ONE-DIMENSIONAL SEADY FLOW OF A PLASMA

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Abstract

We have dealt with an one-dimensional plasma flow across a confining magnetic field. From the results, one may find that plasma can flow only up to a distance, which is different from a conventional fluid. The conservation of mass must be hold, so that an absorber is required to be present at this distance. The wall of vessel enclosing plasma is nothing else but the absorber of plasma.

§ 1. Introduction

In connection with a confinement of high density plassma by a magnetic field, it is important to study a steady flow of plasma across the confining magnetic field. The steady plasma velocity across a magnetic field, due to the collisions, is usually given by[1)]

$$\vec{v} = -\frac{\eta}{B^2}\nabla P, \tag{1}$$

where $\vec{v}$ is the velocity, η the resistivity, $\vec{B}$ the strength of magnetic field and P the plasma pressure. The above equation is derived assuming

$$\vec{j} \times \vec{B} = \nabla P, \tag{2}$$

where $\vec{j}$ is the density of electric current. Such an assumption seems to be wrong in a viewpoint of hydropynamics. Indded, the equation (1) and the equation of continuity lead to that velocity of plasma becomes infinitely large, as it will be shown in the later. For any steady flow, we can not neglect the non-linear term $(\vec{v}\cdot\nabla)\vec{v}$ in the equation of motion[3)]. In this reseanch, we want to discuss the steady flow of plasma.

§ 2. The model of one-dimensional plasma flow

Suppose a fully ionized plasma, being in a steady steady state, moving normal to a magnetic field directed along the z axis of rectangular coordinates, and drifting in the x axis towards the material wall on which charges are recombining. We can assume that such a system is uniform in the y direction, and hence any veriable quantity present in the equations of plasma fluid is a function of x only.

§ 3. The binary-collision theory

We shall start with the equation (1) offered by L. Spitzer Jr, namely

$$v = -\frac{\eta}{B^2}\frac{dP}{dx},$$

and the equation of continuity, which is

$$\frac{d}{dx} nv = 0.$$

The pressure P is equal to $n(kT_i + kT_e)$, where T_i and T_e are the temperatures of ions and electrons. We assume T_i and T_e constant and B also. Then we can solve n or v as a function of x as follows,

$$n = n_0\left(1 - \frac{x}{d}\right)^{1/2}. \tag{3}$$

with

$$d = n_0\eta(kT_e \times kT_i)/2\, v_0 B^2. \tag{4}$$

Here we have chosen a set of boundary conditions as $n = n_0$ and $v = v_0$ at $x = 0$.

The above equation (3) shows $n = 0$ at $x = d$. Thus d is an upper limit of x and therefore we can no more find any plasma in the range xd. We know from Eq. (3) $v \to \infty$ as $x \to d$. Such a difficulty is also found in the Schottky ambipolar diffusion theory on the well-known positive column of a gaseous discharge[5]. In the theory, that difficulty is removed by the bact that generation of charges in the positive column balances loss of charges on the wall[6], leaving a problem whether the ion temperature near the wall should be set equal to the room temperature or determined by Bohm's criterion leading to the sheath formation[2]. In the next paragraph we shall show that Bohm's criterion is automatically derived by taking the hydrodynamic non-linear term $(\vec{v}\cdot\nabla)\vec{v}$ into account.

§ 4. The hydrodynamic theory

A set of exact equations of plasma fluid are

$$\frac{d}{dx} nv = 0, \tag{5}$$

$$mnv\frac{dv}{dx} = jB - \frac{dP}{dx}, \tag{6}$$

$$\eta j = -vB, \tag{7}$$

$$E = -\frac{1}{en}\left(mnv\frac{dv}{dx} + \frac{dP_i}{dx}\right). \tag{8}$$

where $\vec{v}$ is the flow velocity whose components are $(v, 0, 0)$, m the mass of an ion, $\vec{j}$ the density of electric current $(=(0, j, 0))$, $\vec{B}$ the strength of magnetic field $(=(0, 0, B))$, $\vec{E}$ the strength of electric field $(=E, 0, 0))$, e the unit of electric charge and P_i the pressure of ions. Furthermore, the magnetic field B is related to the following equation,

$$\frac{dB}{dx} = -\mu_0 j, \tag{9}$$

where μ_0 is the permeability in vacuum.

We have assumed the x comdonent of $\vec{j}$ zero since the electric current can not flow into the insulated wall. Such a situation require E=finite to make j_x be zero. That is, a space charge sheath is set up, and therefore we have to take the electrostatic force $\rho^*\vec{E}$ (where ρ^* is the charge accumulation) and the space charge current $\rho^*\vec{v}$ into account. However we neglect these terms in the above equations and shall discuss effects of these terms upon the results obtained by the neglection of them in the later.

The equations (5) and (6) can be integrated as follows,

$$nv = \text{const.}, \tag{10}$$

$$mnvv + \frac{B^2}{2\mu_0} + \frac{nv}{v}(kT_i + kT_e) = \text{const.}, \tag{11}$$

using Eq. (9). These equations are the conservations of mass and stress respectively. With a set of boundary conditions $n=n_0$, $v=v_0$, $B=B_0$ at $x=0$, and the following notations

$$N = n/n_0, \qquad y = v/v_0, \qquad z = B/B_0,$$
$$\gamma = 2\mu_0 m n_0 v_0^2/B_0^2 \text{ and } \beta = 2\mu_0 n_0 (kT_e + kT_i)/B_0^2,$$

the equations (10) and (11) become

$$Ny = 1, \tag{12}$$

$$z^2 = 1 - \gamma(y-1) + \beta\left(1 - \frac{1}{y}\right). \tag{13}$$

The function $(z(y))^2$ has a maximum at

$$y = y_p = v_p/v$$

where $v_p = (kT_i + kT_e)/m)^{1/2}$. We are assuming the temperatures T_i and T_e constant. The maximum value of z, denoting it by z_m, is given by the following equation,

$$z_m^2 = 1 + (\gamma^{1/2} - \beta^{1/2})^2 \geq 1.$$

When $\beta = 0$, then $z_m^2 = (1+\gamma)$, and when $\beta = 1$, then $z_m^2 = 1 + (1-\gamma)^{1/2})^2$.

Since $z^2 \geq 0$ is required, we can find a permissible range of y which is of course depend on β and γ. Let y_1 and y_2 be the lower limit and the upper limit of the range. Then $y_1 < 1$ and $y_2 > y_p$ are valid.

With Eqs. (5)-(7), it follows

$$\left(\frac{y_p^2}{y^2} - 1\right)\frac{dy}{d\xi} = z^2 y, \tag{14}$$

where $\xi = 2\mu_0 x/\gamma\eta$. The function z^2 is never negative for $1 < y < y_2$, so that $dy/d\xi$

is positive when $1<y<y_p$ and negative when $y_p<y<y_2$. When $y=y_p$, then $d\xi/dy=0$ and $d^2\xi/dy^2<0$. That is, ξ takes a maximum at $y=y_p$. Now the difficulty $v\to\infty$ appeared in the preceding paraph has been removed.

The plasma flux nv must be constant at every point of x. However the above result shows that the upper limit of x is present, and therefore it is quite natural to conclude that wall of vessel will be existent at $x=x_m$, where x_m is the upper limit of x. Furthermore, it is interest that the velocity of plasma at $x=x_m$ is equal to the thermal velocity of plasma $((kT_i+kT_e)/m)^{1/2}$. It is noted that x_m is of the order of the magnitude of $(\gamma\eta/2\mu_0)y_p^2=d$, as it is easily seen from Eq. (14).

Now we shall regard in Eq. (8), which is rewritten as follows,

$$E=-\left(\frac{B_0^2}{en_0\eta}\right)\left(\frac{dv}{d\xi}\right)\left(y-\frac{y_i^2}{y}\right) \tag{15}$$

where $y_i^2=kT_i/mv_0^2$. From the equation, E is positive for $y_1<y<y_i$, (where $y<y_i$ is identical to $v<(kT_i/m)^{1/2}$), $E<0$ for $y_i<y<y_p$, E=infinite for $y=y_p$ and $E>0$ for $y_p<y<y_2$.

The electric potential is, using the relation $E=-dV/dx$, given by the following equation,

$$V=\left(\frac{mv_0^2}{e}\right)\left(\frac{1-y^2}{2}+y_i^2\ \ln\ y\right),$$

where the integral constant was determined by $V=0$ at $y=1$ (or at $x=0$). Let V_f be the value of V at $y=y_p$ (or at $x=x_m$). Then it follows

$$2eV_f=-(kT_i+kT_e)+(kT_i)\ln((kT_i+kT_e)/mv_0^2),$$

provided that $y_p^2\gg1$.

§ 5. Discussions and Conclusion

In § 4, we have found that x has a maximum, $_nx_m$. The Plasma is accelarared up to the thermal velocity independent of megnetic field, at which just $x=x_m$. This result seems to be expressing Bohm's criterion.

The magnetic field B as a function of velocity takes a maximum at the thermal velocity, which means that the magnetic field a satunction of x also has a maximum value at $x=x_m$.

In thehydrodynamic theory, one may find a difficulty that $E\to\infty$ at $x=x_m$, arising from the assumption that $n_e=n_i$. If, however, the electrostatic force ρ^*E (or $e(n_i-n_e)E$) is taken into account, then the difficulty will certainly be removed. It has been shown by Tonks and Langmuir[4] that the plane at which E infinite marks the formation of a space charge sheath. If the sheath thickness is much smaller than the plasma dimension, then x gives the plane at which plasma absorber of wall is present to hold the continuity of plasma flux. However, the velocity of plasma at $x=x_m$ may be different from the thermal velocity due to $n_e\neq n_i$, and depend on the strength of magnetic field.

We can temporarily determined the position of the sheathplasma boundary, although the boundary is not sharply defined. That is, $\varepsilon_0E^2=P$. This temporal condition can be calculated as a function of v or y using the results in § 4, from which we ma be able to find the velocity at the boundary (Note that the plasma assumption $n_e = n_i$ is no more valid when $\varepsilon_0 E^2/2 \geq P.$). But more exact solution will be obtained by making use of a two-fluid model of plasma than a fluid model treated here.

We shall now summarize our results:

(1) Any plasma can flow across a confining magnetic field up to a distance roughly given by Eq. (4), which requires an absorber or a wall to maintain the consevation of mass.

(2) The binary-collsion theory in which Eq. (1) is valid has a difficulty that the velocity of plasma is infinite at the distance $x=d$ (see Eq. (4)), and the electric field is also.

(3) The hydrodynamic theory in which the non-linear term $(\vec{v}\cdot\nabla)\vec{v}$ is taken into account shows that at $x=x_m$, at which the absorber is present, the velocity is equal to the thermal velocity of plasma. The theory as same as the binary collision theory has a difficulty that the electric field is infinite at $x=x_m$, which is arising from the assumption of electrical neutrality.

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