ADAPTIVE OPTIMALIZING CONTROL UTILIZING OPEN LOOP TYPE OPTIMUM-CONTROLLER AND THE SENSITIVITY COEFFICIENTS

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The mathematical studies of optimal control processes are based on many idealized assumptions. When disturbances that are not known even statisically are considered, the optimum-controller must be readjusted every time the disturbances occur and the parameters of the plant must be compensated so as to maintain the deviation from the expected optimal state within some allowable limit. The control scheme can be realized only by using a high speed digital computer on its feebdack path. This study presents all the mathematical conditions that are necessary to the scheme and accessible to modern high speed computers. The magnitude of disturbances (plant parameter variation) is estimated by solving an equation of observation that relates the parameter disturbances to the plant dynamic sensitivity coefficients. The deviation of the performance index (PI) is also calculated by means of the coefficients of PI variation. To a given margin for the deviation of PI, one simple procedure by which the deviation can be reduced to a value within the margin is proposed as an example.

1. Introduction

One of the many idealized assumptions of the optimal control problems is to assume that the disturbances are known before the control begins. This is a very different attitude from the classical automatic control which endeavors to conquer unknown disturbances. If the technical language "automation" is used as the classical meaning, the currently prevailing optimum-control processes are given unreserved criticism that they are not "automation". Fig. 1 shows that

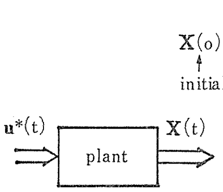

FIG. 1. Open loop optimum-control

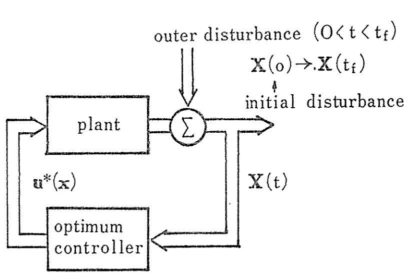

FIG. 2. Closed loop optimum-control

the optimum-control processes are realized essestially in open loop systems.

However, in such time optimal problems where one must adjust the positions and velocities of point mass-system to specified final values[1], closed-loop (type optimum-control) laws $\mathbf{u}^*(\mathbf{x})$ can be obtained without including the initial position $\mathbf{x}(0)$ explicitly by eliminating time variable t from the open-loop (type optimum-control) laws $\mathbf{u}^*(t)$ and trajectories $\mathbf{x}^*(t)$. Therefore as shown in Fig. 2, the optimum-control state can automatically be kept for outer disturbances (this terminology is used for such disturbances that wind deviates a flying body from its nominal trajectory) for $0<t<t_f$ (final time).

As these results were considered important, the closed-loop laws have been sought laboriously by many researchers in many kinds of optimum-control problems.

However, when some disturbances occur in plant parameters (parameter disturbances), the situation governed by such closed-loop laws is no longer optimal and moreover there is no guarantee that the state can stay near the optimum subsequently. For example, in the above mentioned time optimal problems of point mass-system, any little parameter disturbances make t_f mathematically infinite. Recalling that ordinary little outer and parameter disturbances can be conquered and the desired state can almost be realized in classical automatic control, we can not talk about both processes such that they have equivalent significance in enginerring.

As another example in a group of problems known as Kalman's regulator[2], the closed-loop system shown in Figs. 3 and 4 can be realized[3)4)]. As the former opti-

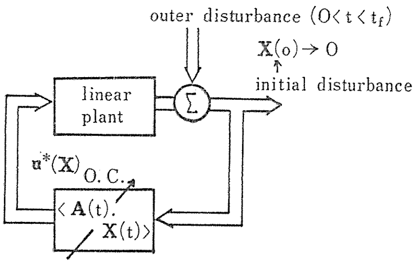

FIG. 3. One of closed loop optimum controllers for linear plant

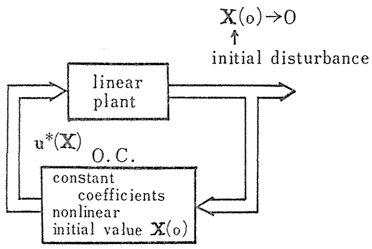

FIG. 4. Another type controller for the same plant as FIG. 3.

mum-control law is a linear combination of instantaneous state vector components with variable time coefficients, the outer disturbances can be controlled optimally, but there must be no mistake in changing $A(t)$ as specified. The latter optimum-control law in Fig. 4 is a stationary nonlinear control that includes the initial point $\mathbf{x}(0)$ explicitly in it. Therefore the outer disturbances can not be controlled. All the above systems have no optimally control ability for parameter disturbances.

Concluding that by using such direct feedback schemes as shown in Fig. 2~4 even suboptimally controlled states can not be realized for the outer and parameter disturbances, we will consider a more complicated control process having a digital computer on its feedback path in the following section.

There the disturbances are assumed to be known very roughly in their characteristics, for instance, the kind of physical quantity or the maximum-order of absolute magnitude, etc., but unknown in detail.

2. Optimal Readjustment of Controller

The readjusting device A for the optimum-controller shown in Fig. 5 plays a leading role. This is composed mainly of an electronic digital computer and measuring instrumants of the state vector. It is assumed that before beginning control ($t \leqq 0$) no outer disturbences have occurred and the state transition equation of the plant (including effects of control signal) and all its parameter values are known (the plant identification has been accomplished). In this situation the digital computer computes the optimal control law $\mathbf{u}^*(t)$ and trajectory $\mathbf{x}^*(t)$ by use of one of the currently available optimization mathematical techniques and stores $\mathbf{x}^*(t)$ in its memory.

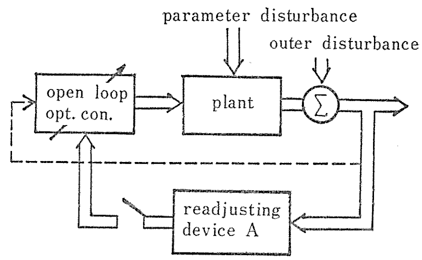

FIG. 5. A system having an open loop controller and its readjusting device.

The measuring instruments of state vector are assumed to be able to detect the variation of all the state vectors when their magnitudes change beyond the instruments sensitivity. Comparing the measured trajectory with the stored one, mechanism A determines the variation of disturbances and the time of its occurence t_j whenever a deviation of trajectory is observed. Then, calculating the position in state space at the time t_j+T, A finds the new optimum-control so that it can operate after t_j+T by making use of the calculated position and the identified parameters. Then at t_j+T the optimum-controller will be changed. Here T is a time reguired to the above readjusting process and assumed to be constant in the present article.

The feedback mechanism of the dotted line in Fig. 5 shows the direct feedback process stated in section 1, but this is not used because of the existance of feedback through mechanism A and moreover because the direct feedback mechanism sometimes makes the calculation inaccessible that is necessary to readjust the optimal controller (stated later). So the optimum-controller in Fig. 5 is of open loop type in which the successive operations to the plant are programed sequentially.

3. Optimal Control Problem and Disturbances

The optimum-control problem and the disturbances to be discussed will be characterized as follows: The state transition equation is

$$\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}, \mathbf{u}, \alpha) \tag{1}$$

Here, $\mathbf{x}$ is an n state vector, $\mathbf{u}$ is an r control vector ($r \leqq n$) and $\mathbf{u} \in U$, U is an open or closed set of r dimentional Eucridian space. α is a p parameter vector and the assumption $p=n$ is made for simplicity but the letter p will be unchanged. In the cases $p<n$ and $p>n$, the treatment in the case of $p=n$ can be applied almost unchanged with some additional considerations as stated later. $\mathbf{f}(\mathbf{x}, \mathbf{u}, \alpha)$ is assumed differentiable with respect to the arguments.

The performance index is $Q=\int_{t_0}^{t_f} F(\mathbf{x}^*, \mathbf{u}^*)\, dt$ in which t_0 is specified but t_f is sometimes specified and sometimes not. $F(\mathbf{x}, \mathbf{u})$ is also differentiable with respect to $\mathbf{x}$ and $\mathbf{u}$. The initial position $\mathbf{x}(t_0)=\mathbf{x}(0)$, is given and for the final point $\mathbf{x}(t_f)$ an m $(<n)$ dimentional manifold where $\mathbf{x}(t_f)$ terminates is specified. It often happens that no trajectories disturbed by parameter variations can reach the final state given as a point in finite time.

The disturbances are assumed to happen at $t_1, t_2, \cdots, t_k$ $(t_0<t_1\cdots t_k<t_f)$, stepping by small magnitudes, as shown in Fig. 6, and $\boldsymbol{\alpha}$ for $t_0 \leqq t < t_1$ $[t_0, t_1>$ (here [indicates the boundary closed, >open), $[t_1, t_2>, \ldots, [t_k, t_f]$ are $\boldsymbol{\alpha}^0, \boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^k$. Excluding $\boldsymbol{\alpha}^0$, we do not know the magnitude of them even statistically. Let $\Delta\boldsymbol{\alpha}^0=\boldsymbol{\alpha}^1-\boldsymbol{\alpha}^0$, $\Delta\boldsymbol{\alpha}^1=\boldsymbol{\alpha}^2-\boldsymbol{\alpha}^1$, $\cdots, \Delta\boldsymbol{\alpha}^{k-1}=\boldsymbol{\alpha}^k-\boldsymbol{\alpha}^{k-1}$. The times $t_1, t_2, \cdots, t_k$ are also not known previously, but the time intervals $t_{j+1}-t_j$ between neighboring times are assumed to be larger than the time T. That is, the disturbances are supposed to have longer periods of occurence than T. Fig. 6 also shows that the optimum-control laws used for intervals $[t_0, t+T>, [t_1+T, t_2+T>, \cdots, [t_k+T, t_f]$ are $\mathbf{u}^0, \mathbf{u}^1, \cdots, \mathbf{u}^k$.

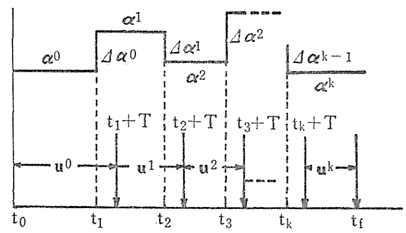

FIG. 6. Parametar disturbance.

4. For $[t_1, t_1+T]$

$\mathbf{u}^0(t)$ is optimal for $[t_0, t_1>$, and not for $[t_1, t_1+T]$. However the optimum-control law is not known for the latter and so $\mathbf{u}^0(t)$ is applied unchanged.† As stated in section 2 some deviation from the predicted optimum trajectory may be observed in this interval. In order to calculate the parameter variations $\Delta\boldsymbol{\alpha}^0$, using the observed deviations we introduce dynamic sensitivities $\partial\mathbf{x}/\partial\boldsymbol{\alpha}$ which are limited values of ratios of trajectory variations to parameter changes.

For $[t_0, t_1>$, the state transition equation is

$$\dot{\mathbf{x}}=\mathbf{f}\{\mathbf{x}, \mathbf{u}^0(t, \boldsymbol{\alpha}^0), \boldsymbol{\alpha}^0\} \qquad (2),$$

while, for $[t_1, t_1+T]$

$$\dot{\mathbf{x}}=\mathbf{f}\{\mathbf{x}, \mathbf{u}^0(t, \boldsymbol{\alpha}^0), \boldsymbol{\alpha}^1\} \qquad (3),$$

that is, the open loop control law for the preceding time interval is used unchanged in spite of the parameter variations in the plant for $[t_1, t_1+T]$. A trajectory that may be realized if no parameter variation happens for $[t_0, t_f]$ describes a different path from the trajectory determined by Eq. (3) after t_1.

The dynamic sensitivity coefficients can be obtained from the following variational equation[5]:

$$\frac{d}{dt}\left(\frac{\partial\mathbf{x}}{\partial\boldsymbol{\alpha}^1}\right)^D_{\boldsymbol{\alpha}^0}=\left(\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right)_{\boldsymbol{\alpha}^0}\left(\frac{\partial\mathbf{x}}{\partial\boldsymbol{\alpha}^1}\right)^D_{\boldsymbol{\alpha}^0}+\left(\frac{\partial\mathbf{f}}{\partial\boldsymbol{\alpha}^1}\right)^D_{\boldsymbol{\alpha}^0} \qquad (4).$$

† In the following the optimum symbol * is omitted.

Here the dynamic sensitivity coefficients,

$$\left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\alpha}^1}\right)_{\alpha^0}^D = \begin{bmatrix} \frac{\partial x_1}{\partial \alpha_1^1} & \cdots & \frac{\partial x_1}{\partial \alpha_p^1} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial \alpha_1} & \cdots & \frac{\partial x_n}{\partial \alpha_p^1} \end{bmatrix}_{\alpha^0}^D \tag{5}$$

are an $n \times p$ matrix and the upper suffix D, is used to indicate the case where the plant parameters change while the controller parameters are held constant.

In Eq. (4) coefficient matrices are

$$(\partial \mathbf{f}/\partial \mathbf{x})_{\alpha^0} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\alpha^0} \tag{6}$$

and

$$\left(\frac{\partial \mathbf{f}}{\partial \boldsymbol{\alpha}^1}\right)_{\alpha^0} = \begin{bmatrix} \frac{\partial f_1}{\partial \alpha_1^1} & \cdots & \frac{\partial f_1}{\partial \alpha_p^1} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial \alpha_1^1} & \cdots & \frac{\partial f_n}{\partial \alpha_p^1} \end{bmatrix}_{\alpha^0} \tag{7}.$$

Here Eq. (6) is an $n \times n$ matrix and Eq. (7) is an $n \times p$ matrix and their values are evaluated in the limit $(\boldsymbol{\alpha}^1 \to \boldsymbol{\alpha}^0)$, that is, along the optimal trajectory in the case of $\boldsymbol{\alpha} = \boldsymbol{\alpha}^0$ for $[t_0, t_f]$.

For the existance of all the elements in Eqs. (6) and (7), it is assumed that $\mathbf{f}(\mathbf{x}, \mathbf{u}, \boldsymbol{\alpha})$ is a differentiable function of the arguments in their defined domain. In the case where $\mathbf{u}$ is a closed loop control law $\mathbf{u}^*(\mathbf{x})$ of a Bang-Bang type, $\mathbf{f}$ does not satisfy this condition about $\mathbf{x}$. Therefore from Eq. (4), the dynamic sensitivity can not be obtained but in the case where the controller is of open loop Bang-Bang type, this may be possible. One example is shown in Appendix I.

As all the elements in Eqs. (6) and (7) are functions of time, Eq. (4) is a linear differential equation with variable coefficients and under its specified initial conditions, the solutions exist and can be determined by use of an electronic digital computer. If the parameter variations do not alter the order of the state transition equation, the initial values of Eq. (4) are all zero. When considering $\mathbf{x}(0)$ as parameters, and including them into $\boldsymbol{\alpha}$, we can make the initial values for $\partial x_i/\partial x_j(0)$ to be 1 in the case of $i=j$, and zero, in the case of $i \neq j$. If the order of the state transition equation is altered by parameter variations, the initial values of Eq. (4) must be calculated seperately[7].

Next we will try to consider the parameter sensitivity of trajectory variation between the case where the controller is changed from $\mathbf{u}(t, \alpha^0)$ to $\mathbf{u}(t, \alpha^1)$ instantly when the plant parameters change at t_1 and the case where no parameter variation occur in the plant and the optimum controller $\mathbf{u}(t, \alpha^0)$ is used successively. After t_1, the next two state transition equations may be considered :

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\{\mathbf{x}, \mathbf{u}^1(t, \alpha^1), \alpha^1\} \\ \dot{\mathbf{x}} &= \mathbf{f}\{\mathbf{x}, \mathbf{u}^0(t, \alpha^0), \alpha^0\} \end{aligned} \qquad (8)'.$$

A variational equation

$$\frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \alpha^1}\right)^I_{\alpha^0} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\alpha^0}\left(\frac{\partial \mathbf{x}}{\partial \alpha^1}\right)^I_{\alpha^0} + \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right)_{\alpha^0}\left(\frac{\partial \mathbf{u}}{\partial \alpha^1}\right)_{\alpha^0} + \left(\frac{\partial \mathbf{f}}{\partial \alpha^1}\right)_{\alpha^0} \qquad (8)$$

is established. These dynamic sensitivity $(\partial \mathbf{x}/\partial \alpha^1)^I_{\alpha^0}$ is concerned with the difference of the trajectories on which optimal control efforts have been continued. The supperscript I is used to denote this case. If the coefficient terms on the right side of Eq. (8) exist and are integrable, the solution of Eq. (8) can be obtained in the same way as Eq. (4). These sensitivity coefficients will be utilized in section 6. The dynamic sensitivity corresponding to any other time interval $[t_j, t_j+T]$ can be treated in the same way as in this section.

5. Plant Identification Using Dynamic Sensitivity Coefficients $(\partial \mathbf{x}/\partial \alpha)^D$

The mathematical procedure that estimates parameter variation $\Delta\alpha$ by using the dynamic sensitivity coefficients and by measuring the deviation $\Delta\mathbf{x}$ of the trajectory from the nominal one is called an inverse problem of the sensitivity for the dynamic system[8]. If the assumption is made that the same number of state variables as the plant parameters be observable, the following simple method of estimation to $\Delta\alpha$ can be formulated. Another method for a more general situation is shown in Appendix II.

If at some measured time instants $t_1+\tau_1$ and $t_1+\tau_2$,— though $t_1+\tau_1$ and $t_1+\tau_2$ are measured quantities, but t_1 is as yet not known, so τ_1 and τ_2 are unknown parameters, but their difference is assumed known—the measuring instrument of trajectory in mechanism A can observe the trajectory deviations $\Delta\mathbf{x}(\tau_1)$ and $\Delta\mathbf{x}(\tau_2)$ which are larger than the instrument sensitivity, the following relations

$$\left(\frac{\partial \mathbf{x}}{\partial \alpha^1}\right)^D_{\substack{\alpha^0 \\ t=\tau_1}} \cdot \Delta\alpha^0 = \Delta\mathbf{x}(\tau_1) \qquad (9)$$

and

$$\left(\frac{\partial \mathbf{x}}{\partial \alpha^1}\right)^D_{\substack{\alpha^0 \\ t=\tau_2}} \cdot \Delta\alpha^0 = \Delta\mathbf{x}(\tau_2) \qquad (10)$$

are established. The difference between τ_1 and τ_2 is

$$\tau_2 - \tau_1 = h_1 \qquad (11)$$

Eqs. (9) and (10) are composed of n linear equations with respect to $\Delta\alpha^0 = (\alpha^1 - \alpha^0) = (\Delta\alpha_1^0, \Delta\alpha_2^0, \cdots, \Delta\alpha_n^0)$. Using $n+1$ equations such as Eqs. (9) and (11) and one equation taken arbitrarily from Eq. (10), we can determine all the elements of

$\Delta\boldsymbol{\alpha}^0$ and τ_1, τ_2.

As $(\partial\mathbf{x}/\partial\boldsymbol{\alpha}^1)^D_{\alpha^0,\tau_1}$ is an $n\times n$ square matrix and, hence, if it is assumed not to be singular, $\Delta\boldsymbol{\alpha}^0$ is determined by Eq. (9) such as

$$\Delta\boldsymbol{\alpha}^0=\left(\frac{\partial\mathbf{x}}{\partial\boldsymbol{\alpha}^1}\right)^{D-1}_{\substack{\alpha^0\\ \tau_1}}\cdot\Delta\mathbf{x}(\tau_1) \qquad (12).$$

An equation may be taken from Eq. (10), for example.

$$\sum_{i=1}^{n}\left(\frac{\partial x_1}{\partial\alpha_i^1}\right)^{D}_{\substack{\alpha_i^0\\ \tau_2}}\cdot\Delta\alpha_i^0=\Delta x_1(\tau_2) \qquad (13).$$

Substituting Eqs. (11) and (12) into Eq. (13) :

$$\sum_{i=1}^{n}\left(\frac{\partial x_1}{\partial\alpha_i^1}\right)^{D}_{\substack{\alpha_i^0\\ \tau_1+h_1}}\cdot\left(\sum_{j=1}^{n}d_{ij}(\tau_1)\Delta x_j((\tau_1))\right)=\Delta x_1(\tau_1+h_1) \qquad (14).$$

Here $d_{ij}(\tau_1)$ are elements in ith law and jth column of $(\partial\mathbf{x}/\partial\boldsymbol{\alpha}^1)^{D-1}_{\alpha^0,\tau_1}$.

As Eq. (14) is an equation of τ_1, if its one significant root can be determined by use of a digital computer, all the components of $\Delta\boldsymbol{\alpha}^0$ can be obtained from it using Eq. (12) and the plant will again be in an identified state. As the time instant $t_1+\tau_1$ has already been measured, τ_1 also determines the time t_1 and then the trajectory for $[t_1, t_1+T]$ and the point $\mathbf{x}(t_1+T)$ can be obtained from Eq. (3).

Thus all the necessary conditions have been given that enable the computer to calculate the optimum-control $\mathbf{u}^1(t)$ for the plant after t_1+T, the dynamic sensitivity coefficients and the coefficients of performance index sensitivity that will be stated in the next section. At t_1+T, $\mathbf{u}^0$ will be changed to $\mathbf{u}^1$. For other time intervals $[t_2, t_2+T]$, ...$[t_k, t_k+T]$, the same method as stated in this section may be applied and all the necessary factors to readjust the optimum-controller at the time can be obtained.

6. Deviation ΔQ of Performance Index (PI)

Concerning the deviation ΔQ, many kinds of deviations may be suggested and one example is shown in Fig. 7. This figure is a projection of the trajectory on

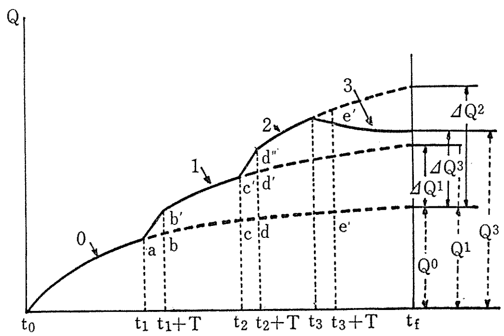

FIG. 7. Deviation of performance index

$Q-t$ plane, where the trajectory is drawn in $n+2$ dimentional space constructed by n state vectors, Q and t. $Q\Delta$ in Fig. 7 is a deviation from Q^0 that is a value of PI in the situation having no disturbances for $[t_0, t_f]$. The deviation may be considered at time t_f and Q^0 can be expressed by

$$Q^0 = \int_{t_0}^{t_f} F\{\mathbf{x}(t, \alpha^0), \mathbf{u}(t, \alpha^0)\}dt = Q^0(\alpha^0, \alpha^0, \mathbf{x}^0_{t_0})|_{t_0}^{t_f} \qquad (15).$$

In the last expression of Eq. (15), parameters α^i, α^j written in order designate plant-parameters by the first one and controller-parameters by the second (this conventional expression will be used through the present article). $\mathbf{x}^0_{t_0}$ shows a state-space position corresponding to the lower integration limit of PI. Upper suffix 0 of $\mathbf{x}^0_{t_0}$ shows that $\mathbf{x}^0_{t_0}$ is on the trajectory No. 0 as shown in Fig. 7.

No. 0 trajectory corresponds to the case where no disturbances occur for $[t_0, t_f]$ and trajectory No. 1 deviates from No. 0 by disturbances at t_1 and is compensated optimally at t_1+T and is not disturbed after then. A branch point of two trajectories is assumed to be on the smaller numbered trajectory. Upper suffix of Q indicates the frequency of disturbances for $[t_0, t_f]$.

6.1. A Case of One Disturbance (at t_1) for $[t_0, t_f]$

As the situations differ for three time intervals $[t_0, t_1]$, $[t_1, t_1+T]$, $[t_1+T, t_f]$, the integration of PI may be divided by the three parts as follows :

$$\begin{aligned} Q^1 &= \int_{t_0}^{t_1} F\{\mathbf{x}(t, \alpha^0, \alpha^0), \ \mathbf{u}(t, \alpha^0)\}\,dt + \int_{t_1}^{t_1+T} F\{\mathbf{x}(t, \alpha^1, \alpha^0), \ \mathbf{u}(t, \alpha^0)\}dt \\ &\quad + \int_{t_1+T}^{t_f} F\{\mathbf{x}(t, \alpha^1, \alpha^1), \ \mathbf{u}(t, \alpha^1)\}dt \\ &= Q^1(\alpha^0, \alpha^0, \mathbf{x}^0_{t_0})|_{t_0}^{t_1} + Q^1(\alpha^1, \alpha^0, \mathbf{x}^0_{t_1})|_{t_1}^{t_1+T} + Q^1(\alpha^1, \alpha^1, \mathbf{x}^1_{t_1+T})|_{t_1+T}^{t_f} \end{aligned} \qquad (16).$$

Thus,

$$\begin{aligned} \Delta Q^1 = Q^1 - Q^0 &= [Q^1(\alpha^1, \alpha^0, \mathbf{x}^0_{t_1}) - Q^0(\alpha^0, \alpha^0, \mathbf{x}^0_{t_1})]_{t_1}^{t_1+T} \\ &\quad + [Q^1(\alpha^1, \alpha^1, \mathbf{x}^1_{t_1+T}) - Q^0(\alpha^0, \alpha^0, \mathbf{x}^0_{t_1+T})]_{t_1+T}^{t_f} \end{aligned} \qquad (17).$$

Here $\mathbf{x}^0_{t_1}$, $\mathbf{x}^0_{t_1+T}$ and $\mathbf{x}^1_{t_1+T}$ are the point a, b and b' in Fig. 7.

The first difference of PI in Eq. (17) corresponds to the situations that the plant parameters change and nevertheless the same controller is being used, while the second differnce corresponds to the situations that the controller is readjusted optimally at the different phase points $\mathbf{x}^1_{t_1+T}$ and $\mathbf{x}^0_{t_1+T}$, adapting to the plant parameter variation. Therefore approximating each term in Eq. (17) by the first order term of Taylor's series expansion to obtain its numerical value, upper suffix D and I will be given to the expansion coefficients $\partial Q/\partial \alpha$ for the same reason as explained in section 4. In the following treatment, $< \ >$ expresses inner product of the vector quantity.

Thus ΔQ^1 in Eq. (17) can be expressed as :

$$\Delta Q^1 = \left\langle\left(\frac{\partial Q}{\partial \alpha^1}\right)^D_{\alpha^1=\alpha^0} \cdot (\alpha^1 - \alpha^0)\right\rangle \Big|_{t_1}^{t_1+T} + \Big[\left\langle\left(\frac{\partial Q}{\partial \alpha^1}\right)^I_{\alpha^0} \cdot (\alpha^1 - \alpha^0)\right. + \left\langle\left(\frac{\partial Q}{\partial \mathbf{x}^1_{t_1+T}}\right)^I_{\mathbf{x}^1_{t_1+T}=\mathbf{x}^0_{t_1+T}} \cdot (\mathbf{x}^1_{T_1+t} - \mathbf{x}^0_{t_1+T})\right\rangle\Big]^{t_f}_{t_1+T} \quad (18).$$

The relations between PI sensitivity coefficients and dynamic sensitivity coefficients are

$$\left(\frac{\partial Q}{\partial \alpha^1}\right)^D_{\alpha^0} = \frac{\partial}{\partial \alpha^1}\int_{t_0}^{t_f} F(\mathbf{x},\ \mathbf{u})dt = \int_{t_0}^{t_f}\left(\frac{\partial F}{\partial \mathbf{x}}\right)_{\alpha^0} \cdot \left(\frac{\partial \mathbf{x}}{\partial \alpha^1}\right)^D_{\alpha^0} dt \quad (19)$$

and

$$\left(\frac{\partial Q}{\partial \alpha^1}\right)^I_{\alpha^0} = \frac{\partial}{\partial \alpha^1}\int_{t_0}^{t_f} F(\mathbf{x},\ \mathbf{u})dt = \int_{t_0}^{t_f}\left\{\left(\frac{\partial F}{\partial \mathbf{x}}\right)_{\alpha^0}\left(\frac{\partial \mathbf{x}}{\partial \alpha^1}\right)^I_{\alpha^0} + \left(\frac{\partial F}{\partial \mathbf{u}}\right)_{\alpha^0}\left(\frac{\partial \mathbf{u}}{\partial \alpha^1}\right)_{\alpha^0}\right\}dt \quad (20).$$

As the dynamic sensitivity coefficients are given by Eqs. (4) or (8), and $\partial \mathbf{u}/\partial \alpha$ can be determined because $\mathbf{u}$ is known, and the quantities in Eq. (19) or (20) can be calculated. Of course, the integrand functions in Eqs. (19) and (20) are assumed to be integrable. The magnitudes of the quantities in Eqs. (19) and (20) are often nearly equal as shown in Appendix III.

6. 2. A Case of Two Disturbances (at t_1, t_2) for $[t_0,\ t_f]$

Q^2 may be written by

$$Q^2 = Q^2(\alpha^0,\ \alpha^0,\ \mathbf{x}^0_{t_0})\big|_{t_0}^{t_1} + Q^2(\alpha^1,\ \alpha^0,\ \mathbf{x}^0_{t_1})\big|_{t_1}^{t_1+T} + Q^2(\alpha^1,\ \alpha^1,\ \mathbf{x}^1_{t_1+T})\big|_{t_1+T}^{t_2} + Q^2(\alpha^2,\ \alpha^1,\ \mathbf{x}^1_{t_2})\big|_{t_2}^{t_2+T} + Q^2(\alpha^2,\ \alpha^2,\ \mathbf{x}^1_{t_2+T})\big|_{t_2+T}^{t_f} \quad (21)$$

and thus ΔQ^2 can be expressed by

$$\begin{aligned}\Delta Q^2 = Q^2 - Q^0 &= [Q^2(\alpha^1,\ \alpha^0,\ \mathbf{x}^0_{t_1}) - Q^0(\alpha^0,\ \alpha^0,\ \mathbf{x}^0_{t_1})]_{t_1}^{t_1+T} \\ &+ [Q^2(\alpha^1,\ \alpha^1,\ \mathbf{x}^1_{t_1+T}) - Q^0(\alpha^0,\ \alpha^0,\ \mathbf{x}^0_{t_1+T})]_{t_1+T}^{t_2} \\ &+ [Q^2(\alpha^2,\ \alpha^1,\ \mathbf{x}^1_{t_2}) - Q^0(\alpha^0,\ \alpha^0,\ \mathbf{x}^0_{t_2})]_{t_2}^{t_2+T} \\ &+ [Q^2(\alpha^2,\ \alpha^2,\ \mathbf{x}^1_{t_2+T}) - Q^0(\alpha^0,\ \alpha^0,\ \mathbf{x}^0_{t_2+T})]_{t_2+T}^{t_f} \quad (22).\end{aligned}$$

Here $\mathbf{x}^0_t$, $\mathbf{x}^0_{t+T}$, $\mathbf{x}^1_{t+T}$, $\mathbf{x}^0_{t_2}$, $\mathbf{x}^1_{t_2}$, $\mathbf{x}^0_{t_2+T}$ and $\mathbf{x}^1_{t_2+T}$ are the points a, b, b′, c, c′, d and d′ in Fig. 7. In comparing Eq. (22) with Eq. (17), another expression of ΔQ^2 may be obtained as follows :

$$\begin{aligned}\Delta Q^2 = \Delta Q^1 &- [Q^1(\alpha^1,\ \alpha^1,\ \mathbf{x}^1_{t_1+T}) - Q^0(\alpha^0,\ \alpha^0,\ \mathbf{x}^0_{t_1+T})]_{t_2}^{t_f} \\ &+ [\text{the 3rd and 4th term of Eq. (22)}] \quad (23).\end{aligned}$$

The second term of Eq. (23) is subtracted from the total because it was added excessively when alternating Eq. (22) to Eq. (23). Therefore it is conventionally called an "extra term $R(Q^1,\ Q^0)\big|_{t_2}^{t_f}$". It can be expanded only by the quantities already known as ;

$$R(Q^1,\ Q^0)\,|_{t_2}^{t_f} = [Q^1(\alpha^1,\ \alpha^1,\ x^1_{t_1+T}) - Q^0(\alpha^0,\ \alpha^0,\ x^0_{t_1+T})]_{t_2}^{t_f}$$
$$= [Q^1(\alpha^1,\ \alpha^1,\ x^1_{t_2}) - Q^0(\alpha^0,\ \alpha^0,\ x^0_{t_2})]_{t_2}^{t_f}$$
$$= \left\{\left\langle\left(\frac{\partial Q}{\partial \alpha^1}\right)^I_{\alpha^0}\cdot(\alpha^1-\alpha^0)\right\rangle + \left\langle\left(\frac{\partial Q}{\partial x^I_{t_2}}\right)^1_{x^0_{t_2}}\cdot(x^1_{t_2}-x^0_{t_2})\right\rangle\right\}_{t_2}^{t_f} \qquad (24).$$

Extra terms that will emerge in section 6. 3 and others can be treated as stated here.

The third term in Eq. (22) can be expanded as follows :

The third term

$$= [Q^2(\alpha^2,\ \alpha^1,\ x^1_{t_2}) - Q^0(\alpha^0,\ \alpha^0,\ x^0_{t_2})]_{t_2}^{t_2+T}$$
$$= [Q^2(\alpha^2,\ \alpha^1,\ x^1_{t_2}) - Q^1(\alpha^1,\ \alpha^1,\ x^1_{t_2})]_{t_2}^{t_2+T} + [Q^1(\alpha^1,\ \alpha^1,\ x^1_{t_2}) - Q^0(\alpha^0,\ \alpha^0,\ x^0_{t_2})]_{t_2}^{t_2+T}$$
$$= \left\langle\left(\frac{\partial Q}{\partial \alpha^2}\right)^D_{\alpha^1}\cdot(\alpha^2-\alpha^1)\right\rangle\Big|_{t_2}^{t_2+T} + \left\{\left\langle\left(\frac{\partial Q}{\partial \alpha^1}\right)^I_{\alpha^0}\cdot(\alpha^1-\alpha^0)\right\rangle + \left\langle\left(\frac{\partial Q}{\partial x^1_{t_2}}\right)^I_{x^0_{t_2}}\cdot(x^1_{t_2}-x^0_{t_2})\right\rangle\right\}_{t_2}^{t_2+T} \qquad (25).$$

The fourth term in Eq. (22) can be expanded like the second term in Eq. (17). Thus ΔQ^2 can be evaluated by use of the series expansion of all known quantities, such as PI sensitivity coefficients, variations of plant parameters and of the trajectories. That is

$$\Delta Q^2 = \Delta Q^1 - \{\text{Eq. (24)}\} + \{\text{Eq. (25)}\} + \left\{\left\langle\left(\frac{\partial Q}{\partial \alpha^2}\right)^I_{\alpha^0}\cdot(\alpha^2-\alpha^0)\right\rangle + \left\langle\left(\frac{\partial Q}{\partial x^1_{t_2+T}}\right)^I_{x^0_{t_2+T}}\cdot(x^1_{t_2+T}-x^0_{t_2+T})\right\rangle\right\}_{t_2+T}^{t_f} \qquad (26).$$

6. 3. *A Case of Three Disturbances (at t_1, t_2, t_3) for $[t_0,\ t_f]$*

ΔQ^4 can be written like ΔQ^2 in Eq. (22), such as :

$$\begin{aligned}\Delta Q^3 = Q^3 - Q^0 = &[Q^3(\alpha^1,\ \alpha^0,\ x^0_{t_1}) - Q^0(\alpha^0,\ \alpha^0,\ x^0_{t_1})]_{t_1}^{t_1+T}\\ &+[Q^3(\alpha^1,\ \alpha^1,\ x^1_{t_1+T}) - Q^0(\alpha^0,\ \alpha^0,\ x^0_{t_1+T})]_{t_1+T}^{t_2}\\ &+[Q^3(\alpha^2,\ \alpha^1,\ x^1_{t_2}) - Q^0(\alpha^0,\ \alpha^0,\ x^0_{t_2})]_{t_2}^{t_2+T}\\ &+[Q^3(\alpha^2,\ \alpha^2,\ x^2_{t_2+T}) - Q^0(\alpha^0,\ \alpha^0,\ x^0_{t_2+T})]_{t_2+T}^{t_3}\\ &+[Q^3(\alpha^3,\ \alpha^2,\ x^2_{t_3}) - Q^0(\alpha^0,\ \alpha^0,\ x^0_{t_3})]_{t_3}^{t_3+T}\\ &+[Q^3(\alpha^3,\ \alpha^3,\ x^3_{t_3+T}) - Q^0(\alpha^0,\ \alpha^0,\ x^0_{t_3+T})]_{t_3+T}^{t_f}\end{aligned} \qquad (27),$$

where $x^2_{t_3}$, $x^0_{t_3+T}$ and $x^3_{t_3+T}$ are the points d″, e and e′ in Fig. 7.

The first four terms in Eq. (27) can be expressed by ΔQ^2 of Eq. (22) and extra term $R(Q^2,\ Q^0)\,|_{t_3}^{t_f}$. The fifth term can be expanded as Eq. (25). Therefore :

$$\Delta Q^3 = \Delta Q^2 - R(Q^2,Q^0)\,|_{t_3}^{t_f} + \left[\left\langle\left(\frac{\partial Q}{\partial \alpha^3}\right)^D_{\alpha^2}(\alpha^3-\alpha^2)\right\rangle + \left\langle\left(\frac{\partial Q}{\partial \alpha^2}\right)^I_{\alpha^0}\cdot(\alpha^2-\alpha^0)\right\rangle + \left\langle\left(\frac{\partial Q}{\partial x^2_{t_3}}\right)^I_{x^0_{t_3}}\cdot(x^2_{t_3}-x^0_{t_3})\right\rangle\right]_{t_3}^{t_3+T}$$

$$+\Big[\Big\langle\Big(\frac{\partial Q}{\partial \alpha^3}\Big)^I_{\alpha^0}\cdot(\alpha^3-\alpha^0)\Big\rangle+\Big\langle\Big(\frac{\partial Q}{\partial \mathrm{x}^3_{t_3+T}}\Big)^I_{\mathrm{x}^0_{t_3+T}}\cdot(\mathrm{x}^3_{t_3+T}-\mathrm{x}^0_{t_3+T})\Big\rangle\Big]^{t_f}_{t_3+T} \qquad (28).$$

6. 4. General Case of k Disturbances (at $t_1, t_2\cdots t_k$) for $[t_0, t_f]$

Thinking along the same line as to formulate Eq. (28), ΔQ^k may be expanded as follows :

$$\Delta Q^k=Q^k-Q^0=\Delta Q^{k-1}-R(Q^{k-1},\ Q^0)\,|^{t_f}_{t_k}+\Big[\Big\langle\Big(\frac{\partial Q}{\partial \alpha^k}\Big)^D_{\alpha^{k-1}}(\alpha^k-\alpha^{k-1})\Big\rangle$$
$$+\Big\langle\Big(\frac{\partial Q}{\partial \alpha^{k-1}}\Big)^I_{\alpha^0}\cdot(\alpha^{k-1}-\alpha^0)\Big\rangle+\Big\langle\Big(\frac{\partial Q}{\partial \mathrm{x}^{k-1}_{t_k}}\Big)^I_{\mathrm{x}^0_{t_k}}\cdot(\mathrm{x}^{k-1}_{t_k}-\mathrm{x}^0_{t_k})\Big\rangle\Big]^{t_k+T}_{t_k}$$
$$+\Big[\Big\langle\Big(\frac{\partial Q}{\partial \alpha^k}\Big)^I_{\alpha^0}\cdot(\alpha^k-\alpha^0)\Big\rangle+\Big\langle\Big(\frac{\partial Q}{\partial \mathrm{x}^k_{t_k+T}}\Big)^I_{\mathrm{x}^0_{t_k+T}}\cdot(\mathrm{x}^k_{t_k+T}-\mathrm{x}^0_{t+kT})\Big\rangle\Big]^{t_f}_{t_k+T} \qquad (29).$$

Thus ΔQ^k can be evaluated by ΔQ^{k-1}, extra term $R(Q^{k-1}, Q^0)|^{t_f}_{t_k}$ and the five Taylor's expansion terms. When kth disturbance occurs, ΔQ^{k-1} has already been known numerically, so the other terms from ΔQ^{k-1} are to be evaluated at the time.

After many disturbances, both the deviation of trajectory from No. 0 and the difference between α^k and α^0 may become so large that the validity of approximation with the first order terms of Taylor's series expansion becomes inaccurate. If a plant parameter compensation device that will be stated in the next section operates to maintain the deviation from No. 0 trajectory within some allowance, the difference between α^k and α^0 may also be kept small and such approximated procedure may be considerably available.

A different deviation of PI from the above stated one can be considered as follows. Let us consider a control such as divine work that optimally changes the control scheme instantly when disturbances occur at $t_1, t_2, \cdots, t_k$. Letting its PI be Q^k_I and the PI of the scheme to readjust the controller optimally refering to the divine work be Q^k, ΔQ^k may be defined by

$$\Delta Q^k=Q^k-Q^k_I \qquad (30).$$

As disturbances occur from some uncontrollable origin, the deviation from Q^k_I in which the best endeavor is always continued is conceived more reasonable than the deviation from Q^0. However the magnitude of Q^k_I is not known in advance, therefore it is not convenient in an optimalizing control synthesis point of view, but, if necessary, ΔQ^k in Eq. (30) can also be evaluated in the same manner as shown in this section.

7. Compensation of Plant Parameter Utilizing Q^j

We assume that some allowance ΔQ_R (absolute value) is given to ΔQ at t_f. If ΔQ^j evaluated after jth disturbance has a value within ΔQ_R, the control condition at the time end may be approved good and needs no compensation. However when ΔQ^j exceeds the allowable limit, the plant parameter must now be compensated according to the magnitude $|\Delta Q^j|-\Delta Q_R$. As the controller has already been readjusted optimally, the only remaining possibility is to compensate the plant parameters.

Fig. 8 shows its block diagram. The plant is fedback through the compensator.

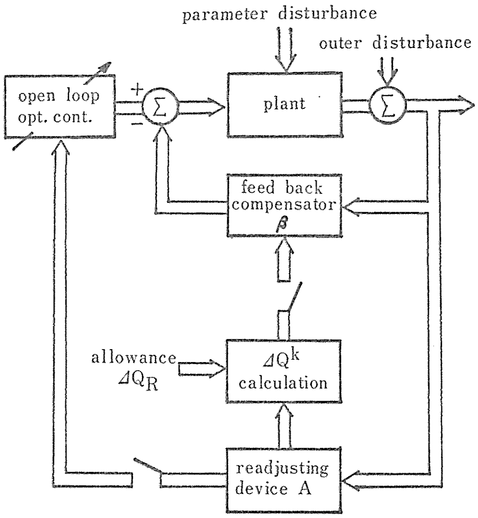

FIG. 8. Adaptive optimalizing control system

It is assumed to be capable of influencing all the plant parameters α independently by means of changing the compensator parameters β. It will be a problem studied henceforth in many points of view to find some adequate modes of changing β. But as the change in β may be treated by the mechanism A in the same manner as the plant disturbances, its magnitude and frequency should be as small as possible.

Let us consider one example of the compensating scheme. Assume, $|\Delta Q^{j-1}| < \Delta Q_R$, and $|\Delta Q^j| > \Delta Q_R$. At a time t_{j+1} when the calculation of duty after t_j+T (this is summarized in section 8) will have been finished, some plant parameter change $\Delta\alpha^j$ for the compensation will be given to the plant (practically to the compensator, but for the present, not considering the magnitude of $\Delta\beta$). The value $\Delta\alpha^j$ may be determined in such a manner as to make $(\Delta Q^{j+1})'$ equal to ΔQ_R. Here, $(\Delta Q^{j+1})'$ would be a value of PI variation if the controller were not readjusted at $t_{j+1}+T$ but successively used to the final time t_f as it was, in spite of the parameter change $\Delta\alpha^j$. In practice, the mechanism A will readjust the controller automatically and optimally at $t_{j+1}+T$, so $|\Delta Q^{j+1}| < |(\Delta Q^{j+1})'|$ and $|\Delta Q^{j+1}|$ will be less than ΔQ_R. That is an allowable situation.

$(\Delta Q^{j+1})'$ can be expanded as

$$(\Delta Q^{j+1})' = \Delta Q^j - R(Q^j,\ Q^0)\,|_{t_{j+1}}^{t_f} + \left\langle \left(\frac{\partial Q}{\partial \alpha^{j+1}}\right)_{\alpha^j}^{D} (\alpha^{j+1} - \alpha^j) \right\rangle |_{t_{j+1}}^{t_f} = \Delta Q_R\,.$$

Therefore, letting

$$\left\langle \left(\frac{\partial Q}{\partial \alpha^{j+1}}\right)_{\alpha^j}^{D} \cdot \Delta\alpha^j \right\rangle |_{t_{j+1}}^{t_f} = \Delta Q_R - \Delta Q^j + R(Q^j,\ Q^0)\,|_{t_{j+1}}^{t_f} = E_R^j \qquad (31),$$

E_R^j is a known quantity after t_j+T.

As there are many ways to determine $\Delta\alpha^j$ so as to satisfy Eq. (31), the following is one example. Calling n elements of $\left(\frac{\partial Q}{\partial \alpha^{j+1}}\right)^D_{\alpha^j} \Big|^{t_f}_{t_{j+1}}$, $q_1, q_2, \ldots, q_n$ in order of absolute magnitude, we write each element of $\Delta\alpha^j$ as $\Delta\alpha_1^j, \Delta\alpha_2^j, \ldots, \Delta\alpha_n^j$ corresponding to q_i $(i=1, 2, \ldots, n)$.

First assuming that $\Delta\alpha_1^j \neq 0$ and $\Delta\alpha_2^j = \Delta\alpha_3^j = \cdots = \Delta\alpha_n^j = 0$ and equating $q_1 \cdot \Delta\alpha_1^j = E_R^j$, we obtain

$$\Delta\alpha_1^j = E_R^j / q_1 \tag{32}$$

Now let us assume that there is such an allowable common limit ρ to the absolute magnitude $\Delta\alpha_i^j$ $(i=1,2,\ldots,n)$ as to maintain the Taylor's series expansion of the first order terms in good approximation. If Eq. (32) results in $|\Delta\alpha_1^j| > \rho$, let

$$\Delta\alpha_1^j = \mathrm{sgn}(E_R^j/q_1) \cdot \rho \quad \text{(sgn denotes the sign function)} \tag{33}$$

and let $\Delta\alpha_2^j \neq 0$. Then from Eqs. (31) and (33), the following relation:

$$q_1 \cdot \Delta\alpha_1^j + q_2 \cdot \Delta\alpha_2^j = q_1 \cdot \mathrm{sgn}(E_R^j/q_1)\rho + q_2 \cdot \Delta\alpha_2^j = E_R^j,$$

is established, but as $\rho>0$, sgn $\Delta\alpha_1^j = \mathrm{sgn}(E_R^j/q_1)$ in Eq. (33), and thus

$$\Delta\alpha_2^j = \frac{E_R^j - q_1 \cdot \mathrm{sgn}(\Delta\alpha_1^j)\,\rho}{q_2} \tag{34}$$

If once again $|\Delta\alpha_2^j| > \rho$ results, $\Delta\alpha_2^j$ must also take a value as

$$\Delta\alpha_2^j = \mathrm{sgn}\left[\frac{E_R^j - q_1 \cdot \mathrm{sgn}(\Delta\alpha_1^j)\,\rho}{q_2}\right] \cdot \rho \tag{35}$$

and letting $\Delta\alpha_3^j \neq 0$, $\Delta\alpha_3^j$ must be determined like the preceeding treatment.

Generally in the case where $|\Delta\alpha_1^j| = |\Delta\alpha_2^j| = \cdots = |\Delta\alpha_{k-1}^j| = \rho$, $\Delta\alpha_k^j \neq 0$ and it is required to determine $\Delta\alpha_k^j$, from Eqs. (31), (33) and (35) etc., it follows that

$$q_1 \cdot \mathrm{sgn}(\Delta\alpha_1^j)\,\rho + q_2 \cdot \mathrm{sgn}(\Delta\alpha_2^j)\,\rho + \cdots + q_k \cdot \Delta\alpha_k^j = E_R^j$$

Therefore

$$\Delta\alpha_k^j = \frac{E_R^j - \{q_1 \cdot \mathrm{sgn}(\Delta\alpha_1^j) + q_2 \cdot \mathrm{sgn}(\Delta\alpha_2^j) + \cdots + q_{k-1} \cdot \mathrm{sgn}(\Delta\alpha_{k-1}^j)\}\,\rho}{q_k} \tag{36}$$

When $|\Delta\alpha_k^j| < \rho$, this calculating process ends, and the other parameter changes are not necessary, that is, $\Delta\alpha_{k+1}^j = \Delta\alpha_{k+2}^j = \cdots = \Delta\alpha_n^j = 0$.

As a given composition of feedback compensator determines the relations between α and β, the conversion from $\Delta\alpha$ to $\Delta\beta$ can be done according to the relation.

In Fig. 8 two samplers are inserted so as to indicate that the devices on the circuit of each sampler do not operate continually but operate only when necessary.

8. Conclusion

According to two different kinds of time interval, the jobs of the electronic computer may be arranged as follows: (1) $[t_{j-1}+T, t_j]$; calculating new optimum-trajectory $x(t)$, dynamic sensitivity coefficients $(\partial x/\partial \alpha^j)^D_{\alpha^{j-1}}$, $(\partial x/\partial \alpha^j)^I_{\alpha^{j-1}}$ and PI sensitivity coefficients $(\partial Q/\partial \alpha^{j-1})^D_{\alpha^0}$, $(\partial Q/\partial \alpha^{j-1})^I_{\alpha^0}$ and storing them in memory. Next using PI sensitivity coefficients, calculating the deviation ΔQ^{j-1} of PI, and comparing it with ΔQ_R and, when finding it out of the allowance, calculating $\Delta\alpha_1^{j-1}$, $\Delta\alpha_2^{j-1}$, . . . $\Delta\alpha_k^{j-1}$, as explained in the preceeding section. (2) $[t_j, t_j+T]$; using measured quantities $x(\tau)$ and the calculated values $(\partial x/\partial \alpha^j)^D_{\alpha^{j-1}}$ in (1), calculating $\Delta\alpha$ and the trajectory for $[t_j, t_j+T]$ and the phase point $x(t_j+T)$ and then obtaining the optimum control u^j that is scheduled after t_j+T.

The former time interval is long enough to compute all the needed factors, but as the latter one is required as small as possible, special high speed computers are necesitated. It has recently been reported that several compartively simple optimum-control problems can be solved by Pontryagin's method and their optimum-trajectories can be drawn on a $X-Y$ recorder in 1 or 2 minutes by use of a hybrid combination of an analog and a digital computer[9]. Considering about these circumstances it is just adequate to our scheme to handle such an optimum-problem that the time interval $[t_0, t_f]$ is of the order of the day and $(t_{j+1}-t_j)$ is of the order of the hour and T is a few minutes.

Our control scheme makes the direct feedback shown by the dotted line in Fig. 5 unnecessary, but as the detail has been given in this report, all the optimalizing control tasks consist of computer computation. This takes more time than the direct feedback mechanism, so if it is insured that only outer disturbances occur, the direct feedback devices may be suited much better than our scheme. But in this case it must be remembered that a closed loop type optimum-controller often has a very complex structure and can not always be realized in practice.

Lastly let us consider a scheme that optimally determines the control and the plant parameters under their restrictions at a time during the interval T for readjustment. To obtain the solutions for this problem under severe restrictions is much more complex than to obtain the optimum-control only. Though formal studies have already begun[10], the required time for the computer to solve this problem has not yet been reported. If the T is much prolonged, so severe compensation of the plant parameters may be required that there might not be an adequate optimal solution to maintain the deviation of PI, ΔQ within the allowance ΔQ_R under the given restrictions. The parameter compensation in this report never aims to the optimal compensation but merely endeavors to keep ΔQ within ΔQ_R.

APPENDIX I

Let us consider a time optimal problem in which a point mass reaches a circle of radius R centered at the orgin of the state space from some specified initial point in the shortest possible time (T) and also calculate the sensitivity of T with respect to the mass (m) variation. The state transition equation, constraint on control signal and the initial point are as follows:

$$\left.\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \frac{1}{m}u\end{aligned}\right\}|u|\leqq M,\ x_1(t_0)=x_1^0,\ x_2(t_0)=x_2^0 \quad (37).$$

Switching Time of Control Signal

To the state variables, x_1, x_2 in Eq. (37), let the auxiliary variables be p_1, p_2. By use of the Pontryagin's method, the optimum-control and the auxiliary variables are given by

$$u^*(t)=M\cdot\text{sgn}\ p_2 \quad (38)$$

and

$$p_1=c_1\ \ (\text{constant}),\qquad p_2=-c_1t+c_2\ \ (c_2\text{: constant}) \quad (39).$$

Therefore the switching of control signal happens at most once and letting the switching time be t_c, it is given from Eqs. (38) and (39) such as

$$t_c=c_2/c_1 \quad (40).$$

Transversality Condition

Let an angle α be as shown in Fig. 9. Then the terminal point (x_1^T, x_2^T) can be expressed as

$$x_1^T=R\cdot\cos\ \alpha,\qquad x_2^T=R\ \sin\ \alpha \quad (41).$$

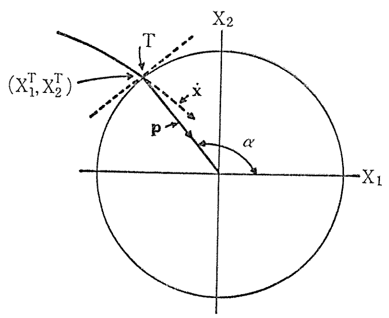

FIG. 9. Situation at the terminal point

Using a positive number ρ, the transversality condition requires the next relations:

$$p_1(T)=-\rho\ \cos\ \alpha,\qquad p_2(T)=-\rho\ \sin\ \alpha \quad (42).$$

From Eqs. (39), and (42)

$$c_1=-\rho\ \cos\ \alpha,\qquad c_2=-\rho\ (\sin\ \alpha+T\ \cos\ \alpha) \quad (43).$$

Therefore from Eq. (40), we obtain

$$T-t_c=-\tan\ \alpha \quad (44).$$

As $T-t_c\geqq 0$ in Eq. (44), α is an angle of the second or fourth quadrant. So the switching happens only when the terminal point lies on the arc in the second or fourth quadrant.

Switching Curve

In Fig. 10, the trajectories (parabola) terminating on the arc in the second quadrant are shown. The figure shows that the trajectories drawn between the parabolas passing through the points $(-1, 0)$ and $(0, 1)$ intersect the switching curve. Let us determine the curve by use of inversed time solutions of Eq. (37). The other switching curve sits at the symmetric position with respect to the origin.

Letting $t=-\tau$ in Eq. (37) and using the fact that the magnitude of u at the

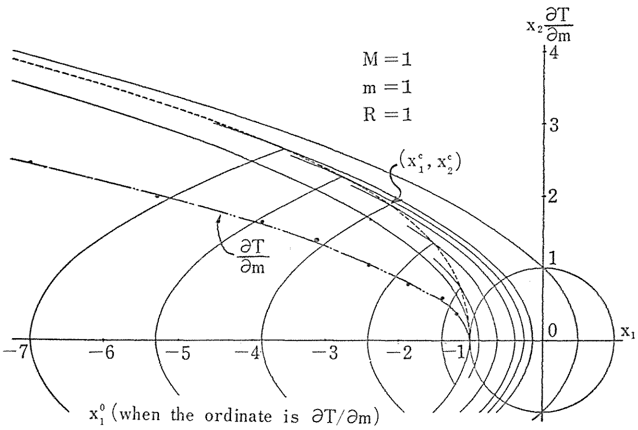

FIG. 10. Optimum-trajectories, switching line and values of PI sensitivity coefficient.

terminal is $-M$, Eq. (37) changes to

$$\left.\begin{aligned}\dot{x}_1 &= -x_2 \\ \dot{x}_2 &= \frac{1}{m}M\end{aligned}\right\} \quad \tau = 0(t = T), \text{ initial point} = \text{Eq. (41)} \tag{45}$$

Therefore the solutions are as follows:

$$x_1 = -\frac{1}{2}\frac{M}{m}\tau^2 - x_2^T\tau + x_1^T \tag{46}$$

$$x_2 = \frac{M}{m}\tau + x_2^T \tag{47}$$

Eq. (47) shows that the difference between ordinates expresses M/m times of elapsed duration. As switching point is reached when $\tau = -\tan\alpha$ as shown in Eq. (44), letting its coordinates be x_1^c, x_2^c, from Eqs. (46) and (47), it follows that

$$x_1^c = -\frac{M}{2m}(\tan\alpha)^2 + R\sin\alpha + R\cos\alpha \tag{48}$$

$$x_2^c = -\frac{M}{m}\tan\alpha + R\sin\alpha \tag{49}$$

Eqs. (48) and (49) are a parameter (α) expression of switching curve and when $m = M = R = 1$, α can be comparatively easily eliminated and

$$x_2^c = \sqrt{1 - \left[\frac{1 + \sqrt{2(1 - x_1^c)}}{2x_1^c - 1}\right]^2 \cdot \left\{1 - \frac{2x_1^c - 1}{1 + \sqrt{2(1 - x_1^c)}}\right\}} \tag{50}$$

is obtained. When we want to obtain the switching curve graphically, points whose ordinates are $x_2^c = \frac{M}{m}(-\tan\alpha) + x_2^T$ may be measured on the trajectory passing through a terminal point (x_1^T, x_2^T). The broken line in Fig. 10 is the switching curve thus obtained and well agrees the one obtained from Eq. (50). If the initial points are in the domain between the broken line and the parabola passing through the point (1, 0) (this parabola is not drawn in Fig. 10 and also in its symmetrical domain with respect to the orgin, the point mass can reach the circle in the shortest possible time without switching.

Sensitivity of T with Respect to m

For simplisity, assume that the initial points lie on the negative x axis outside the terminal circle, that is, their coordinates are $(x_1^0, 0)$. Let unit step function be $U(t)$, then the optimum-control of open loop type can be written as

$$u(t)=MU(t)-2MU(t-t_c) \qquad (51),$$

and as $x_2^0=0$, from Eq. (37)

$$t_c = x_2^c/(M/m) \qquad (52).$$

The variational equation such as Eq. (4) is

$$\frac{d}{dt}\begin{pmatrix} \frac{\partial x_1}{\partial m} \\ \frac{\partial x_2}{\partial m} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial x_1}{\partial m} \\ \frac{\partial x_2}{\partial m} \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{M}{m^2}[U(t)-2U(t-t_c)] \end{pmatrix} \qquad (53),$$

where initial values are zero. Therefore,

$$\frac{\partial x_1}{\partial m} = -\frac{M}{m^2}\left[\frac{1}{2}t^2-(t-t_c)^2\right] \qquad (54)$$

and

$$\frac{\partial x_2}{\partial m} = -\frac{M}{m^2}[t-2(t-t_c)] \qquad (55).$$

As the coordinates of intersection between the trajectory starting from the point $(x_1^0, 0)$ and the switching curve are obtained by using Eq. (50) or graphically, t_c can be determined for each initial point $(x_1^0, 0)$ from Eq. (52). Therefore Eqs. (54) and (55) become known functions of time

Next let the equation of the terminal circle be

$$g(x_1, x_2) = x_1^2+x_2^2-R^2=0 \qquad (56)$$

and the trajectory with parameter m (changed from m_0) be

$$x_1=x_1(t, m), \quad x_2=x_2(t, m) \qquad (57),$$

then the trajectory intersects the circle of Eq. (56) at T (this is T_0 when $m=m_0$). Therefore,

$$g[x_1(T, m), x_2(T, m)]=\{x_1(T, m)\}^2+\{x_2(T, m)\}^2-R^2=0. \qquad (58)$$

Thus,

$$\left(\frac{\partial T}{\partial m}\right)_{m_0} = -\frac{\left[\frac{\partial g}{\partial x_1}\ \frac{\partial x_1}{\partial m} + \frac{\partial g}{\partial x_2}\ \frac{\partial x_2}{\partial m}\right]_{\substack{T=T_0\\m=m_0}}}{\left[\frac{\partial g}{\partial x_1}\ \frac{\partial x_1}{\partial t} + \frac{\partial g}{\partial x_2}\ \frac{\partial x_2}{\partial t}\right]_{\substack{T=T_0\\m=m_0}}} \tag{59}$$

Substituting the values of Eqs. (37), (54) and (55) at T_0 into the Eq. (59), we get

$$\left(\frac{\partial T}{\partial m}\right)_{m_0} = \frac{\frac{M}{m^2}\left[x_1^{T_0}\left\{\frac{1}{2}T_0^2 - (T_0 - t_c)^2\right\} + x_2^{T_0}\{T_0 - 2(T_0 - t_c)\}\right]}{x_2^{T_0}\left[x_1^{T_0} - \frac{M}{m}\right]} \tag{60}$$

where the magnitude T_0 can be also obtained graphically by the elapsed ordinate length. The chain line in Fig. 10 is the result obtained from Eq. (60) for each initial point $(x_1^0, 0)$ in the case of $m=M=R=I$.

APPENDIX II

Along the same way as in section 5, let us consider the situation for the time interval $[t_1, t_1+T]$. There only two observed values $\Delta \mathrm{x}(\tau_1)$, and $\Delta \mathrm{x}(\tau_2)$ were utilized. Now assume that the observed values can continuously be utilized for this interval and let the observed values be $\Delta \mathrm{x}^m(\tau)$. This is a n vector.

As a model of deviation $\Delta \mathrm{x}^c(\tau)$ of trajectory composed of dynamic sensitivity coefficients $(\partial \mathrm{x}/\partial \alpha^1)^D_{\alpha^0}$ and parameter variation $\Delta \boldsymbol{\alpha}^0$ (a p vector), the following linear one may be considered,

$$\Delta x^c(\tau) = \left(\frac{\partial x}{\partial \boldsymbol{\alpha}^1}\right)^D_{\boldsymbol{\alpha}^0} \cdot \Delta \alpha^0 \tag{61}$$

and p elements of $\Delta \boldsymbol{\alpha}^0$ can be determined so as to minimize the next integral value I.

$$I = \int_{\tau_1}^{\tau_2} \sum_{i=1}^{n} (\Delta x_i^m(\tau) + \Delta x_i^c(\tau))^2 dt \tag{62}$$

Substituting each component of Eq. (61) into Eq. (62), we obtain

$$I = \int_{\tau_1}^{\tau_2} \sum_{i=1}^{n} \left[\Delta x_i^m(\tau) + \sum_{j=1}^{p} \frac{\partial x_i}{\partial \alpha_j}(\tau)\cdot\Delta\alpha_j\right]^2 dt \tag{63}$$

where $(\partial x_i/\partial \alpha_j^1)^D_{\alpha_j^0}$ is written as $\partial x_i/\partial \alpha_j$ and $\Delta\alpha_j^0$ as $\Delta\alpha_i$.

As I has a minimum with respect to $\Delta\alpha_j$:

$$\begin{aligned}
\frac{\partial I}{\partial \Delta\alpha_1} &= 2\int_{\tau_1}^{\tau_2}\sum_{i=1}^{n}\left[\left\{\Delta x_i^m(\tau) + \sum_{j=1}^{p}\frac{\partial x_i}{\partial \alpha_j}(\tau)\cdot\Delta\alpha_j\right\}\frac{\partial x_i}{\partial \alpha_1}(\tau)\right]d\tau = 0\\
&\vdots\\
\frac{\partial I}{\partial \Delta\alpha_k} &= 2\int_{\tau_1}^{\tau_2}\sum_{i=1}^{n}\left[\left\{\Delta x_i^m(\tau) + \sum_{i=1}^{p}\frac{\partial x_i}{\partial \alpha_j}(\tau)\cdot\Delta\alpha_j\right\}\frac{\partial x_i}{\partial \alpha_k}(\tau)\right]d\tau = 0\\
&\vdots\\
\frac{\partial I}{\partial \Delta\alpha_p} &= 2\int_{\tau_1}^{\tau_2}\sum_{i=1}^{n}\left[\left\{\Delta x_i^m(\tau) + \sum_{j=1}^{p}\frac{\partial x_i}{\partial \alpha_j}(\tau)\cdot\Delta\alpha_j\right\}\frac{\partial x_i}{\partial \alpha_p}(\tau)\right]d\tau = 0
\end{aligned} \tag{64}$$

If in Eq. (64), the following integral values are considered,

$$-\int_{\tau_1}^{\tau_2}\sum_{i=1}^{n}\left[\Delta x_i^m(\tau)\frac{\partial x_i}{\partial \alpha_k}(\tau)\right]d\tau = b_k \quad (k=1,\cdots,p) \tag{65}$$

and

$$\int_{\tau_1}^{\tau_2}\sum_{i=1}^{n}\left[\frac{\partial x_i}{\partial \alpha_k}(\tau)\frac{\partial x_i}{\partial \alpha_j}(\tau)\right]d\tau = C_{kj} \quad \begin{matrix}(j=1,\cdots,p)\\(k=1,\cdots,p)\end{matrix} \tag{66}$$

b_k $(k=1,\cdots,p)$ and C_{kj} (k and j, $1,\cdots,p$) can be determined by use of electronic computer as the integrand of Eqs. (65) and (66) are known functions of time.

Therefore Eq. (64) can be converted into the next p linear equations with respect to $\Delta\alpha_j$ $(j=1,\cdots,p)$.

$$\begin{matrix} C_{11}\Delta\alpha_1 + C_{12}\Delta\alpha_2 + \cdots + C_{1p}\Delta\alpha_p = b_1 \\ \vdots \\ C_{k1}\Delta\alpha_1 + C_{k2}\Delta\alpha_2 + \cdots + C_{kp}\Delta\alpha_p = b_k \\ \vdots \\ C_{p1}\Delta\alpha_1 + C_{p2}\Delta\alpha_2 + \cdots + C_{pp}\Delta\alpha_p = b_p \end{matrix} \tag{67}$$

Let the coefficient matrix in Eq. (67) be $\mathbf{c}$, then Eq. (67) can be expressed by

$$\mathbf{c}\cdot\Delta\boldsymbol{\alpha} = \mathbf{b} \tag{68}$$

where $b=(b_1, b_2, \cdots, b_p)'$, $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_p)'$ ($'$denotes transpose of a row vector). As $\mathbf{c}$ is a $p\times p$ square matrix, if it is not singular, $\Delta\alpha$ can be obtained as

$$\Delta\boldsymbol{\alpha} = \mathbf{c}^{-1}\cdot\mathbf{b} \tag{69}$$

In the above method, it is noteworthy that the number of component of Δx or $\Delta\alpha$ is arbitrarily n or p.

APPENDIX III

In the section 6, $Q(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^0)$ is the PI when the plant parameter changes from $\boldsymbol{\alpha}^0$ to $\boldsymbol{\alpha}^1$ but the controller parameter $\boldsymbol{\alpha}^0$ does not change and $Q(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^1)$ is the PI when the controller parameter also changes to $\boldsymbol{\alpha}^1$. Then the sensitivity coefficients in Eqs. (19) and (20) are the series expansion coefficients of the first variation of the following total variations.

$$\Delta Q^D = Q(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^0) - Q(\boldsymbol{\alpha}^0, \boldsymbol{\alpha}^0) \tag{70}$$

and

$$\Delta Q^I = Q(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^1) - Q(\boldsymbol{\alpha}^0, \boldsymbol{\alpha}^0) \tag{71}$$

Eq. (71) can be rearranged as

$$\Delta Q^I = [Q(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^1) - Q(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^0)] + [Q(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^0) - Q(\boldsymbol{\alpha}^0, \boldsymbol{\alpha}^0)] = \Delta Q^C + \Delta Q^D \tag{72}$$

Therefore in comparing Eq. (19) with Eq. (20), the variation

$$\varDelta Q^C = Q(\alpha^1, \alpha^1) - Q(\alpha^1, \alpha^0) = -[Q(\alpha^1, \alpha^0) - Q(\alpha^1, \alpha^1)]$$

must be investigated. $-\varDelta Q^c$ expresses the variation of PI from optimal $Q(\alpha^1, \alpha^1)$ to nonoptimal $Q(\alpha^1, \alpha^0)$ in which the controller is not optimal.

Generally in optimal control problems the magnitude of control signal is often restricted to some finite values and when the optimum-control takes the value of restriction, it is not perfectly arbitrary to change its magnitude. But when some values within the restriction are always taken, the optimum-control is in the minimal or maximal situation with respect to arbitrary small control variation in both the magnitude and the time behavior. In these cases the first variation of $\varDelta Q^c$ is zero and the difference between $\varDelta Q^I$ and $\varDelta Q^D$ in Eq. (72) is equal to the second or higher variation of $\varDelta Q^C$. Therefore the values of sensitivity coefficients in Eps. (19) and (20) are considered to be equal. But in the former stated situation, that is when the values of restriction are always used, the stationary condition is not always established with respect to arbitrary magnitude variation of the cotrol. Therefore, in general, the first variation of $\varDelta Q^c$ is not zero, and the difference between Eqs. (19) and (20) must be considered.

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