SOLUTIONS FOR SPHERICAL SHELLS UNDER CONCENTRATED FORCES AND MOMENTS

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(Received October 30, 1965)

1. Introduction

The problem of concentrated actions on a spherical shell has already been discussed by several authors. It appears, however, that most of the previous investigations are on the shallow spherical shell problems. For example, E. Reissner[1] has developed the detailed solutions for the shallow spherical cap under a concentrated radial force at its apex. A. Jahanshahi[2] has obtained the singular solutions for the case of radial as well as tangential concentrated forces applied to the shallow spherical shell.

A notable exception is Koiter's recent investigation on the complete spherical shell under normal point loads at its poles[3]. He showed that the Reissner's results for the shallow spherical cap may be applied in the vicinity of the poles of the complete spherical shell, and that the membrane theory[4] gives an appropriate solution at some distance from the poles. On the other hand, Leckie and Penny[5] have obtained the asymptotic solutions for the concentrated radial and tangential forces and for the concentrated moment, which are valid for all values of colatitude of the spherical shell. Their solutions are suitable for numerical calculations since they are expressed in terms of the modified Bessel functions which have already been tabulated.

In this paper the rigorus solutions are obtained for the following four loadings applied at the apex on the spherical shell;

(1) a concentrated radial force (Fig. 1 a),
(2) a concentrated moment about the polar axis (Fig. 1 b),
(3) a concentrated tangential force (Fig. 1 c),
(4) a concentrated moment about the axis given by $\phi=0$ and $\theta=90°$ (Fig. 1 d).

The solutions are derived from the shell theory based on the Kirchhoff-Love's assumption and on the assumption of zero Poisson's ratio[6]. The solutions are obtained in explicit form in terms of the Legendre functions and are applicable for all values of colatitude of the spherical shell. Since $N^{ij}=N^{ji}$ is satisfied by the Kirchhoff's assumption in the case of the spherical shells, the errors arising from the Love's approximation are expected to be small. Hence we can safely neglect

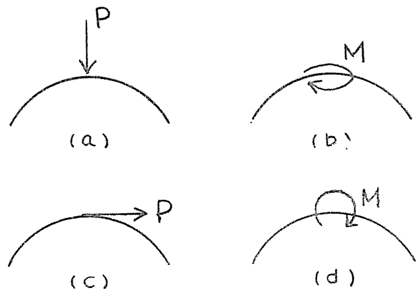


FIG. 1

t^2/a^2 compared with unity (where t is the shell thickness and a is the radius of the middle surface), and let Poisson's ratio be equal to zero, the solutions for the case (1) are identical to those obtained by Koiter. The singularities of the resultant stresses and the displacements at the apex are also investigated in this paper.

2. Fundermental Equations

(1) Equations of Equilibrium and Compatibility and Stress Functions

When Poisson's ratio is equal to zero, the equations of equilibrium and compatibility for the spherical shell can be reduced to the following[6]:

Equilibrium Equation in the Normal Direction and Gauss's Compatibility Equation

$$A_1^{2\cdot\cdot} + (\sin\phi\cos\phi A_2^1)^{\cdot} + \frac{2}{\sin\phi}(\sin\phi A_1^1)^{\cdot\prime} - A_2^{1\prime\prime} + 2i\lambda^2(\sin^2\phi A_2^1 - A_1^2) = 0 \quad (2.1\text{ a})$$

Equilibrium Equations in the Tangential Directions and Mainardi-Codazzi's Compatibility Equations

$$(1-2i\lambda^2)(\sin^2\phi A_1^1)^{\cdot} - \sin^3\phi A_2^{1\prime} + 2i\lambda^2 A_1^{2\prime} = 0 \quad (2.1\text{ b})$$

$$A_1^{2\cdot} + \sin\phi\cos\phi A_2^1 - 2i\lambda^2[(\sin^2\phi A_2^1)^{\cdot} + \cot\phi A_1^2] + (1-2i\lambda^2)A_1^{1\prime} = 0 \quad (2.1\text{ c})$$

where dot and prime denote the derivatives with respect to ϕ and θ, respectively, and $\lambda^2 = \sqrt{3}\,a/t$. A_2^1, A_1^1 and A_1^2 are the mixed tensors composed of the resultant moment tensors and the strain tensors of the middle surface, or the resultant force tensors

$$\left.\begin{aligned} A_2^1 &= M_2^1 + i\frac{Et^2}{\sqrt{12}\,a^2}\frac{\varepsilon_{22}}{\sin\phi} = M_2^1 + i\frac{ta^2}{\sqrt{12}}\frac{N^{22}}{\sin\phi},\\ A_1^1 &= M_1^1 + i\frac{Et^2}{\sqrt{12}\,a^2}\frac{\varepsilon_{12}}{\sin\phi} = M_1^1 + i\frac{ta^2}{\sqrt{12}}\sin\phi N^{12},\\ A_1^2 &= M_1^2 - i\frac{Et^2}{\sqrt{12}\,a^2}\frac{\varepsilon_{11}}{\sin\phi} = M_1^2 - i\frac{ta^2}{\sqrt{12}}\sin^3\phi N^{11}, \end{aligned}\right\} \quad (2.2\text{ a-c})$$

where E is Young's modulus.

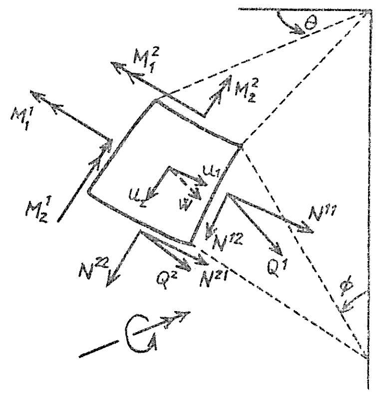
A_2^1, A_1^1 and A_1^2 can be expressed by the stress functions and the displacement components as follows,

$$\left.\begin{aligned} A_2^1 &= \frac{1}{a^2\sin\phi}(\alpha_2^{\cdot} - a\Phi),\\ A_1^1 &= -A_2^2 = \frac{1}{2a^2\sin\phi}(\alpha_1^{\cdot} + \alpha_2' - 2\cot\phi\,\alpha_1),\\ A_1^2 &= -\frac{\sin\phi}{a^2}\left(\frac{\alpha_1'}{\sin^2\phi} + \cot\phi\,\alpha_2 - a\Phi\right). \end{aligned}\right\} \quad (2.3\text{ a-c})$$

where

$$\alpha_j = U_j + i\frac{Et^2}{\sqrt{12}}u_j \quad (j=1,\ 2), \qquad \Phi = W + i\frac{Et^2}{\sqrt{12}}w.$$

$$(U_j,\ W;\ \text{Stress functions}) \quad (2.4)$$



The positive directions of the resultant stress tensors and the displacement components are shown in Fig. 2, and the relations between these tensorial and physical quantities are in the following:

FIG. 2

$$\left.\begin{array}{l}\overline{N}^{11} = a^2 \sin^2 \phi N^{11},\ \overline{N}^{12} = a^2 \sin \phi N^{12},\ \overline{N}^{22} = a^2 N^{22},\ \overline{Q}^1 = a \sin \phi Q^1, \\ \overline{Q}^2 = aQ^2,\ \overline{M}_2^1 = \sin \phi M_2^1,\ \overline{M}_1^1 = M_1^1 = -\overline{M}_2^2 = -M_2^2,\ \overline{M}_1^2 = M_1^2/\sin \phi, \\ \overline{u}_1 = u_1/a \sin \phi,\ \overline{u}_2 = u_2/a,\ \overline{w} = w,\end{array}\right\} \quad (2.5)$$

where the corresponding physical quantities are indicated with the sign $\overline{}$.

(2) *Derivation of the Differential Equations*

Substituting (2.3) into (2.1 a), $[(2.1\,c)^{\cdot} + (2.1\,b)'/\sin^2 \phi]/\sin \phi$, and $[(2.1\,b)/\sin \phi]^{\cdot} - [(2.1\,c)/\sin \phi]'$ we obtain

$$\left.\begin{array}{l}aH(\Phi) + (1+2\,i\lambda^2)(X-2\,a\Phi) = 0, \\ -2\,i\lambda^2 H(X) + (1+2\,i\lambda^2)aH(\Phi) + (1+2\,i\lambda^2)(X-2\,a\Phi) = 0, \\ H(Y) = 0,\end{array}\right\} \quad (2.6\,\text{a-c})$$

where

$$H(A) = A^{\cdot\cdot} + \cot \phi A^{\cdot} + 2\,A + \frac{A''}{\sin^2 \phi}, \quad (2.7)$$

and

$$X = \frac{1}{\sin \phi}(\sin \phi \alpha_2) + \frac{\alpha_1}{\sin^2 \phi}, \qquad Y = \frac{1}{\sin \phi}(\alpha_1^{\cdot} - \alpha_2'). \quad (2.8\,\text{a, b})$$

Subtracting (2.6 b) from (2.6 a) we find

$$H(X) - aH(\Phi) = 0. \quad (2.6\,\text{d})$$

From (2.6 a)

$$aHH(\Phi) + (1+2\,i\lambda^2)[H(X) - 2\,aH(\Phi)] = 0.$$

Using the relation of (2.6 d), the above equation becomes

$$HH(\Phi) - (1+2\,i\lambda^2)H(\Phi) = 0,$$

which can be solved into

$$H(\Phi) = 0, \quad (2.6\,\text{e})$$

and

$$H(\Phi) - (1+2\,i\lambda^2)\Phi = 0. \tag{2.6 f}$$

(2.6 e), (2.6 f) and (2.6 c) are our final equations.

If we define the unknown functions Φ, X and Y in the form

$$\Phi = \Phi_n \cos n\theta, \qquad X = X_n \cos n\theta, \qquad Y = Y_n \sin n\theta, \tag{2.9}$$

(2.6 e), (2.6 f) and (2.6 c) can be rewritten by

$$\left.\begin{aligned} &L(\Phi_n) = 0, \qquad L(\Phi_n) - (1+2\,i\lambda^2)\Phi_n = 0, \\ &L(Y_n) = 0, \end{aligned}\right\} \tag{2.10 a-c}$$

where Φ_n, X_n and Y_n are the functions of ϕ only, and

$$L(A) = A^{\cdot\cdot} + \cot\phi A^{\cdot} + \left(2 - \frac{n^2}{\sin^2\phi}\right)A. \tag{2.11}$$

(2.10 a-c) are satisfied by the Legendre functions. Hence

$$\Phi_n = AP_1^{-n}(\cos\phi) + BQ_1^n(\cos\phi) + CP_\nu^n(\cos\phi) + DQ_\nu^n(\cos\phi), \tag{2.12}$$

$$Y_n = KP_1^{-n}(\cos\phi) + LQ_1^n(\cos\phi), \tag{2.13}$$

where ν is one of the roots of the equation

$$\nu(\nu+1) = 1 - 2\,i\lambda^2. \tag{2.14}$$

Let denote the solutions of (2.10 a) and (2.10 b) by $\Phi_n^{(1)}$ and $\Phi_n^{(2)}$, respectively, then we have from (2.6 a)

$$X_n = 2\,a\Phi_n^{(1)} + a\Phi_n^{(2)}. \tag{2.15}$$

From (2.8 a, b), (2.13) and (2.15) α_1 and α_2 can be described as follows,

$$\alpha_1 = \left[na\left(\Phi_n^{(1)} + \frac{1}{1-2\,i\lambda^2}\Phi_n^{(2)}\right) - \frac{\sin\phi}{2}Y_n^{\cdot}\right]\sin n\theta, \tag{2.16}$$

$$\alpha_2 = \left[-a\left(\Phi_n^{(1)\cdot} + \frac{1}{1-2\,i\lambda^2}\Phi_n^{(2)\cdot}\right) + \frac{n}{2\sin\phi}Y_n\right]\cos n\theta, \tag{2.17}$$

where the solutions for the case $X=Y=0$ are omitted since they cannot satisfy the equations of equilibrium and compatibility.

3. Solution for a Concentrated Radial Force P

Consider the spherical shell subject to a concentrated radial force P at its apex (Fig. 1 a). The solution in this case can be obtained from (2.10 a, b) as $n=0$.

The general solutions of (2.10 a, b) for $n=0$ are

$$\left.\begin{aligned}\Phi_0 &= \Phi_0^{(1)} + \Phi_0^{(2)},\\ \Phi_0^{(1)} &= A\cos\phi + B\Big(\frac{\cos\phi}{2}\log\frac{1+\cos\phi}{1-\cos\phi} - 1\Big),\\ \Phi_0^{(2)} &= CP_\nu(\cos\phi) + DQ_\nu(\cos\phi),\end{aligned}\right\} \qquad (3.1\,\text{a-c})$$

where $P_\nu(\cos\phi)$ and $Q_\nu(\cos\phi)$ are the Legendre functions of order ν, and A, B, C and D are the complex constants of integration which may be expressed in the following form

$$A = C_1 + i\overline{C}_1,\ \ B = i\overline{C}_2,\ \ C = C_3 + i\overline{C}_3,\ \ D = C_4 + i\overline{C}_4, \qquad (3.2)$$

in which the real part of B is omitted since they do not satisfy the relations between the resultant stresses and the displacement components.

The A term in (3.1 b) represents a rigid-body translation along the polar axis, and the corresponding resultant stresses vanish, hence it may now be dropped from our solutions. Since we are interested only in the stresses in the vicinity of the apex, the C term in (3.1 c) which increases as ϕ increases may be dropped, too. We now have to determine the constants B and D. This would be done from the conditions of equilibrium for the vertical forces and from the continuity conditions of the shell:

Equilibrium for the Vertical Forces

$$P = -2\pi a \lim_{\phi\to 0} \sin\phi(\overline{N}^{22}\sin\phi + \overline{Q}^2\cos\phi). \qquad (3.3)$$

Continuity of the Deformation at the Apex

$$\overline{u}_2 = 0, \qquad \overline{u}_2 + \overline{w}^{\cdot} = 0 \qquad \text{at } \phi = 0 \qquad (3.4\,\text{a, b})$$

where (3.4 b) means that the rotation must be zero at the apex from the symmetry.

Substituting (3.1) into (2.9 a) and (2.17), and dropping the A and the C terms from these equations, we have

$$\left.\begin{aligned}\Phi &= B\Big(\frac{\cos\phi}{2}\log\frac{1+\cos\phi}{1-\cos\phi} - 1\Big) + DQ_\nu(\cos\phi),\\ \alpha_2 &= a\Big[B\Big(\frac{\sin\phi}{2}\log\frac{1+\cos\phi}{1-\cos\phi} + \cot\phi\Big) - \frac{D}{1-2i\lambda^2}Q_\nu^{\cdot}(\cos\phi)\Big].\end{aligned}\right\} \qquad (3.5\,\text{a, b})$$

For $\phi\to 0$, $Q_\nu(\cos\phi)$ and its derivative with respect to ϕ are[8]

$$\left.\begin{aligned}Q_\nu(\cos\phi) &= -\log\frac{\phi}{2} + \text{Const.},\\ Q_\nu^{\cdot}(\cos\phi) &= -\frac{1}{\phi} + \frac{1-2i\lambda^2}{2}\phi\log\frac{\phi}{2} + 0(\phi),\end{aligned}\right\} \qquad (3.6)$$

and also from (3.5 a, b), (2.4) and (2.5), (3.4 a, b) become

$$Im\Big(B + \frac{D}{1-2i\lambda^2}\Big) = 0, \qquad Im(B+D) = 0,$$

hence

$$C_4 = -2\lambda^2 \overline{C}_2, \qquad \overline{C}_4 = -\overline{C}_2. \tag{3.7 a, b}$$

We now have only one constant of integration to be determined from the condition of equilibrium for the vertical forces. It is easily shown that the D solutions (the bending disturbance solutions) are self-equilibrating since they satisfy the equilibrium condition for the vertical forces

$$\overline{N}^{22} \sin\phi + \overline{Q}^2 \cos\phi = 0.$$

It follows that only the B solutions (the membrane solutions) must be considered in order to satisfy the equilibrium condition (3.3). $\overline{N}^{22}$ arising from the membrane solutions is

$$\overline{N}^{22} = -\frac{\sqrt{12}}{ta}\frac{\overline{C}_2}{\sin^2\phi}. \tag{3.8}$$

Applying (3.8) to (3.3) we have our last integral constant

$$\overline{C}_2 = \frac{Pt}{4\sqrt{3}\,\pi}. \tag{3.9}$$

Substituting (3.9) and (3.7 a, b) into (3.5 a, b) we obtain

$$\left.\begin{aligned} \Phi &= i\frac{Pt}{4\sqrt{3}\,\pi}\left[\frac{\cos\phi}{2}\log\frac{1+\cos\phi}{1-\cos\phi} - 1 - (1-2i\lambda^2)Q_\nu(\cos\phi)\right], \\ \alpha_2 &= i\frac{Pta}{4\sqrt{3}\,\pi}\left[\frac{\sin\phi}{2}\log\frac{1+\cos\phi}{1-\cos\phi} + \cot\phi + Q_\nu^{\cdot}(\cos\phi)\right], \end{aligned}\right\} \tag{3.10 a,b}$$

where the real and the imaginary parts represent the stress functions and the displacement components, respectively, in view of (2.4).

Substituting (3.10 a, b) into (2.3 a–c) we obtain

$$\left.\begin{aligned} A_2^1 &= -i\frac{Pt}{4\sqrt{3}\,\pi a}\left[\frac{1}{\sin^2\phi} + \cot\phi\, Q_\nu^{\cdot}(\cos\phi)\right]\frac{1}{\sin\phi}, \\ A_1^2 &= -i\frac{Pt}{4\sqrt{3}\,\pi a}\left[\frac{1}{\sin^2\phi} + \cot\phi\, Q_\nu^{\cdot}(\cos\phi) + (1-2i\lambda^2)Q_\nu(\cos\phi)\right]\sin\phi, \end{aligned}\right\} \tag{3.11 a,b}$$

where the real and the imaginary parts represent the resultant moments and the resultant forces, respectively, in view of (2.2).

The solutions obtained in this section are identical to those obtained by Koiter[3], when Poisson's ratio is equal to zero.

4. Solution for a Concentrated Moment M about the Polar Axis

Consider the spherical shell subject to a concentrated moment M about the polar axis (Fig. 1 b). The solution in this case can be derived from (2.1 b) by dropping all the terms containing the derivatives with respect to θ. That is, we obtain

$$(\sin^2\phi A_1^1)^{\cdot} = 0 \tag{4.1}$$

with the solution

$$A_1^1 = \frac{A}{\sin^2\phi}, \tag{4.2}$$

where A is a complex constant of integration. The relation between the real and the imaginary parts of A is determined in order to satisfy the relation $\varepsilon_{12}=a\kappa_{12}$. Let denote the real and the imaginary parts of A by A_r and A_i, respectively, then we have from (4.2) and (2.2 b)

$$M_1^1 = \frac{A_r}{\sin^2\phi}, \qquad N^{12} = \frac{2\sqrt{3}}{ta^2}\frac{A_i}{\sin^3\phi}. \tag{4.3 a, b}$$

Using the elastic law[7], we have from (4.3 a, b)

$$\kappa_{12} = \frac{12\,a^2}{Et^3}\frac{A_r}{\sin\phi}, \qquad \varepsilon_{12} = \frac{2\sqrt{3}\,a^2}{Et^2}\frac{A_i}{\sin\phi}. \tag{4.4 a, b}$$

Substituting (4.4 a, b) into the relation $\varepsilon_{12}=a\kappa_{12}$ we have

$$A_r = \frac{A_i}{2\,\lambda^2}. \tag{4.5}$$

The remaining condition to determine the constant of integration is the equilibrium condition for the moment about the polar axis

$$M = 2\,\pi a \lim_{\phi\to 0} \sin^3\phi(a\overline{N}^{12} + \overline{M}_1^1). \tag{4.6}$$

Applying (4.3 a, b) to (4.6), and from (4.5) we have

$$A_i = \frac{Mt}{4\sqrt{3}\,\pi a^2}. \tag{4.7}$$

Substituting (4.5) and (4.7) into (4.3 a, b) we obtain

$$N^{12} = \frac{M}{2\,\pi a^4}\frac{1}{\sin^3\phi}, \qquad M_1^1 = \frac{Mt^2}{24\,\pi a^3}\frac{1}{\sin^2\phi}. \tag{4.8 a, b}$$

It can be noted that the contribution of M_1^1 to the moment equilibrium is of negligible order since t^2/a^2 is small compared with unity.

5. Solutions for a Concentrated Tangential Force P and for a Concentrated Moment M about the Axis Given by $\phi=0$ and $\theta=90°$

Consider the spherical shell subject to a concentrated tangential force P and a concentrated moment M about the axis given by $\phi=0$ and $\theta=90°$ at its apex (Fig. 1 c, d). The solutions in these cases can be obtained from (2.10 a–c) as $n=1$.

The general solutions of (2.10 a–c) for $n=1$ are

$$\left.\begin{aligned}
&\Phi_1 = \Phi_1^{(1)} + \Phi_1^{(2)},\\
&\Phi_1^{(1)} = A\sin\phi + B\Big(\frac{\sin\phi}{2}\log\frac{1+\cos\phi}{1-\cos\phi} + \cot\phi\Big),\\
&\Phi_1^{(2)} = CP_\nu^1(\cos\phi) + DQ_\nu^1(\cos\phi),\\
&Y_1 = K\sin\phi + L\Big(\frac{\sin\phi}{2}\log\frac{1+\cos\phi}{1-\cos\phi} + \cot\phi\Big).
\end{aligned}\right\} \qquad (5.1\text{ a-d})$$

where A, B, C, D, K and L are the complex constants of integration which may be expressed in the form

$$\left.\begin{aligned}
&A = C_1 + i\overline{C}_1,\ \ B = i\overline{C}_2,\ \ C = C_3 + i\overline{C}_3,\ \ D = C_4 + i\overline{C}_4,\\
&K = C_5 + i\overline{C}_5,\ \ L = \overline{C}_6\Big(\frac{1}{2\lambda^2} + i\Big).
\end{aligned}\right\} \qquad (5.2)$$

The real part of B is omitted as the same reason for the case $n=0$ (Section 3). On the other hand, the real part of L is related to its imaginary part such that the relations among the resultant forces, the rerultant moments, the stress functions and the displacement components are consistently satisfied.

The A and the K terms in (5.1) represent a rigid-body translation and rotation, respectively, and the corresponding resultant stresses vanish, hence they may be dropped from our solutions. Since we are interested only in the stresses in the vicinity of the apex, the C terms in (5.1 c) which increases as ϕ increases may be dropped, too. We now have to determine the constants B, D and L. This would be done from the conditions of equilibrium for the horizontal forces and for the moments, and from the continuity conditions of the shell:

Equilibrium for the Horizontal Forces

$$P = \lim_{\phi\to 0}\int_0^{2\pi}[(\overline{N}^{22}\cos\phi - \overline{Q}^2\sin\phi)\cos\theta - \overline{N}^{12}\sin\theta]\,a\sin\phi\, d\theta \qquad (5.3\text{ a})$$

Equilibrium for the Moments

$$M = \lim_{\phi\to 0}\int_0^{2\pi}[(\overline{N}^{22}\sin\phi + \overline{Q}^2\cos\phi)a\sin\phi\cos\theta + \overline{M}_1^2\cos\theta - \overline{M}_1^1\cos\phi\sin\theta]\,a\sin\phi\, d\theta$$

(5.3 b)

Continuity of the Horizontal Deflection

$$u_1/\sin\theta = -u_2/\cos\theta \qquad \text{at } \phi = 0 \qquad (5.4\text{ a})$$

Continuity of the Vertical Deflection

$$\overline{w}\cos\phi + \overline{u}_2\sin\phi = 0 \qquad \text{at } \phi = 0. \qquad (5.4\text{ b})$$

Substituting (5.1) into (2.9 a), (2.16) and (2.17), and dropping the A, the C and the K terms from these equations, we have

$$\left.\begin{aligned}
\Phi &= \Big[B\Big(\frac{\sin\phi}{2}\log\frac{1+\cos\phi}{1-\cos\phi}+\cot\phi\Big)+DQ_\nu^{\cdot}(\cos\phi)\Big]\cos\theta,\\
\alpha_1 &= a\Big[B\Big(\frac{\sin\phi}{2}\log\frac{1+\cos\phi}{1-\cos\phi}+\cot\phi\Big)+\frac{D}{1-2i\lambda^2}Q_\nu^{\cdot}(\cos\phi)\\
&\quad -\frac{L}{2a}\sin\phi\Big(\frac{\cos\phi}{2}\log\frac{1+\cos\phi}{1-\cos\phi}-1-\frac{1}{\sin^2\phi}\Big)\Big]\sin\theta,\\
\alpha_2 &= -a\Big[B\Big(\frac{\cos\phi}{2}\log\frac{1+\cos\phi}{1-\cos\phi}-1-\frac{1}{\sin^2\phi}\Big)-D\Big\{\frac{\cot\phi}{1-2i\lambda^2}Q_\nu^{\cdot}(\cos\phi)\\
&\quad +Q_\nu(\cos\phi)\Big\}-\frac{L}{2a}\Big(\frac{1}{2}\log\frac{1+\cos\phi}{1-\cos\phi}+\frac{\cos\phi}{\sin^2\phi}\Big)\Big]\cos\theta.
\end{aligned}\right\}\quad (5.5\,\text{a-c})$$

From (5.5 a-c), (2.4), (2.5) and (3.6), (5.4 a, b) become

$$Im\Big(B-\frac{D}{1-2i\lambda^2}+\frac{L}{2a}\Big)=0,\qquad Im(B-D)=0,$$

hence

$$\overline{C}_2=\overline{C}_4,\qquad \overline{C}_2+\frac{\overline{C}_6}{2a}-\frac{1}{2\lambda^2}\Big(C_4+\frac{\overline{C}_4}{2\lambda^2}\Big)=0. \qquad (5.6\,\text{a, b})$$

As in the case of Section 4, the D solutions (the bending disturbance solutions) are self-equilibrating since they satisfy the equilibrium coditions for the tangential forces

$$\int_0^{2\pi}[(\overline{N}^{22}\cos\phi-\overline{Q}^2\sin\phi)\cos\theta-\overline{N}^{12}\sin\theta]\,d\theta=0,$$

and for the moments

$$\int_0^{2\pi}[(\overline{N}^{22}\sin\phi+\overline{Q}^2\cos\phi)a\sin\phi\cos\theta+\overline{M}_1^2\cos\theta-\overline{M}_1^1\cos\phi\sin\theta]\,d\theta=0.$$

It follows that only the B and the L solutions (the membrane solutions) must be considered to satisfy the equilibrium conditions (5.3 a, b). $\overline{N}^{22}$ and $\overline{N}^{12}$ arising from the membrane solutions are

$$\left.\begin{aligned}
\overline{N}^{22} &= -\frac{4\sqrt{3}}{ta}\frac{1}{\sin^3\phi}\Big(\overline{C}_2\cos\phi+\frac{\overline{C}_6}{2a}\Big)\cos\theta,\\
\overline{N}^{12} &= -\frac{4\sqrt{3}}{ta}\frac{1}{\sin^3\phi}\Big(\overline{C}_2+\frac{\overline{C}_6}{2a}\cos\phi\Big)\sin\theta.
\end{aligned}\right\}\quad (5.7\,\text{a, b})$$

Applying (5.7 a, b) to (5.3 a, b) we obtain

$$\left.\begin{aligned}
\overline{C}_2 &= \frac{Pt}{4\sqrt{3}\pi}.\\
\overline{C}_2+\frac{\overline{C}_6}{2a} &= -\frac{Mt}{4\sqrt{3}\pi a}.
\end{aligned}\right\}\quad (5.8\,\text{a, b})$$

(*1*) *Solution for a Concentrated Tangential Force P*

When $M=0$, we have from (5.6 a, b), (5.8 a, b) and (5.2)

$$B=i\frac{Pt}{4\sqrt{3}\,\pi},\quad D=\frac{Pt}{4\sqrt{3}\,\pi}\Big(-\frac{1}{2\lambda^2}+i\Big),\quad L=-\frac{Pta}{2\sqrt{3}\,\pi}\Big(\frac{1}{2\lambda^2}+i\Big). \tag{5.9 a-c}$$

Substituting (5.9 a-c) into (5.5 a-c) we obtain

$$\left.\begin{aligned}
\Phi &= i\frac{Pt}{4\sqrt{3}\,\pi}\Big[\frac{\sin\phi}{2}\log\frac{1+\cos\phi}{1-\cos\phi}+\cot\phi+\Big(1+\frac{i}{2\lambda^2}\Big)Q'_\nu\Big]\cos\theta,\\
\alpha_1 &= i\frac{Pta}{4\sqrt{3}\,\pi}\Big[\frac{1}{2}\Big(1+\cos\phi-\frac{i}{2\lambda^2}\cos\phi\Big)\log\frac{1+\cos\phi}{1-\cos\phi}-1+\frac{i}{2\lambda^2}\\
&\quad-\frac{1}{\sin^2\phi}\Big(1-\cos\phi-\frac{i}{2\lambda^2}\Big)+\frac{i}{2\lambda^2}\frac{1}{\sin\phi}Q'_\nu\Big]\sin\phi\sin\theta,\\
\alpha_2 &= -i\frac{Pta}{4\sqrt{3}\,\pi}\Big[\frac{1}{2}\Big(1+\cos\phi-\frac{i}{2\lambda^2}\Big)\log\frac{1+\cos\phi}{1-\cos\phi}-1-\frac{1}{\sin^2\phi}\Big(1-\cos\phi\\
&\quad+\frac{i}{2\lambda^2}\cos\phi\Big)-\frac{i}{2\lambda^2}\cot\phi Q'_\nu-\Big(1+\frac{i}{2\lambda^2}\Big)Q_\nu\Big]\cos\theta.
\end{aligned}\right\} \tag{5.10 a-c}$$

Substituting (5.10 a-c) into (2.3 a-c) we obtain

$$\left.\begin{aligned}
A^1_2 &= i\frac{Pt}{4\sqrt{3}\,\pi a}\Big[\frac{2}{\sin^3\phi}\Big(1-\cos\phi-\frac{i}{2\lambda^2}\Big)-\frac{i}{2\lambda^2}(1+2\cot^2\phi)Q'_\nu\\
&\quad-\Big(1+\frac{i}{2\lambda^2}\Big)\cot\phi Q_\nu\Big]\frac{\cos\theta}{\sin\phi},\\
A^1_1 &= -i\frac{Pt}{4\sqrt{3}\,\pi a}\Big[\frac{2}{\sin^3\phi}\Big(1-\cos\phi+\frac{i}{2\lambda^2}\cos\phi\Big)+\frac{i}{\lambda^2}\frac{\cos\phi}{\sin^2\phi}Q'_\nu\\
&\quad+\Big(1+\frac{i}{2\lambda^2}\Big)\frac{1}{\sin\phi}Q_\nu\Big]\sin\theta,\\
A^2_1 &= i\frac{Pt}{4\sqrt{3}\,\pi a}\Big[\frac{2}{\sin^3\phi}\Big(1-\cos\phi-\frac{i}{2\lambda^2}\Big)+\Big(1-i\frac{\cot^2\phi}{\lambda^2}\Big)Q'_\nu\\
&\quad-\Big(1+\frac{i}{2\lambda^2}\Big)\cot\phi Q_\nu\Big]\sin\phi\cos\theta.
\end{aligned}\right\} \tag{5.11 a-c}$$

(*2*) *Solution for a Concentrated Moment M*

When $P=0$, we have from (5.6 a, b), (5.8 a, b) and (5.2)

$$B=0,\ D=-\frac{M}{2\pi},\quad L=-\frac{Mt}{2\sqrt{3}\,\pi}\Big(\frac{1}{2\lambda^2}+i\Big). \tag{5.12 a-c}$$

Substituting (5.12 a-c) into (5.5 a-c) we obtain

$$\left.\begin{aligned}
\Phi &= -\frac{M}{2\pi}[Q_\nu^{\cdot}(\cos\phi)]\cos\theta,\\
\alpha_1 &= \frac{Mt}{4\sqrt{3}\,\pi}\left(\frac{1}{2\lambda^2}+i\right)\left[\frac{\cos\phi}{2}\log\frac{1+\cos\phi}{1-\cos\phi}-1-\frac{1}{\sin^2\phi}\right.\\
&\quad\left.-\frac{Q_\nu^{\cdot}}{\sin\phi}\right]\sin\phi\sin\theta,\\
\alpha_2 &= -\frac{Mt}{4\sqrt{3}\,\pi}\left(\frac{1}{2\lambda^2}+i\right)\left[\frac{1}{2}\log\frac{1+\cos\phi}{1-\cos\phi}+\frac{\cos\phi}{\sin^2\phi}+\cot\phi Q_\nu^{\cdot}\right.\\
&\quad\left.+(1-2i\lambda^2)Q_\nu\right]\cos\theta.
\end{aligned}\right\}\quad(5.13\text{ a-c})$$

Substituting (5.13 a-c) into (2.3 a-c) we obtain

$$\left.\begin{aligned}
A_2^1 &= \frac{Mt}{4\sqrt{3}\,\pi a^2}\left(\frac{1}{2\lambda^2}+i\right)\left[\frac{2}{\sin^3\phi}+(1+2\cot^2\phi)Q_\nu^{\cdot}+(1-2i\lambda^2)\cot\phi Q_\nu\right]\frac{\cos\theta}{\sin\phi},\\
A_1^1 &= \frac{Mt}{4\sqrt{3}\,\pi a^2}\left(\frac{1}{2\lambda^2}+i\right)\left[\frac{2\cos\phi}{\sin^3\phi}+\frac{2\cos\phi}{\sin^2\phi}Q_\nu^{\cdot}+\frac{1-2i\lambda^2}{\sin\phi}Q_\nu\right]\sin\theta,\\
A_1^2 &= \frac{Mt}{4\sqrt{3}\,\pi a^2}\left(\frac{1}{2\lambda^2}+i\right)\left[\frac{2}{\sin^3\phi}+(2i\lambda^2+2\cot^2\phi)Q_\nu^{\cdot}\right.\\
&\quad\left.+(1-2i\lambda^2)\cot\phi Q_\nu\right]\sin\phi\cos\theta.
\end{aligned}\right\}$$

(5.14 a-c)

6. Some Properties of the Legendre Functions

In the preceding sections we have obtained the solutions for the spherical shell subject to the concentrated forces and moments as indicated in Fig. 1. The solutions are expressed in terms of the Legendre function of the second kind. In this section some properties of this function are described[8)9)].

The Legendre function of the second kind is expanded into the hypergeometric series as follows,

$$\begin{aligned}
Q_\nu(\cos\phi) &= \frac{\pi}{2}\left[\cot\nu\pi+\frac{\psi(\nu+1)+\psi(-\nu)+C}{\sin\nu\pi}\right]F\left(\nu+1,-\nu,1;\frac{1-\cos\phi}{2}\right)\\
&-\frac{1}{2}\log\left(\frac{1-\cos\phi}{2}\right)F\left(\nu+1,-\nu,1;\frac{1-\cos\phi}{2}\right)-\frac{1}{2}F_1\left(\nu+1,-\nu,1;\frac{1-\cos\phi}{2}\right).\quad(6.1)
\end{aligned}$$

where

$$\left.\begin{aligned}
&F\left(\nu+1,-\nu,1;\frac{1-\cos\phi}{2}\right)\\
&\quad=\sum_{k=0}^{\infty}\frac{(\nu+1)(\nu+2)\cdots(\nu+k)(-\nu)(1-\nu)\cdots(k-1-\nu)}{(k!)^2}\left(\frac{1-\cos\phi}{2}\right)^k.\\
&F_1\left(\nu+1,-\nu,1;\frac{1-\cos\phi}{2}\right)\\
&\quad=\sum_{k=1}^{\infty}\frac{(\nu+1)(\nu+2)\cdots(\nu+k)(-\nu)(1-\nu)\cdots(k-1-\nu)}{(k!)^2}\\
&\qquad\times\left\{\sum_{r=0}^{k-1}\left(\frac{1}{r+\nu+1}+\frac{1}{r-\nu}\right)-\sum_{r=1}^{k}\frac{1}{r}\right\}\left(\frac{1-\cos\phi}{2}\right)^k.
\end{aligned}\right\}$$

(6.2 a, b)

and

$$\psi(z) = \frac{d}{dz}\{\log \Gamma(z)\}; \qquad \text{Gauss's } \psi \text{ function}$$

$$C = 0.577215\cdots \quad ; \qquad \text{Euler's constant.}$$

The coefficient of the first term of (6.1) are rather complicated. However, it can easily be evaluated by the asymptotic expansion of Gauss's ψ functions[9], which results

$$\psi(\nu+1)+\psi(-\nu) = \log(2\lambda^2) - i\left(\text{Arctan } 2\lambda^2 - \frac{1}{6\lambda^2}\right) + 0(\lambda^{-3}), \tag{6.3}$$

and the relations

$$\left.\begin{aligned} \sin \nu\pi &= \sin \nu_r\pi \cosh \nu_i\pi + i \cos \nu_r\pi \sinh \nu_i\pi, \\ \cos \nu\pi &= \cos \nu_r\pi \cosh \nu_i\pi - i \sin \nu_r\pi \sinh \nu_i\pi, \end{aligned}\right\} \tag{6.4}$$

where ν_r and ν_i denote the real and the imaginary parts of ν, respectively, being obtained from (2.14) as follows,

$$\nu_r = \lambda - \frac{1}{2} + \frac{5}{16\lambda} + 0(\lambda^{-2}), \qquad \nu_i = -\lambda + \frac{5}{16\lambda} + 0(\lambda^{-2}). \tag{6.5 a, b}$$

On the other hand, the power series (6.2 a, b) converge for $0 \leq \phi < 180°$. Their convergence is rapid for smaller ϕ and for smaller ν (smaller λ), and is slow for larger ϕ and for larger ν (larger λ).

7. Singularities of the Resultant Stresses and the Displacements

In this section the singularities of the resultant stresses and the displacements at the apex are investigated.

(1) Singularities for the Case of a Concentrated Radial Force P

$$\left.\begin{aligned} \overline{N}^{11} &= -\frac{P}{4\pi a}\left(\log\frac{\phi}{2} - \alpha - 2\lambda^2\beta - 2\right), \quad \overline{N}^{22} = -\frac{P}{4\pi a}\left(\log\frac{\phi}{2} - \alpha - 2\lambda^2\beta + 2\right), \\ \overline{M}^1_2 &= -\frac{P}{4\pi}\left(\log\frac{\phi}{2} - \alpha + \frac{\beta}{2\lambda^2} - \frac{1}{2}\right), \quad \overline{M}^2_1 = \frac{P}{4\pi}\left(\log\frac{\phi}{2} + \alpha - \frac{\beta}{2\lambda^2} - \frac{1}{2}\right), \\ \overline{u}_2 &= -\frac{P\phi}{4\pi Et}\left(\log\frac{\phi}{2} + \alpha + 2\lambda^2\beta + \frac{3}{2}\right), \quad \overline{w} = -\frac{P}{2\pi Et}(\alpha + 2\lambda^2\beta + 1). \end{aligned}\right\} \tag{7.1}$$

(2) Singularities for the Case of a Concentrated Moment M about the Polar Axis

$$\overline{N}^{12} = \frac{M}{2\pi a^2}\frac{1}{\phi^2}, \qquad \overline{M}^1_1 = \frac{Mt^2}{24\pi a^3}\frac{1}{\phi^2}, \qquad \overline{u}_2 = -\frac{M}{2\pi Eta}\frac{1}{\phi}, \tag{7.2}$$

(3) Singularities for the Case of a Concentrated Tangential Force P

$$\left.\begin{aligned}
&\bar{N}^{11} = -\frac{P}{4\pi a}\frac{1}{\phi}, \qquad \bar{N}^{12} = -\frac{P}{4\pi a}\frac{1}{\phi}, \qquad \bar{N}^{22} = \frac{3P}{4\pi a}\frac{1}{\phi},\\
&\bar{M}_2^1 = \frac{Pt^2}{48\pi a^2}\frac{1}{\phi}, \qquad \bar{M}_1^1 = -\frac{Pt^2}{48\pi a^2}\frac{1}{\phi}, \qquad \bar{M}_1^2 = \frac{Pt^2}{16\pi a^2}\frac{1}{\phi},\\
&\bar{u}_1 = -\frac{P}{4\pi Et}\Big(3\log\frac{\phi}{2}+\alpha-\frac{\beta}{2\lambda^2}+\frac{7}{2}\Big), \quad \bar{u}_2 = \frac{P}{4\pi Et}\Big(3\log\frac{\phi}{2}+\alpha-\frac{\beta}{2\lambda^2}+\frac{5}{2}\Big),\\
&\bar{w} = -\frac{P\phi}{2\pi Et}(\alpha+\lambda^2\beta+1).
\end{aligned}\right\} \tag{7.3}$$

(4) *Singularities for the Case of a Concentrated Moment M about the Axis Given by* $\phi=0$ *and* $\theta=90°$

$$\left.\begin{aligned}
&\bar{N}^{11} = -\frac{M}{2\pi a^2}\frac{1}{\phi}, \quad \bar{N}^{12} = \frac{M}{4\pi a^2}\frac{1}{\phi}, \quad \bar{N}^{22} = \frac{3M}{4\pi a^2}\frac{1}{\phi},\\
&\bar{M}_2^1 = -\frac{M}{4\pi a}\frac{1}{\phi}, \quad \bar{M}_1^1 = -\frac{M}{4\pi a}\frac{1}{\phi}, \quad \bar{M}_1^2 = \frac{M}{4\pi a}\frac{1}{\phi},\\
&\bar{u}_1 = -\frac{M}{2\pi Eta}\Big(\log\frac{\phi}{2}-\lambda^2\beta+1\Big), \quad \bar{u}_2 = \frac{M}{2\pi Eta}\Big(\log\frac{\phi}{2}-\lambda^2\beta\Big),\\
&\bar{w} = \frac{3Ma\phi}{\pi Et^3}\Big(\log\frac{\phi}{2}-\alpha+\frac{\beta}{2\lambda^2}-\frac{1}{2}\Big).
\end{aligned}\right\} \tag{7.4}$$

where

$$\alpha+i\beta = \frac{\pi}{2}\Big[\cot\nu\pi+\frac{\phi(\nu+1)+\phi(-\nu)+C}{\sin\nu\pi}\Big]. \tag{7.5}$$

(Description of $\cos\theta$ and $\sin\theta$ is omitted in (7.3) and (7.4).)

8. Concluding Remarks

The present solutions are rigorous ones in the sense that they are derived without making any other approximations than neglecting t/a compared with unity and letting Poisson's ratio be equal to zero. After somewhat troublesome computation of $Q_\nu(\cos\phi)$ according to Section 6, we can calculate the resultant stresses and the displacements for all values of colatitude of the spherical shell. On the other hand, the shallow solutions[1] or the asymptotic solutions[5] are suitable for the numerical use since they are expressed in terms of the modified Bessel functions which have already been tabulated. Our solutions will serve to examine the accuracy and the validity of these practical solutions.

The singularities of the resultant stresses and the displacements at the apex are investigated in the latter part of the present paper. It is observed that, in all cases, the resultant stresses have singularities at the apex. The displacements for a concentrated radial force remain finite, and on the other hand the displacements for a concentrated tangential force and concentrated moments become infinite at $\phi=0$. These results are slightly different from those obtained by Reissner[1] or Leckie and Penny[5]. For example, $\bar{N}^{22}$ for the case of a radial force is negative (compressive) finite value by Reissner or Leckie and Penny, while positive (tensile) infinite in the present solutions. But this discrepancy is limitted to only an extreme vicinity of the apex, and at a slight distance from the apex the latter approaches to the former.

Finally, it should be noticed that the singular solutions derived from the shell theory are applicable to at a distance from the load point not less than several times t, since in the shell theory the transverse shear stresses are assumed to be small which is important in the vicinity of the load point.

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