerated in x-direction and there exists no limit cycle.

(2) Case II; $s<\mu_0$.

If we assume a very large value of u_x, $\frac{du_x}{d\tau}$ becomes negative. Thus, this system has a stable limit cycle with some mean value of u_x, and the smaller the value of μ_0-s, the larger the value of $u_{x\mathrm{mean}}$ is observed. Examples for this case are shown in Fig. 4.

(3) Case III; $s<\mu_0$, with considerably large value of μ_0-s.

Non-slip period is some times observed in this case. Fig. 5 shows such cases for example.

(4) The frictional viscosity.

The cases II and III are somewhat analogous to the motion of a mass through some viscous fluid when it has reached to the stable state. Thus the author call

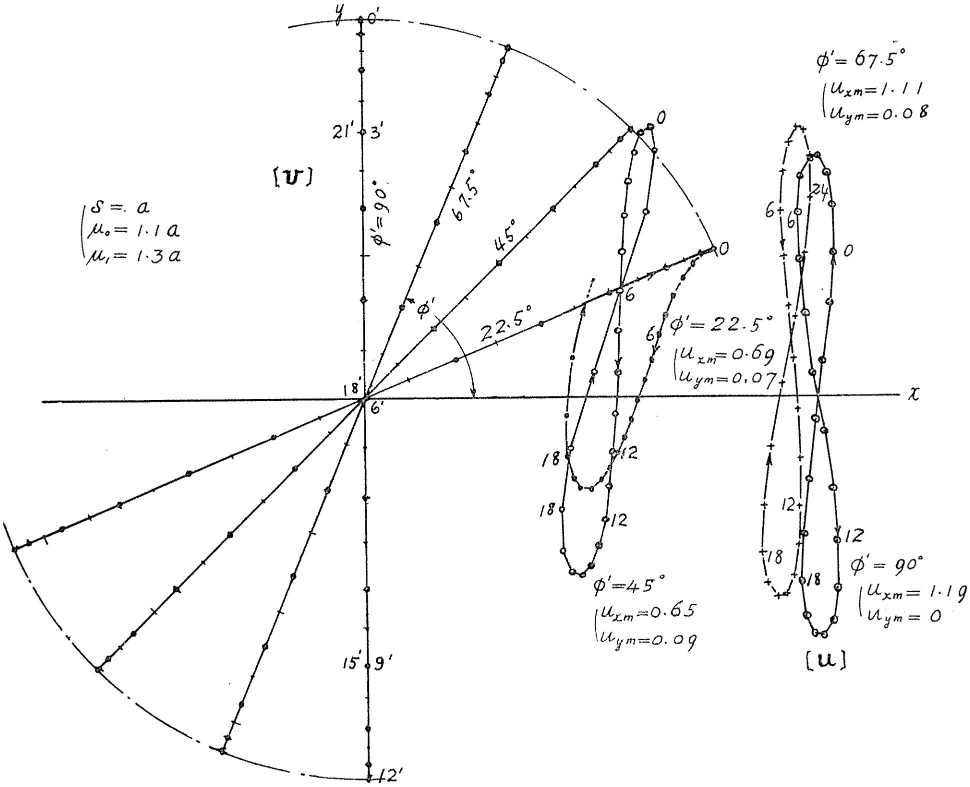

FIG. 4. Limit cycles obtained by the graphical methods ($v=a\cos\tau e^{i\phi'}$, $\phi'=90°$, $67.5°$, $45°$, $22.5°$; $s=a$, $\mu_0=1.1\,a$, $\mu_1=1.3\,a$)

this phenomena the **"Frictional Viscosity"**. The direct calculation of $u_{x\mathrm{mean}}$ and the effective coefficient of viscosity seams to be difficult except for the case of one dimensional problem shown below, but we can easily deduce them from the results of these graphical calculations.

(5) The mean velocity in y-direction.

The mean velocity $u_{y\mathrm{mean}}$ is also observed in general. This means that, in the meaning of average viewpoint, the frictional power is consumed selectively into a specified direction other than that of **s**.

§ 5. One dimensional problem

When the direction of vibration is coincident with that of **s**, or $v_y=u_y=0$, the method of solution shown above is not favourable one owing to the accuracy and technical confusion. A better method is to use velocity curves on time-abscissa. In Figs. 7 and 8, the curve A shows the motion of the base surface $a\cos\tau$ and B, B′, C, C′ the auxiliary curves showing $(s\pm\mu_0)\tau$ and $(s\pm\mu_1)\tau$ respectively. When $u-v>0$ at a time τ_0, the trace of u must be parallel to the curve B′, and, if $s-\mu_0<0$, it crosses the v-curve with no exception at some point (u_1 at time τ_1 in Fig. 7). In the next period, $u-v$ becomes negative and the motion is parallel

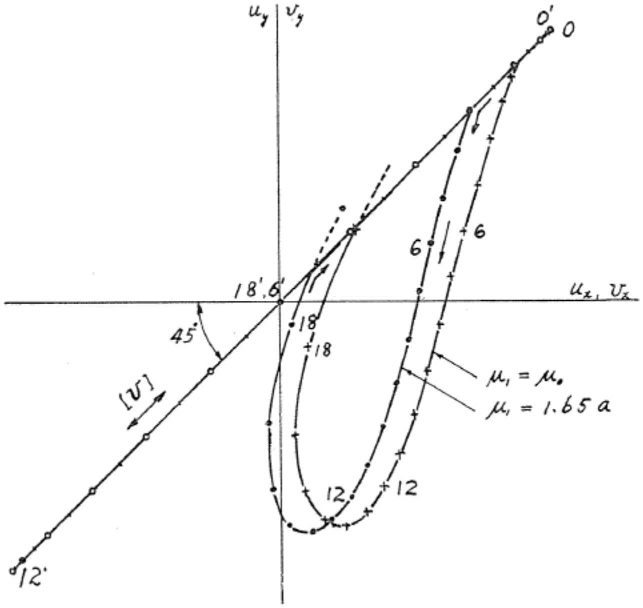

Fig. 5. Limit cycles obtained by the graphical methods
($v=a\cos\tau e^{i\phi'}$, $\phi'=45°$; $s=1.1\,a$, $\mu_0=1.3\,a$)

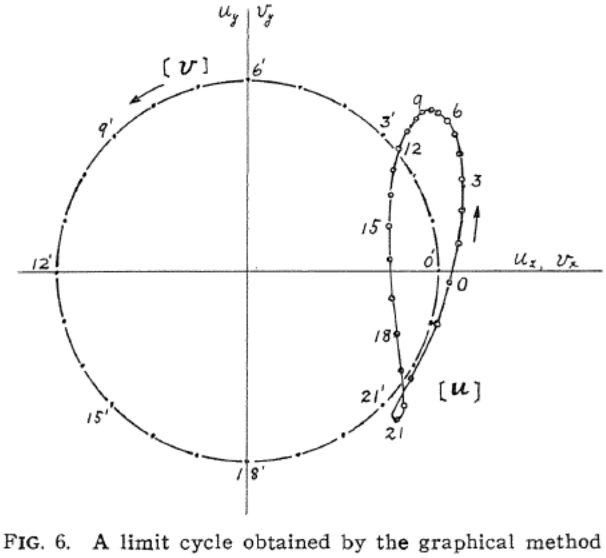

FIG. 6. A limit cycle obtained by the graphical method
($v=ae^{i\tau}$, $s=a$, $\mu_0=1.15\,a$. $u_{xm}=0.908\,a$, $u_{ym}=0.234\,a$)

to the curve B so far as the condition $\frac{dv}{d\tau}<s+\mu_1$ is fulfilled, and it again crosses the v-curve (u_2 at τ_2). Similar process as continued will soon converges to a stational state, a limit cycle shown with a chain line in Fig. 7.

If, however, $\frac{dv}{d\tau}$ at τ_1 or τ_2 lies between $s\pm\mu_1$, the u-curve does not detatch

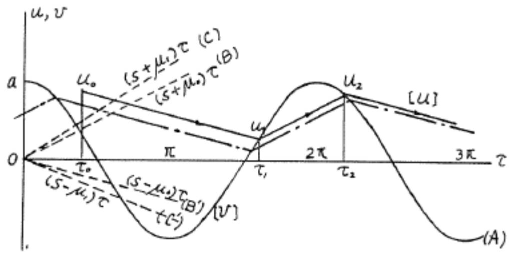

FIG. 7. An example of the graphical solution for one dimensional problems (The chain line shows the limit cycle)

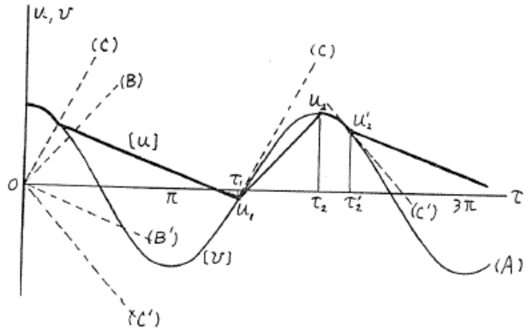

FIG. 8. An example of the graphical solution for one dimensional problems with non-slip periods.

the v-curve until the time τ_2' at which such condition is released. An example of such cases is demonstrated in Fig. 8. At τ_2' the tangent line to the v-curve is parallel to C′. The solutions for one dimensional problems can be mathematically obtained, but we omit description for them because the graphical expressions are more explanatory.