"SUMMED AND DIFFERENTIAL HARMONIC OSCILLATIONS IN NONLINEAR VIBRATORY SYSTEMS"

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CONTENTS

General Introduction

When there are nonlinear spring characteristics in vibratory systems having multiple degree-of-freedom, the so-called "summed and differential harmonic oscillations" take place[1][2] with ordinary sub-harmonic oscillations. Summed and differential harmonic oscillations may be grouped into two classes, as follows:

(1) Summed and differential harmonic oscillations in vibratory systems with unsymmetrical nonlinearity occur when the frequency ω of external periodic force becomes nearly equal to sum of or difference in two natural frequencies p_i and p_j of the system, *i.e.*,

$$\omega \fallingdotseq p_i \pm p_j \qquad (p_i > p_j). \tag{1}$$

At the frequency ω of the external force satisfied Eq. (1), two vibrations of frequencies ω_i and ω_j which are nearly equal to the natural frequencies p_i and p_j respectively, *i.e.*,

$$\omega_i \fallingdotseq p_i, \qquad \omega_j \fallingdotseq p_j. \tag{2}$$

appear simultaneously, and the so-called "summed and differential harmonic oscillations" take place[3].

(2) Summed and differential harmonic oscillations in vibratory systems with symmetrical nonlinear spring characteristics occur when the frequency ω satisffies the following relation:

$$\omega \fallingdotseq |2p_i \pm p_j|, \tag{3}$$

or

$$\omega \fallingdotseq |p_i \pm p_j \pm p_k|. \tag{4}$$

When the relation of Eq. (3) holds, two vibrations of frequencies $\omega_i(\fallingdotseq p_i)$ and $\omega_j(\fallingdotseq p_j)$ appear[4], and when the relation of Eq. (4) is satisfied, three vibrations with frequencies $\omega_i(\fallingdotseq p_i)$, $\omega_j(\fallingdotseq p_j)$ and $\omega_k(\fallingdotseq p_k)$ occur simultaneously[5]

In the present paper, summed and differential harmonic oscillations induced by unsymmetrical and symmetrical nonlinearities are treated in Part A and Part B severally. Summed and differential harmonic oscillations occurring when the frequency ω of external force satisfies the relations of Eqs. (1), (3) and (4) are represented by the expressions $[p_i \pm p_j]$, $[2p_i \pm p_j]$ and $[p_i \pm p_j \pm p_k]$ separately.

Incidentally, when $p_i = p_j$ or $p_i = p_j = p_k$ in Eqs. (1), (3) and (4), summed and differential harmonic oscillations become sub-harmonic oscillations of order 1/2 or 1/3. Accordingly sub-harmonic oscillations are considered as special cases of summed and differential harmonic oscillations. In this paper, sub-harmonic oscillations of order 1/2 and order 1/3 are expressed by $[2p_i]$ and $[3p_i]$ respectively.

Part A. Summed and Differential Harmonic Oscillations Induced by Unsymmetrical Nonlinearity

In Part A, summed and differential harmonic oscillations of $[p_i \pm p_j]$ which are caused by unsymmetrical nonlinear spring characteristics are discussed, and in Chaper I, summed and differential harmonic oscillations appearing in usual vibratory systems with n degree-of-freedom in which rectilinear vibrations occur, are treated, and in Chapter II, summed and differential harmonic oscillations of rotating shaft systems are researched. Generally, in the rotating shaft systems, lateral vibrations of the shaft are not rectilinear vibrations, but whirling motions[6]. Consequently all modes of vibrations of summed and differential harmonic oscillations of the rotating shaft are always whirling motions.

Chapter I. Summed and Differential Harmonic Oscillations $[p_i \pm p_j]$ in Vibratroy Systems with Unsymmetrical Nonlinear Spring Characteristics

1. Introduction

In the present chapter, summed and differential harmonic oscillations of rectilinear vibratroy systems are treated. When multiple degree-of-freedom systems have unsymmetrical nonlinear spring characteristics, this nonlinearity results in occurrence of sub-harmonic oscillations and summed and differential harmonic oscillations. Analytically conditions for possibility of occurrence of summed and differential harmonic oscillations induced by unsymmetrical nonlinearity are discussed, and modes of vibrations which can take place actually and equations of response curves are determined. Further stability problems for summed and differential harmonic oscillations are studied and comparison of obtained theoretical conclusions with experimental results is performed.

2. Summed and differential harmonic oscillations $[p_i \pm p_j]$ in rectilinear vibratory systems[3)]

The equation of motion of n degree-of-freedom systems are given as follows:

$$\sum_{j=1}^{n}\{m_{ij}\ddot{x}_j + c_{ij}\dot{x}_j + k_{ij}x_j + \varphi_i(x_1, x_2, \ldots, x_n)\} = q_i \cos \omega t, \quad (i=1, 2, \ldots, n) \tag{1-1}$$

in which x_i is coordinate; $m_{ij}\ddot{x}_j$ is term of inertia; k_{ij} is spring constant; c_{ij} damping coefficient; φ_i nonlinear term; $q_i \cos \omega t$ external force. Here, c_{ij} and φ_i are small quantities and obviously the relations $m_{ij}=m_{ji}$, $k_{ij}=k_{ji}$ hold here. For brevity, notation $\sum$ is used in place of $\sum_{j=1}^{n}$. The frequency equation of system represented by Eq. (1-1) is

$$\Delta = |(k_{ij} - m_{ij}p^2)| = 0, \tag{1-2}$$

where p is natural frequency, Δ is a determinant consisting of $(k_{ij}-m_{ij}p^2)$. Let the cofactor of Δ be Δ_{ij}, and Δ and Δ_{ij} when $p=p_r$ be Δ_r and $\Delta_{r,ij}$ respectively, then transformation from generalized coordinates x_i into normal coordinates X_i is given by

$$\left.\begin{aligned} x_i &= \sum B_{ij}X_j, \qquad (i=1, 2, \ldots, n) \\ \text{where} \quad B_{ij} &= \Delta_{j,ij}\{\sum_{r,s} m_{rs}\Delta_{j,rj}\Delta_{j,sj}\}^{-1/2} \end{aligned}\right\} \tag{1-3}$$

Inserting Eq. (1-3) into Eq. (1-1), we have

$$\sum\{(\sum_k m_{ik}\Delta_{j,kj})(\sum_{r,s} m_{rs}\Delta_{j,rj}\Delta_{j,sj})^{-1/2}\cdot(\ddot{X}_j + p_j^2 X_j)\} = q_i \cos \omega t - \varphi_i - \sum c_{ij}\dot{x}_j. \tag{1-4}$$

If we denote the determinant consisting of $(\sum_k m_{ik}\Delta_{j,kj})(\sum_{r,s} m_{rs}\Delta_{j,rj}\Delta_{j,sj})^{-1/2}$ by D, and the cofactor of D by D_{ij}, we have the following relation:

$$B_{ij} = \frac{D_{ij}}{D}. \tag{1-5}$$

From the simultaneous linear equations of $(\ddot{X}_s + p_s^2 X_s)$, *i.e.*, from Eq. (1-4), the following equation is given by using of the above relation:

$$\begin{aligned} \ddot{X}_i + p_i^2 X_i &= \sum B_{ij}(q_i \cos \omega t - \varphi_i - \sum_k c_{jk}\dot{x}_k) \\ &= F_i \cos \omega t - \psi_i(X_1, X_2, \ldots, X_n) - \sum C_{ij}\dot{X}_j \\ &\qquad (i = 1, 2, \ldots, n) \end{aligned} \tag{1-6}$$

In Eq. (1-6), the symbols F_i, ψ_i, C_{ij} are as follows:

$$\left.\begin{aligned} &F_i = \sum B_{ji} q_j, \quad \psi_i(X_1, X_2, \ldots, X_n) = \sum B_{ji}\varphi_j(x_1, x_2, \ldots, x_n), \\ &C_{ij} = \sum_{r,s} B_{ri} b_{rs} B_{sj}. \end{aligned}\right\} \tag{1-7}$$

Eq. (1-6) gives the equations of motion represented by normal coordinate X_j.

Kinetic energy T, potential energy V and dissipation function C'' are expressed by

$$\left.\begin{aligned} 2T &= \sum_{r,s} m_{rs}\dot{x}_r\dot{x}_s = \sum \dot{X}_j^2, \\ 2V &= \sum_{r,s} k_{rs} x_r x_s + \sum \Phi_j = \sum(p_j^2 X_j^2 + \Psi_j), \\ 2C'' &= \sum_{r,s} c_{rs}\dot{x}_r\dot{x}_s = \sum_{r,s} C_{rs}\dot{X}_r\dot{X}_s. \end{aligned}\right\} \tag{1-8}$$

If nonlinear terms φ_i in Eq. (1-1) consist of the second and the third powers of coordinates, Φ_j and Ψ_j are quadratic equations of x_j and X_j respectively. Further, the relations

$$\left.\begin{aligned} &\frac{\partial V}{\partial x_i} = \sum k_{ij}x_j + \varphi_i = \sum k_{ij}x_j + \sum_{r,s} {}_i\alpha'_{rs}x_r x_s + \sum_{q,r,s} {}_i\beta'_{qrs}x_q x_r x_s, \\ &\frac{\partial V}{\partial X_i} = p_i^2 X_i + \psi_i = p_i^2 X_i + \sum_{r,s} {}_i\alpha_{rs}X_r X_s + \sum_{q,r,s} {}_i\beta_{qrs}X_q X_r X_s, \\ &\frac{\partial C''}{\partial \dot{x}_i} = \sum c_{ij}\dot{x}_j, \quad \frac{\partial C''}{\partial \dot{X}_i} = \sum C_{ij}\dot{X}_j, \end{aligned}\right\} \tag{1-9}$$

are obtained, where ${}_i\alpha'_{rs}$ and ${}_i\alpha_{rs}$ are coefficients of the second power terms, ${}_i\beta'_{qrs}$ and ${}_i\beta_{qrs}$ are those of the third power terms. If we define

$$\begin{aligned} &{}_i\alpha_{(jj)} = 2{}_i\alpha_{jj}, \quad 2{}_i\beta_{(jjj)} = 3{}_i\beta_{jjj}, \quad {}_i\beta_{(jrr)} = {}_i\beta_{jrr} + \\ &\qquad + {}_i\beta_{rjr} + {}_i\beta_{rrj}, \quad {}_i\beta_{(jrs)} = {}_i\beta_{jrs} + {}_i\beta_{jsr} + {}_i\beta_{sjr} + {}_i\beta_{srj} + {}_i\beta_{rjs} + {}_i\beta_{rsj}, \end{aligned} \tag{1-10}$$

it is found that the following relations hold:

$$\left.\begin{aligned} &{}_i\alpha_{(jj)} = {}_j\alpha_{(ij)}, \quad {}_j\alpha_{(ii)} = {}_i\alpha_{(ij)}, \quad {}_i\alpha_{(jr)} = {}_j\alpha_{(ir)}, \\ &2{}_i\beta_{(jjj)} = {}_j\beta_{(ijj)}, \quad {}_i\beta_{(jii)} = 2{}_j\beta_{(iii)}, \quad {}_i\beta_{(ijj)} = {}_j\beta_{(jii)}, \\ &{}_i\beta_{(jrr)} = {}_j\beta_{(irr)}, \quad 2{}_i\beta_{(jjr)} = {}_j\beta_{(ijr)}, \quad {}_i\beta_{(jrs)} = {}_j\beta_{(irs)}, \end{aligned}\right\} \tag{1-11}$$

Similar reltations are given for ${}_i\alpha'_{ij}$ and ${}_i\beta'_{irs}$. Between coefficients of nonlinear terms in the original equation (1-1) and those in Eq. (1-6) represented by normal coordinates, there are the relations

$$\left.\begin{aligned} {}_m\alpha_{ij} &= \sum_{q,r,s} {}_q\alpha'_{rs} B_{qm} B_{ri} B_{sj}, \\ {}_m\beta_{ijk} &= \sum_{l,q,r,s} {}_l\beta'_{qrs} B_{lm} B_{qi} B_{rj} B_{sk}. \end{aligned}\right\} \tag{1-12}$$

Since the dissipation funtion C'' takes the minimum value zero when all $\dot{X}_j$ are zero, we conclude

$$C_{ii} > 0. \qquad (i = 1, 2, \ldots, n) \tag{1-13}$$

In the neighborhood of the frequency ω of external force which satisfies the following relation:

$$\omega = p_i \pm p_j \qquad (p_i > p_j), \tag{1-14}$$

that is, in the neighborhood of ω where the absolute value of sum of or difference in two natural frequencies p_i and p_j is equal to ω, both amplitudes of two vibrations with frequencies $\omega_i(\fallingdotseq p\ )$ and $\omega_j(\fallingdotseq p_j)$ build up and form the peak of summed and differentical harmonic oscillation. Further the relation

$$\omega_i \pm \omega_j = \omega \qquad (\omega_i > \omega_j) \tag{1-15}$$

is always satisfied.

In the present section, solutions for summed and differential harmonic oscillations will be obtained through perturbation method. Eq. (1-6) is rewritten as follows:

$$\left.\begin{aligned} \ddot{X}_i + \omega_i^2 X_i &= (\omega_i^2 - p_i^2) X_i + F_i \cos \omega t - \psi_i - \sum_r C_{ir} \dot{X}_r, \\ \ddot{X}_j + \omega_j^2 X_j &= (\omega_j^2 - p_j^2) X_j + F_j \cos \omega t - \psi_j - \sum_r C_{jr} \dot{X}_r, \\ \ddot{X}_s + p_s^2 X_s &= F_s \cos \omega t - \psi_s - \sum_r C_{sr} \dot{X}_r, \qquad (s \neq i, j) \end{aligned}\right\} \tag{1-6 a}$$

In Eq. (1-6 a), damping terms, nonlinear terms and detunings $(\omega_i^2 - p_i^2)$, $(\omega_j^2 - p_j^2)$ are small quantities. We put

$$X_r = X_{r0} + \varepsilon X_{r1} + \varepsilon^2 X_{r2} + \cdots, \qquad (r = 1, 2, \ldots, n), \tag{1-16}$$

in which ε is a small parameter. As the solutions of Eq. (1-6 a) of the first approximation, we put

$$\left.\begin{aligned} X_{i0} &= R_i \cos(\omega_i t - \theta_i) + P_i \cos \omega t, \\ X_{j0} &= R_j \cos(\omega_j t - \theta_j) + P_j \cos \omega t, \\ X_{s0} &= P_s \cos \omega t. \end{aligned}\right\} \tag{1-16 a}$$

In Eq. (1-16 a), R_i and R_j are amplitudes of summed and differential harmonic oscillations, and θ_i and θ_j are phase angles. Inserting Eqs. (1-16) and (1-16 a) into Eq. (1-6 a) and referring the relation of Eq. (1-15), the following equations are derived by the condition that resonant terms $\sin \omega_i t$, $\cos \omega_i t$, $\sin \omega_j t$ and $\cos \omega_j t$ should not be contained in the right hand sides of the first and second equation of Eq. (1-6 a):

$$\left.\begin{aligned}
&\lambda_i R_i - \nu_i R_j \cos(\theta_i \pm \theta_j) = (\rho_i R_i^2 + \sigma_i R_j^2) R_i, \\
&\lambda_j R_j - \nu_j R_i \cos(\theta_i \pm \theta_j) = (\rho_j R_j^2 + \sigma_j R_i^2) R_j, \\
&C_{ii} \omega_i R_i + \nu_i R_j \sin(\theta_i \pm \theta_j) = 0, \\
&C_{jj} \omega_j R_j \pm \nu_j R_i \sin(\theta_i \pm \theta_j) = 0,
\end{aligned}\right\} \tag{1-17}$$

where

$$P_i = F_i \cdot (\omega_i^2 - \omega^2)^{-1},\ P_j = F_j \cdot (\omega_j^2 - \omega^2)^{-1},\ P_s = F_s \cdot (p_s^2 - \omega^2)^{-1}, \tag{1-18}$$

$$\left.\begin{aligned}
&\lambda_i = (\omega_i^2 - p_i^2) - \frac{1}{2}\{\sum_{r,s} {}_i\beta_{(irs)} P_r P_s + \sum_r {}_i\beta_{(iir)} P_i P_r\}, \\
&\lambda_j = (\omega_j^2 - p_j^2) - \frac{1}{2}\{\sum_{r,s} {}_j\beta_{(jrs)} P_r P_s + \sum_r {}_j\beta_{(jjr)} P_j P_r, \\
&\nu_i = \frac{1}{2}\sum_r {}_i\alpha_{(jr)} P_r,\quad \nu_j = \frac{1}{2}\sum_r {}_j\alpha_{(ir)} P_r, \\
&\rho_i = \frac{1}{2}\, {}_i\beta_{(iii)},\quad \rho_j = \frac{1}{2}\, {}_j\beta_{(jjj)},\quad \sigma_i = \frac{1}{2}\, {}_i\beta_{(ijj)},\quad \sigma_j = \frac{1}{2}\, {}_j\beta_{(iij)}.
\end{aligned}\right\} \tag{1-19}$$

The upper and lower signs of $\pm$ in Eq. (1-17) correspond to the summed and differential harmonic oscillations respectively. Observing Eqs. (1-17) and (1-19), it is seen that $\sum_r {}_{\substack{i\\j}}\alpha_{\substack{(jr)\\(ir)}} P_r = 0$ leads to $\nu_i = \nu_j = 0$ and $R_i = R_j = 0$, further ${}_{\substack{i\\j}}\beta_{\substack{(ijj)\\(jii)}} = {}_{\substack{i\\j}}\beta_{\substack{(iii)\\(jjj)}} = 0$ leads to $\rho_{\substack{i\\j}} = \sigma_{\substack{i\\j}} = 0$ and $R_i = R_j = 0$. Consequently it is concluded that existence of nonlinear terms of both the second and third powers needs for occurrence of summed and differential harmonic oscillation $[p_i \pm p_j]$. From the third and fourth equations in Eq. (1-17), we have

$$\frac{R_i^2}{R_j^2} = \pm \frac{\nu_i C_{jj} \omega_j}{\nu_j C_{ii} \omega_i}.$$

Considering the relation $\nu_i = \nu_j$ which is obtained from Eqs. (1 11) and (1-19), we attain

$$\frac{R_i^2}{R_j^2} = \pm \frac{C_{jj} \omega_j}{C_{ii} \omega_i}. \tag{1-20}$$

Since $R_i > 0$, $R_j > 0$, $C_{ii} > 0$ and $C_{jj} > 0$ [see Eq. (1-13)], the value $\pm \omega_j / \omega_i$ must be positive. Accordingly only the summed harmonic oscillation can take place because $+\omega_j/\omega_i > 0$, and the differential harmonic oscillation cannot occur because $-\omega_i/\omega_j < 0$. In the rotating shaft system, on the other hand, both summed and differential harmonic oscillations can appear, as is seen later.

3. Response curves of summed harmonic oscillation $[p_i + p_j]$[3)7)]

Since the procedure to obtain five values R_i, R_j, $\theta_i + \theta_j$, ω_i and ω_j from five equations $\omega_i + \omega_j = \omega$ and Eq. (1-17) is quite complicated, then a comparatively simple case of two degree-of-freedom system governed by Eq. (1-21) is treated in the present section.

$$\left.\begin{aligned}&m_{11}\ddot{x}_1+k_{11}x_1+k_{12}x_2+c_{11}\dot{x}_1+\alpha x_1^2+\beta x_1^3=q_1\cos\omega t,\\&m_{22}\ddot{x}_2+k_{21}x_1+k_{22}x_2+c_{22}\dot{x}_2=0.\end{aligned}\right\}\tag{1-21}$$

In Eq. (1-21), an external force and nonlinear terms exist only in the first equation. If $m_{11}=m_{22}=m$ and $k_{11}=k_{22}$ in Eq. (1-21), calculation can be more simplified because the amplitude ratio for free vibration is given by 1 : 1 or 1 : −1. Then Eq. (1-21) is rewritten as follows:

$$\left.\begin{aligned}&\ddot{x}_1+x_1+\gamma x_2+c_{11}\dot{x}_1+\alpha x_1^2+\beta x_1^3=q\cos\omega t,\\&\ddot{x}_2+\gamma x_1+x_2+c_{22}\dot{x}_2=0.\end{aligned}\right\}\tag{1-22}$$

The following transformation

$$x_1=\frac{X_1}{\sqrt{2}}+\frac{X_2}{\sqrt{2}},\qquad x_2=\frac{X_1}{\sqrt{2}}-\frac{X_2}{\sqrt{2}}.\tag{1-3 a}$$

results in

$$\left.\begin{aligned}&\ddot{X}_1+\omega_1^2X_1=(\omega_1^2-p_1^2)X_1-C\dot{X}_1-C_{12}\dot{X}_2-\alpha_0(X_1+X_2)^2-\frac{4}{3}\beta_0(X_1+X_2)^3+F\cos\omega t,\\&\ddot{X}_2+\omega_2^2X_2=(\omega_2^2-p_2^2)X_2-C\dot{X}_2-C_{12}\dot{X}_1-\alpha_0(X_1+X_2)^2-\frac{4}{3}\beta_0(X_1+X_2)^3+F\cos\omega t,\end{aligned}\right\}\tag{1-6 b}$$

$$X_{10}=R_1\cos(\omega_1t-\theta_1)+P\cos\omega t,\quad X_{20}=R_2\cos(\omega_2t-\theta_2)+P\cos\omega t,\tag{1-16 b}$$

$$\left.\begin{aligned}&\{(\omega_1^2-p_1^2)-2\beta_0P^2\}R_1-\alpha_0P\cos(\theta_1+\theta_2)\cdot R_2=\beta_0(R_1^2+2R_2^2)R_1,\\&\{(\omega_2^2-p_2^2)-2\beta_0P^2\}R_2-\alpha_0P\cos(\theta_1+\theta_2)\cdot R_1=\beta_0(R_2^2+2R_1^2)R_2,\\&C\omega_1R_1+\alpha_0P\sin(\theta_1+\theta_2)\cdot R_2=0,\\&C\omega_2R_2+\alpha_0P\sin(\theta_1+\theta_2)\cdot R_1=0,\end{aligned}\right\}\tag{1-17 a}$$

in which

$$\left.\begin{aligned}&F=\frac{q}{\sqrt{2}},\quad \alpha_0=\frac{\alpha}{2\sqrt{2}},\quad \beta_0=\frac{3}{16}\beta,\quad P=\frac{q}{\sqrt{2}}\left(\frac{1}{\omega_1^2-\omega^2}+\frac{1}{\omega_2^2-\omega^2}\right),\\&C=\frac{1}{2}(c_{11}+c_{22}),\quad C_{12}=\frac{1}{2}(c_{11}-c_{22}),\quad p_1^2=1+\gamma,\quad p_2^2=1-\gamma\end{aligned}\right\}\tag{1-23}$$

Solving Eq. (1-17 a), we attain

$$\left.\begin{aligned}\omega_1=\eta_1\omega+\frac{(\eta_2-\eta_1)}{2(1+2\eta_1\eta_2)\omega}\Big[&(1+\eta_1\eta_2)\{\omega^2-(p_1+p_2)^2\}-2\beta_0P^2\\&\pm\sqrt{\frac{1}{\eta_1\eta_2}(\alpha_0^2P^2-\eta_1\eta_2C^2\omega^2)}\Big],\\\omega_2=\eta_2\omega-\frac{(\eta_2-\eta_1)}{2(1+2\eta_1\eta_2)\omega}\Big[&(1+\eta_1\eta_2)\{\omega^2-(p_1+p_2)^2\}-2\beta_0P^2\\&\pm\sqrt{\frac{1}{\eta_1\eta_2}(\alpha_0^2P^2-\eta_1\eta_2C^2\omega^2)}\Big],\end{aligned}\right\}\tag{1-24}$$

$$\left.\begin{aligned} R_1^2 &= \frac{\eta_2}{\beta_0(1+2\eta_1\eta_2)}\Big[\eta_1\eta_2\{\omega^2-(p_1+p_2)^2\}-2\beta_0P^2 \\ &\qquad \pm 2\eta_1\eta_2\sqrt{\frac{1}{\eta_1\eta_2}(\alpha_0^2P^2-\eta_1\eta_2C^2\omega^2)}\Big], \\ R_2^2 &= \frac{\eta_1}{\beta_0(1+2\eta_1\eta_2)}\Big[\eta_1\eta_2\{\omega^2-(p_1+p_2)^2\}-2\beta_0P^2 \\ &\qquad \pm 2\eta_1\eta_2\sqrt{\frac{1}{\eta_1\eta_2}(\alpha_0^2P^2-\eta_1\eta_2C^2\omega^2)}\Big]. \end{aligned}\right\} \quad (1\text{-}25)$$

where

$$\eta_1 = \frac{p_1}{p_1+p_2}, \quad \eta_2 = \frac{p_2}{p_1+p_2}. \quad (1\text{-}26)$$

Clearly ω_1 and ω_2 given by Eq. (1–24) satisfy the relation $\omega_1+\omega_2=\omega$. The sign $\pm$ in Eq. (1–25) of response curves corresyonds to $\pm$ in Eq. (1–24).

By a similar procedure, response curves of sub-harmonic oscillations of order 1/2 are obtained as follows:

$$\left.\begin{aligned} R_i^2 &= \frac{1}{\beta_0}\Big[\Big\{\Big(\frac{\omega}{2}\Big)^2-p_i^2\Big\}-2\beta_0P^2\pm\sqrt{\alpha_0^2P^2-\Big(\frac{C\omega}{2}\Big)^2}\Big], \\ P &= F\Big(-\frac{4}{3\omega^2}+\frac{1}{p_j^2-\omega^2}\Big), \quad (i,j=1,2,\ i\neq j). \end{aligned}\right\} \quad (1\text{-}25\,\text{a})$$

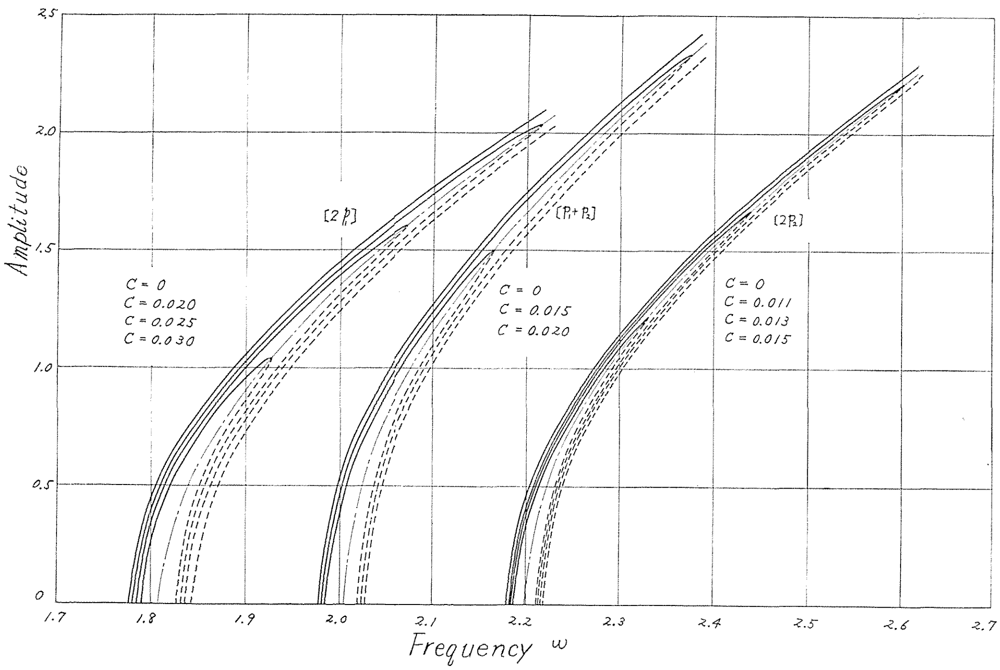

FIG. 1. Response curves of summed harmonic oscillation $[p_1+p_2]$ and sub-harmonic oscillations of order 1/2.
($F=0.372$, $\alpha_0=0.1$, $\beta_0=0.1$, $2p_1=1.788$, $p_1+p_2=1.989$, $2p_2=2.190$)
(F: magnitude of external force, ω: frequency of external force, α_0: coefficient of unsymmetrical nonlinearity, β_0: coefficient of symmetrical nonlinarity, C: coefficient of damping).

The form of Eq. (1-25) is quite similar to that of the equation of response curves of sub-harmonic oscillation of order 1/2.

Since it is impossible to give R_1, R_2, ω_1, ω_2 and $\theta_1+\theta_2$ analytically by using Eq. (1-17 a) and the relation $\omega_1+\omega_2=\omega$, Eqs. (1-24) and (1-25) are determined by the following approximate method. That is, as the first approximation of ω_{10} and ω_{20} satisfying the relation $\omega_{10}+\omega_{20}=\omega$, we adopt

$$\omega_{10}=\eta_1\omega=\frac{p_1}{p_1+p_2}\omega,\quad \omega_{20}=\eta_2\omega=\frac{p_2}{p_1+p_2}\omega, \tag{1-27}$$

which results in Eqs. (1-24) and (1-25). Adoption of Eq. (1-27) does not contradict the experimental results (see *e.g.* Fig. 13).

Amplitudes of summed harmonic and sub-harmonic oscillations given by Eqs. (1-25) and (1-25 a) are graphically represented in Figs. 1, 2 and 4, and for summed harmonic oscillation, sum of amplitudes R_1+R_2 is plotted as amplitude in Figs. 1, 2 and 4, where full lines indicate stable response curves which correspond to stable vibrations, and broken lines represent unstable response curves, and chain lines give the boundaries of stable and unstable zones. Since the coefficient β_0 of symmetrical nonlinearity is positive in Figs. 1, 2, all response curves are of hard spring type.

For a small damping force, jump phenomena take place near the top of response curves, while for a large damping force the jump phenomena can not occur and response curves

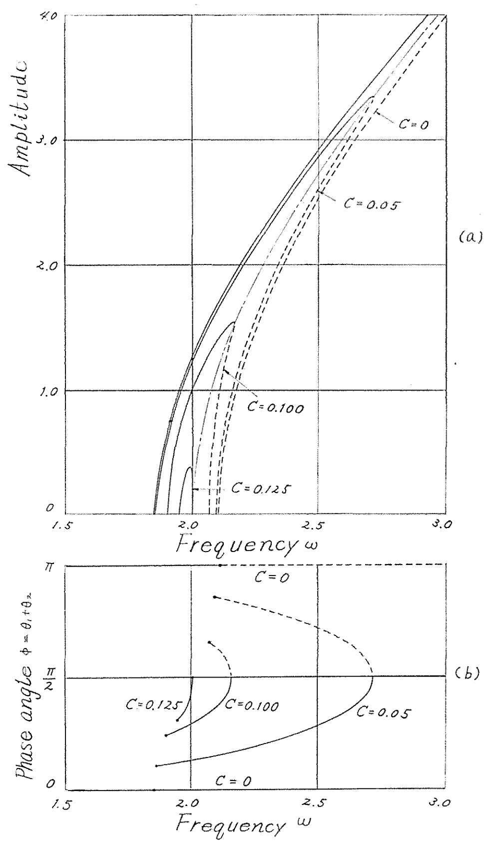

FIG. 2. Response curves and phase angles of summed harmonic oscillation $[p_1+p_2]$. ($F=0.372$, $\alpha_0=0.5$, $\beta_0=0.1$, $p_1+p_2=1.989$)

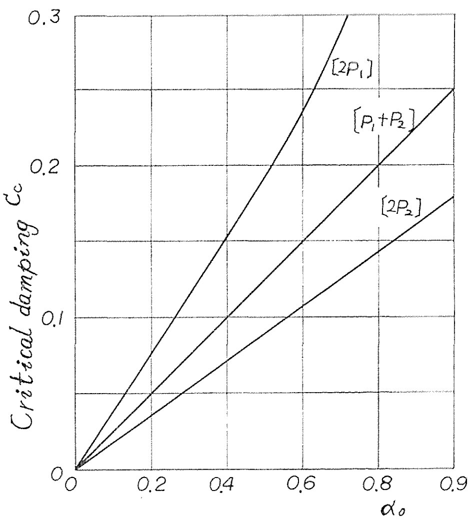

FIG. 3. Critical damping coefficient. ($F=0.372$, $p_1=0.894$, $p_2=1.095$)

are continuous as the curve of $C=0.125$ shown in Fig. 2.

According as the increase of damping force, the maximum value of the response curves falls gradually, and finally it vanishes on the abscissa at a certain value of damping coefficient, *i.e.*, at the critical damping. These critical damping coefficient C_c is derived by

$$R^2(\omega^2,\ C)=0,\ \frac{\partial}{\partial\omega^2}\{R^2(\omega^2,\ C)\}=0, \tag{1-28}$$

in which $R^2(\omega^2, C)$ represents Eq. (1-25) or Eq. (1-25 a). In Fig. 3, the critical damping coefficient C_c determined by Eq. (1-28) is shown. When $|\alpha_0 P|\ll 1$, approximate value of C_c is given as follows:

$$\left.\begin{aligned} C_c &\fallingdotseq \frac{|\alpha_0 P|}{p_1 p_2} \quad \text{for summed harmonic oscillation } [p_1+p_2], \\ C_c &\fallingdotseq \frac{|\alpha_0 P|}{p_i^2} \quad \text{for sub-harmonic oscillation of order } 1/2. \end{aligned}\right\} \tag{1-29}$$

When C_c is somewhat larger, the exact value of C_c given by Eq. (1-28) becomes larger than approximate value of Eq. (1-29). Comparison of C_c among $[2p_1]$, $[2p_2]$ and $[p_1+p_2]$ in Fig. 3 shows that oscillation with low frequency can easily occur, because vibration of lower frequency has smaller value of C_c.

Fig. 4 is response curves of summed harmonic and sub-harmonic oscillations

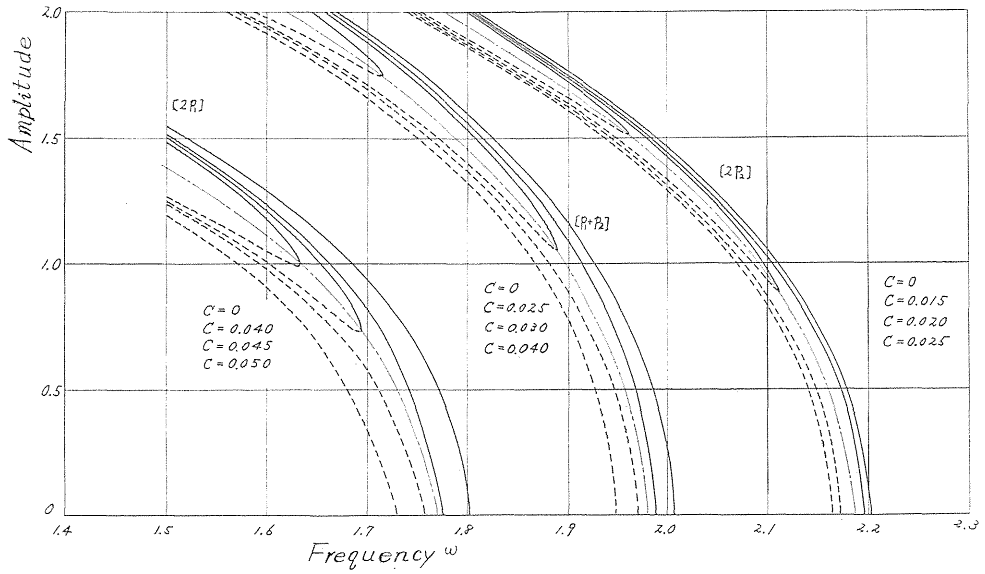

FIG. 4. Response curves of summed harmonic oscillation $[p_1+p_2]$ and sub-harmonic oscillations of order 1/2.

($F=0.372$, $\alpha_0=0.1$, $\beta_0=-0.1$, $p_1=0.894$, $p_2=1.095$)

(F= magnitude of disturbing force, ω= frequency of disturbing force, p_1, p_2= natural frequencies, α_0= coefficient of unsymmetrical nonlinearity, β_0= coefficient of symmetrical nonlinearity, C= coefficient of damping) for negative value of β_0 (coefficient of symmetrical nonlinearity), and all curves are of soft spring type. For $\beta_0<0$, response curves have a lower limit of amplitude when the damping coefficient becomes smoewhat large.

For both cases of $\beta_0>0$ and $\beta_0<0$, frequencies ω_1 and ω_2 of summed harmonic oscillations are shown in Fig. 5, where full and broken lines correspond to stable and unstable vibrations separately, and chain lines give the curues of $\omega_{10}=\eta_1\omega$, $\omega_{20}=\eta_2\omega$ which given by Eq. (1-27).

Phase angles of $\phi=\theta_1+\theta_2$ for summed harmonic oscillations can be obtained through Eq. (1-17 a), and an example of ϕ when $\beta_0>0$ is indicated in the lower figure of Fig. 2, in which full and broken lines mean phase angles of stable and unstable vibrations separately, and phase angles for $C=0$ are represented by straight lines. Boundary between stable and unstable regions is a line of $\phi=\pi/2$ and the lower and the upper parts corresponds to stable and unstable regions when $\beta_0>0$ as shown in Fig. 2; vice versa for $\beta_0<0$.

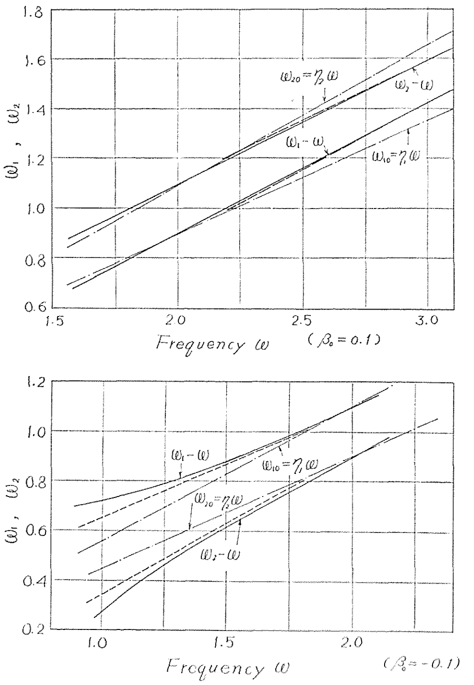

FIG. 5. Frequencies ω_1 and ω_2 of summed harmonic oscillation $[p_1+p_2]$ for both cases of $\beta_0>0$ and $\beta_0<0$.
($F=0.372$, $\alpha_0=1.0$, $C=0$)

4. Stability of summed harmonic oscillation $[p_i+p_j]$

Since the perturbation method used in Sections 2 and 3 is rather inconvenient to stability criteria, the solutions induced by the method of Andronow and Witt[8] is treated in this section. In Eq. (1-16 b), amplitudes R_1, R_2 and phase angles θ_1 and θ_2 are considered as slowly varying functions of time t. Substituting Eq. (1-16 b) into Eq. (1-6 b) and neglecting the terms smaller than the second order of small quantities, we have

$$\left.\begin{aligned}
-2\,\omega_1\frac{dR_1}{dt} &= C\omega_1 R_1+\alpha_0 P\sin(\theta_1+\theta_2)\cdot R_2,\\
-2\,\omega_2\frac{dR_2}{dt} &= C\omega_2 R_2+\alpha_0 P\sin(\theta_1+\theta_2)\cdot R_1,\\
2\,\omega_1 R_1\frac{d\theta_1}{dt} &= \{(\omega_1^2-p_1^2)-2\,\beta_0 P^2\}R_1-\alpha_0 P\cos(\theta_1+\theta_2)\cdot R_2-\beta_0(R_1^2+2\,R_2^2)R_1,\\
2\,\omega_2 R_2\frac{d\theta_2}{dt} &= \{(\omega_2^2-p_2^2)-2\,\beta_0 P^2\}R_2-\alpha_0 P\cos(\theta_1+\theta_2)\cdot R_1-\beta_0(R_2^2+2\,R_1^2)R_2.
\end{aligned}\right\}$$

(1-17 b)

Putting $\frac{dR_1}{dt}=\frac{dR_2}{dt}=\frac{d\theta_1}{dt}=\frac{d\theta_2}{dt}=0$, we obtain the steady state solutions of Eqs. (1-24) and (1-25).

Let

$$R_1=R_{10}+\xi,\ R_2=R_{20}+\zeta,\ \phi=\theta_1+\theta_2=\phi_0+\varphi, \tag{1-30}$$

be solutions which differ slightly from the steady state solutions R_{10}, R_{20} and ϕ_0. Inserting Eq. (1-30) into Eq. (1-17 b) and neglecting all but the linear terms in ξ, ζ and φ, we get

$$\left.\begin{aligned}
-2\omega_1\frac{d\xi}{dt}&=C\omega_1\cdot\xi-\frac{R_{20}}{R_{10}}C\omega_2\zeta+\{\lambda_1-\beta_0(R_{10}^2+2R_{20}^2)\}R_{10}\cdot\varphi,\\
-2\omega_2\frac{d\zeta}{dt}&=-\frac{R_{10}}{R_{20}}C\omega_1\xi+C\omega_2\zeta+\{\lambda_2-\beta_0(2R_{10}^2+R_{20}^2)\}R_{20}\cdot\varphi,\\
\omega_1R_{10}^2\frac{d\varphi}{dt}&=-\beta_0R_{10}(R_{10}^2+2R_{20}^2)\xi-\beta_0R_{20}(2R_{10}^2+R_{20}^2)-C\omega_1R_{10}^2\cdot\varphi,
\end{aligned}\right\} \tag{1-31}$$

where

$$\lambda_1=(\omega_1^2-p_1^2)-2\beta_0P^2,\ \lambda_2=(\omega_2^2-p_2^2)-2\beta_0P^2. \tag{1-19 a}$$

Substituting the assumed solutions

$$\xi=\xi_0e^{st},\ \zeta=\zeta_0e^{st},\ \varphi=\varphi_0e^{st}, \tag{1-32}$$

into Eq. (1-31), we have

$$\left.\begin{aligned}
&\omega_1R_{10}(C+2S)\xi_0-R_{20}C\omega_2\zeta_0+\{\lambda_1-\beta_0(R_{10}^2+2R_{20}^2)\}R_{10}^2\varphi_0=0,\\
&-R_{10}C\omega_1\xi_0+\omega_2R_{20}(C+2S)\zeta_0+\{\lambda_2-\beta_0(2R_{10}^2+R_{20}^2)\}R_{20}^2\varphi_0=0,\\
&\beta_0R_{10}(R_{10}^2+2R_{20}^2)\xi_0+\beta_0R_{20}(2R_{10}^2+R_{20}^2)\zeta_0+\omega_1R_{10}^2(C+S)\varphi_0=0.
\end{aligned}\right\} \tag{1-33}$$

Elimination of ξ_0, ζ_0 and φ_0 in Eq. (1-33) leads to

$$\begin{aligned}
&4\omega_1^2\omega_2R_{10}^2S^3+8\omega_1^2\omega_2R_{10}^2CS^2+\\
&4\Big[\omega_1^2\omega_2C^2R_{10}^2-\frac{\beta_0(\omega_1^2+4\omega_1\omega_2+\omega_2^2)}{\omega_2(\omega_2^2-\omega_1^2)}\{(2\omega_1+\omega_2)\lambda_2-(\omega_1+2\omega_2)\lambda_1\}R_{10}^4\Big]S\\
&-\frac{2C\beta_0\omega_1}{\omega_2(\omega_2^2-\omega_1^2)}(\omega_1^2+4\omega_1\omega_2+\omega_2^2)\{(2\omega_1+\omega_2)\lambda_2-(\omega_1+2\omega_2)\lambda_1\}=0.
\end{aligned} \tag{1-34}$$

Routh-Hurwitz stability criterion for the above equation results in

$$-\beta_0\{(2\omega_1+\omega_2)\lambda_2-(\omega_1+2\omega_2)\lambda_1\}>0. \tag{1-35}$$

Using the relation of Eq. (1-25), Eq. (1-35) can be rewritten as follows:

$$R_{1,2}^2>\frac{\eta_{2,1}}{\beta_0(1+2\eta_1\eta_2)}[\eta_1\eta_2\{\omega^2-(p_1+p_2)^2\}-2\beta_0P^2] \tag{1-36}$$

The expression obtained by replacing inequality with equality furnishes the boundary line between stable and unstable ranges, and it can be verified that the locus of vertical tangent on response curves coincides with this boundary line. Accordingly it is concluded that the situation of stability of summed harmonic osillation $[p_1+p_2]$ is qualitatively analogous to that of harmonic oscillations and sub-harmonic oscillations occurring in a single degree-of-freedom system with nonlinearity.

5. Experimental apparatus and experimental results[3)]

5.1. Experiment I

As shown in Fig. 6, a vibratory body W (120 mm × 120 mm × 50 mm; weight =5.3 kg) is mounted on a leaf spring S (length$=a+b=$265 mm, width=9 mm, thickness=1 mm) at the position $a:b=1:2$. The vibratory system is mounted on a table T which vibrates with amplitude 0.015 mm, then an exciting force is induced. Nonlinear spring characteristics are given by small helical springs C_1, C_2 which are fixed on thread screws S_1, S_2. Since the screws S_1, S_2 are adjusted so that clearances between the vibratory body W and the small springs C_1, C_2 may almost vanish in equilibrium state, unsymmetrical nonlinearity appears in the spring characteristics. The vibratory system shown in Fig. 6 is a two degree-of-freedom system. Then both deflection and inclination angle of W are measured by optical method. By changing frequency ω of the exciting force, the response curves shown in Fig. 7 are obtained experimentally. In the peak $[p_1]$ at $\omega=5$ c/sec and the peak $[p_2]$ at $\omega=8.5$ c/sec, harmonic oscillations induced by exciting force take place and response curves are of hard and soft spring types respectively. In the peak $[2p_1]$ at $\omega=9.5$ c/sec and the peak $[2p_2]$ at $\omega=17$ c/sec, sub-harmonic oscillations of order 1/2 caused by unsymmetrical nonlinear spring characteristics appear, and response curves are of soft spring type. Summed harmonic oscillation of $[p_1+p_2]$ occurs at $\omega=13.5$ c/sec where the relation $p_1+p_2\fallingdotseq\omega$ is satisfied.

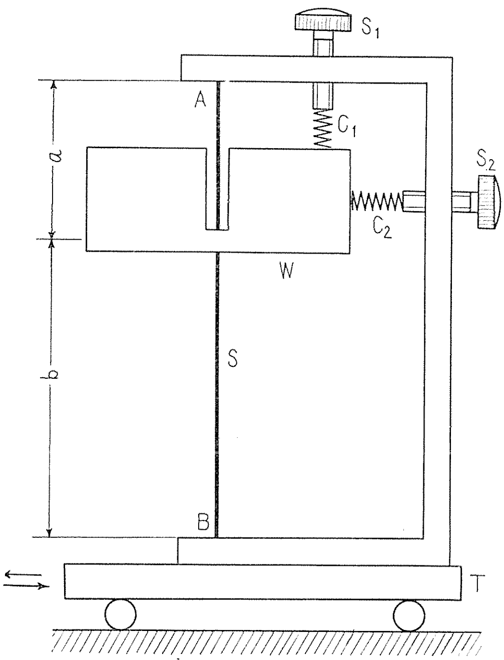

FIG. 6. Apparatus of Experiment I.

Vibratory waves of summed harmonic oscillation in the peak $[p_1+p_2]$ in Fig. 7 are shown in Fig. 8 where fine black vertical lines are marks recorded at each period of the oscillatory table T shown in Fig. 6. In Fig. 8, vibratory waves change periodically between each mark A. At intervals of marks A the table T oscillates 14 times and one vibration (frequency ω_1) oscillates 5 times and the other (frequecy ω_2) 9 times. So we can see $\omega:\omega_1:\omega_2=14:5:9$ and $\omega_1+\omega_2=\omega$.

In the experiment in which the vibratory body W is mounted at location $a:b=1:2.4$ and spring constants of C_1 and C_2 are changed, experimental results

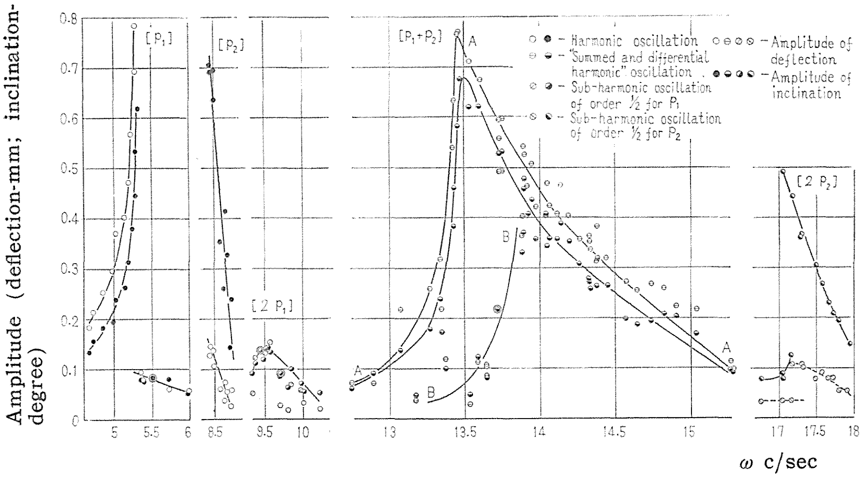

FIG. 7. Response curves of summed harmonic oscillation $[p_1+p_2]$ and sub-harmonic oscillations of order 1/2.
(location of vibratory body $W=a:b=1:2$)

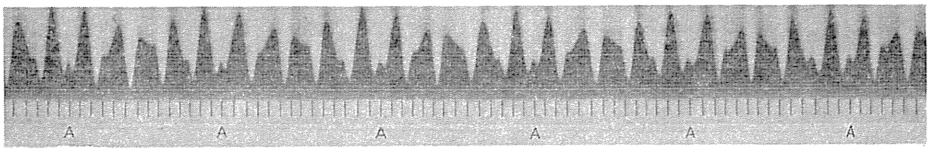

FIG. 8. Vibratory waves of summed harmonic oscillation $[p_1+p_2]$.
($\omega=13.6$ c/sec, $\omega_1 \fallingdotseq 5/14 \cdot \omega$, $\omega_2 \fallingdotseq 9/14 \cdot \omega$)

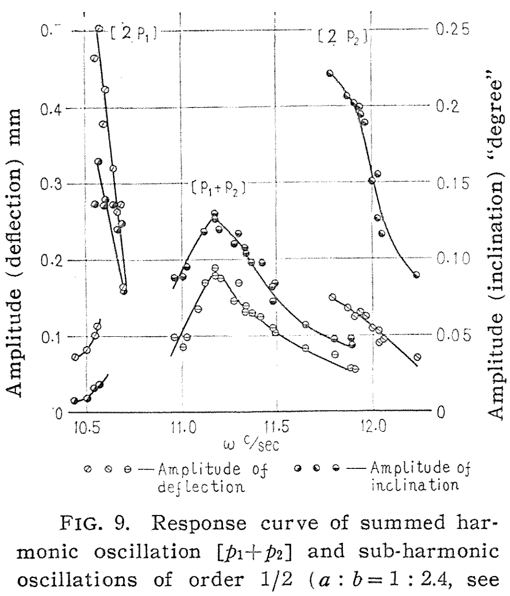

FIG. 9. Response curve of summed harmonic oscillation $[p_1+p_2]$ and sub-harmonic oscillations of order 1/2 ($a:b=1:2.4$, see Fig. 6).

as shown in Fig. 9 are obtained. In Fig. 9, sub-harmonic oscillations at the peaks $[2\,p_1]$ and $[2\,p_2]$ have response curves of soft spring type, and summed harmonic oscillation $[p_1+p_2]$ takes place at $\omega=11.2$ c/sec.

2.5. Experiment II

Experiment II is performed by using a quite different apparatus from that of experiment I. This apparatus is shown in Fig. 10 where the boss B (dia.$=26\ \phi$) is fixed on a free supported horizontal shaft with dia.$=8\ \phi$ and length$=a+b=400$ mm. A disk D (dia. $=360\ \phi$, thickness $=8.5$ mm) is mounted on the boss B at the location $a:b=3:7$. Nonlinear, unsymmetrical spring characteristics are given by backlash ($\fallingdotseq 0.04$ mm)

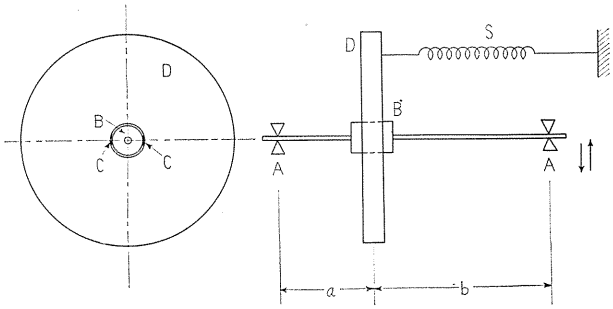

FIG. 10. Apparatus of Experiment II.

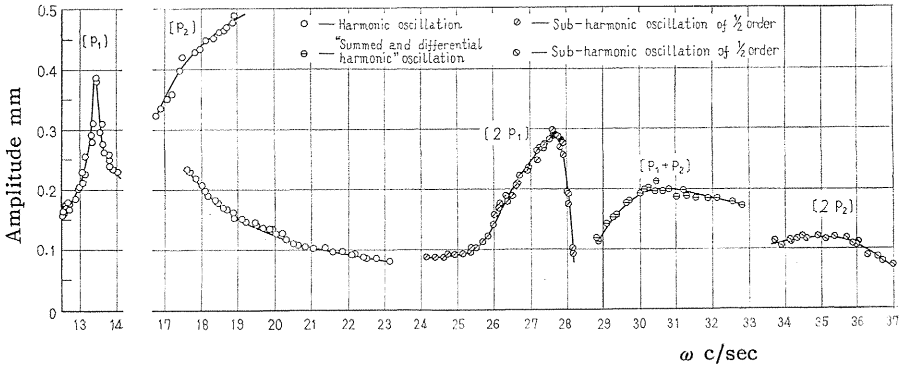

FIG. 11. Response curves of summed harmonic oscillation $[p_1+p_2]$ and sub-harmonic oscillations of order 1/2.

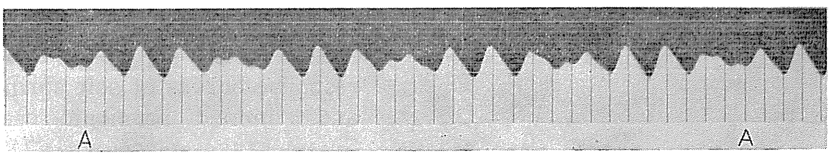

FIG. 12. Vibratory waves of summed harmonic oscillation $[p_1+p_2]$.
($\omega=30.47$ c/sec, $\omega_1 \fallingdotseq 15/34 \cdot \omega$, $\omega_2 \fallingdotseq 19/34 \cdot \omega$)

between disk D and boss B and a long helical spring S which is streched under somewhat weak tension. As one shaft end is fixed on a table oscillating with frequency ω and amplitude 0.05 mm, an exciting force is induced. In order to check the axial displacement of disk D along the shaft, clearance tapes C are inserted between disk D and boss B as shown in the left hand side figure of Fig. 10. By measuring the vertical displacement of disk D by optical method, experimental results as shown in Fig. 11 are obtained. In Fig. 11, the peaks $[p_1]$ and $[p_2]$ are harmonic oscillations and the peaks $[2p_1]$ and $[2p_2]$ are sub-harmonic oscillations of order 1/2. Summed harmonic oscillation $[p_1+p_2]$ occurs between 29-33 c/sec. Vibratory waves of summed harmonic oscillation $[p_1+p_2]$ shown in Fig. 11 are given in Fig. 12. Vibratory waves change periodically at each mark

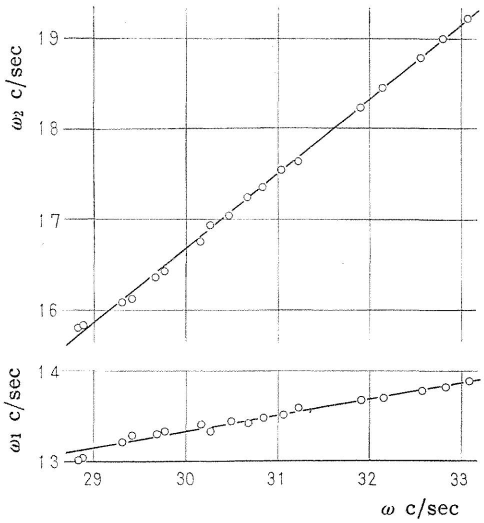

FIG. 13. $\omega_1-\omega$, $\omega_2-\omega$ diagrams.

A, and observing them it is seen that the relations $\omega:\omega_1:\omega_2=34:15:19$ and $\omega_1+\omega_2=\omega$ hold. By analyzing vibratory waves, the frequencies ω_1 and ω_2 of summed oscillation $[p_1+p_2]$ are obtained experimentally, and they are indicated in Fig. 13.

6. Conclusions

Obtained conclusions in the present chapter are as follows:

(1) When the vibratory systems with multiple degree-of-freedom have unsymmetrical nonlinearity in restoring force, there is possibility of occurrence of "summed and differential harmonic oscillations" $[p_i \pm p_j]$.

(2) Theoretical analysis verifies that only summed harmonic oscillation $[p_i+p_j]$ actually takes place, and differential harmonic oscillation $[p_i-p_j]$ does not appear.

(3) Summed harmonic oscillation $[p_i+p_j]$ appears with sub-harmonic oscillation of order 1/2.

(4) Occurrence of summed harmonic oscillation $[p_i+p_j]$ needs both symmetrical and unsymmetrial nonlinearity.

(5) Response curves and frequencies ω_1, ω_2 of summed harmonic oscillation $[p_1+p_2]$ are given by Eqs. (1–25) and (1–24), severally.

(6) Response curves of summed harmonic oscillation $[p_1+p_2]$ are qualitatively analogous to those of sub-harmonic oscillation of order 1/2.

(7) According as positive or negative symmetrical nonlinearity β_0, response curve of summed harmonic oscillation $[p_1+p_2]$ is of hard or of soft spring type.

(8) The stability of summed harmonic oscillation is studied, and the loci of vertical tangent of response curves furnish the boundary between stable and unstable regions.

(9) Between frequencies ω_1, ω_2 of summed harmonic oscillation $[p_1+p_2]$ and frequency ω of external force the relation $\omega_1+\omega_2=\omega$ always holds.

(10) By using two kinds of apparatus, experiments are performed, and occurrence of summed harmonic oscillation $[p_1+p_2]$ is verified, and further obtained theoretical results are proved experimentally.

Chapter II. Summed and Differential Harmonic Oscillations and Sub-Harmonic Oscillations in Rotating Shaft Systems with Unsymmetrical Nonlinear Spring Characteristics[10]

Summed and differential harmonic oscillations treated in the present chapter are not rectilinear vibrations but whirling motions, because gyroscopic action results in whirls of lateral vibrations of the shaft[6]. Accordingly, all free vibrations appearing in the shaft systems are also whirling motions, and natural frequency p means here angular velocity of whirl. Usually frequency takes positive value in rectilinear vibratory systems, however, in the rotating shaft symtems, it takes positive or negative value according as forward or backward precessional whirling motion. In the present chapter notations p_m and p_n which take both positive and negative values are used in place of p_i and p_j used in Chaper I which are always positive.

Sub-harmonic oscillations occurring with summed and differential harmonic oscillations are still whirling motions of forward or backward precession. Since there are few studies[9] for sub-harmonic oscillation of whirl, it is discussed with summed and differential harmonic oscillations in the present chapter.

1. Analysis of summed and differential harmonic oscillations and sub-harmonic oscillations in rotating shaft system

1. 1. Introduction

When a rotating body is supported by single-row radial ball bearings, the unsymmetrical nonlinear spring characteristics appear in the elasticity of the shaft supporting the rotating body[1]. This nonlinearity of spring characteristics results in occurrence of sub-harmonic and summed and differential harmonic oscillations of the rotating shaft[9][11]. In the present section, the authors will discuss analytically conditions for the possibility of occurrence of summed and differential harmonic oscillation, and they will determine the modes of vibration which can take place and equations of response curves. Further the authors will compare the obtained conclusions with the experimental results.

1. 2. Transformation into normal coordinates

The differential equations of a rotating shaft consisting of a light shaft and a rotating body, as shown in Fig. 14, are given as follows:

$$\left.\begin{aligned}
&m\ddot{x}+c_1\dot{x}+\alpha_0 x+\gamma_0\theta_x+\Phi_1=me\omega^2\cos\omega t,\\
&m\ddot{y}+c_2\dot{y}+\alpha_0 y+\gamma_0\theta_y+\Phi_2=me\omega^2\sin\omega t,\\
&I\ddot{\theta}_x+I_p\omega\dot{\theta}_y+c_3\dot{\theta}_x+\gamma_0 x+\delta_0\theta_x+\Phi_3=(I_p-I)\,\tau\omega^2\cos(\omega t+\beta_0),\\
&I\ddot{\theta}_y-I_p\omega\dot{\theta}_x+c_4\dot{\theta}_y+\gamma_0 y+\delta_0\theta_y+\Phi_4=(I_p-I)\,\tau\omega^2\sin(\omega t+\beta_0),
\end{aligned}\right\}\qquad(2\text{-}1)$$

where x, y are displacements of the rotating body in x, y directions (see Fig. 14); θ_x, θ_y are components of inclination angles of the rotating body; m is mass of the rotating body; I_p is polar moment of inertia; I moment of inertia about the axis perpendicular to I_p axis; ω the rotating speed of the shaft; α_0, γ_0 and δ_0 spring constants of the shaft; e eccentricity of the rotating body (static unbalance); τ small deviational angle between the I_p axis and the center line of the shaft (dynamic unbalance); β_0 the angle between directions e and τ; $c_{1,2,3,4}$ coefficients of damping; $\Phi_{1,2,3,4}$ nonlinear terms. For convenience, we introduce the dimensionless quantities as follows:

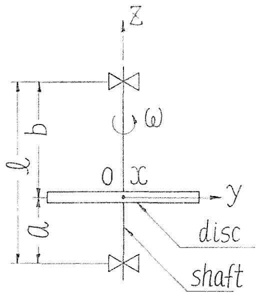

FIG. 14

$$\left.\begin{array}{l} x/e = q_1,\ y/e = q_2,\ \theta_x\cdot\sqrt{(I/m)}/e = q_3,\ \theta_y\cdot\sqrt{(I/m)}/e = q_4,\ I_p/I = I_1, \\ \gamma_0/\alpha_0\cdot\sqrt{(m/I)} = \gamma_0',\ \delta_0 m/(\alpha_0 I) = \delta_0',\ \omega\cdot\sqrt{(m/\alpha_0)} = \omega',\ t\cdot\sqrt{(\alpha_0/m)} = t', \\ \tau/e\cdot\sqrt{(I/m)} = \tau',\ c_{1,2}/\sqrt{(\alpha_0 m)} = c_{1,2}',\ c_{3,4}\cdot\sqrt{(m/\alpha_0)}/I = c_{3,4}', \\ \Phi_{1,2}/(\alpha_0 e) = \varphi_{1,2},\ \Phi_{3,4}\sqrt{(m/I)}/(\alpha_0 e) = \varphi_{3,4}. \end{array}\right\} \quad (2\text{-}2)$$

Substituting Eq. (2-2) into Eq. (2-1) and omitting primes on the dimensionless quantities, we have

$$\left.\begin{array}{l} \ddot{q}_1 + c_1\dot{q}_1 + q_1 + \gamma_0 q_3 + \varphi_1 = \omega^2\cos\omega t, \\ \ddot{q}_2 + c_2\dot{q}_2 + q_2 + \gamma_0 q_4 + \varphi_2 = \omega^2\sin\omega t, \\ \ddot{q}_3 + I_1\omega\dot{q}_4 + c_3\dot{q}_3 + \gamma_0 q_1 + \delta_0 q_3 + \varphi_3 = (I_1 - 1)\tau\omega^2\cos(\omega t + \beta_0), \\ \ddot{q}_4 - I_1\omega\dot{q}_3 + c_4\dot{q}_4 + \gamma_0 q_2 + \delta_0 q_4 + \varphi_4 = (I_1 - 1)\tau\omega^2\sin(\omega t + \beta_0). \end{array}\right\} \quad (2\text{-}3)$$

The frequency equation is

$$(1 - p^2)(\delta_0 + I_1\omega p - p^2) - \gamma_0^2 = (p - p_1)(p - p_2)(p - p_3)(p - p_4) = 0, \quad (2\text{-}4)$$

in which p is the natural frequency, and $p_{1,2,3,4}$ are roots of the frequency equation.

Since the present vibratory system is four degree-of-freedom system with gyroscopic terms $I_1\omega\dot{q}_4$, $-I_1\omega\dot{q}_3$, there are four natural frequencies $p_{1,2,3,4}$ and a whirling motion of forward or backward precession takes place. Rectilinear vibration appears only when $\omega = 0$. Assuming $p_1 > p_2 > p_3 > p_4$, p_1, p_2 are positive and p_3, p_4 negative, and the relations

$$p_1 > 1,\ |p_4| > 1,\ p_2 < 1,\ |p_3| < 1 \quad (2\text{-}5)$$

hold[6].

Hereafter we use notation $\sum$ in place of $\sum_{i=1}^{4}$. Transformation into normal coordinates can be carried out by putting[12]

$$q_1 = -\sum p_i X_i,\ q_2 = \sum x_i,\ q_3 = \sum p_i\kappa_i X_i,\ q_4 = -\sum \kappa_i x_i, \quad (2\text{-}6)$$

where X_i is indefinite integration of x_i, and

$$\kappa_i = (1 - p_i^2)/\gamma_0,\ (\gamma_0 < 0). \quad (2\text{-}7)$$

Transfering $c_i\dot{q}_i$, φ_i of Eq. (2-3) into right hand sides of Eq. (2-3) and inserting Eq. (2-6) into left hand sides of Eq. (2-3), the left hand sides of Eq. (2-3) becomes linear equations of $(\ddot{x}_i+p_i^2x_i)$. Let the determinant consisting of coefficients of these four linear equations be $D=|\alpha_{ij}|$, then

$$\alpha_{1j}=-p_j,\ \alpha_{2j}=1,\ \alpha_{3j}=(\delta_0-\gamma_0^2-\delta_0p_j^2)/(\gamma_0p_j),\ \alpha_{4j}=-(1-p_j^2)/\gamma_0.$$

Let the cofactor of D be A_{ij} and putting

$$\Delta_{ij}=A_{ij}/D, \tag{2-8}$$

we obtain

$$\Delta_{2j}=-p_j\Delta_{1j},\ \Delta_{3j}=-\kappa_j\Delta_{1j},\ \Delta_{4j}=\kappa_jp_j\Delta_{1j}.$$

Using the above relations, we have

$$\begin{aligned}\ddot{x}_i+p_i^2x_i=a_i\sin\omega t+b_i\cos\omega t&-\Delta_{1i}(c_1\ddot{q}_1-p_ic_2\dot{q}_2-\kappa_ic_3\ddot{q}_3+\kappa_ip_ic_4\dot{q}_4)\\&-\Delta_{1i}(\dot{\varphi}_1-p_i\varphi_2-\kappa_i\dot{\varphi}_3+\kappa_ip_i\varphi_4),\qquad(i=1,2,3,4)\end{aligned}\tag{2-9}$$

from the four linear equations of $(\ddot{x}_i+p_i^2x_i)$. In Eq. (2-9),

$$\left.\begin{aligned}a_i&=-\omega^2(\omega+p_i)\,\Delta_{1i}\{1+\kappa_i(1-I_1)\,\tau\cos\beta_0\},\\b_i&=-\omega^2(\omega+p_i)\,\Delta_{1i}\kappa_i(1-I_1)\,\tau\sin\beta_0.\end{aligned}\right\}\tag{2-10}$$

Assuming that nonlinear terms consist of the second and the third powers, and that coordinates q_1, q_3 (x direction) do not couple to q_2, q_4 (y direction) through nonlinear terms, the nonlinear terms are given by

$$\left.\begin{aligned}\varphi_{1,3}&=\alpha_{1,3}q_{1,3}^2+\gamma_{1,3}q_{3,1}^2+\xi_{1,3}q_1q_3+\beta_{1,3}q_{1,3}^3+\delta_{1,3}q_{3,1}^3+\eta_{1,3}q_{1,3}^2q_{3,1}+\zeta_{1,3}q_{1,3}q_{3,1}^2,\\\varphi_{2,4}&=\alpha_{2,4}q_{2,4}^2+\gamma_{2,4}q_{4,2}^2+\xi_{2,4}q_2q_4+\beta_{2,4}q_{2,4}^3+\delta_{2,4}q_{4,2}^3+\eta_{2,4}q_{2,4}^2q_{4,2}+\zeta_{2,4}q_{2,4}q_{4,2}^2.\end{aligned}\right\}\tag{2-11}$$

Since the relations $(q_{1,2}+\gamma_0q_{3,4}+\varphi_{1,2})=\partial V/\partial q_{1,2}$ and $(\gamma_0q_{1,2}+\delta_0q_{3,4}+\varphi_{3,4})=\partial V/\partial q_{3,4}$ (V is the potential energy) hold, we have

$$\xi_{1,3}=2\gamma_{3,1},\ \xi_{2,4}=2\gamma_{4,2},\ \eta_{1,3}=3\delta_{3,1},\ \eta_{2,4}=3\delta_{4,2},\ \zeta_{1,2}=\zeta_{3,4}.\tag{2-12}$$

1. 3. Sub-harmonic oscillations

In the present section, we treat sub-harmonic oscillation of order 1/2 which appears frequently in experiments[9)12)]. Now we will consider sub-harmonic oscillation occurring in the neighborhood of $|p_r|=1/2\cdot\omega$ and will solve approximately by perturbation method. Rewriting the equations of x_r in Eq. (2-9), we get

$$\begin{aligned}\ddot{x}_r+\frac{1}{4}\omega^2x_r&=a_r\sin\omega t+b_r\cos\omega t+\left(\frac{1}{4}\omega^2-p_r^2\right)x_r\\&-\Delta_{1r}(c_1\ddot{q}_1-p_rc_2\dot{q}_2-\kappa_rc_3\ddot{q}_3+\kappa_rp_rc_4\dot{q}_4)-\Delta_{1r}(\dot{\varphi}_1-p_r\varphi_2-\kappa_r\dot{\varphi}_3+\kappa_rp_r\varphi_4).\end{aligned}\tag{2-13}$$

Putting

$$\left.\begin{aligned} q_i &= q_{i0} + \varepsilon q_{i1} + \varepsilon^2 q_{i2} + \cdots, \\ x_i &= x_{i0} + \varepsilon x_{i1} + \varepsilon^2 x_{i2} + \cdots, \\ \therefore \quad \varphi_i &= \varphi_{i0} + \varepsilon \varphi_{i1} + \varepsilon^2 \varphi_{i2} + \cdots, \end{aligned}\right\} \tag{2-14}$$

and inserting Eq. (2-14) into Eqs. (2-9) and (2-13), then assuming that damping terms, non-linear terms and $[(1/4)\omega^2 - p_r^2]$ are as small as the order of ε, we have

$$\left.\begin{aligned} \ddot{x}_{r0} + \frac{1}{4}\omega^2 x_{r0} &= a_r \sin \omega t + b_r \cos \omega t, \\ \ddot{x}_{s0} + p_s^2 x_{s0} &= a_s \sin \omega t + b_s \cos \omega t, \end{aligned}\right\} \tag{2-15}$$

$$\begin{aligned} \ddot{x}_{r1} + \frac{1}{4}\omega^2 x_{r1} = \left(\frac{1}{4}\omega^2 - p_r^2\right) x_{r0} &- \Delta_{1r}(c_1 \dot{q}_{10} + p_r c_2 \dot{q}_{20} - \kappa_r c_3 \dot{q}_{30} + \kappa_r p_r c_4 \dot{q}_4) \\ &- \Delta_{1r}(\dot{\varphi}_{10} - p_r \varphi_{20} - \kappa_r \dot{\varphi}_{30} + \kappa_r p_r \varphi_{40}). \end{aligned} \tag{2-16}$$

Solutions of Eq. (2-15) are

$$\left.\begin{aligned} x_{r0} &= P \cos \frac{1}{2}\omega t + Q \sin \frac{1}{2}\omega t + A_r \sin \omega t + B_r \cos \omega t, \\ x_{s0} &= A_s \sin \omega t + B_s \cos \omega t, \end{aligned}\right\} \tag{2-17}$$

where

$$A_r = -4\, a_r/(3\,\omega^2), \; B_r = -4\, b_r/(3\,\omega^2), \; A_s = a_s/(p_s^2 - \omega^2), \; B_s = b_s/(p_s^2 - \omega^2). \tag{2-18}$$

Values of P and Q are determined by using the conditions that the periodicity of solutions holds, namely, that the terms exciting resonance are not contained in the right-hand side of Eq. (2-16). Inserting Eq. (2-17) into Eq. (2-16), and vanishing the coefficients of terms $\cos \frac{1}{2}\omega t$ and $\sin \frac{1}{2}\omega t$, we get

$$\left[\left\{g(P^2+Q^2) + \frac{1}{\Delta_{1r} p_r}\left(\frac{1}{4}\omega^2 - p_r^2\right) + h\right\} + i_1\right] P + (i_2 + c\omega)\, Q = 0,$$

$$(i_2 - c\omega) P + \left[\left\{g(P^2+Q^2) + \frac{1}{\Delta_{1r} p_r}\left(\frac{1}{4}\omega^2 - p_r^2\right) + h\right\} - i_1\right] Q = 0.$$

From the above equations, we obtain the equation of response curve of sub-harmonic oscillation of order 1/2,

$$R^2 = \frac{\left(\frac{1}{2}\omega\right)^2 - p_r^2}{(-\Delta_{1r} p_r g)} - \frac{h}{g} \pm \frac{1}{g}\sqrt{i_1^2 + i_2^2 - c^2\omega^2}, \tag{2-19}$$

in which R is the amplitude of sub-harmonic oscillation and

$$
\left.
\begin{aligned}
&R^2 = P^2 + Q^2,\\
&g = \frac{3}{4}\{(\beta_1 + \beta_2 + \kappa_r^4 \beta_3 + \kappa_r^4 \beta_4) - 4\kappa_r(\kappa_r^2 \delta_1 + \kappa_r^2 \delta_2 + \delta_3 + \delta_4) + \kappa_r^2 \textstyle\sum \zeta_i\},\\
&h = \sum\left(\frac{3}{2}\beta_i - \kappa_r \eta_i\right)(F_i^2 + f_i^2) + \frac{1}{2}\zeta_1\{\kappa_r^2(F_1^2 + f_1^2) + (F_3^2 + f_3^2)\}\\
&\qquad + \frac{1}{2}\zeta_2\{\kappa_r^2(F_2^2 + f_2^2) + (F_4^2 + f_4^2)\} + (3\kappa_r^2\delta_1 + 3\delta_3 - 2\kappa_r\zeta_1)(F_1F_3 + f_1f_3)\\
&\qquad + (3\kappa_r^2\delta_2 + 3\delta_4 - 2\kappa_r\zeta_2)(F_2F_4 + f_2f_4),\\
&{}^{i_1}_{i_2} = -(\alpha_1 + \kappa_r^2\gamma_1 - \kappa_r\xi_1)^{F_1}_{f_1} + (\alpha_2 + \kappa_r^2\gamma_2 - \kappa_r\xi_2)^{f_2}_{F_2}\\
&\qquad - (\kappa_r^2\alpha_3 - \kappa_r\xi_3 + \gamma_3)^{F_3}_{f_3} + (\kappa_r^2\alpha_4 - \kappa_r\xi_4 + \gamma_4)^{f_4}_{F_4},\\
&c = \frac{1}{2}(c_1 + c_2 + \kappa_r^2 c_3 + \kappa_r^2 c_4),\\
&F_1 = \frac{1}{\omega}\sum p_i A_i,\ f_1 = -\frac{1}{\omega}\sum p_i B_i,\ F_2 = \sum A_i,\ f_2 = \sum B_i,\ F_3 = -\frac{1}{\omega}\sum \kappa_i p_i A_i,\\
&f_3 = \frac{1}{\omega}\sum \kappa_i p_i B_i,\ F_4 = -\sum \kappa_i A_i,\ f_4 = -\sum \kappa_i B_i.
\end{aligned}
\right\}
\tag{2-20}
$$

As seen from Eq. (2-20), g in Eq. (2-19) is coefficient of the third powers in nonlinear terms Eq. (2-11), h is a product of coefficient of the third powers and exciting force, i_1 and i_2 are products of coefficient of the second powers and exciting force, c is sum of damping coefficients. Accordingly, form of Eq. (2-19) is quite similar to that of rectilinear vibratory system with one degree-of-freedom [cf. Eq. (1-25 a)][13]. Obviously sub-harmonic oscillation can take place when both the second and third powers in nonlinearity exist.

When a denominator of the first term in the right hand side of Eq. (2-19), *i.e.*, $(-\Delta_{1r}p_r g)$ is positive, response curves are hard spring type; when negative, soft spring type. By calculation, we conclude that spring characteristics is hard spring type when all β_i, δ_i, η_i and ζ_i in Eq. (2-11) are positive, and soft spring type when all are negative. When $p_r = p_2$ and $p_r = p_3$, we conclude that the hard and soft spring characteristics results in response curves of hard and soft spring type respectively, because of $\kappa_2 < 0$, $\kappa_3 > 0$, $g > 0$ and $-p_i\Delta_{1i} > 0$, then $-p_i\Delta_{1i}g > 0$. For $p_r = p_1$ and $p_r = p_4$; however, since $\kappa_{1,4} > 0$, the above conclusion does not hold. Clearly for a more general case when β_i, δ_i, η_i and ζ_i take both positive and negative values, hard spring characteristics, for instance, results in both hard and soft spring type of response curves. Consequently, one spring characteristic results in hard response curves at one peak and soft response curves at another peak. This fact will be shown experimentally in Section 2.

1.4. Summed and differential harmonic oscillations

Summed and differential harmonic oscillations consist of two vibrations having frequencies $\omega_m \fallingdotseq p_m$ and $\omega_n \fallingdotseq p_n$, and the absolute value of sum of or difference in two frequencies ω_m and ω_n is equal to the rotating speed of the shaft ω which is also the frequency of periodic external force induced by eccentricity e and small deviational angle τ. When ω_m or ω_n is positive or negative, the whirling motion is forward or backward precession respectively. The peaks of these vibrations appear in the neighborhood of the rotating speed

$$\omega \fallingdotseq |p_m \pm p_n|. \tag{2-21}$$

At first, we consider a case when one of p_m and p_n is positive, and here we assume $p_m>0$. Then, we can put

$$p_m \pm p_n \fallingdotseq \omega \qquad (p_m > p_n), \tag{2-21 a}$$

$$p_m \fallingdotseq \omega_m, \ p_n \fallingdotseq \omega_n, \ \omega_m \pm \omega_n = \omega. \tag{2-21 b}$$

Rewriting the equations of x_m and x_n in Eq. (2-9), we have

$$\begin{aligned} \ddot{x}_{m,n} + \omega_{m,n}^2 x_{m,n} &= a_{m,n} \sin \omega t + b_{m,n} \cos \omega t + (\omega_{m,n}^2 - p_{m,n}^2) x_{m,n} \\ &\quad - \Delta_{1m,1n}(c_1 \ddot{q}_1 - p_{m,n} c_2 \dot{q}_2 - \kappa_{m,n} c_3 \ddot{q}_3 + \kappa_{m,n} p_{m,n} c_4 \dot{q}_4) \\ &\quad - \Delta_{1m,1n}(\dot{\varphi}_1 - p_{m,n}\varphi_2 - \kappa_{m,n}\dot{\varphi}_3 + \kappa_{m,n} p_{m,n}\varphi_4). \end{aligned} \tag{2-22}$$

Substituting Eq. (2-14) into Eqs. (2-9) and (2-22), and assuming that damping terms, non-linear terms, $(\omega_m^2 - p_m^2)$ and $(\omega_n^2 - p_n^2)$ are as small as ε, we get

$$\left.\begin{aligned} \ddot{x}_{m0,n0} + \omega_{m,n}^2 x_{m0,n0} &= a_{m,n} \sin \omega t + b_{m,n} \cos \omega t, \\ \ddot{x}_{s0} + p_s^2 x_{s0} &= a_s \sin \omega t + b_s \cos \omega t \end{aligned}\right\} \tag{2-23}$$

$$\begin{aligned} \ddot{x}_{m1,n1} + \omega_{m,n}^2 x_{m1,n1} &= (\omega_{m,n}^2 - p_{m,n}^2) x_{m0,n0} \\ &\quad - \Delta_{1m,1n}(c_1 \ddot{q}_{10} + p_{m,n} c_2 \dot{q}_{20} - \kappa_{m,n} c_3 \ddot{q}_{30} + \kappa_{m,n} p_{m,n} c_4 \dot{q}_4) \\ &\quad - \Delta_{1m,1n}(\dot{\varphi}_{10} - p_{m,n}\varphi_{20} - \kappa_{m,n}\dot{\varphi}_{30} + \kappa_{m,n} p_{m,n}\varphi_{40}). \end{aligned} \tag{2-24}$$

Putting

$$\left.\begin{aligned} x_{m0,n0} &= P_{m,n} \cos \omega_{m,n} t + Q_{m,n} \sin \omega_{m,n} t + A_{m,n} \sin \omega t + B_{m,n} \cos \omega t, \\ x_{s0} &= A_s \sin \omega t + B_s \cos \omega t, \end{aligned}\right\} \tag{2-25}$$

where

$$A_{m,n} = \frac{a_{m,n}}{\omega_{m,n}^2 - \omega^2}, \quad B_{m,n} = \frac{b_{m,n}}{\omega_{m,n}^2 - \omega^2}, \quad A_s = \frac{a_s}{p_s^2 - \omega^2}, \quad B_s = \frac{b_s}{p_s^2 - \omega^2}. \tag{2-26}$$

and considering that terms $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \omega_m t$ and $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \omega_n t$ should not exist in the right-hand side of equations x_{m1} and x_{n1} of Eq. (2-24) respectively, we obtain

$$\left.\begin{aligned} \lambda_m P_m + \mu_m Q_m \pm \nu_m P_n \pm \pi_m Q_n &= (\rho_m R_m^2 + \sigma_m R_n^2) P_m, \\ -\mu_m P_m + \lambda_m Q_m + \pi_m P_n - \nu_m Q_n &= (\rho_m R_m^2 + \sigma_m R_n^2) Q_m, \\ \pm \nu_n P_m + \pi_n Q_m + \lambda_n P_n + \mu_n Q_n &= (\sigma_n R_m^2 + \rho_n R_n^2) P_n, \\ \pm \pi_n P_m - \nu_n Q_m - \mu_n P_n + \lambda_n Q_n &= (\sigma_n R_m^2 + \rho_n R_n^2) Q_n, \end{aligned}\right\} \tag{2-27}$$

where $R_{m,n}$ is the amplitude of summed and differetial harmonic oscillation, *i.e.*,

$$R_{m,n}^2 = P_{m,n}^2 + Q_{m,n}^2. \tag{2-28}$$

Putting

$$P_{m,n}\cos\omega_{m,n}t+Q_{m,n}\sin\omega_{m,n}t=R_{m,n}\sin(\omega_{m,n}t-\psi_{m,n}),$$

namely

$$P_{m,n}=-R_{m,n}\sin\psi_{m,n},\quad Q_{m,n}=R_{m,n}\cos\psi_{m,n},\tag{2-29}$$

Eq. (2-27) becomes

$$\left.\begin{aligned}
&\lambda_m R_m-R_n\{\nu_m\cos(\psi_m\pm\psi_n)\pm\pi_m\sin(\psi_m\pm\psi_n)\}=(\rho_m R_m^2+\sigma_m R_n^2)R_m,\\
&\lambda_n R_n-R_m\{\nu_n\cos(\psi_m\pm\psi_n)\pm\pi_n\sin(\psi_m\pm\psi_n)\}=(\sigma_n R_m^2+\rho_n R_n^2)R_n,\\
&\mu_m R_m-R_n\{\nu_m\sin(\psi_m\pm\psi_n)\mp\pi_m\cos(\psi_m\pm\psi_n)\}=0,\\
&\mu_n R_n-R_m\{\pm\nu_n\sin(\psi_m\pm\psi_n)-\pi_n\cos(\psi_m\pm\psi_n)\}=0,
\end{aligned}\right\}\tag{2-27a}$$

where

$$\begin{aligned}
\lambda_{m,n}=&(\omega_{m,n}^2-p_{m,n}^2(+p_{m,n}\Delta_{1m,1n}\Big[\frac{3}{2}\{(\beta_1-\kappa_{m,n}\delta_3)(F_1^2+f_1^2)+(\beta_2-\kappa_{m,n}\delta_4)(F_2^2+f_2^2)\\
&-\kappa_{m,n}(\delta_1-\kappa_{m,n}\beta_3)(F_3^2+f_3^2)-\kappa_{m,n}(\delta_2-\kappa_{m,n}\beta_4)(F_4^2+f_4^2)\}\\
&+\frac{1}{2}\{-\kappa_{m,n}(\eta_1-\kappa_{m,n}\zeta_3)(F_1^2+f_1^2)-\kappa_{m,n}(\eta_2-\kappa_{m,n}\zeta_4)(F_2^2+f_2^2)\\
&+(\zeta_1-\kappa_{m,n}\eta_3)(F_3^2+f_3^2)+(\zeta_2-\kappa_{m,n}\eta_4)(F_4^2+f_4^2)\}\\
&+(\eta_1-2\kappa_{m,n}\zeta_1+\kappa_{m,n}^2\eta_3)(F_1F_3+f_1f_3)+(\eta_2-2\kappa_{m,n}\zeta_2+\kappa_{m,n}^2\eta_4)(F_2F_4+f_2f_4)\Big],\\
\mu_{m,n}=&p_{m,n}\Delta_{1m,1n}\omega_{m,n}\{c_1+c_2+\kappa_{m,n}^2(c_3+c_4)\}=p_{m,n}\Delta_{1m,1n}\omega_{m,n}c_{m,n},\\
c_{m,n}=&c_1+c_2+\kappa_{m,n}^2(c_3+c_4)>0,\\
\nu_{m,n}=&\frac{\omega_{m,n}p_{n,m}}{\omega_{n,m}}\Delta_{1m,1n}\{-(\alpha_1-\kappa_{m,n}\gamma_3)F_1+\kappa_{n,m}(\gamma_1-\kappa_{m,n}\alpha_3)F_3\\
&+(\gamma_3-\kappa_{m,n}\gamma_1)(\kappa_{n,m}F_1-F_3)\}\pm p_{m,n}\Delta_{1m,1n}\{(\alpha_2-\kappa_{m,n}\gamma_4)f_2\\
&-\kappa_{n,m}(\gamma_2-\kappa_{m,n}\alpha_4)f_4-(\gamma_4-\kappa_{m,n}\gamma_2)(\kappa_{n,m}f_2-f_4)\},\\
\pi_{m,n}=&\pm\frac{\omega_{m,n}p_{n,m}}{\omega_{n,m}}\Delta_{1m,1n}\{-(\alpha_1-\kappa_{m,n}\gamma_3)f_1+\kappa_{n,m}(\gamma_1-\kappa_{m,n}\alpha_3)f_3\\
&+(\gamma_3-\kappa_{m,n}\gamma_1)(\kappa_{n,m}f_1-f_3)\}+p_{m,n}\Delta_{1m,1n}\{(\alpha_2-\kappa_{m,n}\gamma_4)F_2\\
&-\kappa_{n,m}(\gamma_2-\kappa_{m,n}\alpha_4)F_4-(\gamma_4-\kappa_{m,n}\gamma_2)(\kappa_{n,m}F_2-F_4)\},\\
\rho_{m,n}=&-\frac{3}{4}p_{m,n}\Delta_{1m,1n}\Big\{\frac{p_{m,n}^2}{\omega_{m,n}^2}(\beta_1-\kappa_{m,n}\delta_3)+(\beta_2-\kappa_{m,n}\delta_4)\\
&-\kappa_{m,n}^3\frac{p_{m,n}^2}{\omega_{m,n}^2}(\delta_1-\kappa_{m,n}\beta_3)-\kappa_{m,n}^3(\delta_{m,n}-\kappa_{m,n}\beta_4)\\
&-\kappa_{m,n}\frac{p_{m,n}^2}{\omega_{m,n}^2}(\eta_1-\kappa_{m,n}\zeta_3)-\kappa_{m,n}(\eta_2-\kappa_{m,n}\zeta_4)\\
&+\kappa_{m,n}^2\frac{p_{m,n}^2}{\omega_{m,n}^2}(\zeta_1-\kappa_{m,n}\eta_3)+\kappa_{m,n}^2(\zeta_2-\kappa_{m,n}\eta_4)\Big\},\\
\sigma_{m,n}=&-p_{m,n}\Delta_{1m,1n}\Big[\frac{3}{2}\Big\{\frac{p_{n,m}^2}{\omega_{n,m}^2}(\beta_1-\kappa_{m,n}\delta_3)+(\beta_2-\kappa_{m,n}\delta_4)
\end{aligned}$$

$$
\begin{aligned}
&- \kappa_{m,n}\kappa_{n,m}^2 \frac{p_{n,m}^2}{\omega_{n,m}^2}(\delta_1 - \kappa_{m,n}\beta_3) - \kappa_{m,n}\kappa_{n,m}^2(\delta_2 - \kappa_{m,n}\beta_4)\Big\} \\
&- \left(\frac{1}{2}\kappa_{m,n} + \kappa_{n,m}\right)\frac{p_{n,m}^2}{\omega_{n,m}^2}(\eta_1 - \kappa_{m,n}\zeta_3) \\
&- \left(\frac{1}{2}\kappa_{m,n} + \kappa_{n,m}\right)(\eta_2 - \kappa_{m,n}\zeta_4) \\
&+ \left(\kappa_{m,n} + \frac{1}{2}\kappa_{n,m}\right)\kappa_{n,m}\frac{p_{n,m}^2}{\omega_{n,m}^2}(\zeta_1 - \kappa_{m,n}\eta_3) \\
&+ \left(\kappa_{m,n} + \frac{1}{2}\kappa_{n,m}\right)\kappa_{n,m}(\zeta_2 - \kappa_{m,n}\eta_4)\Big].
\end{aligned}
\tag{2-30}
$$

Values of F_i and f_i are given by Eq. (2-20). As seen from Eqs. (2-27) and (2-27 a), summed and differential harmonic oscillations can take place only when both the second and the third terms of nonlinearity exist. The upper and lower signs of $\pm$ and $\mp$ in Eqs. (2-27) and (2-27 a) correspond to summed and differential harmonic oscillation respectively.

The frequency of sub-harmonic oscillation is $1/\nu$ (ν: integer) times as large as that of exciting force. Accordingly it is determined when ω is given. However, even if ω is given, ω_m and ω_n of summed and differential harmonic oscillation cannot be determined from only one relation $|\omega_m \pm \omega_n| = \omega$. In order to determine ω_m and ω_n, we must solve the equations of motion. Namely, amplitudes R_m, R_n, frequencies ω_m, ω_n and phase angle $\phi_m \pm \phi_n$ are given by five equations of Eq. (2-27) or Eq. (2-27 a) and $|\omega_m \pm \omega_n| = \omega$. Generally speaking ω/ω_m and ω/ω_n are not integers. Consequently individual phase angles ϕ_m and ϕ_n are meaningless, and there is a physical meaning only in the quantity $\phi_m \pm \phi_n$.

From the third and fourth equations of Eq. (2-27 a), we have

$$
\frac{R_m^2}{R_n^2} = \pm \frac{\mu_n\{\nu_m \sin(\phi_m \pm \phi_n) \mp \pi_m \cos(\phi_m \pm \phi_n)\}}{\mu_m\{\nu_n \sin(\phi_m \pm \phi_n) \mp \pi_n \cos(\phi_m \pm \phi_n)\}}.
$$

From Eqs. (2-21 b) and (2-30), we get

$$
\frac{R_m^2}{R_n^2} \fallingdotseq \pm \left(\frac{p_m^2 \Delta_{1\,m}^2 c_m}{p_n^2 \Delta_{1\,n}^2 c_n}\right)\frac{\omega_m}{\omega_n}. \tag{2-31}
$$

Value $\pm \dfrac{\omega_m}{\omega_n}$ must be positive, because the inside of ( ) is positive and R_m, R_n are real numbers.

Since $\omega_m \fallingdotseq p_m > 0$ as we assumed,

(1) when $\omega_n > 0$, summed harmonic oscillation can take place because $+\omega_m/\omega_n > 0$, and differential harmonic oscillation cannot occur because $-\omega_m/\omega_n < 0$,

(2) when $\omega_n < 0$, only the differential harmonic oscillation can appear, because $-\omega_m/\omega_n > 0$, $+\omega_m/\omega_n < 0$.

When both p_m and p_n are negative and $p_m < p_n (|p_m| > |p_n|)$, we have

$$
p_m \pm p_n \fallingdotseq -\omega,\quad p_m \fallingdotseq \omega_m,\quad p_n \fallingdotseq \omega_n,\quad \omega_m \pm \omega_n = -\omega. \tag{2-21 c}
$$

By calculation, we conclude that we must use $-f_{1,3}$ and $-F_{2,4}$ in place of $f_{1,3}$

and $F_{2,4}$ in Eq. (2–30) and others are the same as when $p_m>0$. Accordingly even for $p_m<0$, $p_n<0$, Eq. (2–31) holds good and only the summed harmonic oscillation can take place. Consequently when both two vibrations are forward precessional motions (p_1, p_2), or both are backward precessional motions (p_3, p_4), only the summed harmonic oscillation can occur. When one is forward and another is back ward, only the differential harmonic oscillation can appear.

To summarize, since p_1 and p_2 are forward and p_3, p_4 backward, the types having possibility of occurrence are

$$\left.\begin{array}{l} p_1+p_2=\omega,\ \ p_1-p_3=\omega,\ \ p_1-p_4=\omega, \\ p_2-p_3=\omega,\ p_2-p_4=\omega,\ |p_3+p_4|=\omega, \end{array}\right\} \tag{2-32}$$

and the following types

$$\left.\begin{array}{l} p_1-p_2=\omega,\ \ p_1+p_3=\omega,\ \ p_1+p_4=\omega, \\ p_2+p_3=\omega,\ \ p_2+p_4=\omega,\ \ p_3-p_4=\omega, \end{array}\right\} \tag{2-33}$$

do not take place. When the magnitude of I of moment of interia is comparable with value of I_p of polar moment of inertia, the natural frequency p_1 is much larger than ω, then there is no rotating speed ω satisfying the relations $p_1+p_2=\omega$, $p_1-p_3=\omega$ and $p_1-p_4=\omega$. Therefore only three types $p_2-p_3=\omega$, $p_2-p_4=\omega$ and $|p_3+p_4|=\omega$ can occur. Experimental results using the rotating body of $I=1/2\cdot I_p$ agree with the above conclusion[11].*

Since the procedure to obtain R_m, R_n, ω_m and ω_n from Eq. (2–27) or Eq. (2–27 a) is quite complicated, we shall treat a comparatively simple case in which $r_0=0$ and there are nonlinear terms θ_y^2 and θ_y^3 only in y direction. Equations of motion are

$$\left.\begin{array}{l} I\ddot{\theta}_x+I_p\omega\dot{\theta}_y+c_3\dot{\theta}_x+\delta_0\theta_x=(I_p-I)\,\tau\omega^2\cos\omega t, \\ I\ddot{\theta}_y-I_p\omega\dot{\theta}_x+c_4\dot{\theta}_y+\delta_0\theta_y+\alpha\theta_y^2+\beta\theta_y^3=(I_p-I)\,\tau\omega^2\sin\omega t, \end{array}\right\} \tag{2-1 a}$$

where $c_3=c_4$. Putting

$$\left.\begin{array}{l} I_p/I=I_1,\ \ \theta_x/\{\tau(1-I_1)\}=q_x,\ \ \theta_y/\{\tau(1-I_1)\}=q_y,\ \ \omega/\sqrt{(\delta_0/I)}=\omega', \\ \sqrt{(\delta_0/I)}\cdot t=t',\ c_{3,4}/\sqrt{(\delta_0 I)}=c,\ \alpha\tau(1-I_1)/\delta_0=\alpha',\ \beta\tau^2(1-I_1)^2/\delta_0=\beta', \end{array}\right\} \tag{2-2 a}$$

and inserting Eq. (2-2 a) into Eq. (2-1 a) and omitting primes, we have

$$\left.\begin{array}{l} \ddot{q}_x+I_1\omega\dot{q}_y+c\dot{q}_x+q_x=-\omega^2\cos\omega t, \\ \ddot{q}_y-I_1\omega\dot{q}_x+c\dot{q}_y+q_y+\alpha q_y^2+\beta q_y^3=-\omega^2\sin\omega t. \end{array}\right\} \tag{2-3 a}$$

The frequency equation is

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* In bibliography (11) and Fig. 17, there is one exception. That is, there is one peak appearing at $p_2+p_3=\omega$. At this peak, however, ω where sub-harmonic oscillation of order 1/2 occurs, incidentally coincides with ω at which the relation $\omega=p_2+p_3$ holds. Vibrations appearing there are not summed and differential harmonic oscillation, but sub-harmonic oscillation of order 1/2.

$$p^2 - I_1\omega p - 1 = (p-p_1)(p-p_2) = 0, \tag{2-4 a}$$

and $p_1>0$, $p_2<0$. Accordingly only the differential harmonic oscillation of type $p_1-p_2=\omega$ can take place. Putting

$$q_x = -p_1X_1 - p_2X_2, \quad q_y = x_1 + x_2, \tag{2-6 a}$$

then

$$\left.\begin{aligned} \ddot{x}_1 + p_1^2 x_1 &= \frac{1}{p_1-p_2}\{-(\omega+p_1)\omega^2 \sin \omega t + c(\ddot{q}_x - p_1\dot{q}_y) - p_1(\alpha q_y^2 + \beta q_y^3)\}, \\ \ddot{x}_2 + p_2^2 x_2 &= \frac{1}{p_1-p_2}\{(\omega+p_2)\omega^2 \sin \omega t + c(-\ddot{q}_x + p_2\dot{q}_y) + p_2(\alpha q_y^2 + \beta q_y^3)\}. \end{aligned}\right\} \tag{2-9a}$$

Further, for brevity, we introduce

$$\left.\begin{aligned} &\omega/(p_1-p_2) = \omega', \ (p_1-p_2)t = t', \ c/(p_1-p_2) = c', \\ &\alpha/(p_1-p_2)^2 = \alpha', \ \beta/(p_1-p_2)^2 = \beta', \ p_{1,2}/(p_1-p_2) = p'_{1,2}. \end{aligned}\right\}$$

If we omit the primes, Eq. (2-9 a) becomes

$$\left.\begin{aligned} \ddot{x}_1 + p_1^2 x_1 &= -(\omega+p_1)\omega^2 \sin \omega t + c(\ddot{q}_x - p_1\dot{q}_y) - p_1(\alpha q_y^2 + \beta q_y^3), \\ \ddot{x}_2 + p_2^2 x_2 &= (\omega+p_2)\omega^2 \sin \omega t + c(-\ddot{q}_x + p_2\dot{q}_y) + p_2(\alpha q_y^2 + \beta q_y^3). \end{aligned}\right\} \tag{2-9 b}$$

Here

$$p_1 - p_2 = 1 \fallingdotseq \omega, \ p_1 \fallingdotseq \omega_1, \ p_2 \fallingdotseq \omega_2, \ \omega_1 - \omega_2 = \omega. \tag{2-21 d}$$

Putting

$$x_{10,20} = P_{1,2}\cos \omega_{1,2}t + Q_{1,2}\sin \omega_{1,2}t + A_{1,2}\sin \omega t, \tag{2-25 a}$$

we have by the similar procedure

$$\left.\begin{aligned} &\lambda_1 R_1 - \alpha A R_2 \sin(\psi_1-\psi_2) = \frac{3}{4}\beta(R_1^2 + 2R_2^2)R_1, \\ &\lambda_2 R_2 + \alpha A R_1 \sin(\psi_1-\psi_2) = -\frac{3}{4}\beta(2R_1^2 + R_2^2)R_2, \\ &2c\omega_1 R_1 - \alpha A R_2 \cos(\psi_1-\psi_2) = 0, \\ &2c\omega_2 R_2 + \alpha A R_1 \cos(\psi_1-\psi_2) = 0, \end{aligned}\right\} \tag{2-27 b}$$

in which

$$\left.\begin{aligned} A &= A_1 + A_2 = \frac{\omega^2(\omega+p_1)}{\omega^2-p_1^2} - \frac{\omega^2(\omega+p_2)}{\omega^2-p_2^2}, \\ \lambda_1 &= (\omega_1^2 - p_1^2)p_1^{-1} - \frac{3}{2}\beta A^2, \\ \lambda_2 &= (\omega_2^2 - p_2^2)p_2^{-1} + \frac{3}{2}\beta A^2. \end{aligned}\right\} \tag{2-30 a}$$

From the third and fourth equations of Eq. (2-27 b), we obtain

$$R_1^2/R_2^2 = -\omega_2/\omega_1. \tag{2-31 a}$$

Using Eq. (2–31 a) and the first and the second equations of Eq. (2–27 b), we get

$$\begin{matrix} R_1^2 \\ R_2^2 \end{matrix} = \begin{matrix} \omega_2 \\ -\omega_1 \end{matrix}\left[\frac{4\{p_1\omega_1(\omega_2^2-p_2^2) - p_2\omega_2(\omega_1^2-p_1^2)\}}{3\beta p_1 p_2 \omega(\omega_1+\omega_2)} + \frac{2A^2}{\omega} \right]. \tag{2-34}$$

Eliminating R_1 and R_2 from Eqs. (2–27 b) and (2–31 a), we have

$$\omega_1\omega_2\{\lambda_1(\omega-\omega_2)+\lambda_2(\omega+\omega_1)\}^2 = -(\alpha^2A^2+4c^2\omega_1\omega_2)(\omega_1+\omega_2)^2\omega^2. \tag{2-35}$$

From Eqs. (2–30 a) and (2–35),

$$\begin{aligned} &[\omega_1^2\omega_2^2(\omega+\omega_1)^2(\omega-\omega_2)^2\{p_2(\omega_1^2-p_1^2)(\omega-\omega_2)+p_1(\omega_2^2-p_2^2)(\omega+\omega_1)\} \\ &+\frac{3}{2}\beta p_1p_2\omega^4(\omega_1+\omega_2)\{\omega_1(\omega-\omega_2)(\omega+p_1)+\omega_2(\omega+\omega_1)(\omega+p_2)\}^2]^2 \\ &= -p_1^2p_2^2\omega^2\omega_1\omega_2(\omega_1+\omega_2)^2(\omega+\omega_1)^2(\omega-\omega_2)^2[\alpha^2\omega^2\{\omega_1(\omega-\omega_2)(\omega+p_1) \\ &+\omega_2(\omega+\omega_1)(\omega+p_1)\}^2+4c^2\omega_1^3\omega_2^3(\omega+\omega_1)^2(\omega-\omega_2)^2]. \end{aligned} \tag{2-36}$$

Determining ω_1 and ω_2 by Eq. (2-36) and $\omega_1-\omega_2=\omega$, we can obtain R_1 and R_2. It is, however, impossible to give ω_1 and ω_2 analytically by using Eq. (2-36), so we shall determine them approximately. As the first approximation ω_{10} and ω_{20} satisfying the relation $\omega_{10}-\omega_{20}=\omega$, experimental results lead to

$$\omega_{10} \fallingdotseq p_1\omega, \qquad \omega_{20} \fallingdotseq p_2\omega. \tag{2-37}$$

When we neglect higher powers of ε, we can use ω_{10} and ω_{20} in place of ω_1 and ω_2 in Eq. (2–35), except ω_1 and ω_2 in λ_1 and λ_2. Then we get from Eq. (2–35),

$$\left.\begin{aligned} \omega_1 &= p_1\omega - \frac{p_1+p_2}{6\omega}\left\{(\omega^2-1)+\frac{3}{2}\beta A^2 \pm \sqrt{\frac{-1}{p_1p_2}(\alpha^2A^2+4c^2p_1p_2\omega^2)}\right\}, \\ \omega_2 &= p_2\omega - \frac{p_1+p_2}{6\omega}\left\{(\omega^2-1)+\frac{3}{2}\beta A^2 \pm \sqrt{\frac{-1}{p_1p_2}(\alpha^2A^2+4c^2p_1p_2\omega^2)}\right\}, \\ A &= \frac{-\{p_1(1-p_2)(\omega+p_1)+p_2(1+p_1)(\omega+p_2)\}}{p_1p_2(1+p_1)(1-p_2)}. \end{aligned}\right\} \tag{2-38}$$

Clearly ω_1 and ω_2 given by Eq. (2–38) satisfy the relation $\omega_1-\omega_2=\omega$. Equations of response curves

$$\left.\begin{aligned} R_1^2 &= -\frac{4p_2}{9\beta}\left\{(\omega^2-1)-3\beta A^2 \pm \sqrt{\frac{-1}{p_1p_2}(\alpha^2A^2+4c^2p_1p_2\omega^2)}\right\}, \\ R_2^2 &= \frac{4p_1}{9\beta}\left\{(\omega^2-1)-3\beta A^2 \pm \sqrt{\frac{-1}{p_1p_2}(\alpha^2A^2+4c^2p_1p_2\omega^2)}\right\}, \end{aligned}\right\} \tag{2-39}$$

are given by Eqs. (2–34) and (2–38). The sign $\pm$ in Eq. (2–38) corresponds to $\pm$ in Eq. (2–39). The form of Eq. (2–39) is quite similar to the equation of response curves of sub-harmonic oscillation of order 1/2.

1.5. Conclusions

The vibratory system treated in the present section is a nonlinear system with multiple degree-of-freedom and gyroscopic terms. Obtained conclusions for such a system may be summarized as follows:

(1) Equation of response curves of sub-harmonic oscillation of order 1/2 having a mode of whirling motion is quite similar to that of rectilinear vibration system with one degree-of-freedom system.

(2) Hard spring characteristics does not always result in response curves of hard spring type. One spring characteristics can result in both response curves of hard and soft spring types.

(3) When both the two vibrations in summed and differential harmonic oscillation are forward precessional motions or backward, only the summed type can occur. When one is forward and the other is backward, only the differential type can take place.

(4) Equations of response curves of summed and differential harmonic oscillations are similar to those of sub-harmonic oscillations of order 1/2.

Even for a simple system Eq. (2-1 a), we cannot determine analytically ω_1 and ω_2 by using Eq. (2-36). Accordingly appropriateness for Eqs. (2-38) and (2-39) entirely depends on that of Eq. (2-37). For a more general case, it may be rather easy and direct to obtain response curves from experiments[14], as shown in Section 2.

2. Experiments of summed and differential harmonic oscillations and sub-harmonic oscillations in rotating shaft system

2.1. Introduction

Analytical results of summed and differential harmonic oscillations and sub-harmonic oscillations in rotating shaft systems which are obtained in the previous section are verified experimentally in this section.

As is concluded in the previous section, there are three kinds of critical speeds of summed and differential harmonic oscillations $[p_2-p_3]$, $[p_2-p_4]$ and $[p_3+p_4]$ which appear at the rotating speeds

$$\omega \fallingdotseq p_2-p_3,\ \ \omega \fallingdotseq p_2-p_4,\ \ \omega \fallingdotseq |p_3+p_4| \qquad (2\text{-}40)$$

respectively. In Experiment A, occurrence of these three kinds of critical speeds of summed and differential harmonic oscillations is verified experimentally, and in Experiment B, shapes of response curves of summed and differential harmonic oscillations and sub-harmonic oscillations are studied for various magnitudes of eccentricity e and various nonlinearity of spring characteristics.

2.2. Experimental apparatus

Experimental apparatus is shown in Fig. 15 where the shaft mounting one disk is supported by deep-grooved sigle-row ball bearings with inner diameter of 10 mm. Distances a and b from both shaft ends to disk are different from each other, *i.e.*, $a \neq b$, and there is gyroscopic action in this system. The disk is driven by a V-belt, power supplied by a 15 *H.P.D.C.* motor with speed variations of from 0 to 6000 r.p.m., shunt controled. In order to secure the safety and to remove the disturbances from the belt, the spring coupling S consisting of a helical spring is inserted between the pulley V and the shaft. The guard ring G is equipped to check the increase of deflections of the shaft. The whirling motion is obtained by projections of motion of disk with relation to rectangular coordinates OA (x-direction) and OB (y-direction), that is, the motions of the disk edge at points

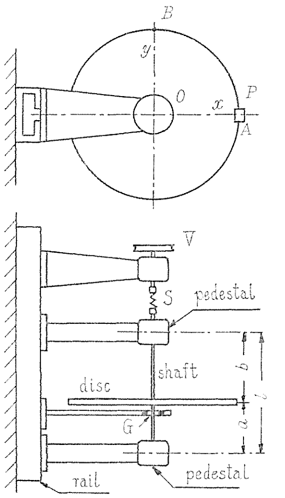


FIG. 15. Experimental apparatus.

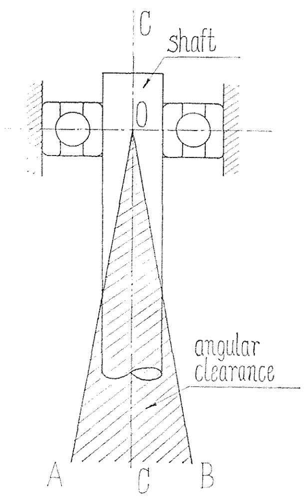


FIG. 16. Angular clearance of deep-grooved single-row ball bearing.

A and B (Fig. 15) are recorded on oscillographic paper by the optical method. A small piece of paper P is attached to one point on the disk edge, the light is intercepted by this paper at each revolution of the shaft and thus the angular speed of rotation of the shaft is recorded.

As shown in Fig. 16, deep-grooved single-row ball bearings have the so-called "angular clearance" $\angle AOB$, and when the inclination angle of the elastic deflection curve of the shaft is smaller than angular clearance at the shaft end, the shaft is supported freely, and the shaft becomes a fixed shaft when the inclination angle increases beyond the angular clearance. It is impossible for both bearing pedestals to be in exact alignment, and the equilibrium position of the shaft is not located at the middle of the angular clearance. Accordingly, the shaft has various kinds of nonlinear, unsymmetrical spring characteristics[1)] which result in summed and differential harmonic oscillations and sub-harmonic oscillations of order 1/2.

In the present section the following expressions are used to represent modes of vibrations. Notation $[+m\omega]$ means the motion having a frequency of m times as large as that of angular velocity ω, where the positive sign + represents a whirling motion of forward precession. On the other hand, negative sign − means a backward precession, hence notation $[-n\omega]$ denotes a whirling motion of backward precession with whirling speed $n\omega$.

2.3. Experiment A[11)]

In experimental apparatus of Experiment A, a vertical shaft (dia. = 11.72 ϕ, length = 506.3 mm) mounting a disk (dia. = 482.8 mm, thickness = 5.22 mm, weight $W=7.804$ kg, moment of inertia $I=1.114$ kg cm sec^2, and $I_p=2I$) is supported by deep-grooved single-row ball bearings with angular clearance 0.3°. The location of the disk is $a:b=1:4$.

2.3.1. Response curves of critical speeds and vibratory waves of summed and differential harmonic oscillations. Obtained response curves of Experiment A is illustrated in Fig. 17, where summed and differential harmonic oscillations $[p_2-p_4]$, $[p_3+p_4]$ and $[p_2-p_3]$ take place at $\omega=\omega_{24}$ (Peak I), $\omega=\omega_{34}$ (Peak III) and $\omega=\omega_{23}$ (Peak V), respectively, and sub-harmonic oscillation of forward precession $[2p_2]$ appears at $\omega=\omega_{1/2}$ (Peak II). Peak VI at $\omega=\omega_c$ is the major critical speed where the forced vibration $[+\omega]$ caused by unbalance in rotating body occurs, and in Peak IV, synchronous backward precession $[-\omega]$ which is induced by difference

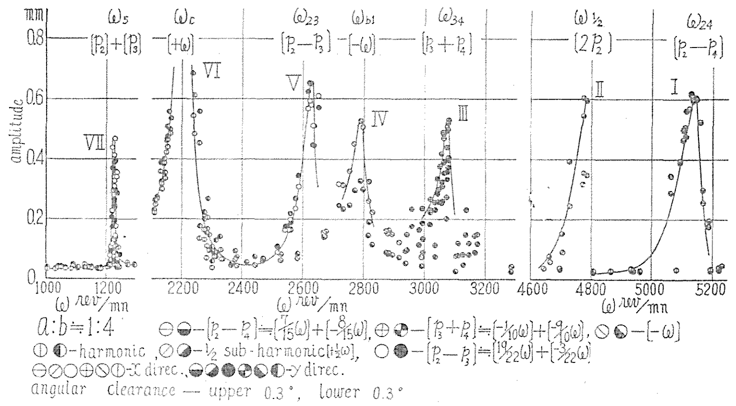


FIG. 17. Response curves of summed and differential harmonic oscillations of Experiment A (vertical shaft; shaft dia. = 11.72 mm, shaft length = 506.3 mm; disk dia. = 482.8 mm, weight of disk = 7.804 kg, thickness of disk = 5.22 mm, $I = 1.114$ kg cm sec^2, $I_p \fallingdotseq 2I$).

of shaft rigidity in x and y directions.[15)16)]

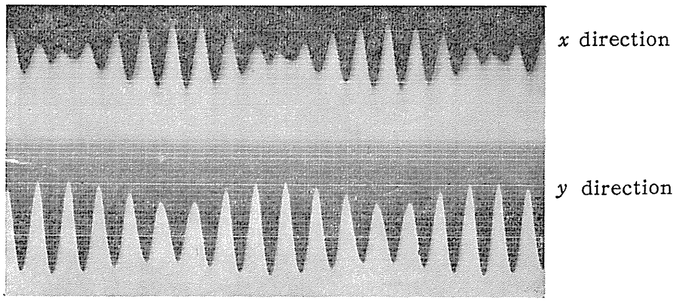
Vibratory waves of summed and differential harmonic oscillations are shown in Fig. 18, where vertical white lines are rotating marks recorded at each revolution by a small piece of paper P in Fig. 15, and fine white horizontal lines are furnished by a scale of 1.0 mm put on the slit for measuring the amplitudes of vibrations. Observing the vibratory waves in Fig. 18, it is seen that sum of absolute values of two frequencies of summed and differential harmonic oscillations is always equal to the rotating speed of the shaft ω, and the relation of Eq. (2-21 b), *i.e.*, the relation of $\omega_m \pm \omega_n = \omega$ is satisfied.

2.3.2. Locations of critical speeds and modes of vibrations of summed and differential harmonic oscillations. Using the quantities in Eq. (2-1) having dimensions, frequency equation of the rotating shaft system becomes

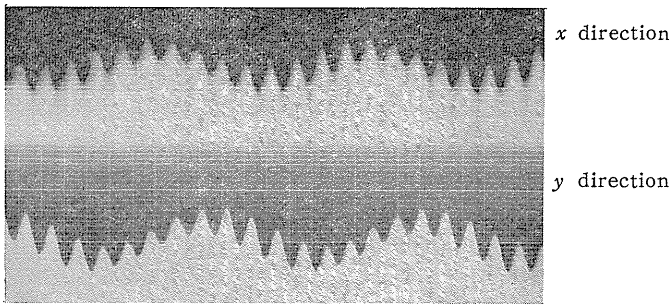
$$(\alpha_0 - mp^2)(\delta_0 + I_p \omega p - Ip^2) - \gamma_0^2 = (p - p_1)(p - p_2)(p - p_3)(p - p_4) = 0 \quad (2\text{-}4\,\mathrm{a})$$

Dimensionless expression of Eq. (2-4 a) is Eq. (2-4). In Fig. 19, natural frequencies p_2, p_3, p_4 of experimental apparatus of Experiment A are shown by full line curves. Chain line curves are natural frequencies p_1, p_2, p_3, p_4 when the shaft is supported freely by double-row self-aligning ball bearings.

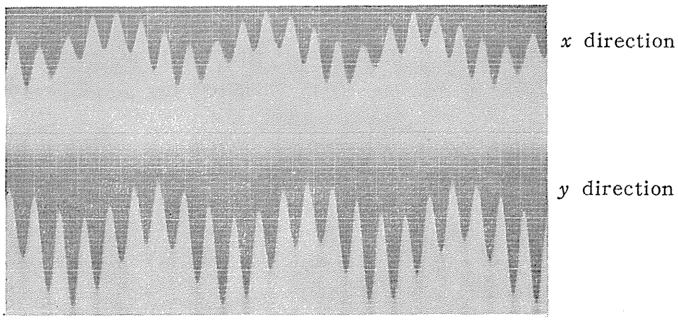
Where critical speeds of summed and differential harmonic oscillations occur, whatever modes of vibrations may have, they must be determined by using $p-\omega$ diagram as Fig. 19. Writing curves $|p_m \pm p_n|$ (m, $n = 1, 2, 3, 4$) against ω, and obtaining intersecting point where curve $|p_m \pm p_n|$ and straight line $\omega = p$ cross, we are given critical speed from abscissa of this intersecting point. Because at the rotating speed ω determined by this intersecting point we have $|p_m \pm p_n| = \omega$. In Fig. 19, for example, a point A_{24} is the intersecting point of curve $|p_2 - p_4| - \omega$ with line $\omega = p$, and the abscissa of point A_{24} takes the value of $\omega_{24} = 5130$ r.p.m. The values of $p_2 = 2380$ r.p.m. $= 6.96/15 \cdot \omega$ and $p_4 = -2750$ r.p.m. $= -8.04/15 \cdot \omega$ are given from ordinates of points a and b respectively where a vertical line $\omega_{24} = 5130$



(a) $[p_2-p_4]$ at $\omega=\omega_{24}$ (Peak I) ($\omega:\omega_2:\omega_4 \fallingdotseq 15:7:-8$, $\omega=5113.0$ r.p.m.)



(b) $[p_3+p_4]$ at $\omega=\omega_{34}$ (Peak III) ($\omega:\omega_3:\omega_4 \fallingdotseq 10:-1:-9$, $\omega=3062.2$ r.p.m.)



(c) $[p_2-p_3]$ at $\omega=\omega_{23}$ (Peak V) ($\omega:\omega_2:\omega_3 \fallingdotseq 22:19:-3$, $\omega=2624.1$ r.p.m.)

FIG. 18. Vibratory waves of summed and differential harmonic oscillations $[p_2-p_4]$, $[p_3+p_4]$ and $[p_2-p_3]$ in Experiment A.

r.p.m. intersects curves $p_2-\omega$ and $p_4-\omega$. Thus we can determine that at the rotating speed ω_{24} ($=5130$ r.p.m.), two vibrations with frequencies 2380 r.p.m. ($\fallingdotseq 7/15\cdot\omega$) and -2750 r.p.m. ($\fallingdotseq -8/15\cdot\omega$) build up and form the critical speed of Peak I shown in Fig. 17. Similarly, in Fig. 19, a point A_{34} is the intersecting point of curve $|p_3+p_4|-\omega$ with line $\omega=p$, and the abscissa of point A_{34} determines the critical speed ω_{34} ($=3065$ r.p.m.) of Peak III in Fig. 17; ordinates of points c and d give $p_3=-305$ r.p.m. $\fallingdotseq -1/10\cdot\omega$ and $p_4=-2760$ r.p.m. $\fallingdotseq -9/10\cdot\omega$ respectively, which agree with experimental results. For the summed and differential oscillation $[p_2-p_3]$ at ω_{23}, the rotating speed ω_{23} and modes of vibrations are determined in the same manner.

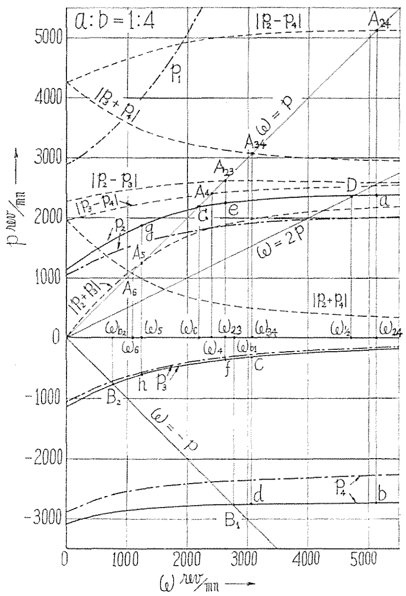


FIG. 19. Natural frequency $p-\omega$ diagrams of the rotating shaft system of Experiment A.

2.4. Experiment B[14]

2.4.1. Experimental apparatus and experimental results. In Experiment B, the horizontal shaft system as shown in Fig. 20 is used. A disk of diameter = 481.6 mm, thickness = 5.51 mm and weight = 7.868 kg is mounted on a horizontal shaft of diameter = 11.89 mm, length = 508.3 mm, and the location of the disk is $a:b=3:7$.

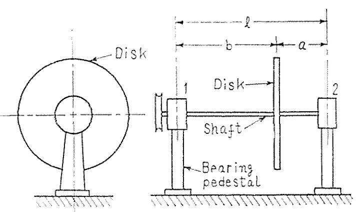


FIG. 20. Experimental apparatus of rotating shaft system used in Experiment B.

Nonlinearity of spring characteristics is changed according as the magnitude of angular clearance of ball bearings varies. Further reassembling of bearing pedestals results in change of nonlinearity, because degree of out of alignment of bearing center lines of both pedestals is changed[1]. Experiments using ball bearings with angular clearances 0.35° and 0.30° (ball bearings *I*) or those having angular clearances 0.3° and 0.6° (ball bearings *II*) are performed under different conditions of *a* or *b*, of assembling of bearing pedestals. In the present section, a certain experiment is specified, for instance, notation *II-a-2*. Notation *II-a-2* means that ball bearings with angular clearances 0.3° and 0.6° are used in bearing pedestals *1* and *2* (see Fig. 20) respectively, *i.e.* ball bearings *II* are used, and the experiment is carried out under the assembling condition *a*. Since the last numeral (*e.g.* *2* in *II-a-2*) denotes magnitude of eccentricity, then the same numeral corresponds to the same magnitude of unbalance.

The vibratory system as shown in Fig. 20 has two natural frequencies p_1 and p_2 ($p_1>0$, $p_2>0$, $p_1>p_2$) of forward precession and two natural frequencies p_3 and p_4 ($p_4<p_3<0$, $|p_4|>|p_3|$) of backward precession. In the present experimental apparatus, the major critical speed ω appears at $\omega=1440$ r.p.m. (for ball bearings *I*) and $\omega=1360$ r.p.m. (for ball bearings *II*), where the rotating speed of shaft ω is equal to the natural frequency of forward precession p_2; the critical speed $\omega_{1/2}$ of sub-harmonic oscillation $[2p_2]$ of order 1/2 having mode of forward precession $[+1/2\cdot\omega]$ takes place at $\omega=2870$ r.p.m. (for ball bearings *II*), where $1/2\cdot\omega=p_2$; the peaks of $\omega_{-1/2}$ and $\omega'_{-1/2}$ of backward precessions $[-1/2\cdot\omega]$ of sub-harmonic oscillations [$2p_3$] and [$2p_4$] appear at $\omega=1380$ r.p.m. where $-1/2\cdot\omega=p_3$ and at $\omega=3500$ r.p.m. where $-1/2\cdot\omega=p_4$ respectively when ball bearings *I* are used, and they take place at $\omega=1380$ r.p.m. and $\omega=3300$ r.p.m. respectively for ball bearings *II*; the critical speed ω_{23} of summed and differential harmonic oscillation [p_2-p_3] occurs at $\omega=2020$ r.p.m. (for *I*) and $\omega=1960$ r.p.m. (for *II*), where the relation $\omega=p_2-p_3$ is held; the peak of ω_{24} takes place at $\omega=3250$ r.p.m. (for *I*) and at $\omega=3100$ r.p.m. (for *II*), where $\omega=p_2-p_4$. Since large angular clearance of ball bearings results in decrease of stiffness of shaft[1], then the peaks when ball bearings *II* having larger angular clearances than ball bearings *I* are used, appear at lower rotating speeds than when ball bearings *I* are used.

I. The critical speeds of sub-harmonic oscillations [$2p_3$] and [$2p_4$] of order 1/2 having mode of backward precession [$-1/2\cdot\omega$]. This kind of critical speeds takes place at two rotating speeds $\omega_{-1/2}$ and $\omega'_{-1/2}$ because there are two values of ω at which the natural frequencies p_3, p_4 are equal to $-1/2\cdot\omega$.

I-1. The critical speed $\omega_{-1/2}$ of sub-harmonic oscillation [$2p_3$]. In this section the magnitude of unbalance of rotating body is represented by the maximum amplitude A_m at the major critical speed ω_c. If the deflection of shaft increases more than about 1.00 mm, there is a risk of breaking the apparatus and therefore A_m for fairly large unbalance cannot be obtained actually. Accordingly, when the magnitude of unbalance is large, A_m at the major critical speed ω_c is estimated by measuring the amplitude near ω_c (see Fig. 26). The value A_m with under line means value thus estimated.

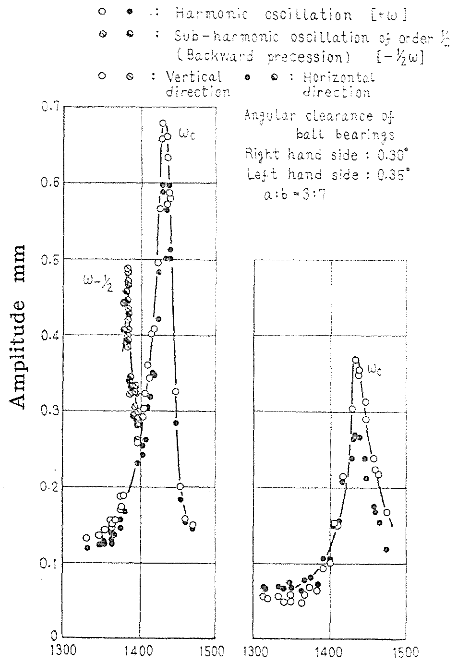
In the present experimental apparatus, the critical speed $\omega_{-1/2}$ occurs near the major critical speed ω_c as shown in Fig. 21 where experimental results using ball bearings *I* (angular clearances 0.30° and 0.35°) are given. When the maximum amplitude A_m is 0.68 mm (*I-a-3*), the peak $\omega_{-1/2}$ can take place; on the other hand, it does not appear when magnitude of unbalance decreases and A_m is equal to 0.37 mm (*I-a-5*). The appearance or non-appearance of the critical speed $\omega_{-1/2}$ is shown in Tables 1 and 2 where the mark ○ means appearance, × non-appearance. As we see in Fig. 21 and Tables 1 and 2, the critical speed $\omega_{-1/2}$ can only appear when the magnitude of unbalance is somewhat large.

TABLE 1. Appearance of the peak $\omega_{-1/2}$ (*I-a* series) of sub-harmonic oscillation [$2p_3$]

Experiment No.	*I-a*-1	*I-a*-2	*I-a*-3	*I-a*-4	*I-a*-5	*I-a*-6
A_m (mm)	0.76	0.71	0.68	0.62	0.37	0.34
the peak $\omega_{-1/2}$	○	○	○	○	×	×

TABLE 2. Appearance of the peak $\omega_{-1/2}$ (*I-b* series) of sub-harmonic oscillation [$2p_3$]

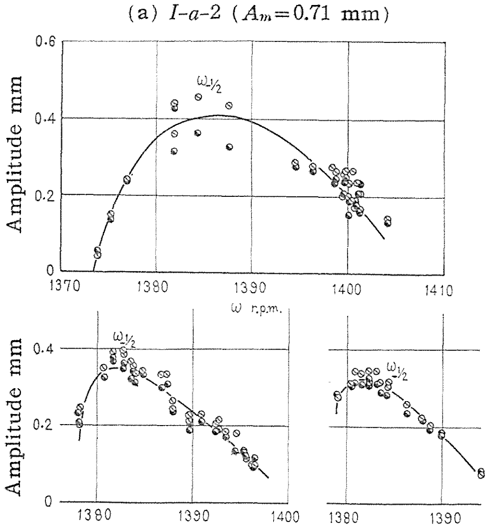
Experiment No.	*I-b*-5	*I-b*-6	*I-b*-9	*I-b*-10	*I-b*-11	*I-b*-12	*I-b*-13	*I-b*-14	*I-b*-15
A_m (mm)	$\underline{1.43}$	$\underline{1.30}$	0.83	0.80	0.71	0.48	0.40	0.28	0.21
the peak $\omega_{-1/2}$	○	○	×	○	×	×	×	×	×



ω r.p.m. (a) *I-a*-3 (A_m=0.68 mm)

ω r.p.m. (b) *I-a*-5 (A_m=0.37 mm)

FIG. 21. The peak of $\omega_{-1/2}$ of sub-harmonic oscillation [$2p_3$] having mode of [$-1/2\cdot\omega$] (*I-a* series).



ω r.p.m. (b) *I-a*-3 (A_m=0.68 mm)

ω r.p.m. (c) *I-a*-4 (A_m=0.62 mm)

FIG. 22. Response curves at the critical speed $\omega_{-1/2}$ of sub-harmonic oscillation [$2p_3$] with mode [$-1/2\cdot\omega$] (*I-a* series).

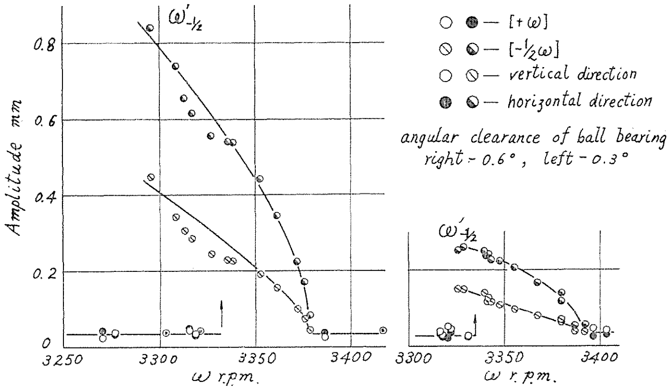
For experiments of *II* series when ball bearings *II* with angular clearnaces 0.3° and 0.6° are used, the circumstance is quite similar and the peak $\omega_{-1/2}$ can appear only when A_m is large.

Analyzing the vibratory waves on oscillographic paper, and separating vibration [$-1/2\cdot\omega$] from vibration [$+\omega$], components of amplitude of vibration [$-1/2\cdot\omega$] are plotted against ω in Fig. 22, in which the scale of abscissa ω is enlarged. Fig. 22 shows that the shape of the response curves of the peak $\omega_{-1/2}$ is of soft spring type.

I-2. The critical speed $\omega_{-1/2}$ of sub-harmonic oscillation [$2p_4$]. Experimental results of *II-a* series are shown in Fig. 23, in which response curves of soft spring type are obtained. In Fig. 24 (a), experiments of *I-a* series are given. Comparing Fig. 23 and Fig. 24 (a), it is clear that amplitude at the peak $\omega'_{-1/2}$ increases with A_m and the peak cannot appear when A_m is small.

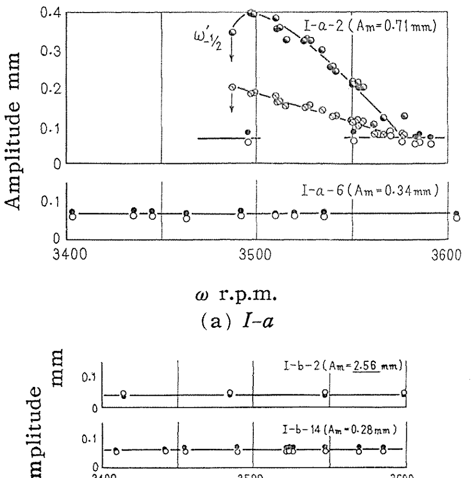
Incidentially, the peak $\omega'_{-1/2}$ does not appear in *I-b* series, even if A_m is as large as 2.56 mm, as shown in Fig. 24 (b).

II. The critical speeds $\omega_{1/2}$ of sub-harmonic oscillation [$2p_2$] having mode of forward precession [$+1/2\cdot\omega$]. Since the value of ω satisfying the relation



(a) *II-a*-1 (A_m=1.90 mm) (b) *II-a*-2 (A_m=0.68 mm)

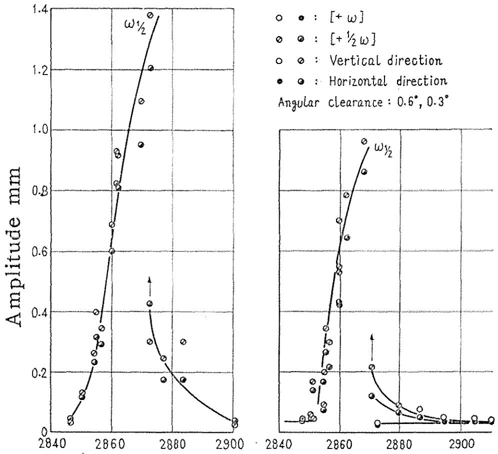
Fig. 23. Response curves at the critical speed $\omega'_{-1/2}$ of sub-harmonic oscillation [$2p_4$] of backward precession [$-1/2\cdot\omega$] (*II-a* series).



ω r.p.m.
(a) *I-a*

ω r.p.m.
(b) *I-b*

FIG. 24. Response curves at the critical speed $\omega'_{-1/2}$ of sub-harmonic oscillation [$2p_4$] with mode of [$-1/2\cdot\omega$] (*I-a* and *I-b* series).



ω r.p.m. ω r.p.m.
(a) *II-a*-1 (b) *II-a*-2
(A_m=1.90 mm) (A_m=0.68 mm)

FIG. 25. Response curves at the critical speed $\omega_{1/2}$ of sub-harmonic oscillation [$2p_2$] of forward precession [$+1/2\cdot\omega$] (*II-a* series).

$p=1/2\cdot\omega$ is unique, then there is only one critical speed of [$+1/2\cdot\omega$].

Experimental results of *II-a* series are shown in Fig. 25, where response curves of hard spring type are obtained. Here amplitude of [$+1/2\cdot\omega$] increases with A_m.

III. The critical speeds of summed and differential harmonic oscillations. When the present experimental apparatus is used, there are only six values of the rotating speeds ω of the shaft at which the relation $|p_m \pm p_n| = \omega$ $(m \neq n)$ holds, because the value p_1 is very large and there is no value of ω at which $|p_1 \pm p_n| = \omega$ can be satisfied as mentioned previously. Consequently, there are six values of ω where relations $\omega = |p_2 \pm p_3|$, $\omega = |p_2 \pm p_4|$ and $\omega = |p_3 \pm p_4|$ can hold respectively. It is, however, theoretically proved as discussed before that the critical speed of summed and differential harmonic oscillations can take place only at ω where relations $\omega = |p_2 - p_3|$, $\omega = |p_2 - p_4|$ and $\omega = |p_3 + p_4|$ hold, and it does not occur at ω where $\omega = |p_2 + p_3|$, $\omega = |p_2 + p_4|$ and $\omega = |p_3 - p_4|$ are satisfied. Accordingly, only three critical speeds ω_{23}, ω_{24} and ω_{34}, where $\omega = |p_2 - p_3|$, $\omega = |p_2 - p_4|$ and $\omega = |p_3 + p_4|$ are satisfied respectively, can actually appear. Experimental results verify the above fact[1] as is shown in Fig. 17, and only three peaks of summed and differential harmonic oscillation are obtained experimentally at ω_{23}, ω_{24} and ω_{34}.

Since the peak of the critical speed ω_{34} is very small for the present apparatus, then two critical speeds ω_{23} and ω_{24} are treated in this section.

III-1. The critical speed ω_{23} of summed and differential harmonic oscillation $[p_2 - p_3]$. The critical speed ω_{23} takes place at ω where the relation $\omega = |p_2 - p_3|$ holds, and a whirling motion of forward precession having frequency p_2 and that of backward precession with frequency $|p_3|$ build up and form the peak of the critical speed ω_{23}.

Response curves of the critical speed ω_{23} with those at the major critical speed ω_c are shown in Fig. 26. Here the numerals in the left hand side figure where peaks ω_c are given correspond to the numerals within the notation specifying the experiment (*e.g. I-b-5*). For instance, numeral *6* of peak ω_c corresponds to *6* in *I-b-6*. As we see in Fig. 26 the peak ω_{23} decreases as A_m increases, and the shape of response curves is continuous without jump phenomena. In Fig. 27 the relation between the maximum amplitude B_m at ω_{23} and the maximum amplitude A_m at the major critical speed is shown. The appearance or non-appearance of the critical speed ω_{23} is shown in Table 3 which is obtained by experiments of *I-b* series. As we see in Fig. 26, Fig. 27 and Table 3, the critical speed ω_{23} cannot occur when A_m is somewhat large.

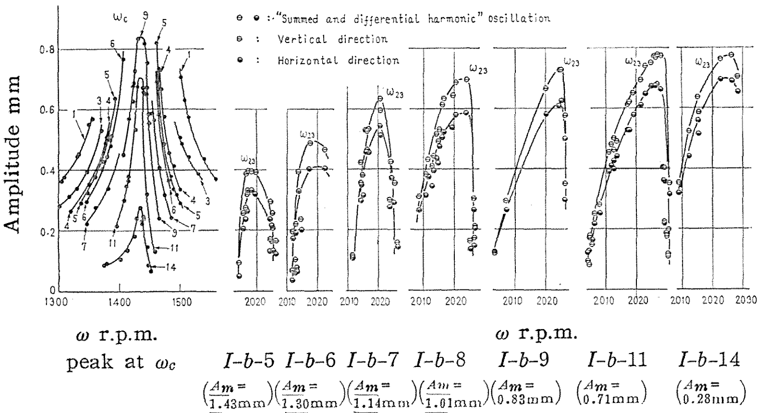


FIG. 26. Response curves at the critical speed ω_{23} of summed and differential harmonic oscillation $[p_2 - p_3]$ and peaks at the major critical speed ω_c (*I-b* series) ($p_2 \fallingdotseq 8/11 \cdot \omega$, $p_3 \fallingdotseq -3/11 \cdot \omega$).

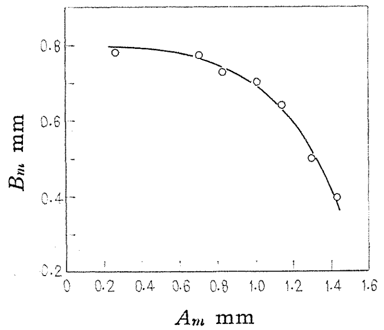


FIG. 27. Relation between B_m and A_m (*I-b* series).
(A_m: maximum amplitude at the major critical speed ω_c, B_m: maximum amplitude at the critical speed ω_{23} of summed and differential harmonic oscillation $[p_2-p_3]$).

III-2. The critical speed ω_{24} of summed and differential harmonic oscillation $[p_2-p_4]$. At the critical speed ω_{24} where the relation $\omega=|p_2-p_4|$ is satisfied, a forward precession with frequency p_2 and a backward precession having frequency $|p_4|$ take place and result in the peak ω_{24}. Experimental results of *II-b-1* (A_m=0.58 mm) are shown in Fig. 28 in which the response curve of the peak ω_{24} is continuous. For *II-a-1* (A_m=1.90 mm) and *II-a-2* (A_m=0.68 mm) in which the assembling conditions of pedestal and nonlinearity are different from those of *II-b-1*, similar continuous response curves are obtained. For *I-b-14* in which ball bearings are different from those in *II* series, the shape of response curve is soft spring type with jump phenomena as shown in Fig. 29.

TABLE 3. Appearance of the critical speed ω_{23} of summed and differential harmonic oscllation $[p_2-p_3]$ (*I-b* series)

Experiment No.	*I-b-1*	*I-b-2*	*I-b-3*	*I-b-4*	*I-b-5*	*I-b-6*	*I-b-7*	*I-b-8*	*I-b-9*	*I-b-11*	*I-b-14*
A_m mm	3.04	2.56	1.99	1.59	1.43	1.30	1.14	1.01	0.83	0.71	0.28
Peak ω_{23}	×	×	×	×	○	○	○	○	○	○	○

For response curves of soft spring type as well as continuous response curves, the amplitudes of vibration at ω_{24} are not related with the magnitude of A_m.

Vibratory waves at the critical speed ω_{24} are shown in Fig. 30, which is obtained in the experiment *II-b-1*.

2.5. Conclusions

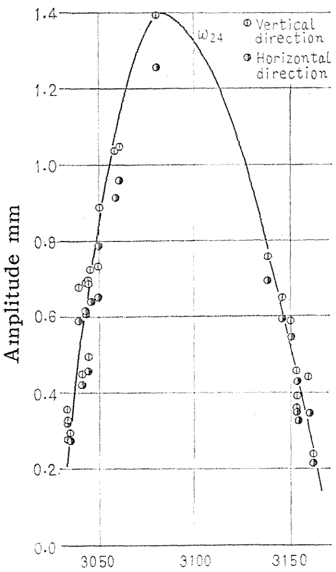
Obtained conclusions through experimental results of Experiment A and Experiment B may be summarized as follows:

(1) There are three kinds of the critical speeds of summed and differential harmonic oscillations $[p_2-p_3]$, $[p_3+p_4]$, $[p_2-p_4]$ in the rotating shaft system.

(2) Sub-harmonic oscillations of order 1/2 *i.e.*, $[\pm 1/2\cdot\omega]$ take place with summed and differential harmonic oscillations.

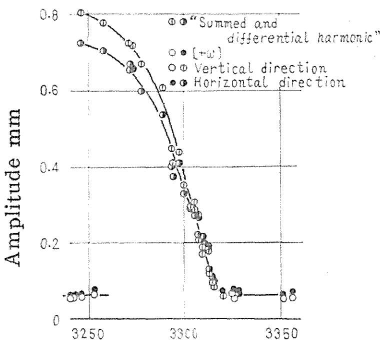
(3) The locations of the critical speeds and modes of vibrations of summed and differential harmonic oscillations can be determined through natural frequency p-rotating speed ω diagram.

(4) The absolute values of sum of or difference in two frequencies of summed and differential harmonic oscillations are always equal to the rotating speed of the shaft ω.



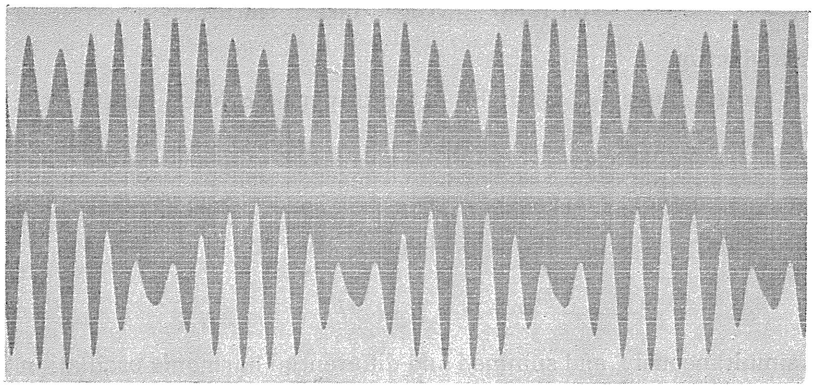
ω r.p.m.
$p_2 \fallingdotseq 8/15 \cdot \omega$, $p_4 \fallingdotseq -7/15 \cdot \omega$
$A_m = 0.58$ mm (*II-b*-1)

FIG. 28. Response curve at the critical speed ω_{24} of summed and differentical harmonic oscillation $[p_2 - p_4]$ (*II-b*-1).



ω r.p.m.
$p_2 \fallingdotseq 15/28 \cdot \omega$, $p_4 \fallingdotseq -13/28 \cdot \omega$
$A_m = 0.28$ mm (*I-b*-14)

FIG. 29. Response curve at the critical speed ω_{24} of summed and differential harmonic oscillation $[p_2 - p_4]$ (*I-b*-14).



$p_2 \fallingdotseq 8/15 \cdot \omega$, $p_4 \fallingdotseq -7/15 \cdot \omega$, $\omega = 3060.5$ r.p.m. (*II-b*-1)
(upper: vertical direction, lower: horizontal direction)

FIG. 30. Vibratory waves at the critical speed ω_{24} of summed and differential harmonic oscillation $[p_2 - p_4]$ (*II-b*-1, see Fig. 28).

(5) Amplitude of sub-harmonic oscillations of order 1/2 appearing at $\omega_{1/2}$, $\omega_{-1/2}$ and $\omega'_{-1/2}$ increases with the maximum amplitude A_m at the major critical speed ω_c.

(6) Shapes of response curves of sub-harmonic oscillation of forward preces-

sion $[+1/2\cdot\omega]$ at $\omega_{1/2}$ are of hard spring type, and response curves of soft spring type are obtained for the critical speeds $\omega_{-1/2}$ and $\omega'_{-1/2}$ of backward precession $[-1/2\cdot\omega]$.

(7) All response curves of summed and differential harmonic oscillation at the critical speed ω_{23} are continuous response curves.

(8) Amplitude at ω_{23} decreases as A_m increases.

(9) At the critical speed ω_{24}, response curves are continuous curves or those of soft spring type, according as nonlinear spring characteristics change.

(10) Amplitudes at the critical speed ω_{24} are not related to the magnitude of A_m.

Part B. Summed and Differential Harmonic Oscillations Induced by Symmetrical Nonlinear Spring Characteristics

In Part A, summed and differential harmonic oscillations $[p_i \pm p_j]$ and $[p_m \pm p_n]$ occurring in multiple degree-of-freedom systems with unsymmetrical nonlinearity are treated. When the vibratory systems with multiple degree-of-freedom have symmetrical nonlinearity in restoring forces, summed and differential harmonic oscillations $[2p_i \pm p_j]$ and $[p_i \pm p_j \pm p_k]$ take place. In Chapter III, the summed and differential harmonic oscillation $[2p_i \pm p_j]$ is discussed and, in Chaper IV the summed and differential harmonic oscillation $[p_i \pm p_j \pm p_k]$ is studied.

Chapter III. Summed and Differential Harmonic Oscillations $[2p_i \pm p_j]$ in Vibratory Systems with Symmetrical Nonlinear Spring Characteristics

1. Introduction

When the frequency of ω of the external periodic force satisfies the relation

$$\omega \fallingdotseq |2p_i \pm p_j|, \tag{3-1}$$

in the vibratory systems with symmetrical nonlinear spring characteristics, two vibrations of frequencies of ω_i, ω_j which are given by

$$\omega_i \fallingdotseq p_i, \qquad \omega_j \fallingdotseq p_j, \tag{3-2}$$

build up simultaneously, and summed and differential harmonic oscillation $[2p_i \pm p_j]$ takes place.

The summed and differential harmonic oscillation $[2p_i \pm p_j]$ reduces to sub-harmonic oscillation of order 1/3, provided $p_i = p_j$. In the vibratory systems with symmetrical nonlinearity, the summed and differential harmonic oscillation $[2p_i \pm p_j]$ appears with sub-harmonic oscillation of order 1/3 or 1/5, as is shown in experiments in later.

In the present chapter, conditions for the possibility of occurrence of summed and differential harmonic oscillation $[2p_i \pm p_j]$ are analytically discussed, and the modes of vibrations which can actually take place, equations of response curves are determined, and graphical representations of response curves and discussion of stability problem for summed and differential harmonic oscillation of $[2p_i \pm p_j]$ are performed. Further the obtained analytical conclusions are verified by experimental results.

2. Summed and differential harmonic oscillations $[2p_i \pm p_j]$[4]

In the summed and differential oscillation $[2p_i \pm p_j]$, between frequencies ω, ω_i and ω_j, there is the relation

$$|2\omega_i \pm \omega_j| = \omega. \tag{3-3}$$

All analytical treatments from Eq. (1-1) to Eq. (1-13) and Eqs. (1-6 a), (1-16) and (1-16 a) in Chapter I can be still applied to the summed and differential oscillation $[2p_i \pm p_j]$. For two frequencies ω_i and ω_j, there are the following three cases:

(I) case of summed harmonic oscillation $[2p_i + p_j]$

(i) $2\omega_i + \omega_j = \omega$,

$$\therefore \quad \omega_i = \omega - \omega_i - \omega_j, \qquad \text{(a)}$$

$$\omega_j = \omega - 2\omega_i, \qquad \text{(b)}$$

(II) case of differential harmonic oscillation $[2p_i - p_j]$

(ii) $2\omega_i - \omega_j = \omega$,

$$\therefore \quad \omega_i = \omega - \omega_i + \omega_j, \qquad \text{(c)}$$

$$\omega_j = 2\omega_i - \omega, \qquad \text{(d)}$$

(iii) $\omega_j - 2\omega_i = \omega$,

$$\therefore \quad \omega_i = -\omega - \omega_i + \omega_j, \qquad \text{(e)}$$

$$\omega_j = \omega + 2\omega_i. \qquad \text{(f)}$$

Inserting Eqs. (1-16) and (1-16 a) into the right hand side of Eq. (1-6 a) and referring the above relations (a)~(f), the following equations are attained through the condition that resonant terms $\sin \omega t$, $\cos \omega t$, $\sin \omega_j t$ and $\cos \omega_j t$ should not be contained in the right hand sides of the first and the second equations of Eq. (1-6 a).

$$\left.\begin{aligned}
&\lambda_i + \pi_i R_j \cos(2\theta_i \pm \theta_j) = \rho_i R_i^2 + \sigma_i R_j^2, \\
&\lambda_j R_j + \pi_j R_i^2 \cos(2\theta_i \pm \theta_j) = (\rho_j R_j^2 + \sigma_j R_i^2) R_j, \\
&C_{ii}\omega_i - \pi_i R_j \sin(2\theta_i \pm \theta_j) = 0, \\
&\mp C_{jj}\omega_j R_j + \pi_j R_i^2 \sin(2\theta_i \pm \theta_j) = 0,
\end{aligned}\right\} \tag{3-4}$$

in which the upper sign and the lower sign correspond to summed and differential harmonic oscillations respectively and R_i, R_j are amplitudes of summed and differential harmonic oscillations, and further

$$\left.\begin{aligned} &\lambda_{\substack{i\\j}} = (\omega_{\substack{i\\j}}^2 - p_{\substack{i\\j}}^2) - \frac{1}{2}\Big\{\sum_{r,s}{}_{\substack{i\\j}}\beta_{\substack{(irs)\\(jrs)}} P_r P_s + \sum_{r}{}_{\substack{i\\j}}\beta_{\substack{(irs)\\(jrs)}} P_r P_s\Big\},\\ &\rho_{\substack{i\\j}} = \frac{1}{2}{}_{\substack{i\\j}}\beta_{\substack{(iii)\\(jjj)}},\quad \sigma_{\substack{i\\j}} = \frac{1}{2}{}_{\substack{i\\j}}\beta_{\substack{(ijj)\\(jii)}},\\ &\pi_i = -\frac{1}{2}\Big({}_i\beta_{(iij)}P_i + {}_i\beta_{(ijj)}P_j + \frac{1}{2}\sum_r {}_i\beta_{(ijr)}P_r\Big),\\ &\pi_j = -\frac{1}{2}\Big({}_j\beta_{(iii)}P_i + \frac{1}{2}{}_j\beta_{(iij)}P_j + \frac{1}{2}\sum_r {}_j\beta_{(iir)}P_r\Big),\\ &P_i = F_i\cdot(\omega_i^2-\omega^2)^{-1},\ P_j = F_j\cdot(\omega_j^2-\omega^2)^{-1},\ P_r = F_r\cdot(\omega_r^2-\omega^2)^{-1}. \end{aligned}\right\} \tag{3-5}$$

From the relation of Eq. (1-11), we have

$$\pi_i = 2\,\pi_j. \tag{3-6}$$

Observing Eq. (3-4), it is seen that, when coefficients of symmetrical nonlinearity, *i.e.*, of nonlinearity of third powers ${}_i\beta_{(iii)}$ etc. vanish, ρ_i, ρ_j, σ_i, σ_j become zero and $\rho_{\substack{i\\j}} = \sigma_{\substack{i\\j}} = 0$ leads to $R_i = R_j = 0$. And further unsymmetrical nonlinearity has no connection with Eq. (3-4). Consequently, it can be concluded that the summed and differential harmonic oscillation $[2p_i + p_j]$ is caused by symmetrical nonlinearity and not by unsymmetrical nonlinearity, and if there is no symmetrical nonlinearity in spring forces, the summed and differential harmonic oscillation $[2p_i + p_j]$ cannot appear.

Referring the relation $\pi_i = 2\pi_j$, the third and fourth equations of Eq. (3-4) lead to

$$\frac{R_i^2}{R_j^2} = \pm\frac{C_{jj}\,\omega_j\,\pi_i}{C_{ii}\,\omega_i\pi_j} = \pm 2\frac{C_{jj}\,\omega_j}{C_{ii}\,\omega_i}. \tag{3-7}$$

Since C_{ii} and C_{jj} take positive values as shown in Eq. (1-13), R_i and R_j can be real only for summed harmonic oscillation. Accordingly, only summed harmonic oscillation can take place and differential harmonic oscillation does not occur.

3. Response curves of summed harmonic oscillation $[2p_i + p_j]$[7)]

As the procedure to obtain five values R_i, R_j, ω_i, ω_j and $2\theta_i + \theta_j$ through the five relations $2\omega_i + \omega_j = \omega$ and Eq. (3-4) is quite complicated, then a rather simple case of two degree-of-freedom system given by Eq. (1-22) is also treated here. Eq. (1-22) can be transformed to Eq. (1-6 a) and we have, in place of Eq. (3-4), the following rather simple equations.

$$\left.\begin{aligned} &\{(\omega_1^2 - p_1^2) - 2\beta_0 P^2\} - 2\beta_0 P R_2\cos(2\theta_1 + \theta_2) = \beta_0(R_1^2 + 2R_2^2),\\ &\{(\omega_2^2 - p_2^2) - 2\beta_0 P^2\}R_2 - \beta_0 P R_1^2\cos(2\theta_1 + \theta_2) = \beta_0(R_2^2 + 2R_1^2)R_2,\\ &C\omega_1 + 2\beta_0 P R_2\sin(2\theta_1 + \theta_2) = 0,\\ &C\omega_2 R_2 + \beta_0 P R_1^2\sin(2\theta_1 + \theta_2) = 0, \end{aligned}\right\} \tag{3-8}$$

where C, β_0 and P are given by Eq. (1-23). From the third and fourth equations of Eq. (3-8), we attain

$$\frac{R_2^2}{R_1^2} = \frac{\omega_1}{2\,\omega_2}. \tag{3-9}$$

Eliminating term of cos from the first and second equations of Eq. (3-8) and using the relation of Eq. (3-9), we get

$$R_1^2 = \frac{2\,\omega_2(\lambda_1\omega_2 - \lambda_2\omega_1)}{\beta_0(2\,\omega_2^2 - 2\,\omega_1\omega_2 - \omega_1^2)},\quad R_2^2 = \frac{\omega_1(\lambda_1\omega_2 - \lambda_2\omega_1)}{\beta_0(2\,\omega_2^2 - 2\,\omega_1\omega_2 - \omega_1^2)}, \tag{3-10}$$

where

$$\lambda_1 = (\omega_1^2 - p_1^2) - 2\,\beta_0 P^2,\ \lambda_2 = (\omega_2^2 - p_2^2) - 2\,\beta_0 P^2. \tag{3-11}$$

From the first and third equations of Eq. (3-8), we have

$$\{\lambda_1 - \beta_0(R_1^2 + 2\,R_2^2)\}^2 + C^2\omega_1^2 = 4\,\beta_0^2 P^2 R_2^2\{\cos^2(2\,\theta_1 + \theta_2) + \sin^2(2\,\theta_1 + \theta_2)\} = 4\,\beta_0^2 P^2 R_2^2.$$

Substitution of Eq. (3-10) into the above equation results in

$$\begin{aligned}\omega_1\{(\omega_1 + 4\,\omega_2)(\omega_1^2 - p_1^2) - 2(\omega_1 + \omega_2)(\omega_2^2 - p_2^2) + 2(\omega_1 - 2\,\omega_2)\beta_0 P^2\}^2 \\ + 4\,\beta_0 P^2(2\,\omega_2^2 - 2\,\omega_1\omega_2 - \omega_1^2)\{\omega_1(\omega_2^2 - p_2^2) - \omega_2(\omega_1^2 - p_1^2) - 2\,\beta_0 P^2(\omega_1 - \omega_2)\} \\ + C^2\omega_1(2\,\omega_2^2 - 2\,\omega_1\omega_2 - \omega_1^2)^2 = 0.\end{aligned} \tag{3-12}$$

Since P is a function of only ω_1 and ω_2 as is seen from Eq. (1-23), then Eq. (3-12) contains only ω_1 and ω_2. Accordingly if the frequencies ω_1 and ω_2 can be determined by the relation $2\,\omega_1 + \omega_2 = \omega$ and Eq. (3-12), the amplitudes of R_1 and R_2 of summed harmonic oscillation can be furnished as functions of only ω by Eq. (3-10). However, it is impossible that ω_1 and ω_2 are given analytically through Eq. (3-12). Therefore the frequencies ω_1 and ω_2 are given by the following approximate procedure. As a first approximate values of ω_1 and ω_2, we put

$$\omega_{10} = \frac{p_1}{2\,p_1 + p_2}\omega = \eta_1\omega,\quad \omega_{20} = \frac{p_2}{2\,p_1 + p_2}\omega = \eta_2\omega, \tag{3-13}$$

where

$$\eta_1 = \frac{p_1}{2\,p_1 + p_2},\qquad \eta_2 = \frac{p_2}{2\,p_1 + p_2}. \tag{3-14}$$

Adoption of Eq. (3-13) does not contradict the experimental results (see Fig. 42). Putting

$$\omega_1 = \eta_1\omega + \varepsilon_1,\quad \omega_2 = \eta_2\omega + \varepsilon_2, \tag{3-15}$$

where ε_1, ε_2 are small quantities as β_0. Since the relation $2\,\omega_1 + \omega_2 = \omega$ holds, we have

$$\varepsilon = \varepsilon_1 = -\frac{1}{2}\varepsilon_2, \tag{3-16}$$

and

$$\omega_1 = \eta_1\omega + \varepsilon,\quad \omega_2 = \eta_2\omega - 2\,\varepsilon. \tag{3-17}$$

Substituting Eq. (3-17) into Eq. (3-12), and neglecting all smaller quantities than the second powers, we attain

$$\left.\begin{aligned} \omega_1 &= \eta_1\omega + \frac{1}{\omega}[f_2(\omega^2-\Omega^2)+\beta_0 f_3 P^2 \pm \frac{f_5}{2f_1^2}\sqrt{D}\,] \\ \omega_2 &= \eta_2\omega - \frac{2}{\omega}[f_2(\omega_2-\Omega^2)+\beta_0 f_3 P^2 \pm \frac{f_5}{2f_1^2}\sqrt{D}\,] \end{aligned}\right\} \quad (3\text{-}18)$$

$$\left.\begin{aligned} R_1^2 &= \frac{2\eta_2}{\beta_0 f_1^2}[\eta_1\eta_2 f_1(\omega^3-\Omega^2)-\beta_0 f_6 P^2 \pm 3\eta_1\eta_2\sqrt{D}\,] \\ R_2^2 &= \frac{\eta_1}{\beta_0 f_1^2}[\eta_1\eta_2 f_1(\omega^2-\Omega^2)-\beta_0 f_6 P^3 \pm 3\eta_1\eta_2\sqrt{D}\,] \end{aligned}\right\} \quad (3\text{-}19)$$

where

$$\left.\begin{aligned} &\eta_{1,2} = p_{1,2}/\Omega,\quad \Omega = 2p_1+p_2, \\ &D = 4\eta_2 f_1\beta_0 P^2(\omega^2-\Omega^2) - f_4\beta_0^2P^4 - f_1^2C^2\omega^2 \\ &f_1 = \eta_1^2 + 8\eta_1\eta_2 + 4\eta_2^2 \\ &f_2 = \frac{1}{2f_1}(2\eta_2^3 + 2\eta_2^2\eta_1 - 4\eta_1^2\eta_2 - \eta_1^3) \\ &f_3 = \frac{1}{f_1^2}(14\eta_2^3 + 6\eta_2^2\eta_1 - 9\eta_2\eta_1^2 - \eta_1^3) \\ &f_4 = \frac{4}{\eta_1}(16\eta_2^3 + 3\eta_2^3\eta_1 + 20\eta_1^2\eta_2 + 2\eta_1^3) \\ &f_5 = 2\eta_2^2 - 2\eta_1\eta_2 - \eta_1^2 \\ &f_6 = 16\eta_2^3 + 22\eta_2^2\eta_1 + 20\eta_2\eta_1^2 + 2\eta_1^3 \end{aligned}\right\} \quad (3\text{-}20)$$

The frequencies and the amplitudes of summed harmonic oscillation $[p_1+2p_2]$ can be given by exchanging subscripts 1 and 2 in the above equations.

By the similar procedure, the amplitudes of sub-harmonic oscillations of order 1/3 are obtained as follows:

$$R_i^2 = \frac{1}{\beta_0}\left[\left\{\left(\frac{\omega}{3}\right)^2 - p_i^2\right\} - \frac{3}{2}\beta_0 P^2 \pm \sqrt{\beta_0 P^2\left\{\left(\frac{\omega}{3}\right)^2 - p_i^2\right\} - \frac{7}{4}\beta_0^2 P^4 - \left(C\frac{\omega}{3}\right)^2}\,\right]$$

where

$$P = f_0\left(-\frac{9}{8\omega^2} + \frac{1}{p_j^2-\omega^2}\right) \qquad (i, j = 1, 2 \quad i \neq j)$$

$$(3\text{-}21)$$

In Eq. (3-20), f_1, f_4 and f_6 are positive, for $\eta_1>0$ and $\eta_2>0$ are always satisfied. By comparing Eq. (3-19) with Eq. (3-21), it is easily seen that response curves of sub-harmonic oscillations of order 1/3 are qualitatively analogous to those of summed and differential harmonic oscillations $[2p_1+p_2]$.

Fig. 31 shows response curves numerically calculated from Eqs. (3-19) and (3-21) for various values of the damping coefficients C. For summed harmonic oscillations, sum of amplitudes R_1+R_2 is plotted. Since β_0 is positive in Fig. 31, response curves are of hard spring type. Full lines in Fig. 31 indicate stable parts of the response curves, which correspond to stable vibration, broken lines

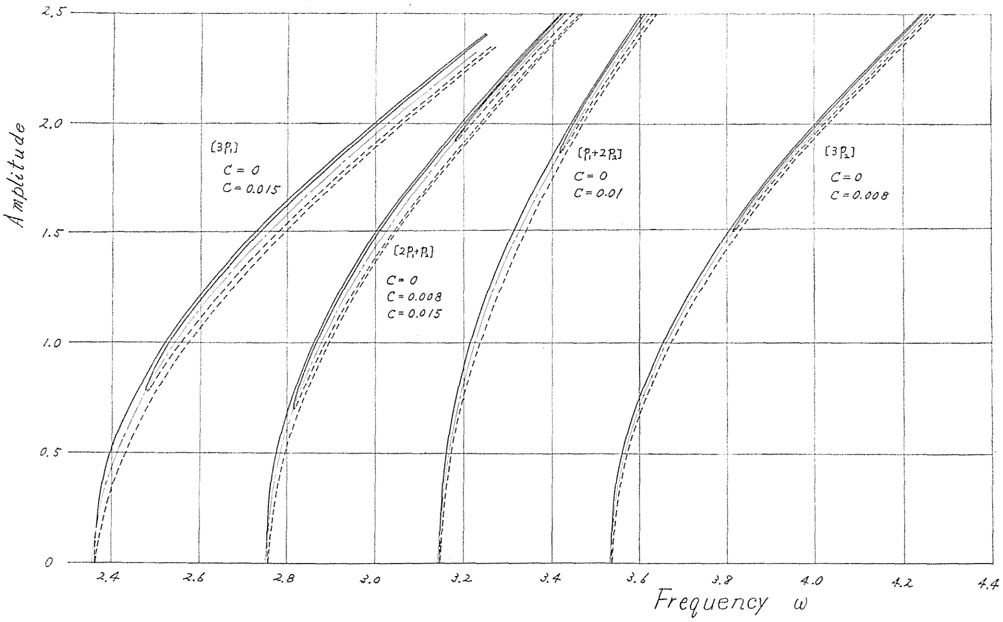


FIG. 31. Response curves of summed harmonic oscillations $[2p_1+p_2]$ and $[p_1+2p_2]$ and sub-harmonic oscillations of order 1/3 with modes of $[3p_1]$ and $[3p_2]$.

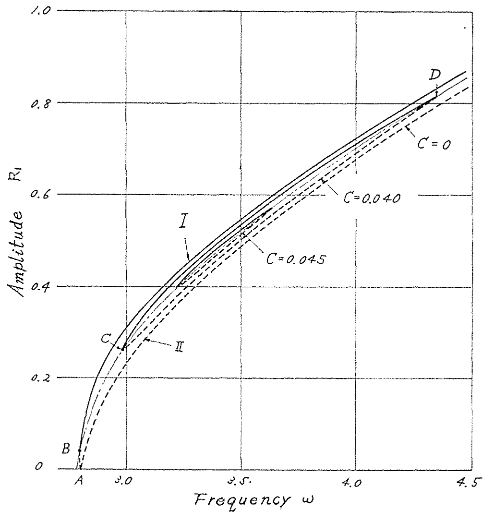
($F=0.432$, $p_1=0.785$, $p_2=1.177$, $\beta_0=0.1$), (ω: frequency of external force, F: magnitude of external force, β_0: coefficient of symmetrical nonlinearity, C: coefficient of damping).

represent unstable parts and chain lines are the boundaries between stable and unstable zones. As shown in Fig. 32 (a), response curves of summed harmonic oscillation $[2p_1+p_2]$ are closed curves when a damping force exists, thus these curves have the maximum and minimum amplitudes.

Response curves of $[3p_1]$, $[3p_2]$ and $[p_1+2p_2]$ have analogous characters to the vibration of $[2p_1+p_2]$ which is shown in Fig. 32 (a). In Fig. 32, if $C=0$, the abscissa becomes a tangent to the response curve I at a point A $\left[\omega=(2p_1+p_2)\sqrt{1+\frac{\beta_0 f_6' P^2}{p_1 p_2 f_1}},\quad f_6'=16\eta_2^3+40\eta_2^2\eta_1+20\eta_2\eta_1^2+2\eta_1^3\right]$ which locates in an unstable range. The stable parts of response curves are always in the region given by Eq. (3-22).

$$\left.\begin{array}{ll} R_i^2>\dfrac{P^2}{2} & \text{(for sub-harmonic oscillation } [3p_i]) \\ \left.\begin{array}{l} R_1^2>\left(\dfrac{2\eta_2}{\eta_1}\right)\left(\dfrac{3\eta_1\eta_2}{f_1}\right)P^2 \\ R_2^2>\left(\dfrac{3\eta_1\eta_2}{f_1}\right)P^2 \end{array}\right\} & \text{(for summed harmonic oscillation } [2p_1+p_2]) \end{array}\right\} \qquad (3\text{-}22)$$

Therefore these modes of vibrations can occur provided initial conditions lead to the situations in which the restricted conditions Eq. (3-22) are satisfied.



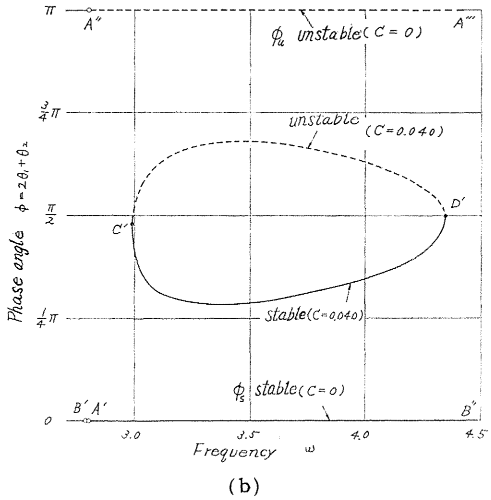


FIG. 32. Summed harmonic oscillation $[2p_1+p_2]$
(a) Response curves
(b) Phase angles
($F=0.432$, $\beta_0=1.0$, $p_1=0.785$, $p_2=1.177$, $2p_1+p_2=2.746$)

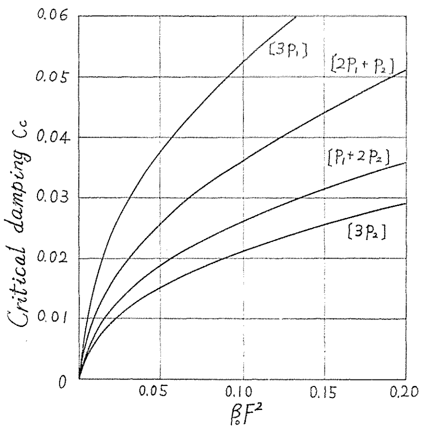


FIG. 33. Critical damping coefficients C_c for summed harmonic oscillations $[2p_1+p_2]$ and $[p_1+2p_2]$ and sub-harmonic oscillations of order 1/3 with modes of $[3p_1]$ and $[3p_2]$. ($p_1=0.785$, $p_2=1.177$, F = magnitude of disturbing force, β_0= coefficients of symmetrical nonlinearity).

For $\beta_0<0$, as is shown in Fig. 34, response curves are of soft spring type and they have lower limits as in case of $\beta_0>0$, however, about the upper limits the situation is different from case of $\beta_0>0$. Since the approximate procedure adopting here is the so-called resonant analysis, obtained results are reliable only in the vicinity of the resonant point. For this reason, only the response curves near the resonant point are shown in Fig. 34 as well as in Fig. 4.

When $\beta_0>0$, an upper and a lower limit of the response curve of summed harmonic oscillations $[2p_i+p_j]$ in Figs. 31 and 32 (a) approach each other with increase of damping force, and finally they converge to a point at a certain damping coefficient C_c, which is defined as critical damping, because occurrences of summed harmonic oscillations $[2p_i+p_j]$ and sub harmonic oscillations of order 1/3 are impossible for $C\geqq C_c$. The critical damping C_c can be determined from Eq. (3-23)

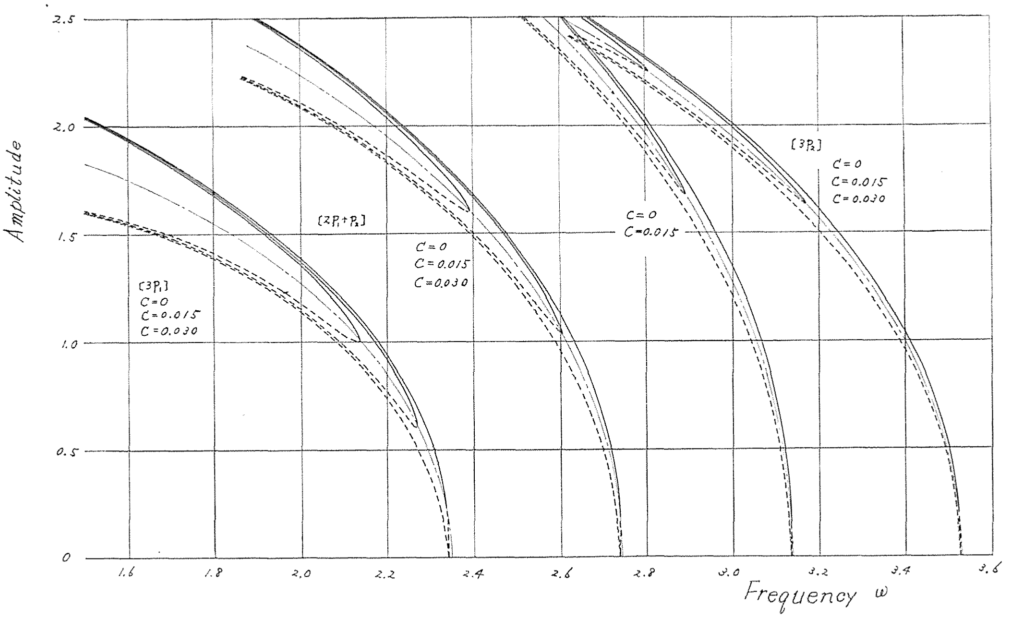


FIG. 34. Response curves of summed harmonic oscillations $[2p_1+p_2]$ and $[p_1+2p_2]$ and sub-harmonic oscillations of order 1/3 with modes of $[3p_1]$ and $[3p_2]$.
(magnitude of disturbing force $F=0.432$, coefficient of symmetrical nonlinearity $\beta_0=-0.1$, $p_1=0.785$, $p_2=1.177$, C=damping coefficient, ω=frequency of disturbing force).

$$D(\omega^2, C)=0, \quad \frac{\partial D(\omega^2, C)}{\partial \omega^2}=0 \tag{3-23}$$

where D represents the formula within radical of Fqs. (3-19) and (3-21). Fig. 33 shows results of numerical calculation of Eq. (3-23). It is concluded from Fig. 33 that nonlinearity $\beta_0 \cdot F^2$ increases with magnitude of C_c, and that the vibration having lower resonant frequency can occur more easily than the vibration with higher resonant frequency.

Fig. 35 shows $\omega_1-\omega$ and $\omega_2-\omega$ curves, where ω_1 and ω_2 are frequencies of summed harmonic oscillations of mode $[2p_1+p_2]$ and $[p_1+2p_2]$ and ω is the forcing frequency. Fig. 35 (a) and (b) correspond to cases $\beta_0>0$ and $\beta_0<0$ and full and broken line curves correspond to stable and unstable vibrations, respectively. Neglecting terms smaller than the first order of ε in Eq. (3-18), we have

$$\omega_{i0}=\eta_i\omega \qquad (i=1, 2) \tag{3-24}$$

which is shown by chain lines in Fig. 35. Stable and unstable branches of $\omega_1-\omega$ curves are in pair and the upper branch corresponds to stable solutions and the lower to unstable solutions in case $\beta_0>0$, and vice versa for $\omega_2-\omega$ curves. When $\beta_0<0$, the situation is reversed. When $\beta_0>0$ and $C\doteqdot 0$, $\omega_i-\omega$ curves become closed curves.

If $\beta_0>0$, the relations $\omega_1<\omega_{10}$ and $\omega_2>\omega_{20}$ hold, and if $\beta_0<0$, these relations are reversed.

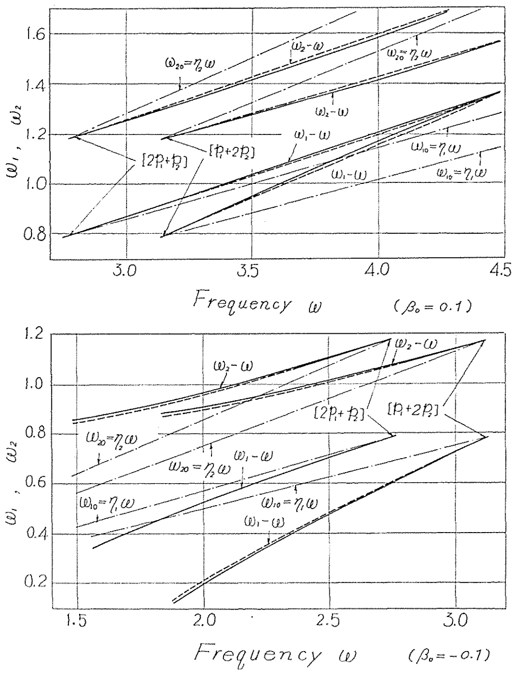


FIG. 35. Frequencies of ω_1, ω_2 of summed harmonic oscillations $[2p_1+p_2]$ and $[p_1+2p_2]$.
($F=0.432$, $C=0$, $p_1=0.785$, $p_2=1.177$)

Phase angle $\phi=2\theta_1+\theta_2$ of summed harmonic oscillation $[2p_1+p_2]$ is indicated in Fig. 32 (b). In the figure, full and broken lines represent phase angle for stable and unstable solutions respectively. If $\beta_0>0$ and $C\neq 0$, these curves are closed and point C in Fig. 32 (a) corresponds to point C' in Fig. 32 (b) and phase angle is slightly smaller than $\pi/2$ at the point C'. The point D in Fig. 32 (a) where response curve intersects with the boundary line of stable and unstable regions corresponds to the point D' in Fig. 32 (b) and phase angle is just $\pi/2$ at the point. The phase angle ϕ_s of stable solution for $C=0$ is always equal to zero [$B'B''$ line in Fig. 32 (b)]; and phase angle ϕ_u of unstable solution between A and B in Fig. 32 (a) for $C=0$ is zero as shown by $A'B'$ in Fig. 32 (b); and for the frequency region higher than the point A of curve II in Fig. 32 (a), phase angle ϕ_u of unstable solution when $C=0$ is equal to π [$A''A'''$ line in Fig. 32 (b)].

For $\beta_0<0$, phase angle is within $\left(\frac{3}{2}\sim 2\right)\pi$ for stable solution; and for unstable solution, it is partly in region $3/2\cdot\pi<\phi_u$ and almost in the region $3/2\cdot\pi>\phi_u>\pi$.

The phase angle of summed harmonic oscillation $[p_1+2p_2]$ and sub-harmonic oscilations of order 1/3 are qualitatively analogous to that of summed harmonic oscillation $[2p_1+p_2]$.

4. Stability of summed harmonic oscillation $[2p_i+p_j]$

In order to discuss the stability problem, the prosedure to the solutions along the method of Andronow and Witt is introduced here. In Eq. (1-16b), we assume that amplitudes R_1, R_2 and phase angles θ_1 and θ_2 are slowly varying functions of time t. Thus substituting Eq. (1-16 b) into Eq. (1-6 b) and neglecting the terms of higher order of small quantities, we attain

$$\left.\begin{aligned}
-2\,\omega_1\frac{dR_1}{dt} &= C\omega_1 R_1 + 2\,R_1R_2\beta_0 P\sin(2\,\theta_1+\theta_2)\\
-2\,\omega_2\frac{dR_2}{dt} &= C\omega_2 R_2 + R_1^2\beta_0 P\sin(2\,\theta_1+\theta_2)\\
2\,\omega_1R_1\frac{d\theta_1}{dt} &= \lambda_1R_1 - \beta_0R_1(R_1^2+2\,R_2^2) - 2\,R_1R_2\beta_0P\cos(2\,\theta_1+\theta_2)\\
2\,\omega_2R_2\frac{d\theta_2}{dt} &= \lambda_2R_2 - \beta_0R_2(2\,R_1^2+R_2^2) - R_1^2\beta_0P\cos(2\theta_1+\theta_2)
\end{aligned}\right\} \tag{3-25}$$

Putting $\frac{dR_1}{dt}=\frac{dR_2}{dt}=\frac{d\theta_1}{dt}=\frac{d\theta_2}{dt}=0$, we obtain the steady state soultions of Eqs. (3-18) and (3-19) from the above equation. Let

$$R_1 = R_{10}+\xi,\ R_2 = R_{20}+\zeta,\ \phi = \phi_0+\varphi \tag{3-26}$$

be solutions which differ slightly from the steady state solutions R_{10}, R_{20} and φ_0 of vibration of mode $[2\,p_1+p_2]$. Substituting Eq. (3-26) into Eq. (3-25) and neglecting all but the linear terms in ξ, ζ and φ, we obtain

$$\left.\begin{aligned}
-2\,\omega_1R_{10}\frac{d\xi}{dt} &= -2\,C\omega_2R_{20}\zeta + \{\lambda_1-\beta_0(R_{10}^2+2\,R_{20}^2)\}R_{10}^2\varphi\\
-2\,\omega_2R_{20}\frac{d\zeta}{dt} &= -2\,C\omega_1R_{20}\xi + C\omega_2R_{10}\zeta + \{\lambda_2-\beta_0(2\,R_{10}^2+R_{20}^2)\}R_{10}R_{20}\varphi\\
\omega_1R_{10}^2\frac{d\varphi}{dt} &= -\{\lambda_1+\beta_0(R_{10}^2+2\,R_{20}^2)\}R_{10}\xi - \{\lambda_2+\beta_0(2\,R_{10}^2+R_{20}^2)\}R_{20}\zeta - 3\,C\omega_2R_{10}^2\varphi
\end{aligned}\right\} \tag{3-27}$$

Substituting the assumed solutions

$$\xi=\xi_0e^{st},\ \zeta=\zeta_0e^{st},\ \varphi=\varphi_0e^{st} \tag{3-28}$$

into Eq. (3-27), we have

$$\left.\begin{aligned}
&2\,\omega_1R_{10}S\cdot\xi_0 - 2\,C\omega_2R_{20}\cdot\zeta_0 + \{\lambda_1-\beta_0(R_{10}^2+2\,R_{20}^2)\}R_{10}^2\cdot\varphi_0 = 0\\
&-2\,C\omega_2R_{20}\cdot\xi_0 + R_{10}\omega_2(C+2\,S)\cdot\zeta_0 + \{\lambda_2-\beta_0(2\,R_{10}^2+R_{20}^2)\}R_{10}R_{20}\cdot\varphi_0 = 0\\
&\{\lambda_1+\beta_0(R_{10}^2+2\,R_{20}^2)\}R_{10}\cdot\xi_0 + \{\lambda_2+\beta_0(2\,R_{10}^2+R_{20}^2)\}R_{20}\cdot\zeta_0\\
&+\omega_2R_{20}^2(3\,C+2\,S)\cdot\varphi_0 = 0
\end{aligned}\right\} \tag{3-29}$$

If ξ_0, ζ_0 and φ_0 have solutions which are not trivial, the following equation must be satisfied.

$$\left.\begin{aligned}
&4\,\omega_1^2\omega_2R_{10}^4S^3 + 16\,\omega_1\omega_2^2CR_{10}^2R_{20}^2S^2 + \frac{\omega_1R_{10}^4}{(2\,\omega_2^2-2\,\omega_1\omega_2-\omega_1^2)^2}\times\\
&[C\omega_1\omega_2(2\,\omega_2^2-2\,\omega_1\omega_2-\omega_1^2)^2 - \{\lambda_1\omega_2(8\,\omega_2^2+4\,\omega_1\omega_2-\omega_1^2)\\
&-\lambda_2\omega_1(2\,\omega_1^2+10\,\omega_1\omega_2+2\,\omega_2^2)\}\{(2\,\omega_1+2\,\omega_2)\lambda_2-(\omega_1+4\,\omega_2)\lambda_1\}]S\\
&-\frac{C\omega_1R_{10}^4}{(2\,\omega_2^2-2\,\omega_1\omega_2-\omega_1^2)^2}[3\,C^2\omega_1\omega_2(2\,\omega_2^2-2\,\omega_1\omega_2-\omega_1^2)^2\\
&+\{\lambda_1\omega_2(8\,\omega_2^2+4\,\omega_1\omega_2-\omega_1^2)-\lambda_2\omega_1(2\,\omega_1^2+10\,\omega_1\omega_2+2\,\omega_2^2)\}\\
&\times\{(2\,\omega_1+2\,\omega_2)\lambda_2-(\omega_1+4\,\omega_2)\lambda_1\}] = 0
\end{aligned}\right\} \tag{3-30}$$

Applying the Routh-Hurwitz theorem, the condition under which real parts of all roots of Eq. (3–30) are negative is

$$\left.\begin{aligned} &-3C^2\omega_1\omega_2(2\omega_2^2-2\omega_1\omega_2-\omega_1^2)^2+\{\lambda_1\omega_2(8\omega_2^2+4\omega_1\omega_2-\omega_1^2)\\ &-\lambda_2\omega_1(2\omega_1^2+10\omega_1\omega_2+2\omega_2^2)\}\{(2\omega_1+2\omega_2)\lambda_2-(\omega_1+4\omega_2)\lambda_1\}>0 \end{aligned}\right\} \quad (3\text{-}31)$$

From Eqs. (3–19) and (3–20), stability criterion Eq. (3–31) can be written as follows:

$$\left\{\pm\beta_0\frac{R_{10}^2 f_1 f_5^2}{2\eta_2}\sqrt{D}\right\}>0 \quad (3\text{-}32)$$

Referring Eq. (3–19), Eq. (3–32) can be further rewritten as follows:

$$R_1^2>\frac{2\eta_2}{\beta_0 f_1^2}[\eta_1\eta_2 f_1\{\omega^2-(2p_1+p_2)^2\}-\beta_0 f_6 P^2] \quad (3\text{-}33)$$

The expression obtained by replacing inequality of Eq. (3–33) with equality gives the boundary line between stability and unstable regions and on this line the value D vanishes, so that the curve is locus of vertical tangent on response curves. The upper part of the locus is stable region and the lower part unstable. For example in Fig. 31 chain lines are boundaries of stable region and full and broken lines correspond to stable and unstable parts of response curves respectively.

For summed harmonic oscillation $[p_1+2p_2]$, the similar results are obtained by exchanging subscripts 1 and 2.

5. Experimental apparatus and experimental rsults[4)]

As shown in Fig. 36, the boss E of diameter=26 ϕ is fixed on a free supported horizontal shaft of diameter=8 ϕ and length $a+b$=400 mm. A disk D of diameter =360 mm and thickness=8.5 mm is mounted on the boss E at the location $a:b$= 3 : 7. Symmetrical nonlinearity in restoring force is furnished by backlash of about 0.04 mm between disk D and boss E. In order to obtain symmetrical nonlinearity, a long helical spring S shown in Fig. 10 is not used here. Since one shaft end is fixed on a table oscillating vertically with frequency ω and amplitude 0.05 mm, an external periodic force is given in the vibratory system. To check the axial displacement of disk D along the shaft, clearance tapes C of thickness=0.02 mm and width=3 mm are inserted between disk D and boss E as is shown in the left hand side figure of Fig. 36.

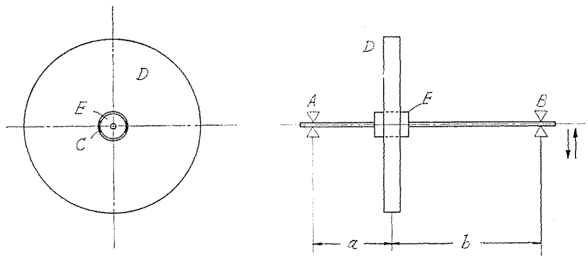


FIG. 36. Experimental apparatus.

Experimental results are illustrated in Figs. 37 and 38. After the experiment, results of which are shown in Fig. 37, the experimental apparatus is disassembled and reassembled again, and the experimental results in Fig. 38 are given again by the succesive experiment. Accordingly between two experiments of Figs. 37 and 38, there is difference in symmetrical nonlinearity of spring characteristics.

The vibratory system shown in Fig. 36 is two degree-of-freedom system because of $a \neq b$, and it has two natural frequencies of p_1 and p_2 ($p_1 < p_2$). Therefore summed harmonic oscillations $[2p_1+p_2]$, $[p_1+2p_2]$ and two sub-harmonic oscillations $[3p_1]$, $[3p_2]$ can take place at $\omega = 2p_1+p_2$, $= p_1+2p_2$, $= 3p_1$ and $= 3p_2$ separately. In Fig. 37, summed harmonic oscillations $[2p_1+p_2]$ and $[p_1+2p_2]$ having response curves of hard spring type appear in the neighborhoods of $\omega = 46$ c/sec and $\omega = 49$ c/sec respectively. And sub-harmonic oscillation of order 1/3 of $[3p_1]$ occurring in the neighborhood of $\omega = 40$ c/sec has response curve of hard spring type, while sub-harmonic oscillation $[3p_2]$ appearing in the neighborhood of $\omega = 50$ c/sec

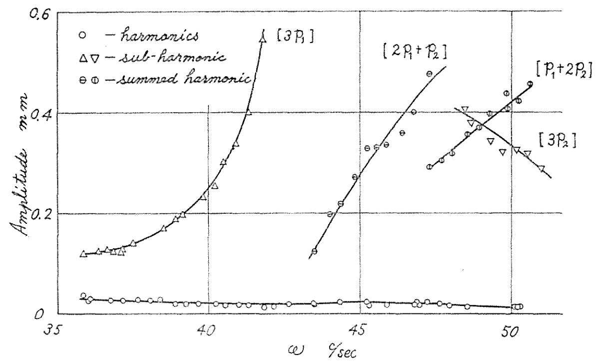


FIG. 37. Response curves of summed harmonic oscillations $[2p_1+p_2]$ and $[p_1+2p_2]$ and sub-harmonic oscillations of order 1/3 with modes of $[3p_1]$ and $[3p_2]$.
(the location of disk $a:b=3:7$)

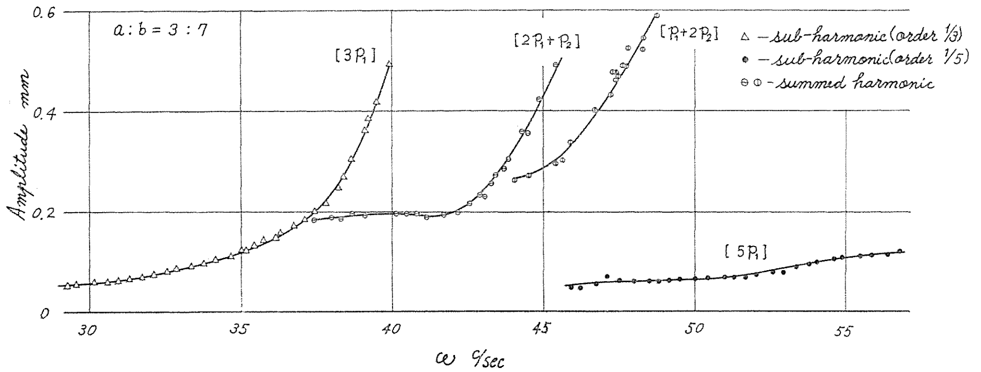


FIG. 38. Response curves of summed harmonic oscillations $[2p_1+p_2]$ and $[p_1+2p_2]$ and sub-harmonic oscillations of order 1/3 with mode of $[3p_1]$ and order 1/5 having mode of $[5p_1]$ (the location of disk $a:b=3:7$).

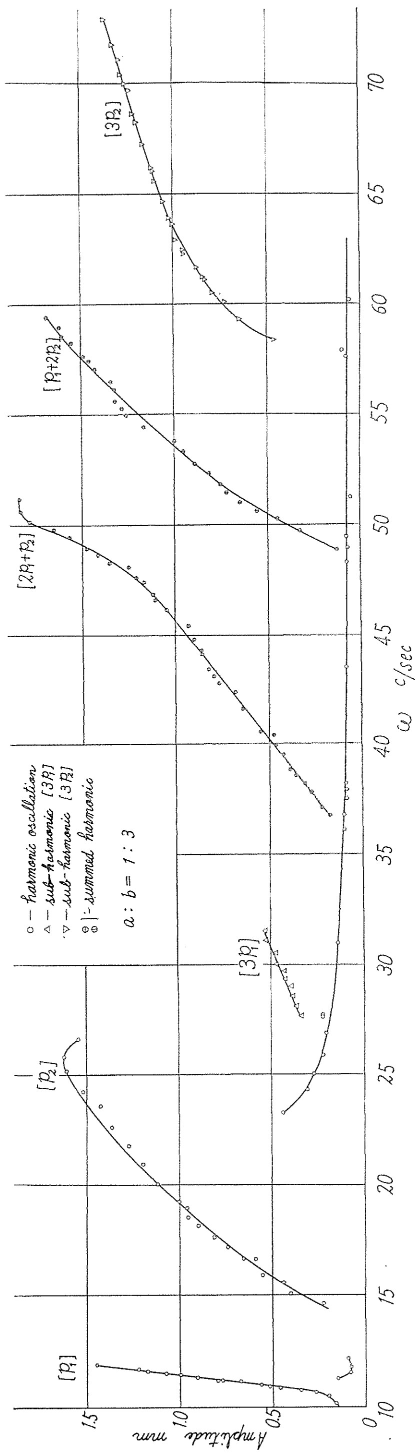
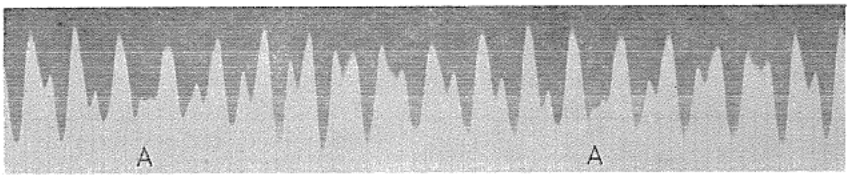


FIG. 39. Response curves of summed harmonic oscillations $[2p_1+p_2]$ and $[p_1+2p_2]$ and sub-harmonic oscillations of order 1/3 with modes of $[3p_1]$, $[3p_2]$ (angular clearance in ball bearing=0.24°, the location of disk $a:b=1:3$).

has response curve of soft spring type. In Fig. 37, it is noticeable that response curves of summed harmonic oscillation $[p_1+2p_2]$ and sub-harmonic oscillation $[3p_2]$ cross each other and further there is a range of ω where both summed harmonic oscillations $[2p_1+p_2]$ and $[p_1+2p_2]$ exist. Which oscillation appears, it is entirely depends on how to give initial conditions. Once one oscillation takes place, it lasts until the frequency ω of external force changes beyond the range of its oscillation. In Fig. 37, over the range of ω from 36 c/sec to 50 c/sec, there are harmonic oscillations of frequency ω with small amplitudes shown by symbol ○. When any disturbance is not given to the vibratory system, only these harmonic oscillations appear along response curve of small amplitudes. Accordingly, occurrences of summed harmonic and sub-harmonic oscillations need some impulse. This fact is caused by backlash between disk and boss which furnishes non-linearity.

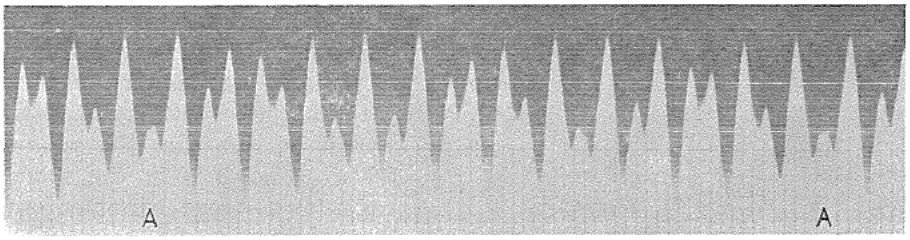
Similar experimental results with Fig. 37 are shown in Fig. 38 where sub-harmonic oscillation $[5p_1]$ of order 1/5 with rather small amplitudes appears.

In experiment shown in Fig. 39, angular clearance in ball bearing is adopted in place of backlash between disk and boss. In experimental apparatus furnishing results in Fig. 39, the location of disk is 1 : 3, and a deep-grooved single-row ball bearing with angular clearance of 0.24° is inserted tightly between disk and boss, and symmetrical nonlinearity is given by the angular clearance. In order to check the revolution of disk, a weight of 0.355 kg is attached at one point on the disk edge. Thus the angular position in which the weight rests below is kept during experiment. In Fig. 39, all response curves of summed harmonic oscillations and sub-harmonic oscillations as well as those of harmonic oscillations



$\omega = 46.11$ c/sec $\quad \omega_1/\omega = 10/39 \quad \omega_2/\omega = 19/39$

FIG. 40. Vibratory waves of summed harmonic oscillation $[2p_1+p_2]$ ($\omega:\omega_1:\omega_2 = 39:10:19$, $2\omega_1+\omega_2=\omega$).



$\omega = 57.58$ c/sec $\quad \omega_1/\omega = 14/64 \quad \omega_2/\omega = 25/64$

FIG. 41. Vibratory waves of summed harmonic oscillation $[p_1+2p_2]$ ($\omega:\omega_1:\omega_2 = 64:14:25$, $\omega_1+2\omega_2=\omega$).

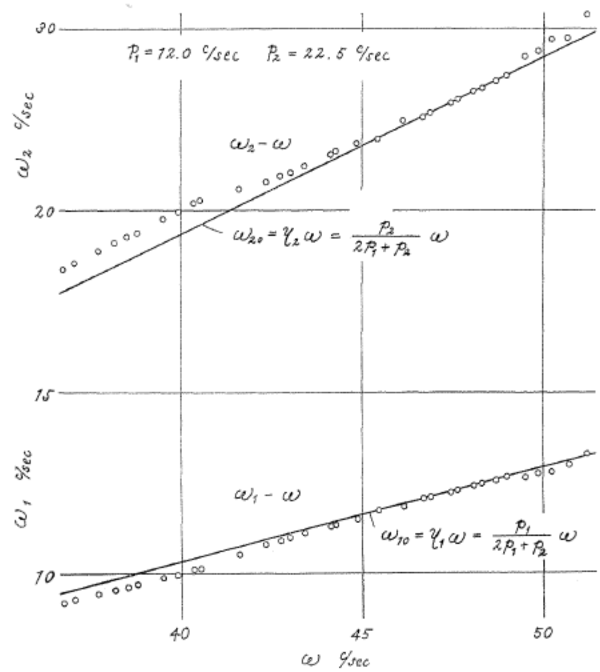


FIG. 42. Frequencies ω_1, ω_2 of summed harmonic oscillation $[2p_1+p_2]$ (experiment shown in Fig. 39).

[p_1] and [p_2] take the shape of hard spring type.

Vibratory waves in the experiment of Fig. 39 are shown in Figs. 40 and 41, where vertical fine black lines are the frequency marks recorded at each vibration of oscillating table on which the shaft end is fixed, and one interval between frequency marks gives one period $2\pi/\omega$ of the external force. Between marks A and A in Fig. 40, summed harmonic oscillation [$2p_1+p_2$] does the same vibratory waves over again, and at intervals of marks A, number of black line intervals is 39 and the vibration with the lower frequency ω_1 oscillates 10 times and the vibration of the higher frequency ω_2 19 times, thus it is seen that the relation $2\omega_1+\omega_2=\omega$ holds. In the vibratory waves of summed harmonic oscillation [p_1+2p_2] in Fig. 41, $\omega:\omega_1:\omega_2=64:14:25$ at intervals of marks A and the relation $\omega_1+2\omega_2=\omega$ is still satisfied.

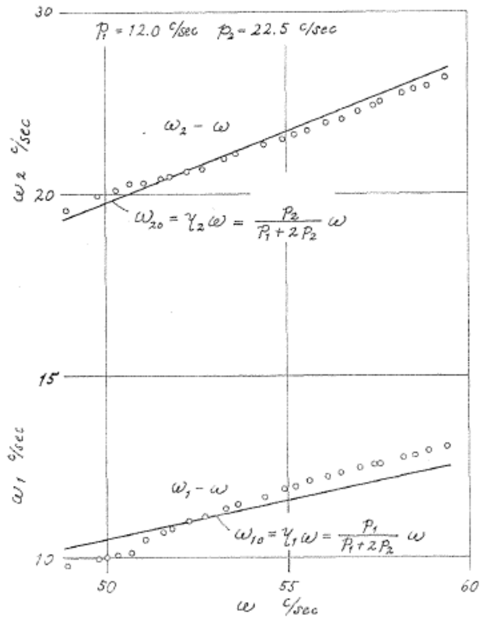


FIG. 43. Frequencies ω_1, ω_2 of summed harmonic oscillation [p_1+2p_2] (experiment shown in Fig. 39).

The frequencies ω_1 and ω_2 of summed harmonic oscillations [$2p_1+p_2$] and [p_1+2p_2] shown in Fig. 39 are plotted against the frequency ω of the external force in Figs. 42 and 43, where full line curves represent the curves of $\omega_{10}=\eta_1\omega$ and $\omega_{20}=\eta_2\omega$. It is seen that adoption of Eq. (3-13) is reasonable.

6. Conclusions

Obtained conclusions in Chaper III may be summarized as follows:

(1) When the vibratory system with multiple degree-of-freedom has symmetrical nonlinearity in spring force, there is possibility of occurrence of "summed and differential harmonic oscillations" [$2p_i\pm p_j$].

(2) Theoretical analysis verifies that only summed harmonic oscillations [$2p_i+p_j$] actually occur and differential harmonic oscillations [$2p_i-p_j$] cannot take place.

(3) Summed harmonic oscillation [$2p_i+p_j$] appears with sub-harmonic oscillation of order 1/3.

(4) Occurrence of summed harmonic oscillation [$2p_i+p_j$] needs symmetrical nonlinearity, but not unsymmetricial nonlinearity.

(5) Response curves and frequencies ω_1, ω_2 of summed harmonic oscillation of [$2p_1+p_2$] are given by Eqs. (3-19) and (3-18), severally.

(6) Response curves of summed harmonic oscillations [$2p_1+p_2$] and [p_1+2p_2] are qualitatively analogous to those of sub-harmonic oscillation of order 1/3.

(7) According as positive or negative symmetrical nonlinearity β_0, response

curves of summed harmonic oscillations $[2p_1+p_2]$ and $[p_1+2p_2]$ are of hard or soft spring type.

(8) The stability of summed harmonic oscillations $[2p_1+p_2]$ and $[p_1+2p_2]$ is studied, and the loci of vertical tangent of response curves furnish the boundary between stable and unstable regions.

(9) Between frequencies ω_1, ω_2 of summed harmonic oscillations $[2p_1+p_2]$, $[p_1+2p_2]$ and frequency ω of disturbing force, the relations $2\omega_1+\omega_2=\omega$, $\omega_1+2\omega_2=\omega$ always hold.

(10) Occurrence of summed harmonic oscillations of both $[2p_1+p_2]$ and $[p_1+2p_2]$ are verified experimentally, and further obtained theoretical results are proved through experiments.

Chapter IV. Summed and Differential Harmonic Oscillations $[p_i \pm p_j \pm p_k]$ in Vibratory Systems with Symmetrical Nonlinear Spring Characteristics

1. Introduction

The response curves and the stability of summed and differential harmonic oscillations $[p_i+p_j]$ and $[2p_i+p_j]$ occurring in the nonlinear vibratory systems with multiple degree-of-freedom have been investigated in the previous chapters. In this chapter, the authors study summed and differential harmonic oscillations $[p_i \pm p_j \pm p_k]$ which can occur in the vibratory system with three or more degree-of-freedom. The vibrations of $[p_i \pm p_j \pm p_k]$ have possibility of its appearance when the following relations between the frequency ω of a periodic external force and the frequencies ω_i, ω_j, ω_k of summed and differential harmonic oscillation $[p_i+p_j+p_k]$ hold:

$$\omega \fallingdotseq |p_i \pm p_j \pm p_k| \tag{4-1}$$

and

$$\left.\begin{aligned} &\omega = |\omega_i \pm \omega_j \pm \omega_k| \qquad (1 \leqq i,\ j,\ k \leqq n) \\ &\omega_i \fallingdotseq p_i, \quad \omega_j \fallingdotseq p_j, \quad \omega_k \fallingdotseq p_k \end{aligned}\right\} \tag{4-2}$$

where $p_1, p_2, \ldots, p_n$ are the natural frequencies of the vibratory system with n degree-of-freedom. In the present chapter, it is verified analytically and experimentally that only summed harmonic oscillation $[p_i+p_j+p_k]$ can take place and all differential types $[p_i+p_j-p_k]$, $[p_i-p_j-p_k]$ etc. cannot occur; and further the response curves and frequencies of summed harmonic oscillation $[p_i+p_j+p_k]$ are calculated and expressed graphically.

2. Summed and differential harmonic oscillations $[p_i \pm p_j \pm p_k]$ in rectilinear vibratory systems[5)]

Eq. (1-6) can be rewritten as follows:

$$\left.\begin{aligned} &\ddot{X}_i + \omega_i^2 X_i = (\omega_i^2 - p_i^2) X_i + F_i \cos \omega t - \psi_i - \sum_r C_{ir} \dot{X}_r, \quad (i = 1, 2, 3) \\ &\ddot{X}_s + p_s^2 X_s = F_s \cos \omega t - \psi_s - \sum_r C_{sr} \dot{X}_r, \qquad (s = 4, 5, \ldots, n) \end{aligned}\right\} \tag{4-3}$$

Along the purterbation methods, the solutions of Eq. (4-3) are developed in power series with respect to a small parameter ε as shown in Eq. (1-16). Substituting Eq. (1-16) into Eq. (4-3) and comparing the terms having the same power of ε, we obtain

$$\left.\begin{aligned} &\ddot{X}_{i0}+\omega_i^2X_{i0}=F_i\cos\omega t \qquad (i=1,2,3)\\ &\ddot{X}_{s0}+p_s^2X_{s0}=F_s\cos\omega t \qquad (s=4,5,\ldots,n)\\ &\ddot{X}_{i1}+\omega_i^2X_{i1}=(\omega_i^2-p_i^2)X_{i0}-\psi_{i0}-\sum_r C_{ir}\dot{X}_{r0}\\ &\ddot{X}_{s1}+p_s^2X_{s1}=-\psi_{s0}-\sum_r C_{sr}\dot{X}_{r0}\\ &\cdots\cdots\cdots \end{aligned}\right\} \tag{4-4}$$

where $\psi_{i0}=\psi_i(X_{10},\ X_{20},\ \ldots,\ X_{n0})$.

The first approximate solution of Eq. (4-4) may be as follows:

$$\left.\begin{aligned} X_{i0}&=R_i\cos(\omega_it-\theta_i)+P_i\cos\omega t\\ X_{s0}&=R_s\cos(p_st-\theta_s)+P_s\cos\omega t \end{aligned}\right\} \tag{4-5}$$

Substituting Eq. (4-5) into Eq. (4-4), and refferring Eq. (4-2), and rejecting the resonant term, we have

$$\left.\begin{aligned} &(\omega_i^2-p_i^2)R_i-Q_i'R_i-(3/4)_i\beta_{(iii)}R_i^3-(1/2)R_i\{_i\beta_{(ijj)}R_j^2+{}_i\beta_{(ikk)}R_k^2\}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad=P_j'R_jR_k\cos\phi\\ &-(\pm\omega_i)C_{ii}R_i=P_j'R_jR_k\sin\phi \quad (i\neq j\neq k,\ i,j,k=1,2,3)\\ &R_s=0 \qquad (s=4,5,\ldots,n) \end{aligned}\right\} \tag{4-6}$$

where

$$\left.\begin{aligned} &P_i'=1/4\sum_{m=1}^{n}\{_i\beta_{(jkm)}P_m\} \qquad (i=1,2,3)\\ &Q_i'=1/2\{\sum_{r,s}{}_i\beta_{(irs)}P_rP_s+\sum_{r}{}_i\beta_{(irr)}P_iP_r\} \quad (r,\ s\neq 1,2,3)\\ &P_i=\frac{F_i}{\omega_i^2-\omega^2},\quad P_{r,s}=\frac{F_{r,s}}{p_{r,s}^2-\omega^2},\quad \phi=\theta_1\pm\theta_2\pm\theta_3 \end{aligned}\right\} \tag{4-7}$$

In Eq. (4-6) and Eq. (4-7), the positive and negative signs correspond to the summed and differential harmonic oscillations $[p_i+p_j+p_k]$ and $[p_i-p_j-p_k]$ etc. respectively. When the nonlinear coefficients of the third order vanish, the summed and differential harmonic oscillations $[p_1\pm p_2\pm p_3]$ cannot occur because of $R_1=R_2=R_3=0$. Consequently it is seen that existence of the nonlinear restoring force of the third order results in appearance of these vibrations. From Eq. (4-6) we have the ratios of R_1^2, R_2^2 and R_3^2 as follows:

$$\rho_1=\frac{R_1^2}{R_2^2}=\frac{\pm C_{22}\omega_2}{C_{11}\omega_1},\quad \rho_2=\frac{R_2^2}{R_3^2}=\frac{\pm C_{33}\omega_3}{\pm C_{22}\omega_2},\quad \rho_3=\frac{R_3^2}{R_1^2}=\frac{C_{11}\omega_1}{\pm C_{33}\omega_3} \tag{4-8}$$

Since the damping coefficients C_{ii} is positive, it is found that there are the following four cases:

$$\begin{array}{lll} \text{summed type} & [p_1+p_2+p_3] & \rho_1>0,\ \rho_2>0,\ \rho_3>0 \\ \text{differential type} \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. & \begin{array}{l} [p_1+p_2-p_3] \\ [p_1-p_2+p_3] \\ [p_1-p_2-p_3] \end{array} & \begin{array}{l} \rho_1>0,\ \rho_2<0,\ \rho_3<0 \\ \rho_1<0,\ \rho_2<0,\ \rho_3>0 \\ \rho_1<0,\ \rho_2>0,\ \rho_3<0 \end{array} \end{array} \quad (4\text{-}9)$$

Even if only one of ρ_1, ρ_2 and ρ_3 is negative, the amplitudes of summed and differential harmonic oscillations $[p_1 \pm p_2 \pm p_3]$ do not become real number. Accordingly it is concluded that only the summed type in Eq. (4-9), *i.e*, the summed harmonic oscillation $[p_1+p_2+p_3]$ can occur and all differential types cannot take place.

3. Response curves of summed harmonic oscillation $[p_i+p_j+p_k]$ in a simplified system

In this section, the response curves of summed harmonic oscillation $[p_1+p_2+p_3]$ are analytically obtained. For brevity, the rather simple system shown in Fig. 44 *i.e.*, a simplified three degree-of-freedom system whose equations of motion are represented by

$$\left. \begin{array}{l} 3\,m\ddot{x}_1 + 3\,K_0 x_1 - K x_2 = -\beta x_1^3 - c\dot{x}_1 + q\cos\omega t \\ 2\,m\ddot{x}_2 - K x_1 + 2\,K_0 x_2 - K x_3 = 0 \\ 6\,m\ddot{x}_3 - K x_2 + 6\,K_0 x_3 = 0 \end{array} \right\} \quad (4\text{-}10)$$

is studied here. In Eq. (4-10), K_0 and K are the spring constant, m the mass, x_1, x_2 and x_3 the displacements of vibratory bodies and $q\cos\omega t$ the periodic external force. The natural frequencies of the vibratory system are

$$p_1 = \sqrt{\frac{2\,K_0 - K}{2\,m}}, \quad p_2 = \sqrt{\frac{K_0}{m}}, \quad p_3 = \sqrt{\frac{2\,K_0 + K}{2\,m}}. \quad (4\text{-}11)$$

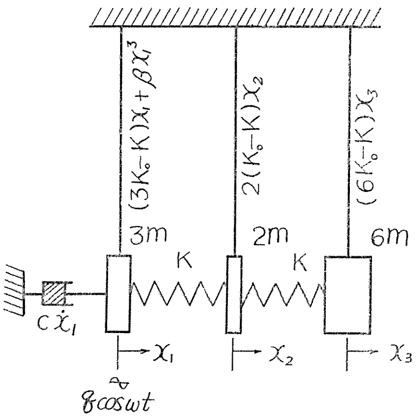


FIG. 44. The simplified vibratory system with three degree-of-freedom ($p_1=0.7071$, $p_2=1.000$, $p_3=1.2247$, $\Omega=p_1+p_2+p_3=2.9318$).

For Eq. (4–10), the transformation from generalized coordinates $x_{1,2,3}$ to normal coordinates $X_{1,2,3}$ is performed by the following equations:

$$\left.\begin{aligned} x_1 &= X_s(X_1+X_2+X_3)/3, \\ x_2 &= X_s(X_1-X_3)/2, \\ x_3 &= X_s(X_1-2X_2+X_3)/6, \end{aligned}\right\} \tag{4-12}$$

in which X_s is the static deflection by force q, as shown in Eq. (4–13). The dimensionless quantities

$$\left.\begin{aligned} &p_1' = p_1/p_0,\ p_2' = p_2/p_0 = 1,\ p_3' = p_3/p_0,\ p_0 = \sqrt{K_0/m},\ \omega' = \omega/p_0, \\ &t' = p_0 t,\ C_0 = c/(9\sqrt{K_0 m}),\ \beta_0 = \beta X_s^2/(108 K_0),\ X_s = q/(3K_0),\ F_0 = 1 \end{aligned}\right\} \tag{4-13}$$

are used here. Substituting Eqs. (4–12) and (4–13) into Eq. (4–10) and omitting primes on the dimensionless quantities, we attain

$$\left.\begin{aligned} \ddot{X}_1 + p_1^2 X_1 &= -C_0(\dot{X}_1+\dot{X}_2+\dot{X}_3) - (4/3)\beta_0(X_1+X_2+X_3)^3 + F_0\cos\omega t \\ \ddot{X}_2 + p_2^2 X_2 &= -C_0(\dot{X}_1+\dot{X}_2+\dot{X}_3) - (4/3)\beta_0(X_1+X_2+X_3)^3 + F_0\cos\omega t \\ \ddot{X}_3 + p_3^2 X_3 &= -C_0(\dot{X}_1+\dot{X}_2+\dot{X}_3) - (4/3)\beta_0(X_1+X_2+X_3)^3 + F_0\cos\omega t \end{aligned}\right\} \tag{4-14}$$

The first approximate solutions of Eq. (4–14) become

$$\left.\begin{aligned} X_i &= R_i\cos(\omega_i t - \theta_i) + P_i\cos\omega t \\ P_i &= \frac{F_0}{\omega_i^2-\omega^2} \qquad (i=1,2,3) \end{aligned}\right\} \tag{4-15}$$

The amplitudes R_i, the phase angles θ_i and the frequencies ω_i are determined by the following equations:

$$\left.\begin{aligned} &(\omega_1^2-p_1^2)R_1 - \beta_0 R_1^3 - 2\beta_0 R_1(R_2^2+R_3^2) - 2\beta_0 P^2 R_1 = 2\beta_0 P R_2 R_3\cos\phi \\ &(\omega_2^2-p_2^2)R_2 - \beta_0 R_2^3 - 2\beta_0 R_2(R_3^2+R_1^2) - 2\beta_0 P^2 R_2 = 2\beta_0 P R_3 R_1\cos\phi \\ &(\omega_3^2-p_3^2)R_3 - \beta_0 R_3^3 - 2\beta_0 R_3(R_1^2+R_2^2) - 2\beta_0 P^2 R_3 = 2\beta_0 P R_1 R_2\cos\phi \\ &-C_0\omega_1 R_1 = 2\beta_0 P R_2 R_3\sin\phi \\ &-C_0\omega_2 R_2 = 2\beta_0 P R_3 R_1\sin\phi \\ &-C_0\omega_3 R_3 = 2\beta_0 P R_1 R_2\sin\phi \end{aligned}\right\} \tag{4-16}$$

where

$$P = P_1+P_2+P_3 = F_0\left(\frac{1}{\omega_1^2-\omega^2} + \frac{1}{\omega_2^2-\omega^2} + \frac{1}{\omega_3^2-\omega^2}\right) \tag{4-16 a}$$

And the ratios of R_i^2 are

$$R_1^2/R_2^2 = \omega_2/\omega_1,\quad R_2^2/R_3^2 = \omega_3/\omega_2,\quad R_3^2/R_1^2 = \omega_1/\omega_3 \tag{4-17}$$

From Eq. (4–16) and Eq. (4–17), we have

$$\left.\begin{aligned} R_i^2 &= \frac{\omega_j \omega_k(\lambda_i \omega_j - \omega_i \lambda_j)}{\beta_0(\omega_j - \omega_i)(2\,\omega_i \omega_j + \omega_j \omega_k + \omega_k \omega_i)} \\ \lambda_i &= \omega_i^2 - p_i^2 - 2\,\beta_0 P^2 \qquad (i, j, k = 1, 2, 3,\ i \neq j \neq k) \end{aligned}\right\} \quad (4\text{-}18)$$

where

and

$$\left.\begin{aligned} &\{\lambda_i \omega_j \omega_k - \beta_0 R_i^2(2\,\omega_i\omega_j + \omega_j\omega_k + 2\,\omega_i\omega_k)\}^2 + (C_0 \omega_i \omega_j \omega_k)^2 \\ &\quad = \omega_i^2 \omega_j \omega_k (2\,\beta_0 P^2) R_i^2 \qquad (i, j, k = 1, 2, 3,\ i \neq j \neq k) \end{aligned}\right\} \quad (4\text{-}19)$$

Putting

$$\left.\begin{aligned} &\omega_i = \eta_i \omega + \varepsilon \omega_{i1} + \varepsilon^2 \omega_{i2} + \cdots \qquad (i = 1, 2, 3) \\ &\omega_{1n} + \omega_{2n} + \omega_{3n} = 0 \qquad (n = 1, 2, \ldots) \\ &\eta_i = p_i/\Omega,\ \Omega = p_1 + p_2 + p_3 \end{aligned}\right\} \quad (4\text{-}20)$$

and inserting Eq. (4–20) into Eq. (4–18) and Eq. (4–19), the frequencies ω_i ($i = 1, 2, 3$) and the amplitudes R_i are given as follows:

$$\left.\begin{aligned} \omega_1 &= \eta_1 \omega + \frac{\eta_1}{3\,\omega f_3 f_4^2}[f_3 f_4\{f_4(\eta_2 + \eta_3 - 2\,\eta_1) + f_1 f_{10}\}(\omega^2 - \Omega^2) \\ &\quad + \{f_4^2(2\,\eta_2\eta_3 - \eta_2\eta_1 - \eta_1\eta_3) - f_5 f_{10}\}(2\,\beta_0 P^2) \pm 3 f_{10} f_3 \sqrt{D}] \\ \omega_2 &= \eta_2 \omega + \frac{\eta_2}{3\,\omega f_3 f_4^2}[f_3 f_4\{f_4(\eta_3 + \eta_1 - 2\,\eta_2) + f_1 f_{20}\}(\omega^2 - \Omega^2) \\ &\quad + \{f_4^2(2\,\eta_3\eta_1 - \eta_3\eta_2 - \eta_2\eta_1) - f_5 f_{20}\}(2\,\beta_0 P^2) \pm 3 f_{20} f_3 \sqrt{D}] \\ \omega_3 &= \eta_3 \omega + \frac{\eta_3}{3\,\omega f_3 f_4^2}[f_3 f_4\{f_4(\eta_1 + \eta_2 - 2\,\eta_3) + f_1 f_{30}\}(\omega^2 - \Omega^2) \\ &\quad + \{f_4^2(2\,\eta_1\eta_2 - \eta_3\eta_2 - \eta_1\eta_3) - f_5 f_{30}\}(2\,\beta_0 P^2) \pm 3 f_{30} f_3 \sqrt{D}] \end{aligned}\right\} \quad (4\text{-}21)$$

$$\left.\begin{aligned} R_i^2 &= \frac{\eta_i \eta_k}{\beta_0 f_4^2}[f_1 f_3 f_4(\omega^2 - \Omega^2) - f_5(2\,\beta_0 P^2) \pm 3 f_3 \sqrt{D}] \\ &\qquad (i, j, k = 1, 2, 3 \qquad i \neq j \neq k) \end{aligned}\right\} \quad (4\text{-}22)$$

where

$$\left.\begin{aligned} &D = 2 f_1 f_3 f_4(2\,\beta_0 P^2)(\omega^2 - \Omega^2) - f_6(2\,\beta_0 P^2)^2 - f_4^2(C_0 \omega)^2 \\ &\Omega = p_1 + p_2 + p_3 \\ &f_1 = \eta_1 + \eta_2 + \eta_3 = 1,\ f_2 = \eta_1\eta_2 + \eta_2\eta_3 + \eta_3\eta_1,\ f_3 = \eta_1\eta_2\eta_3 \\ &f_4 = f_2^2 + 2 f_1 f_3,\ f_5 = f_2 f_4 - 9 f_3^2,\ f_6 = 2 f_2 f_4 - 9 f_3^2 \\ &f_{10} = 2(\eta_2 - \eta_1)(\eta_1\eta_3 + \eta_2\eta_3 + 2\,\eta_1\eta_2)\eta_3 + (\eta_3 - \eta_2)(\eta_1\eta_2 + \eta_1\eta_3 + 2\,\eta_2\eta_3)\eta_1 \\ &f_{20} = 2(\eta_3 - \eta_2)(\eta_2\eta_1 + \eta_3\eta_1 + 2\,\eta_2\eta_3)\eta_1 + (\eta_1 - \eta_3)(\eta_2\eta_3 + \eta_2\eta_1 + 2\,\eta_3\eta_1)\eta_2 \\ &f_{30} = 2(\eta_1 - \eta_3)(\eta_3\eta_2 + \eta_1\eta_2 + 2\,\eta_3\eta_1)\eta_2 + (\eta_2 - \eta_1)(\eta_3\eta_1 + \eta_3\eta_2 + 2\,\eta_1\eta_2)\eta_3 \end{aligned}\right\} \quad (4\text{-}23)$$

Summed harmonic oscillations $[p_1+p_2+p_3]$ have the characters similar to sub-harmonic oscillations of order 1/3 and summed harmonic oscillations $[2p_i+p_j]$. Fig. 45 shows the response curves of the summed harmonic oscillation $[p_1+p_2+p_3]$ given by Eq. (4–22). In Fig. 45, one third of sum of amplitudes $[R_1+R_2+R_3]/3$ are plotted against ω. In the figure, full lines represent the response curves for

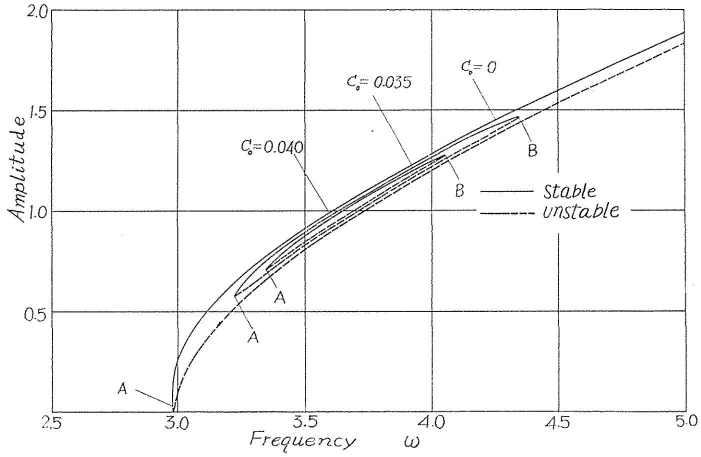


FIG. 45. Response curves of summed harmonic oscillation $[p_1+p_2+p_3]$ ($p_1=0.7071$, $p_2=1$, $p_3=1.2247$, $\Omega=p_1+p_2+p_3=2.9318$, $\beta_0=0.1$) (p_1, p_2, p_3=natural frequency, Ω=resonant frequency, ω=frequency of external force, β_0=coefficient of symmetrical nonlinearity, C_0= damping coefficient).

stable vibration and broken lines the response curves for unstable vibration. The natural frequencies of this system are $p_1=0.7071$, $p_2=1.0000$, $p_3=1.2247$, then the resonant frequency of the summed harmonic oscillation is $\Omega=p_1+p_2+p_3=2.9318$. The coefficient β_0 of symmetrical nonlinearity is 0.1 and the response curves are of hard spring type. At the points A, B, jump phenomena take place. Accordingly, only when some appropriate initial conditions are given, this mode of vibration *i.e.* summed harmonic oscillation $[p_1+p_2+p_3]$ can occur.

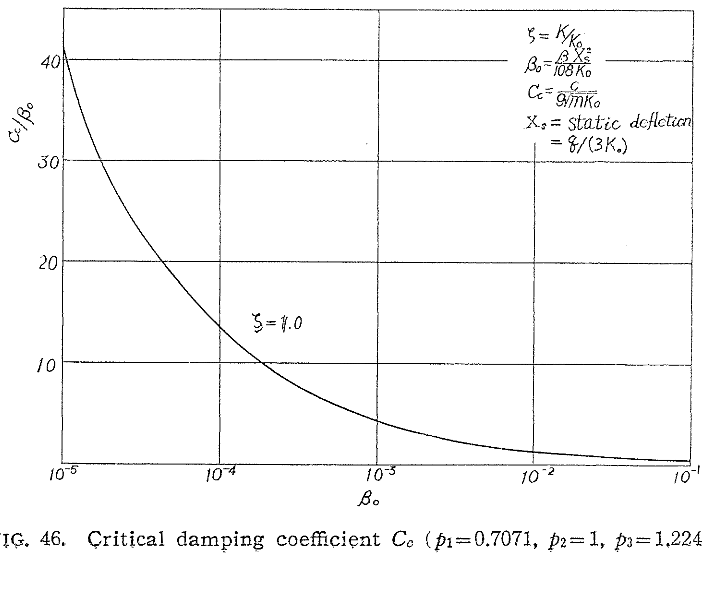


FIG. 46. Critical damping coefficient C_c ($p_1=0.7071$, $p_2=1$, $p_3=1.2247$),

The region of frequency in which the summed harmonic oscillation $[p_1+p_2+p_3]$ can occur is reduced with increase of magnitude of damping coefficient and shrinks to a point at a certain value of damping coefficient which is defined as critical damping coefficient C_c. Summed harmonic oscillation $[p_1+p_2+p_3]$ cannot appear when damping coefficient is larger than the critical damping C_c, which is determined by solving the following equations:

$$\left.\begin{aligned} &D(\omega^2)=0,\quad \frac{\partial D(\omega^2)}{\partial(\omega^2)}=0 \\ &D=2f_1f_3f_4(\omega^2-\Omega^2)(2\beta_0P^2) \\ &\qquad -f_6(2\beta_0P^2)^2-f_4(C_0\omega)^2 \end{aligned}\right\} \tag{4-24}$$

The calculated results are shown in Fig. 46. It is easily seen from Fig. 46 that the critical damping C_c increases with the coefficient β_0 of symmetrical nonlinearity.

The frequency region in which summed harmonic oscillation $[p_1+p_2+p_3]$ can take place varies with the value of damping ratio C_0/C_c, as shown in Fig. 47. The figure shows that the region becomes more narrow with increase of C_0/C_c and when $C_0=C_c$ the region vanishes.

The effect of the nonlinear restoring force is shown in Fig. 48 in which only stable parts of the

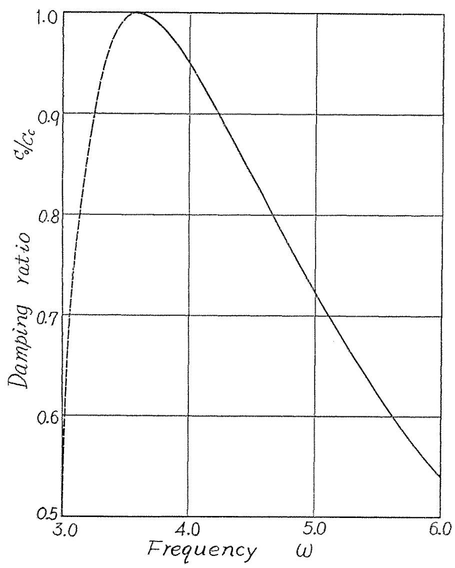


FIG. 47. The region in which summed harmonic oscillation $[p_1+p_2+p_3]$ appears ($\Omega=p_1+p_2+p_3=2.9318$, $\beta_0=0.001$, $C_c=1.293\times10^{-4}$).

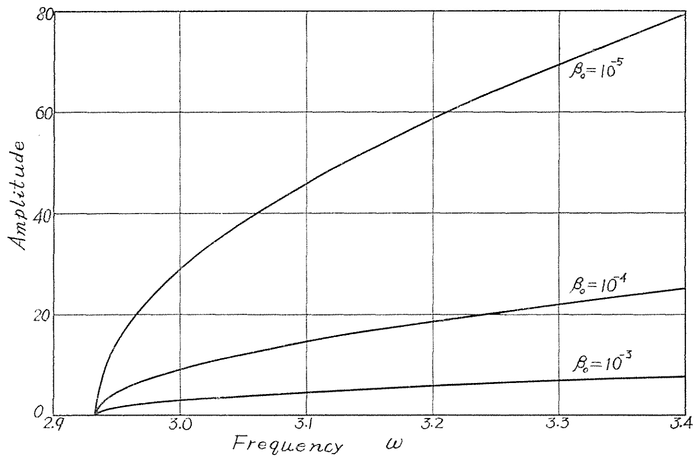


FIG. 48. Response curves of summed harmonic oscillation $[p_1+p_2+p_3]$ for various magnitudes of coefficient β_0 of symmetrical nonlinearity ($p_1=0.7071$, $p_2=1$, $p_3=1.2247$, $C_0=0$).

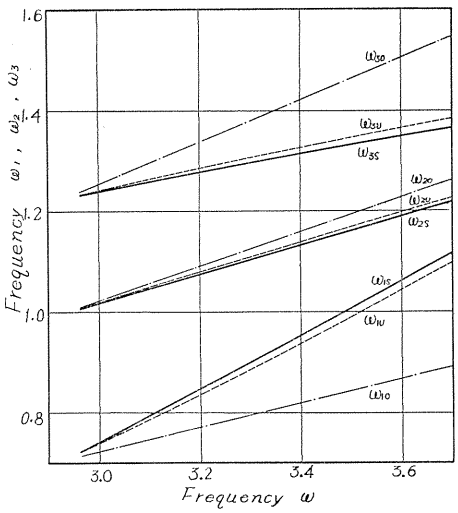


FIG. 49. Frequencies ω_1, ω_2 and ω_3 of summed harmonic oscillation $[p_1+p_2+p_3]$ ($p_1=0.7071$, $p_2=1$, $p_3=1.2247$, $\beta_0=0.1$, $C_0=0$).

response curves for $C_0=0$ are indicated. The amplitude of vibration becomes larger for the smaller nonlinear coefficient β_0.

In Fig. 49, frequencies ω_1, ω_2 and ω_3 of summed harmonic oscillations of $[p_1+p_2+p_3]$ are plotted to the frequency of the external force ω. The full lines are $\omega_{is}=\omega_{i0}+\varepsilon\omega_{i1s}$ which give the frequencies of stable oscillations, and the dotted lines are curves of $\omega_{iu}=\omega_{i0}+\varepsilon\omega_{i1u}$ which represent frequencies of unstable oscillations, and the chain lines are $\omega_{i0}=\eta_i\omega$ in Eq. (4-20). In the figure, the change of ω_i to ω is almost linear.

4. Experimental apparatus and experimental results

Fig. 50 shows the experimental apparatus consisting of the steel strips K_{11}, K_{22} and K_{33} (1 mm thickness and 10 mm width), the mass m_1, m_2 and m_3 attached to an end of each steel strips, and the coil springs K_{12} and K_{23} connecting the mass. The nonlinearity of restoring force is given by the two steel strips N_1, N_2 slightly curved as shown in Fig. 50. The periodic external force is furnished by horizontal oscillations of the frame. The values of several coefficients of the vibratory system are as follow: weights of the vibratory bodies are $m_1g=321.4$ g, $m_2g=218.2$ g and $m_3g=105.8$ g, spring constants $K_{11}=40.3$ g/mm, $K_{22}=19.2$ g/mm, $K_{33}=21.2$ g/mm, $K_{12}=K_{23}=1.105$ g/mm and $K_{31}=0$, damping coefficients $c_{11}=4.14\times10^{-3}$ g sec/mm, $c_{22}=1.25\times10^{-3}$ g sec/mm (or $c_{22}=3.36\times10^{-3}$ g sec/mm), $c_{33}=0.409\times10^{-3}$ g sec/mm, $c_{12}=c_{23}=0.321\times10^{-3}$ g sec/mm and $c_{31}=0$ and the natural frequencies $p_1=6.11$ c/sec, $p_2=7.56$ c/sec and $p_3=9.08$ c/sec. The nonlinear characteristics of restoring force are shown in Fig. 51, where (a) and (b) curves correspond to force-displacement relation of x_1 and x_2 respectively. The spring characteristics, precisely speaking, not $(kx+\beta x^3)$ type but rather piecewise linear and symmetrical to the origin.

In Figs. 52 and 53 the amplitudes obtained by the experiment are plotted against the frequency ω of the external force. The symbols ○, ◑ and ● in the figures represent amplitude of x_1, x_2 and x_3 separately. The left hand side figure of Fig. 52 shows the response curves of the harmonic oscillations near the resonant frequencies p_1, p_2 and p_3. The response curves are of hard spring type and the jump phenomena occur. The right hand side figure of Fig. 52 and Fig. 53 show the response curves of summed harmonic oscillation $[p_1+p_2+p_3]$, where the latter corresponds to case of the larger damping coefficent $c_{22}=3.36\times10^{-3}$ g sec/mm.

When the initial condition is appropriately set up, the summed oscillation $[p_1+p_2+p_3]$ can occur in the region of slightly higher frequency than sum of the natural frequencies $p_1+p_2+p_3=22.75$ c/sec.

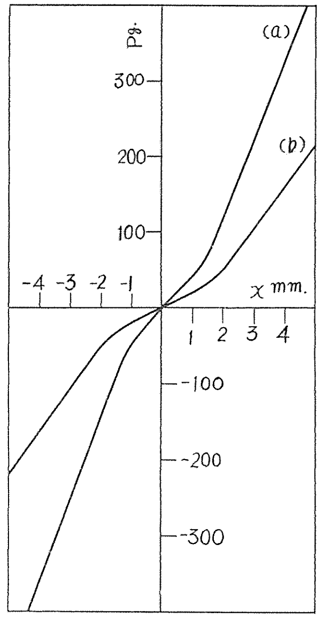


FIG. 51. Spring characteristics.

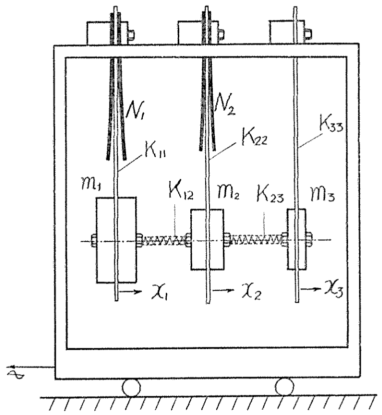


FIG. 50. Experimental apparatus ($p_1=6.10$ c/sec, $p_2=7.57$ c/sec, $p_3=9.08$c/sec, $\Omega=p_1+p_2+p_3=22.75$ c/sec)

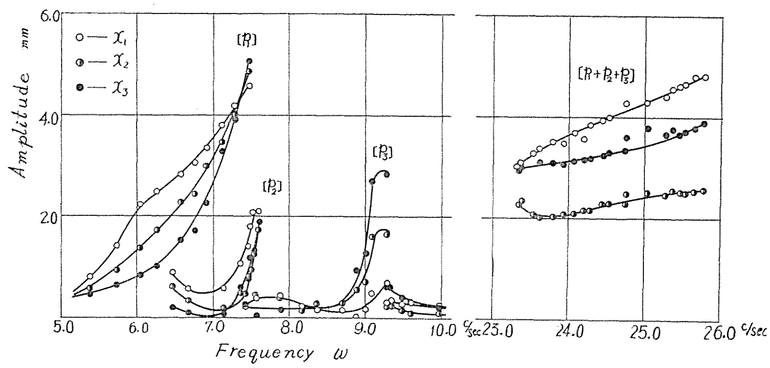


FIG. 52. Response curves of harmonic oscillations $[p_1]$, $[p_2]$, $[p_3]$ and summed harmonic oscillation $[p_1+p_2+p_3]$ ($p_1=6.11$ c/sec, $p_2=7.56$ c/sec, $p_3=9.08$ c/sec, $c_{22}=1.25\times 10^{-6}$ kg sec/mm, $\Omega=p_1+p_2+p_3=22.75$ c/sec).

All response curves are discontinuous because of jump phenomena, and damping coefficient becomes larger, the region in which the summed harmonic oscillation of $[p_1+p_2+p_3]$ can take place becomes smaller.

Fig. 54 is an example of the oscillatory wave obtained by experiment and each wave includes three components of frequencies ω_1, ω_2 and ω_3. The upper,

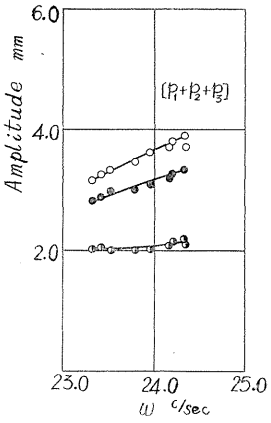
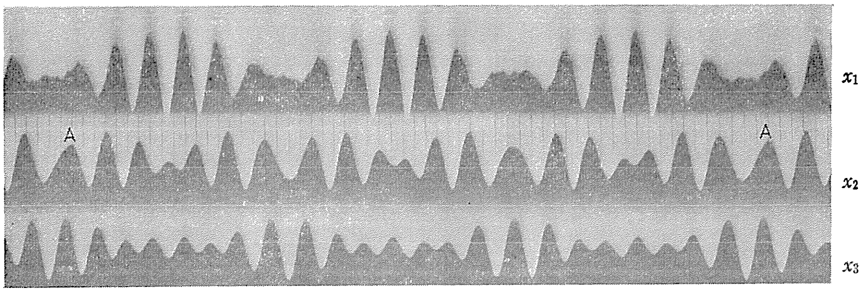


FIG. 53. Response curves of summed harmonic oscillation $[p_1+p_2+p_3]$ ($p_1=6.11$ c/sec, $p_2=7.56$ c/sec, $p_3=9.08$ c/sec, $\Omega=p_1+p_2+p_3=22.75$ c/sec, $c_{22}=3.36\times 10^{-6}$ kg sec/mm).



$\omega=24.172$ c/sec, $\omega_1=17/60\cdot\omega$, $\omega_2=20/60\cdot\omega$, $\omega_3=23/60\cdot\omega$

FIG. 54. Vibratory waves of summed harmonic oscillation $[p_1+p_2+p_3]$ ($\omega_1:\omega_2:\omega_3:\omega=17:20:23:60$).

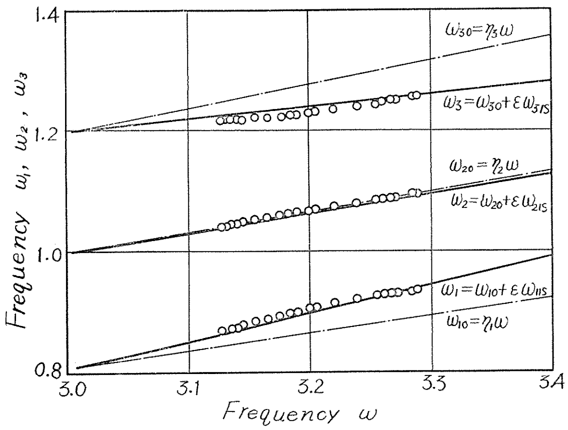


FIG. 55. Frequencies of ω_1, ω_2 and ω_3 of summed harmonic oscillation $[p_1+p_2+p_3]$ ($\eta_1=0.2681$, $\eta_2=0.3327$, $\eta_3=0.3992$).

middle and lower oscillatory waves in Fig. 54 correspond to x_1, x_2 and x_3 respectively. The vertical black lines in Fig. 54 indicate one period of the external force and the vibratory wave changes periodically at each marks A. Between a mark A and the next A the summed harmonic oscillation $[p_1+p_2+p_3]$ vibrates 17, 20 and 23 times separately, while 60 periods of the external force are recorded. Thus it is seen that the relations

$$\omega_1+\omega_2+\omega_3=\frac{17}{60}\omega+\frac{20}{60}\omega+\frac{23}{60}\omega=\omega=24.172 \text{ c/sec} \qquad (4\text{-}25)$$

are satisfied. The relations of the frequencies ω_1, ω_2 and ω_3 to ω are shown in Fig. 55, where dimensionless quantities $\omega_1'=\omega_1/p_2$, $\omega_2'=\omega_2/p_2$ and $\omega_3'=\omega_3/p_2$ ($p_2=$ 7.56 c/sec is one of the natural frequencies) are employed as ω_1, ω_2 and ω_3 severally. In Fig. 55, the full lines represent frequencies ω_1, ω_2 and ω_3 theoretically calculated by Eq. (4-21), the chain lines are $\omega_{i0}=\eta_i\omega$ and the experimental results are represented by the symbol ○.

5. Conclusions

In the present chapter, the following conclusions are obtained:

(1) When the vibratory systems with three or more degree-of-freedom have symmetrical nonlinearity in spring forces, there is possibility of occurrence of "summed and differential harmonic oscillations" $[p_i\pm p_j\pm p_k]$.

(2) Through theoretical analysis, it is cleared up that only summed harmonic oscillation $[p_i+p_j+p_k]$ can appear and all kinds of differential harmonic oscillations do not take place.

(3) Characters of summed harmonic oscillation $[p_i+p_j+p_k]$ are quite similar to those of summed harmonic oscillation $[2p_i+p_j]$ and sub-harmonic oscillation of order 1/3.

(4) Occurrence of summed harmonic oscillation $[p_i+p_j+p_k]$ needs symmetrical nonlinearity.

(5) Between frequencies ω_i, ω_j, ω_k and the frequency ω of disturbing force, the relation of $\omega_i+\omega_j+\omega_k=\omega$ always holds.

(6) Occurrence of summed harmonic oscillation $[p_i+p_j+p_k]$ are secured experimentally, and obtained theoretical results are verified through experiments.

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