# ON THE JET MIXING WHICH CONSISTS OF LAMINAR AND TURBULENT REGIONS

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**Summary**: Flows of the two-dimensional and the circular jet are analysed, where they are laminar in the upstream region and are turbulent in the downstream region. The method is based on the extension of Görtler's theory. The similar solutions are obtained in the respective cases.

## 1. Introduction

The problems of mixing in free boundary flows, such as free jet boundaries, jets and wakes, have been dealt with by many investigators, in either case where the flows are laminar or tubulent everywhere[1) 2)]. As we know, however, there still remains unsolved the case where the flows are laminar in the upstream region, whereas turbulent in the downstream region.

The preceding paper by the author[3)] offered an attempt for attacking the last case in a free jet boundary, though the method might be practical rather than theoretical. The fundamental concept of the theory is essentially an extension of Görtler's method[4)] for the fully developed turbulent flow into the laminar-turbulent combined flow.

Generally speaking, the shearing stress in the laminar flow is due to molecular viscosity and it is expressed by the equation

$$\tau_l = \rho\nu \frac{\partial u}{\partial y} \tag{1}$$

where $\rho$ denotes the density, $\nu$ the kinematic viscosity, $u$ the time mean velocity in the $x$-direction, $x$ and $y$-axes being taken to be parallel and perpendicular to the main flow respectively. On the other hand, in the turbulent flow there arises another kind of shearing stress caused by turbulent fluctuation which is expressed analogously to Eq. (1) by use of the virtual kinematic viscosity $\varepsilon$ in place of $\nu$, as

$$\tau_t = \rho\varepsilon \frac{\partial u}{\partial y}\,. \tag{2}$$

If we assume that the shearing stress is given by Eq. (1) alone in the upstream region and is given by sum of Eqs. (1) and (2) in the downstream region, we can write down the unified expression for the shearing stress $\tau$ which is applicable to the upstream laminar region as well as to the downstream region, as

$$\tau = \tau_l + \tau_t \cdot \mathscr{U}(x-a) = \rho\{\nu + \varepsilon \cdot \mathscr{U}(x-a)\}\frac{\partial u}{\partial y}. \tag{3}$$

Here, the unit function

$$\mathscr{U}(x-a) = \begin{cases} 0 & \text{if} \quad x<a \\ 1 & \text{if} \quad x>a \end{cases} \tag{4}$$

is introduced, and $a$ denotes the $x$-coordinate of the point of transition from laminar to turbulent flow, which should be determined experimentally.

On the basis of the assumption of Eq. (3), together with some further assumptions, we previously developed an analysis of free jet boundary flow and the process of which can be successfully applied to jets or wakes in both two-dimensional and axially symmetrical (circular) cases, only with the slight alterations due to the differences among the flow patterns. The present paper deals with the cases of two-dimensional and circular jets.

## 2. Two-dimensional Jet

In the first place, we consider the problem of a two-dimensional jet issuing from a long narrow slit and mixing with the surrounding fluid at rest. For the calculational convenience, it is assumed that the width of the slit is infinitesimally small so that the fluid velocity at the exit becomes necessarily infinite in order to retain a finite volume and momentum flow rate. It is also assumed that the pressure remains constant everywhere, which is legitimately granted because of the boundary layer nature of jet. The $x$-axis is taken to be coincident with the jet axis, with its origin lying in the slit.

The basic equations of two-dimensional jet in incompressible fluid are written as follows:

eqn. of continuity;
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{5}$$

eqn. of motion;
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial \tau}{\partial y}, \tag{6}$$

with the boundary conditions

$$\left.\begin{aligned} y=0; \qquad & \frac{\partial u}{\partial y}=0, \qquad v=0 \\ y=\pm\infty; \qquad & u=0, \qquad \frac{\partial u}{\partial y}=0, \end{aligned}\right\} \tag{7}$$

and the momentum condition

$$J = \rho\int_{-\infty}^{\infty} u^2 dy = \text{const.} \tag{8}$$

where $v$ denotes the $y$-component of the mean velocity and $J$ is the total momentum in the $x$-direction.

In order to integrate Eq. (6), it is necessary to find some functional dependence of the virtual kinematic viscosity $\varepsilon$ on the parameters of the main flow. In his analysis of the turbulent flow, Görtler used Prandtl's simpler equation of

$$\varepsilon = \kappa_1 b(u_{max} - u_{min}) \tag{9}$$

where $b$ denotes the width of the mixing region, and $\kappa_1$ is a nondimensional constant and subscripts of $u$ stand for the maximum and the minimum values in the cross-section considered. According to Prandtl, the width of the two-dimensional turbulent jet increases linearly with the distance from the exit, and the center-line velocity $u_{max}$ decreases inversely proportional to the square root of the distance. Of course, $u_{min}$ vanishes in the present case.
Consequently Eq. (9) becomes

$$\varepsilon = \kappa_1 b u_{max} \propto x \cdot x^{-1/2}$$

or

$$\varepsilon = \kappa x^{1/2} \tag{10}$$

where $\kappa$ is an empirical dimensional constant different from $\kappa_1$.

In the case under consideration, however, the above equations may fail to apply because of the presence of the upstream laminar region preceding to the turbulent region. In fact, $b$ and $u_{max}$ are no more proportional to $x$ and $x^{-1/2}$ respectively. Nevertheless, we assume Eq. (10) still holds, only with $x$ replaced by $x-a$ in consideration that $\varepsilon$ should vanish at $x=a$, so that

$$\varepsilon = \kappa(x-a)^{1/2}. \tag{10'}$$

Putting Eq. (10′) into Eq. (3), we obtain

$$\tau = \rho\{\nu + \kappa(x-a)^{1/2} \cdot \mathscr{U}(x-a)\}\frac{\partial u}{\partial y}. \tag{11}$$

When $x \gg a$, the second term in the bracket generally prevails over the first term, so Eq. (11) tends to

$$\tau \approx \rho\kappa x^{1/2}\frac{\partial u}{\partial y} = \tau_t,$$

which means that the virtual kinematic viscosity becomes substantially coinci-

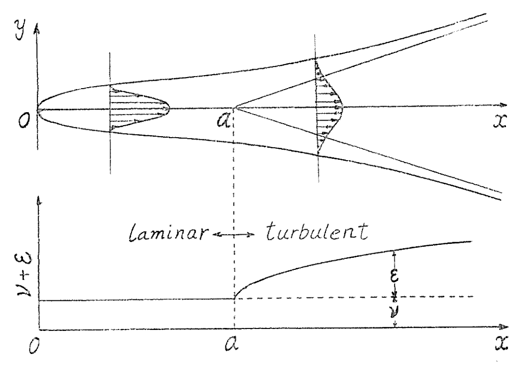


FIG. 1. Flow battern of a two-dimensional jet and variation of the virtual kinematic viscosity.

dent with that of Görtler far downstream. This fact offers some plausibility, though not any thoretical justification at all, for use of Eq. (11). The variation of the virtual kinematic viscosity with the distance is shown schematically in Fig. 1, as well as the sketch of the flow pattern.

Using Eq. (11), the equation of motion (6) becomes

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\{\nu+\kappa(x-a)^{1/2}\cdot\mathscr{U}(x-a)\}\frac{\partial^2 u}{\partial y^2}. \tag{12}$$

Here, we introduce a stream function

$$\Psi=G(x)f(\eta) \quad \text{with} \quad \eta=\frac{y}{Y(x)} \tag{13}$$

from which

$$\left.\begin{aligned} u&=\frac{\partial\Psi}{\partial y}=\frac{G(x)}{Y(x)}f'(\eta)\\ v&=-\frac{\partial\Psi}{\partial x}=-G'(x)f(\eta)+\frac{G(x)Y'(x)}{Y(x)}\eta f'(\eta) \end{aligned}\right\} \tag{14}$$

where the prime denotes the differentiation with respect to each argument in the parenthesis.

Putting Eq. (14) into Eq. (8), we obtain

$$J=\rho\frac{G(x)^2}{Y(x)}\int_{-\infty}^{\infty}f'(\eta)^2 d\eta=\text{const.} \tag{15}$$

Hence we have

$$\frac{G(x)^2}{Y(x)}=\text{const.}=\frac{1}{c} \tag{16}$$

or

$$Y(x)=cG(x)^2. \tag{16'}$$

Consequently, Eq. (14) becomes

$$\left.\begin{aligned} u&=\frac{1}{cG(x)}f'(\eta)\\ v&=-G'(x)\{f(\eta)-2\eta f'(\eta)\}. \end{aligned}\right\} \tag{17}$$

Substituting Eq. (17) into the basic equations, we see that the equation of continuity (5) is satisfied automatically, and the equation of motion (6) reduces to

$$f'''(\eta)+\frac{cG(x)^2G'(x)}{\nu+\kappa(x-a)^{1/2}\cdot\mathscr{U}(x-a)}\{f(\eta)f''(\eta)+f'(\eta)^2\}=0 \tag{18}$$

with the boundary conditions

$$\left.\begin{aligned} \eta&=0; & f&=0, & f'&=1\\ \eta&=\infty; & f'&=0, & f''&=0. \end{aligned}\right\} \tag{19}$$

If we may put the coefficient in front of the bracket into any constant number, say

$$\frac{cG(x)^2G'(x)}{\nu+\kappa(x-a)^{1/2}\cdot\mathscr{U}(x-a)}=2, \tag{20}$$

Eq. (18) becomes the following ordinary differential equation with respect to $\eta$ alone,

$$f''+2(ff''+f'^2)=0 \tag{21}$$

with the boundary conditions of Eq. (19). Eq. (21) is exactly the same equation as those for the laminar case derived by Schlichting and Bickley and also for the turbulent case derived by Görtler, of which solution is known to be

$$f=\tanh\eta. \tag{22}$$

Now Eq. (20) can be easily integrated into

$$G=\left[\frac{1}{c}\{6\,\nu x+4\,\kappa(x-a)^{3/2}\cdot\mathscr{U}(x-a)\}\right]^{1/3} \tag{23}$$

so that the existence of the similar solution, Eq. (22), is assured. In order to determine the constant $c$, Eq. (22) is introduced into the momentum condition (15), whence we have

$$\begin{aligned}\frac{1}{c}=\frac{G^2}{Y}&=\frac{J}{\rho}\left(\int_{-\infty}^{\infty}f'^2d\eta\right)^{-1}\\&=\frac{J}{\rho}\left[\int_{-\infty}^{\infty}(1-\tanh^2\eta)^2d\eta\right]^{-1}=\frac{J}{\rho}\left\{\left[\tanh\eta-\frac{1}{3}\tanh^3\eta\right]_{-\infty}^{\infty}\right\}^{-1}\\&=\frac{J}{\rho}\left(\frac{4}{3}\right)^{-1}=\frac{3}{4}\frac{J}{\rho}.\end{aligned} \tag{24}$$

Substituting this into Eq. (23), we obtain

$$G=\left(\frac{J}{\rho}\right)^{1/3}\left\{\frac{9}{2}\nu x+3\kappa(x-a)^{3/2}\cdot\mathscr{U}(x-a)\right\}^{1/3}. \tag{25}$$

Finally, from Eqs. (16′), (17), (24), (25) and with $K=\dfrac{J}{\rho}$ (kinematic momentum), we have the systam of the solutions

$$\left.\begin{aligned}u&=\frac{3}{4}K^{2/3}\left\{\frac{9}{2}\nu x+3\,\kappa(x-a)^{3/2}\cdot\mathscr{U}(x-a)\right\}^{-1/3}(1-\tanh^2\eta),\\v&=-\frac{3}{2}K^{1/3}\left\{\frac{9}{2}\nu x+3\,\kappa(x-a)^{3/2}\cdot\mathscr{U}(x-a)\right\}^{-2/3}\{\nu+\kappa(x-a)^{1/2}\cdot\mathscr{U}(x-a)\}\\&\qquad\times[\tanh\eta-2\,\eta(1-\tanh^2\eta)],\end{aligned}\right\} \tag{26}$$

the center-line velocity

$$u_{\max}(x)=u_{\eta=0}(x)=\frac{3}{4}K^{2/3}\left\{\frac{9}{2}\nu x+3\kappa(x-a)^{3/2}\cdot\mathscr{U}(x-a)\right\}^{-1/3}, \tag{27}$$

and the width of the jet

$$Y(x)=\frac{4}{3}K^{-1/3}\left\{\frac{9}{2}\nu x+3\kappa(x-a)^{3/2}\cdot\mathscr{U}(x-a)\right\}^{2/3}. \tag{28}$$

In order to obtain good insight into the physical aspects of the flow, it is convenient to rewrite Eqs. (26)-(28) in dimensionless form as follows,

$$\left.\begin{aligned}
\frac{u}{u_{\max\,x=a}}&=\frac{u_{\max}(x)}{u_{\max}(a)}(1-\tanh^2\eta)=\left\{\frac{x}{a}+\frac{2}{3}\frac{\kappa a^{1/2}}{\nu}\left(\frac{x}{a}-1\right)^{3/2}\cdot\mathscr{U}\left(\frac{x}{a}-1\right)\right\}^{-1/3}\\
&\qquad\times(1-\tanh^2\eta),\\
\frac{v}{u_{\max\,x=a}}&=-\left(\frac{16}{9}\right)^{1/3}\left(\frac{\nu^2}{Ka}\right)^{1/3}\left\{\frac{x}{a}+\frac{2}{3}\frac{\kappa a^{1/2}}{\nu}\left(\frac{x}{a}-1\right)^{3/2}\cdot\mathscr{U}\left(\frac{x}{a}-1\right)\right\}^{-2/3}\\
&\qquad\times\left\{1+\frac{\kappa a^{1/2}}{\nu}\left(\frac{x}{a}-1\right)^{1/2}\cdot\mathscr{U}\left(\frac{x}{a}-1\right)\right\}\{\tanh\eta-2\eta(1-\tanh^2\eta)\},
\end{aligned}\right\} \tag{29}$$

$$\frac{u_{\max}}{u_{\max\,x=a}}=\left\{\frac{x}{a}+\frac{2}{3}\frac{\kappa a^{1/2}}{\nu}\left(\frac{x}{a}-1\right)^{3/2}\cdot\mathscr{U}\left(\frac{x}{a}-1\right)\right\}^{-1/3}, \tag{30}$$

$$\frac{Y}{a}=48^{1/3}\left(\frac{\nu^2}{Ka}\right)^{1/3}\left\{\frac{x}{a}+\frac{2}{3}\frac{\kappa a^{1/2}}{\nu}\left(\frac{x}{a}-1\right)^{3/2}\cdot\mathscr{U}\left(\frac{x}{a}-1\right)\right\}^{2/3}. \tag{31}$$

It is seen from the above equations that $u$ is dependent on one parameter $\frac{\kappa a^{1/2}}{\nu}$, whereas $v$ and $Y$ depend on the second parameter $\frac{\nu^2}{Ka}$ besides $\frac{\kappa a^{1/2}}{\nu}$. The variations of the center-line velocity $u_{\max}$ and the width of the jet $Y$ with the distance from the exit are shown plotted in Figs. 2 and 3 respectively. Fig. 4 contains the plots of the distributions of the mean velocity components from Eq. (29), as well as those of the circular jet from Eq. (57) given later.

The transverse velocity at $\eta=\infty$ is

$$\frac{v_\infty}{u_{\max}}=-\left(\frac{16}{9}\right)^{1/3}\left(\frac{\nu^2}{Ka}\right)^{1/3}\left\{\frac{x}{a}+\frac{2}{3}\frac{\kappa a^{1/2}}{\nu}\left(\frac{x}{a}-1\right)^{3/2}\cdot\mathscr{U}\left(\frac{x}{a}-1\right)\right\}^{-2/3}$$

$$\times\left\{1+\frac{\kappa a^{1/2}}{\nu}\left(\frac{x}{a}-1\right)^{1/2}\cdot\mathscr{U}\left(\frac{x}{a}-1\right)\right\}, \tag{32}$$

and the volume rate of discharge per unit height of slit becomes

$$Q=\int_{-\infty}^{\infty}u\,dy,\qquad \text{or}$$

$$Q=G\int_{-\infty}^{\infty}f'd\eta=G[\tanh\eta]_{-\infty}^{\infty}=2G$$

$$=2K^{1/3}\left\{\frac{9}{2}\nu x+3\kappa(x-a)^{3/2}\cdot\mathscr{U}(x-a)\right\}^{1/3}. \tag{33}$$

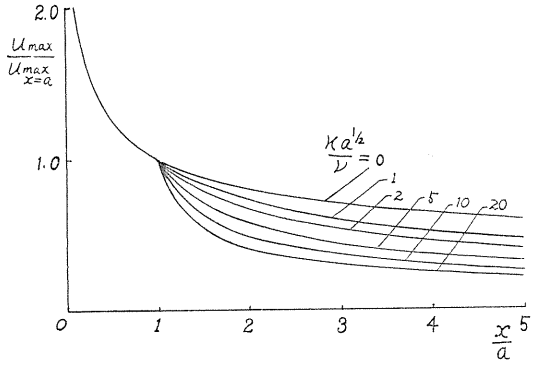


FIG. 2. Variation of the center-line velocity in a two-dimensional jet. Eq. (30).

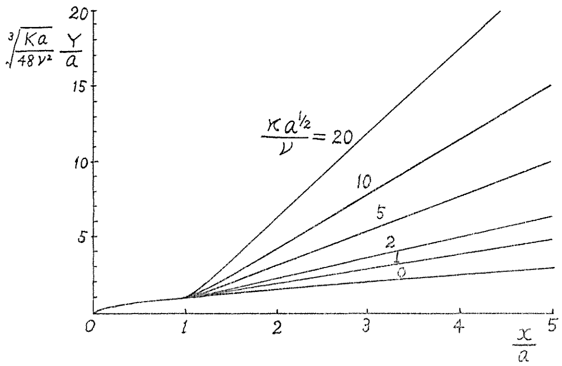


FIG. 3. Variation of the width of a two-dimensional jet along the jet axis. Eq. (31).

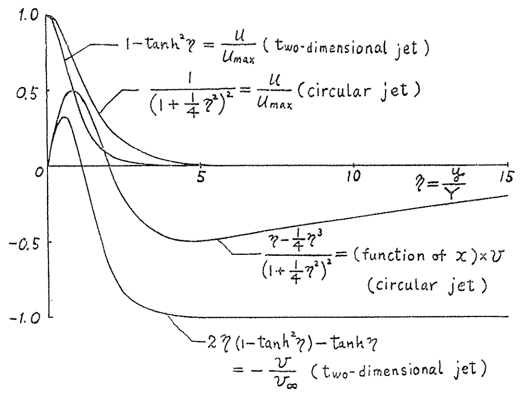


FIG. 4. Velocity distributions in a two-dimensional and circular jet from Eqs. (29) and (57) respectively.

## 3. Circular Jet

The problem of the circular jet issuing from infinitesimally small circular hole into the fluid at rest, is carried out in analogous way to the two-dimensional case. The cylindrical coordinate system is taken with its $x$-axis in the axis of jet and $y$-axis in the radial direction, with their mean velocity components denoted by $u$ and $v$ respectively.

Then we have the basic equations of circular jet in incompressible fluid

eqn. of continuity;
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{v}{y}=0, \tag{34}$$

eqn. of motion;
$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{\rho}\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial \tau}{\partial y}\right), \tag{35}$$

with the boundary conditions

$$\left.\begin{aligned} y=0\ ; &\quad \frac{\partial u}{\partial y}=0, \quad v=0 \\ y=\infty\ ; &\quad u=0, \quad \frac{\partial u}{\partial y}=0, \end{aligned}\right\} \tag{36}$$

and the momentum condition

$$J=2\pi\rho\int_0^\infty u^2 y\,dy=\text{const.} \tag{37}$$

In the case of the fully developed turbulent circular jet, Prandtl's evaluation shows that the width of the jet $b$ is directly proportional and the center-line velocity $u_{\max}$ is inversely proportional to the distance $x$ respectively so that, from Eq. (9), $\varepsilon$ reduces to some constant $\varepsilon_0$.

If this assumption is admitted to hold in the present case also, Eq. (3) results in

$$\tau=\rho\{\nu+\varepsilon_0\cdot\mathscr{U}(x-a)\}\frac{\partial u}{\partial y}, \tag{38}$$

and hence, Eq. (35) becomes

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\{\nu+\varepsilon_0\cdot\mathscr{U}(x-a)\}\left(\frac{\partial^2 u}{\partial y^2}+\frac{1}{y}\frac{\partial u}{\partial y}\right). \tag{39}$$

The variation of $\nu$ and $\varepsilon$ is shown Fig. 5.

Again introducing the stream function

$$\Psi=G(x)f(\eta) \qquad \text{with} \qquad \eta=\frac{y}{Y(x)}. \tag{40}$$

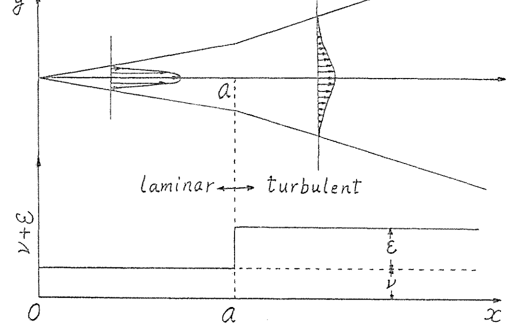


FIG. 5. Flow pattern of a circular jet and variation of the virtual kinematic viscosity.

we have

$$\left.\begin{aligned} u &= \frac{1}{y}\frac{\partial \Psi}{\partial y} = \frac{G(x)}{Y(x)^2}\frac{f'(\eta)}{\eta}. \\ v &= -\frac{1}{y}\frac{\partial \Psi}{\partial x} = -\frac{G'(x)}{Y(x)}\frac{f(\eta)}{\eta} + \frac{Y'(x)G(x)}{Y(x)^2}f'(\eta) \end{aligned}\right\} \tag{41}$$

which satisfy the equation of continuity (34) automatically. Inserting the first equation of (41) into the momentum condition (37), we obtain

$$J = 2\pi\rho\frac{G(x)^2}{Y(x)^2}\int_0^\infty \frac{f'(\eta)^2}{\eta}d\eta = \text{const.} \tag{42}$$

Thus

$$\frac{G(x)^2}{Y(x)^2} = \text{const.} = c^2 \tag{43}$$

or

$$G(x) = cY(x). \tag{43'}$$

Substituting this into Eq. (41), we obtain

$$\left.\begin{aligned} u &= \frac{c}{Y(x)}\frac{f'(\eta)}{\eta}. \\ v &= -\frac{cY'(x)}{Y(x)}\left\{\frac{f(\eta)}{\eta} - f'(\eta)\right\} \end{aligned}\right\} \tag{44}$$

which transform the equation of motion (39) into

$$f''' - \frac{f''}{\eta} + \frac{f'}{\eta^2} + \frac{cY'(x)}{\nu + \varepsilon_0 \cdot \mathscr{U}(x-a)}\left\{\frac{ff''}{\eta} + \frac{f'^2}{\eta} - \frac{ff'}{\eta^2}\right\} = 0. \tag{45}$$

together with the boundary conditions

$$\left.\begin{aligned} &\eta = 0\ ; \qquad f = 0, \qquad f' = 0, \qquad f'' = 0 \\ &\eta = \infty\ ; \qquad f' = \text{finite}. \end{aligned}\right\} \tag{46}$$

If we may put

$$\frac{cY'(x)}{\nu + \varepsilon_0 \mathscr{U} \cdot (x-a)} = \text{constant number, say} = 1, \tag{47}$$

Eq. (45) reduces to the following ordinary differential equation for $f(\eta)$

$$f''' - \frac{f''}{\eta} + \frac{f'}{\eta^2} + \left(\frac{ff''}{\eta} + \frac{f'^2}{\eta} - \frac{ff'}{\eta^2}\right) = 0. \tag{48}$$

As was the case in the two-dimensional jet, we obtain once more the identical equation with those for the laminar case derived by Schlichting and for the turbulent case by Görtler.

After integrating this three times, we obtain a particular solution

$$f=\frac{\eta^2}{1+\frac{1}{4}\eta^2} \tag{49}$$

which satisfies the boundary conditions (46).

Eq. (47) is integrated to give

$$Y=\frac{1}{c}\{\nu x+\varepsilon_0(x-a)\cdot\mathscr{U}(x-a)\} \tag{50}$$

where the continuity of $Y$ at $x=a$ is taken into account.

The scale factor $c$ can now be determined with the aid of the momentum condition (42), together with Eqs. (43) and (49). Thus we have

$$J=2\pi\rho c^2\int_0^\infty \frac{4\eta^2}{\left(1+\frac{1}{4}\eta^2\right)^4}\frac{d\eta}{\eta}$$

$$=2\pi\rho c^2\left(\frac{8}{3}\right)=\frac{16}{3}\pi\rho c^2$$

so that

$$c=\left(\frac{3}{16\pi}\frac{J}{\rho}\right)^{1/2}=\left(\frac{3}{16\pi}K\right)^{1/2}. \tag{51}$$

Insertion of this value into Eq. (50) gives

$$Y=\left(\frac{16}{3}\pi\right)^{1/2}K^{-1/2}\{\nu x+\varepsilon_0(x-a)\cdot\mathscr{U}(x-a\}. \tag{52}$$

From this and Eq. (51), we are led to the final form of the solution

$$\left.\begin{aligned}
u&=\frac{3\pi}{8}\frac{K}{\{\nu x+\varepsilon_0(x-a)\cdot\mathscr{U}(x-a)\}}\frac{1}{\left(1+\frac{1}{4}\eta^2\right)^2},\\
v&=\frac{1}{4}\sqrt{\frac{3}{\pi}}K^{1/2}\frac{\{\nu+\varepsilon_0\cdot\mathscr{U}(x-a)\}}{\{\nu x+\varepsilon_0(x-a)\cdot\mathscr{U}(x-a)\}}\frac{\left(\eta-\frac{1}{4}\eta^3\right)}{\left(1+\frac{1}{4}\eta^2\right)^2},\\
\eta&=\sqrt{\frac{3}{16\pi}}K^{1/2}\frac{y}{\{\nu x+\varepsilon_0(x-a)\cdot\mathscr{U}(x-a)\}}.
\end{aligned}\right\} \tag{53}$$

The center-line velocity is given by

$$u_{\max}=u_{\eta=0}=\frac{3}{8\pi}\frac{K}{\{\nu x+\varepsilon_0(x-a)\cdot\mathscr{U}(x-a)\}}. \tag{54}$$

If we rewrite the above equations in dimensionless form, we have

$$\frac{Y}{a}=\left(\frac{16}{3}\pi\right)^{1/2}(K^{-1/2}\nu)\left\{\frac{x}{a}+\frac{\varepsilon_0}{\nu}\left(\frac{x}{a}-1\right)\cdot\mathscr{U}\left(\frac{x}{a}-1\right)\right\}, \tag{55}$$

$$\frac{u_{\max}}{u_{\max\,x=a}}=\left\{\frac{x}{a}+\frac{\varepsilon_0}{\nu}\left(\frac{x}{a}-1\right)\cdot\mathscr{U}\left(\frac{x}{a}-1\right)\right\}^{-1}, \tag{56}$$

$$\left.\begin{aligned}\frac{u}{u_{\max\,x=a}}&=\left\{\frac{x}{a}+\frac{\varepsilon_0}{\nu}\left(\frac{x}{a}-1\right)\cdot\mathscr{U}\left(\frac{x}{a}-1\right)\right\}^{-1}\frac{1}{\left(1+\frac{1}{4}\eta^2\right)^2},\\ \frac{v}{u_{\max\,x=a}}&=\left(\frac{4}{3}\pi\right)^{1/2}(K^{-1/2}\nu)\frac{\left\{1+\frac{\varepsilon_0}{\nu}\cdot\mathscr{U}\left(\frac{x}{a}-1\right)\right\}}{\left\{\frac{x}{a}\frac{\varepsilon_0}{\nu}\left(\frac{x}{a}-1\right)\cdot\mathscr{U}\left(\frac{x}{a}-1\right)\right\}}\frac{\left(\eta-\frac{1}{4}\eta^3\right)}{\left(1+\frac{1}{4}\eta^2\right)^2}.\end{aligned}\right\} \tag{57}$$

Again, $u$ is dependent on only one parameter $\varepsilon_0/\nu$, whereas $v$ and $Y$ are dependent on two parameters $\varepsilon_0/\nu$ and $K^{-1/2}\nu$. The variations of $u_{\max}$ and $Y$ with $x$ are shown plotted in Figs. 6 and 7 respectively, and the distributions of the mean velocty components have been already plotted in Fig. 4.

The volume flow rate $Q=2\pi\int_0^\infty uydy$ is calculated to be

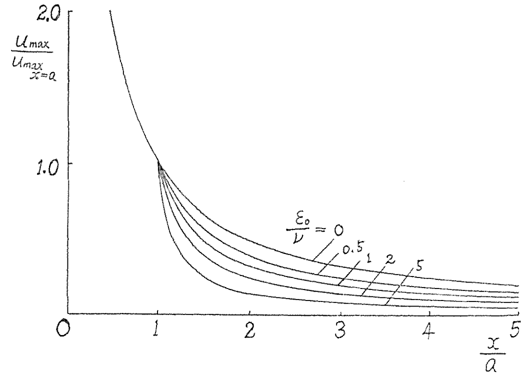


FIG. 6. Variation of the center-line velocity in a circular jet. Eq. (56).

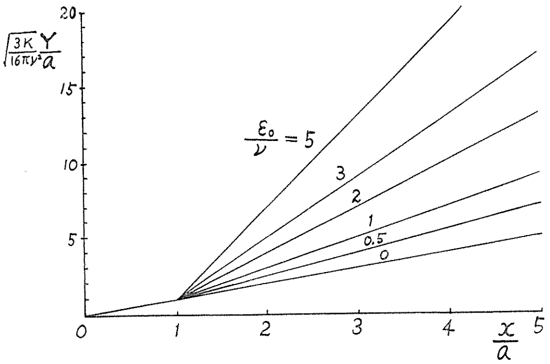


FIG. 7. Variation of the width of a circular jet along the jet axis. Eq. (55).

$$Q = 4\pi\nu\left\{x + \frac{\varepsilon_0}{\nu}(x-a)\cdot\mathscr{U}(x-a)\right\}\int_0^\infty \frac{\eta d\eta}{\left(1+\frac{1}{4}\eta^2\right)}$$

$$= 8\pi\nu\left\{x + \frac{\varepsilon_0}{\nu}(x-a)\cdot\mathscr{U}(x-a)\right\}. \tag{58}$$

## 4. Conclusion

The similar solutions were obtained in the laminar-turbulent combined jet both in the two-dimensional and in the axially symmetrical case. It is seen from Eqs. (26)-(28) and Eqs. (52)-(54) that, in the upstream region where $x<a$, they are exactly identical with those for the laminar jet, while in the far downstream region where $x \gg a$, they become eventually coincident with those for the fully developed turbulent jet as $x\to\infty$. This statement, however, does not at once provide any physical verification of the theory. It can only be said that we have found some mathematical means which make it possible to combine two similar solutions suitably at the point of transition.

In order to see whether the solution is the physically realistic one or not, some experiments would be required, where we may determine the unknown parameters in the theory, *i. e.* the distance of the point of transition $a$ and the intensity of the turbulence $\kappa$ or $\varepsilon_0$.

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