ON THE MIXING OF THE FREE JET BOUNDARY WHICH INCLUDES LAMINAR, AND TURBULENT FLOW REGION IN TANDEM

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Summary

A free jet boundary which includes laminar, and turbulent flow region in tandem is analysed, based on the extension of Görtler's method for the free turbulent flow. A similar solution is obtained for the velocity distribution. The variations of the width of the mixing zone are plotted for various parameters.

1. Introduction

A free jet boundary is formed between two uniform streams which move at different velocities in the same direction. The flow of the mixing zone is laminar only when the Reynolds number of it is small. It is generally expected that the flow will be laminar in the upstream region, then, through the transition region, it will become turbulent far downstream. The x-direction is taken to be parallel to, and y be normal to that of two uniform streams, which meet at $x=0$ with the velocities of U_1 and U_2 respectively, where $U_1 > U_2$ is assumed.

The analysis of the free jet boundary has been performed by many research workers in both laminar and turbulent case. W. Tollmien[3] first analysed the problem of the turbulent mixing of a half-jet, as well as that of a two-dimensional and a circular jet, by making use of Prandtl's mixing length theory which gives the following expression for a turbulent shearing stress

$$\tau_t = \rho l^2 \left|\frac{\partial u}{\partial y}\right| \frac{\partial u}{\partial y}$$

where l denotes the mixing length, ρ the density, u the mean velocity component in the x-direction.

Later on, H. Görtler[4] took up these problems again, being based on Reichardt's constant exchange coefficient hypothesis with some suggestions from Prandtl which is only applicable to the case of free turbulent flow. It assumes that the turbulent shearing stress τ_t may be expressed as

$$\tau_t = \rho \varepsilon \frac{\partial u}{\partial y} \tag{2}$$

introducing the turbulent exchange coefficient, or the virtual kinematic viscosity

$$\varepsilon = \kappa_1 b (u_{max} - u_{min}) \tag{3}$$

where b denotes the width of the mixing zone, κ_1 a dimensionless number to be determined experimentally. u_{max} and u_{min} are the maximum and the minimum mean velocity in the section respectively. It follows from Eq. (3) that ε remains constant over the whole width of every cross-section. Following Prandtl, it is further assumed that the width of the mixing zone associated with a free jet boundary is proportional to the distance from the point where two streams meet; that is

$$b = \text{const.} \times x = cx. \tag{4}$$

Putting Eq. (4) into Eq. (3), we get

$$\varepsilon = \kappa(u_{max} - u_{min})x \tag{5}$$

where $\kappa = \kappa_1 c$ is also an empirical constant.

Started from their different hypotheses, Tollmien and Görtler obtained the results which were similar to each other, though the latter was improved in some respects. Their theories are, of course, valid only to the fully developed turbulent flow where the upstream laminar region is negligibly small.

A theory which takes account of the upstream laminar flow together with the downstream turbulent flow is not yet developed. The present paper offers some approach in this direction, which will be discussed in the following sections.

2. Assumptions for the Shearing Stress

Since no theoretical treatment is yet available for the turbulent flow, and especially for the flow in the transition region, we simply put an assumption that the shearing stress is composed of two different parts, the laminar shearing stress and the turbulent shearing stress; that is

$$\tau = \tau_l + \tau_t. \tag{6}$$

τ_l is given as

$$\tau_l = \rho\nu\frac{\partial u}{\partial y} \tag{7}$$

using the kinematic viscosity, ν, which remains constant over the whole region. τ_t is understood to be zero in the upstream laminar region, extending from $x=0$ to $x=a$, the point where the turbulent mixing is assumed to begin. Beyond this region, we adopt the assumption of Eq. (2) for the turbulent shearing stress, considering that Eq. (2) is more amenable to mathematical treatment than Eq. (1), because of the resemblance in the expressions of τ_l and τ_t.

In the present case, however, some difficulties arise, when we want to apply Eq. (3) or Eq. (5) to ε. It was previously assumed that the shearing stress in the turbulent region is composed of τ_l and τ_t, instead of τ_t alone as in the preceding case, so that the expression of ε must be necessarily different from that of Eq. (3). Moreover, the width b cannot be considered to be proportional to the distance x any longer, for the turbulent region is preceded by the upstream laminar region.

But it is remarked that τ_l is much smaller than τ_t in the fully developed turbulent flow. As a result, we can approximately put τ is equal to τ_t far downstream, and consequently, the width will be increasing almost linearly with x there.

According to the above considerations, we make further assumption that we are allowed to use Eq. (3) for ε, with a modification of the quantity b into the width between two asymptotic lines which the boundaries of the mixing zone will approach as x increases as shown in Fig. 1. We define the point of intersection of the asymptotic lines with the x-axis as the point at which turbulent mixing originates, and it was already designated by a. Now ε is taken as zero when $x<a$, and can be expressed by Eq. (5) when $x>a$ only replacing x by $x-a$, it is possible to write

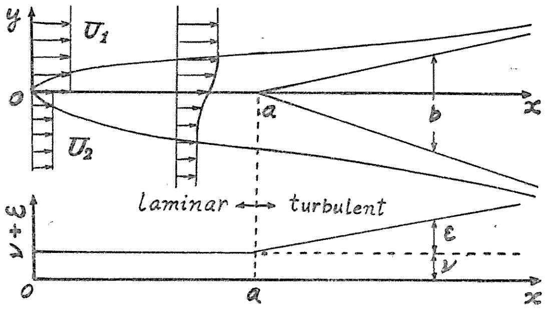

FIG. 1. Flow pattern of a free jet boundary, and variation of the kinematic viscosity

$$\varepsilon = \kappa(u_{\max} - u_{\min})(x-a)\cdot \mathscr{U}(x-a) \tag{8}$$

using the unit function $\mathscr{U}(x-a)$ defined by

$$\mathscr{U}(x-a) = \begin{cases} 0 & \text{if} \quad x<a \\ 1 & \text{if} \quad x>a. \end{cases} \tag{9}$$

Substituting Eqs. (7), (2) and (8) into Eq. (6), we obtain

$$\tau = \rho\{\nu + \kappa(u_{\max} - u_{\min})(x-a)\cdot \mathscr{U}(x-a)\}\frac{\partial u}{\partial y}. \tag{10}$$

3. Basic Equations and Their Solution

The basic equations of free jet boundary flow in incompressible fluid are as follows:

eqn. of continuity; $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{11}$$

eqn. of motion; $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial \tau}{\partial y}, \tag{12}$$

with the boundary conditions

$$\left.\begin{array}{ll} y=\infty; & u=U_1, \quad v=0 \\ y=-\infty; & u=U_2. \end{array}\right\} \tag{13}$$

The pressure term has been dropped in the eqn. of motion, because it is permissible to assume that the pressure remains constant over the whole field.

Substituting Eq. (10) into Eq. (12), we get

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\{\nu+\kappa(U_1-U_2)(x-a)\cdot\mathscr{U}(x-a)\}\frac{\partial^2 u}{\partial y^2} \tag{14}$$

where u_{max} and u_{min} are replaced by U_1 and U_2 respectively.

In order to obtain a similar solution, we introduce a new variable η and a stream function Ψ as

$$\eta=\frac{y}{Y(x)},\quad \Psi=G(x)f(\eta) \tag{15}$$

where $Y(x)$ and $G(x)$ are some functions of x, and $f(\eta)$ is that of η. The velocity components are given by

$$\left.\begin{aligned} u&=\frac{\partial\Psi}{\partial y}=\frac{G(x)}{Y(x)}f'(\eta)=U_1f'(\eta)\\ v&=-\frac{\partial\Psi}{\partial x}=-G'(x)f(\eta)+\frac{G(x)Y'(x)}{Y(x)}\eta f'(\eta)\\ &=U_1Y'(\eta f'-f)\end{aligned}\right\} \tag{16}$$

so that Ψ satisfies the eqn. of continuity (11) naturally, where the prime denotes the differentiation with respect to each argument, and the relation

$$f'(\infty)=1,\quad\text{or}\quad \frac{G}{Y}=U_1 \tag{17}$$

is assumed.

Substituting Eq. (16) into Eq. (14), we obtain the following differential equation for η,

$$f'''(\eta)+\frac{U_1Y(x)Y'(x)}{\nu+\kappa(U_1-U_2)(x-a)\cdot\mathscr{U}(x-a)}f(\eta)f''(\eta)=0. \tag{18}$$

Eq. (18) becomes independent of x, provided the coefficient of the second term is a constant. As we can take any constant, we put

$$\frac{U_1Y(x)Y'(x)}{\nu+\kappa(U_1-U_2)(x-a)\cdot\mathscr{U}(x-a)}=2 \tag{19}$$

for the sake of computational convenience.

Putting Eq. (19) into Eq. (18), we have

$$f'''(\eta)+2f(\eta)f''(\eta)=0 \tag{20}$$

with the boundary conditions

$$\left.\begin{aligned}\eta&=\infty;\quad f'=1,\quad f=0\\ \eta&=-\infty;\quad f'=\frac{U_2}{U_1}.\end{aligned}\right\} \tag{21}$$

Eq. (20) has been solved by use of a digital computer and the results are shown in Fig. 2. Fig. 3 shows that the smaller the difference in the velocities of two uniform streams, the closer the velocity distribution approaches to Görtler's approximate solution.

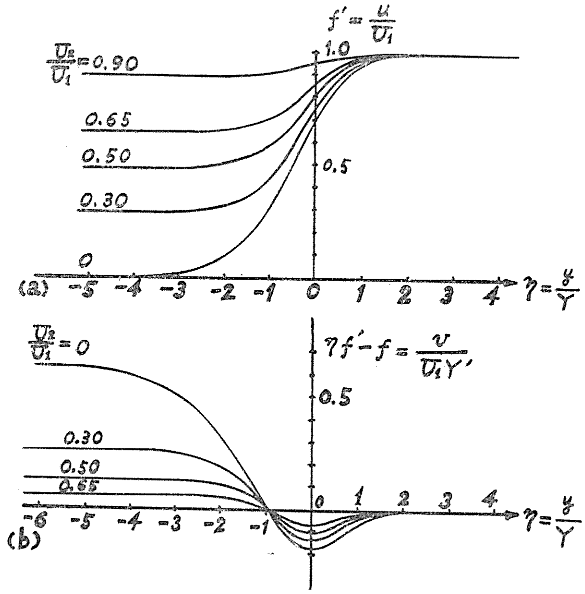

FIG. 2. Distribution of velocity components in the mixing zone of a free jet boundary: (a) x-component; (b) y-component

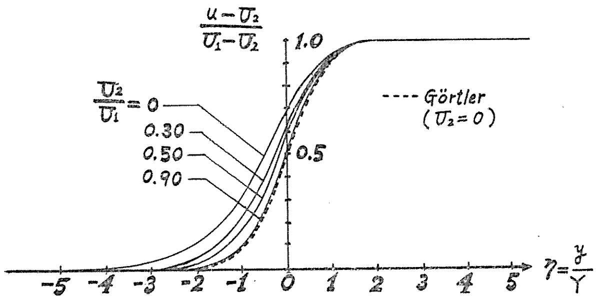

FIG. 3. Effect of the velocity ratio of two uniform streams on the free jet boundary flow, compared with Görtler's approximate solution

Eq. (19) can be integrated into

$$Y = 2\,U_1^{-1/2}\{\nu x + \frac{\kappa}{2}(U_1 - U_2)(x-a)^2 \cdot \mathscr{U}(x-a)\}^{1/2}. \tag{22}$$

Hence we get from Eq. (17)

$$G = 2\,U_1^{1/2}\{\nu x + \frac{\kappa}{2}(U_1 - U_2)(x-a)^2 \cdot \mathscr{U}(x-a)\}^{1/2}. \qquad (23)$$

Substituting Eqs. (22) and (23) into Eqs. (15) and (16), we have the final forms of the solution.

$$\left.\begin{aligned}
\eta &= \frac{U^{1/2}y}{2\{\nu x + \frac{\kappa}{2}(U_1 - U_2)(x-a)^2 \cdot \mathscr{U}(x-a)\}^{1/2}},\\
\Psi &= 2U_1^{1/2}\{\nu x + \frac{\kappa}{2}(U_1 - U_2)(x-a)^2 \cdot \mathscr{U}(x-a)\}^{1/2} \cdot f(\eta),\\
u &= \frac{\partial \Psi}{\partial y} = U_1 f'(\eta),\\
v &= -\frac{\partial \Psi}{\partial x} = U_1^{1/2}\{\nu + \kappa(U_1 - U_2)(x-a) \cdot \mathscr{U}(x-a)\}\\
&\quad \times \{\nu x + \frac{\kappa}{2}(U_1 - U_2)(x-a)^2 \cdot \mathscr{U}(x-a)\}^{-1/2}(\eta f' - f).
\end{aligned}\right\} \qquad (24)$$

4. The Width of the Mixing Zone

$Y(x)$ is regarded as a measure of the width of the mixing zone. Eq. (22) is made dimensionless by using a kind of Reynolds number with respect to a, as

$$\frac{1}{2}\sqrt{\frac{U_1 a}{\nu}}\frac{Y}{a} = \left\{\frac{x}{a} + \frac{\kappa}{2}\frac{U_1 a}{\nu}\left(1 - \frac{U_2}{U_1}\right)\left(\frac{x}{a} - 1\right)^2 \cdot \mathscr{U}\left(\frac{x}{a} - 1\right)\right\}^{1/2}. \qquad (25)$$

This equation is shown in Fig. 4 for various values of $\frac{\kappa}{2}\frac{U_1 a}{\nu}\left(1 - \frac{U_2}{U_1}\right)$ against x/a, which shows that the relative width Y/a multiplied by the root of Reynolds number increases with the strength of turbulence κ, Reynolds number and the relative velocity difference between two uniform streams $(U_1 - U_2)/U_1$.

In particular, if $U_2 = 0$, and the specified value of $\kappa = 0.001372$ from the measurements by H. Reichardt[5] is used, then the variation of the relative width Y/a is dependent on only one parameter $U_1 a/\nu$, and it approaches asymptotically to the

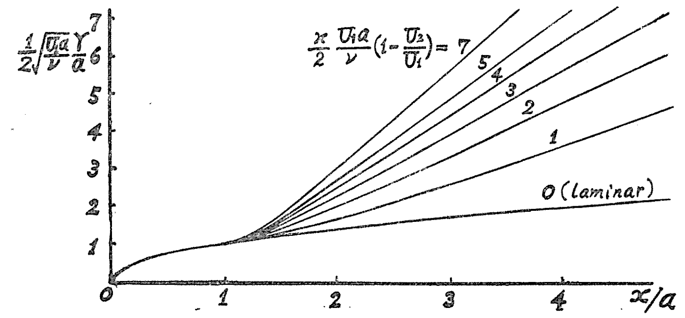

FIG. 4. Variation of the width of the mixing zone in a free jet boundary along the main stream direction. Eq. (25)

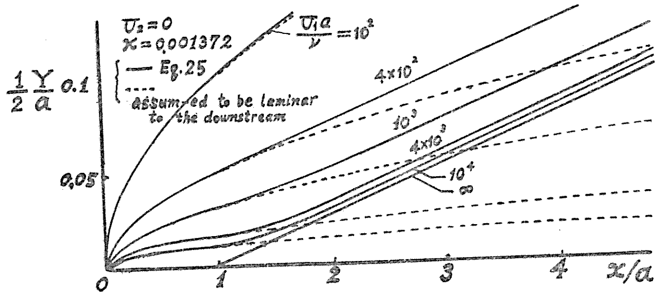

FIG. 5. Variation of the width of the mixing zone in a free jet boundary along the main stream direction; $U_2=0$, $\kappa=0.001372$

straight lines which are all parallel to the limiting line for $U_1a/\nu\rightarrow\infty$, as shown in Fig. 5.

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