STRESSES AROUND REINFORCED CIRCULAR HOLES ARRANGED IN A ROW IN AN INFINITE SHEET

STRESSES IN A REINFORCED SHEET. PART I

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1. Introduction

This paper presents a solution for the stress distributions around reinforced circular holes arranged in a row infinitely and spaced equally in an infinite sheet as shown in Fig. 1. The sheet is assumed to be subjected to a uniform stress system or a bending type stress system at infinity. All reinforcements around holes are assumed to have the equal extensional rigidity and not to have the flexural one. This simple problem might be useful to know the stress distributions around the reinforced windows, *e.g.* of civil aircrafts, arranged in a row.

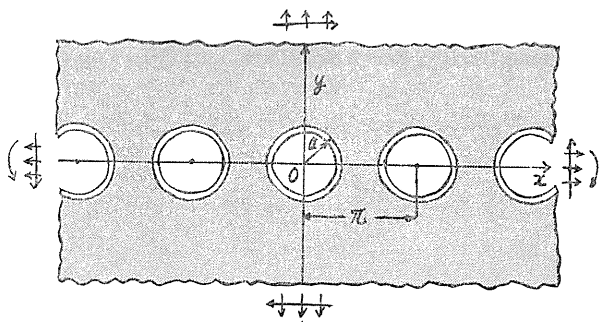

FIG. 1

Howland[1] and Atsumi[2] presented solutions for the stress distributions in a sheet containing an infinite row of unreinforced circular holes. The former dealt with the case that the sheet is subjected to a simple tensile stress system at infinity and the latter a shearing stress or a bending type stress system there. The present paper may confirm also their solutions numerically as a limiting case that the reinforcements become to zero.

(Notation)

a : radius of a circular hole
t : thickness of the sheet
A : cross-sectional area of a reinforcement
n, s : co-ordinates normal and tangential to a hole
ξ : angle between the x axis and the tangent to a hole

u, v : displacements of a point in the x, y directions
T : tention in a reinforcement
$\sigma_x, \sigma_y, \tau_z$: Cartesian stress components
σ_n, σ_s, τ: stress components referred to a pair of axes n and s
f_1, f_2 : tensile stresses at infinity in the x, y directions, respectively
S : shear stress at infinity referred to the x, y directions
p : constant defining the magnitude of a bending stress distribution at infinity

$z=x+iy,\ \bar{Z}=x-iy$

$$Q=\frac{2}{at/A+1+\nu}$$

ν : Poisson's ratio

$$F(n, m)\equiv\frac{(2a)^{m+n}}{m+n}B\Big(\frac{m+n}{2}\Big)\frac{1}{m!\ n!}$$

$$B(q)=\frac{2(2q)!}{(2\pi)^{2q}}\sum_{r=1}^{\infty}\frac{1}{r^{2q}}\quad \text{(Bernoulli's number)}$$

2. Basic Formulae for Plane Stress and Boundary Conditions

We have the following equations wellknown for the stress and displacement components.

$$\begin{aligned}&\sigma_x+\sigma_y=2[\varphi'(z)+\overline{\varphi'(z)}],\\&\sigma_y-\sigma_x+2i\tau_z=2[\bar{z}\varphi''(z)+\psi'(z)],\\&2G(u+iU)=\frac{3-\nu}{1+\nu}\varphi'(z)-z\overline{\varphi'(z)}-\overline{\psi(z)},\end{aligned}\tag{1}$$

where $\varphi(z)$ and $\psi(z)$ are functions of analytic in the region considered.

Then, it can be shown that the equations of equilibrium of an element of the reinforcement (Fig. 2) yield,[3]

$$T/t=ie^{-i\xi}[\varphi(\zeta)+\zeta\overline{\varphi'(\zeta)}+\overline{\psi(\zeta)}],\tag{2}$$

where $\zeta=-iae^{i\xi}$ in our case.

Since T should be real, we obtain the first condition to be satisfied on the boundaries, *i.e.*

$$T/at=\frac{\bar{\zeta}}{a^2}(\varphi(\zeta)+\zeta\overline{\varphi'(\zeta)}+\overline{\psi(\zeta)})=\frac{\zeta}{a^2}(\overline{\varphi(\zeta)}+\bar{\zeta}\varphi'(\zeta)+\psi(\zeta)),\tag{3}$$

Using one of the equilibrium conditions of the element, compatibility conditions on the boundaries give

$$T/A=\sigma_s-\nu\sigma_n=\sigma_s+\sigma_n-(1+\nu)T/at,$$

then we have the second condition on the bounaries

$$T/at=Q(\varphi'(\zeta)+\overline{\varphi'(\zeta)}).\tag{4}$$

Furthermore we find the following relations concerning the stresses in the sheet adjacent to reinforcements.

$$\sigma_n = Q(\varphi'(\zeta) + \overline{\varphi'(\zeta)}),$$
$$\sigma_s = (2 - Q)(\varphi'(\zeta) + \overline{\varphi'(\zeta)}), \qquad (5)$$
$$\tau = d\sigma_s/d\xi.$$

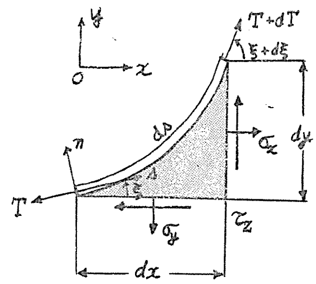

FIG. 2

3. Expressions of Complex Potentials $\varphi(z)$ and $\psi(z)$

Considering the periodicity of the stresses in the direction x and the stress conditions at infinity, we can take

$$\left.\begin{aligned}\varphi(z) &= \frac{\alpha^*}{2}z^2 + \alpha z + \alpha_0 + \sum_{n=1}^{\infty}\alpha_n\frac{d^{n-1}}{dz^{n-1}}\cot z,\\ \psi(z) &= \frac{\gamma^*}{2}z^2 + \gamma z + \gamma_0 + \sum_{n=1}^{\infty}(\gamma_n - n\alpha_n)\frac{d^{n-1}}{dz^{n-1}}\cot z - \sum_{n=1}^{\infty}\alpha_n z\frac{d^n}{dz^n}\cot z.\end{aligned}\right\} \qquad (6)$$

where α^*, γ^*, α, γ, α_0, γ_0 and α_n, γ_n are complex constants to be determined by the boundary conditions and the stress states at infinity, but it is convenient for us to put

$$\alpha_0 = aA_0,\ \gamma_0 = aC_0,\ \alpha_n = a^{n+1}A_n/(n-1)!,\ \gamma_n = a^{n+1}C_n/(n-1)!. \qquad (7)$$

4. The Sheet under Tension at Infinity (Case 1)

Consider the case that the sheet is subjected at infinity to uniform tensile stresses defined as

$$\sigma_{x\infty} = f_1,\ \sigma_{y\infty} = f_2,\ \tau_{z\infty} = 0.$$

When it is assumed that each reinforcement is symmetric about both axes parallel to x and y, consideration of symmetry of the stress distribution requires the following matters.

(i) α^*, γ^*, α_0, $\gamma_0 = 0$, $\quad A_n = C_n = 0$. (n; even)

(ii) α, γ, A_n, C_n $\quad$ (n; odd) are all real.

From the conditions at infinity, we obtain

$$\alpha = (f_1 + f_2)/4, \quad \gamma = -(f_1 - f_2)/2. \qquad (8)$$

Multiplying the boundary conditions (3), (4) by $\frac{1}{2\pi}e^{-im\xi}d\xi$ and integrating from 0 to 2π, we have linear simultaneous equations for unknown coefficients A_n, C_n as follows,

$$C_1 = -2(1-Q)\alpha + 4(1-Q)\sum_{n=0}^{\infty} A_{n+1}F(n, 2),$$

$$C_{m+1} = (1-Q)(m-1)A_{m-1} + (m+2)(1-(m+1)Q)\sum_{n=0}^{\infty} A_{n+1}F(n, m+2),$$

$$\gamma\delta(m-2) + (1+Q(m-1))A_{m-1} - (1-Q)\sum_{n=0}^{\infty} A_{n+1}(m+1)(m+2)F(n, m+2)$$
$$+\sum_{n=0}^{\infty} A_{n+1}m(n+m)F(n, m) - \sum_{n=0}^{\infty} C_{n+1}mF(n, m) = 0 \qquad (9)$$

Note that the integrations in the above procedure are easily performed, remembering a fundamental expression;

$$\cot z = \frac{1}{z} - \sum_{n=1}^{\infty}\frac{2^{2n}B(n)z^{2n-1}}{(2n)!} \quad (|z|<\pi/2).$$

Eliminating C's from above set, we obtain the equations of A's to be solved numerically.

Since A's are determined and subsequently C's also, the expressions for the stress distribution around holes may be easily calculated by the expression,

$$\varphi'(\zeta) + \overline{\varphi'(\zeta)} = 2\{\alpha - 2\sum_{n=0}^{\infty} A_{n+1}F(n, 2) - \sum_{m}(-1)^{m/2}\cos m\xi[(m-1)A_{m-1}$$
$$+\sum_{n=0}^{\infty} A_{n+1}(m+1)(m+2)F(n,m+2)]\}. \qquad (10)$$

5. The Sheet under Shear at Infinity (Case II)

Consider next the case that the sheet is subjected at infinity to simple shear, thus

$$\sigma_{x\infty} = \sigma_{y\infty} = 0 \qquad \tau_{z\infty} = S$$

In this case, it is required

(i) α^*, γ^*, α_0, γ_0 and $A_n = C_n = 0$ (n: even)

(ii) α, γ and A_n, C_n (n: odd) are all purely imaginary, and we can put from conditions at infinity,

$$\alpha = 0, \quad \gamma = iS \qquad (11)$$

Then in the same manner as previous section, we have also the relations between A's and C's

$$C'_1 = 0,$$

$$C'_{m+1} = (1-Q)(m-1)A'_{m-1} - (1-(m+1)Q)\sum_{n=0}^{\infty} A'_{n+1}(m+2)F(n, m+2),$$

$$S\delta(m-2) - (1+Q(m-1)A'_{m-1} - (1-Q)\sum_{n=1}^{\infty} A'_{n+1}(m+1)(m+2)F(n, m+2)$$
$$+\sum_{n=0}^{\infty} A'_{n+1}m(n+m)F(n, m) - \sum_{n=0}^{\infty} C'_{n+1}mF(n, m) = 0. \qquad (12)$$

TABLE 1

	a/b	A/at	A_1	A_3	A_5	A_7	A_9	A_{11}
f_1	0.15	0	0.409563	-0.213155^{-2}	-0.610597^{-4}	-0.169140^{-5}	-0.450007^{-7}	-0.968347^{-10}
		0.25	0.315377	-0.763943^{-3}	-0.165284^{-4}	-0.369364^{-6}	-0.826972^{-8}	-0.782880^{-10}
		0.5	0.278686	-0.494042^{-3}	-0.103403^{-4}	-0.227325^{-6}	-0.504740^{-8}	-0.702060^{-10}
		1	0.247026	-0.319972^{-3}	-0.668860^{-5}	-0.147384^{-6}	-0.328534^{-8}	-0.626189^{-10}
		2	0.224662	-0.221219^{-3}	-0.473135^{-5}	-0.105726^{-6}	-0.238170^{-8}	-0.567631^{-10}
	0.25	0	0.311948	-0.110967^{-1}	-0.800061^{-3}	$-0\ 549343^{-4}$	-0.352260^{-5}	-0.175859^{-7}
		0.25	0.256277	-0.444520^{-2}	-0.253607^{-3}	-0.148905^{-4}	-0.868994^{-6}	-0.208268^{-7}
		0.5	0.234583	-0.304437^{-2}	-0.171341^{-3}	-0.101071^{-4}	-0.599381^{-6}	-0.215229^{-7}
		1	0.215845	-0.210567^{-2}	-0.120555^{-3}	-0.726036^{-5}	-0.441028^{-6}	-0.220050^{-7}
		2	0.202594	-0.155123^{-2}	-0.921852^{-4}	-0.570032^{-5}	-0.354567^{-6}	-0.222993^{-7}
f_2	0.15	0	-0.475031	0.129035^{-2}	0.458851^{-4}	0.141044^{-5}	0.399099^{-7}	0.895192^{-10}
		0.25	-0.345971	0.500318^{-3}	0.128587^{-4}	0.312521^{-6}	0.735518^{-8}	0.718747^{-10}
		0.5	-0.295692	0.362940^{-3}	0.858629^{-5}	0.200523^{-6}	0.461602^{-8}	0.656553^{-10}
		1	-0.252306	0.282526^{-3}	0.619920^{-5}	0.139945^{-6}	0.316515^{-8}	0.608571^{-10}
		2	-0.221659	0.241414^{-3}	0.499204^{-5}	0.109682^{-6}	0.244585^{-8}	0.579454^{-10}
	0.25	0	-0.472871	0.642659^{-2}	0.629715^{-3}	0.502260^{-4}	0.355572^{-5}	0.194183^{-7}
		0.25	-0.331723	0.293508^{-2}	0.208764^{-3}	0.136746^{-4}	0.855343^{-6}	0.214107^{-7}
		0.5	-0.276570	0.228596^{-2}	0.149434^{-3}	0.950688^{-5}	0.591419^{-6}	0.218209^{-7}
		1	-0.228894	0.188735^{-2}	0.114343^{-3}	0.708865^{-5}	0.438557^{-6}	0.220932^{-7}
		2	-0.195166	0.167259^{-2}	0.955279^{-4}	0.579328^{-5}	0.355959^{-6}	0.222500^{-7}

	a/b	A/at	A_2	A_4	A_6	A_8	A_{10}	A_{12}
S	0.15	0	0.116485^{1}	0.470742^{-2}	0.145584^{-3}	0.420937^{-5}	0.115271^{-6}	0.253640^{-9}
		0.25	0.812320	0.159356^{-2}	0.367907^{-4}	0.851608^{-6}	0.195055^{-7}	0.187615^{-9}
		0.5	0.687487	0.104258^{-2}	0.229297^{-4}	0.517636^{-6}	0.116906^{-7}	0.164494^{-9}
		1	0.584786	0.709816^{-3}	0.151664^{-4}	0.338124^{-6}	0.759505^{-8}	0.145527^{-9}
		2	0.514864	0.532114^{-3}	0.112019^{-4}	0.248472^{-6}	0.557527^{-8}	0.132637^{-9}
	0.25	0	1.569572	0.424757^{-1}	0.351455^{-2}	0.273845^{-3}	0.203046^{-4}	0.122437^{-6}
		0.25	1.005756	0.139738^{-1}	0.876800^{-3}	0.554119^{-4}	0.347593^{-5}	0.899388^{-7}
		0.5	0.827679	0.919760^{-2}	0.555792^{-3}	0.346042^{-4}	0.216178^{-5}	0.817876^{-7}
		1	0.688428	0.635359^{-2}	0.377203^{-3}	0.234304^{-4}	0.147010^{-5}	0.758356^{-7}
		2	0.597118	0.484270^{-2}	0.285800^{-3}	0.178015^{-4}	0.112385^{-5}	0.720985^{-7}

where written as $A_n = iA'_n$, $C_n = iC'_n$ for brevity.

The stresses around a hole are now obtained with these values for the coefficients,

$$\varphi'(\zeta)+\overline{\varphi'(\zeta)}$$
$$= -2\sum_{m=2}^{\infty}(-1)^{2/m}\sin m\xi[(m-1)A'_{m-1}-\sum_{n=0}^{\infty}A'_{n+1}(m+1)(m+2)F(n,\ m+2)] \tag{13}$$

6. The Sheet under Simple Bending at Infinity (Case III)

Consider lastly the case that the sheet is subjected at infinity to simple bending, *i.e.*,

$$\sigma_{x\infty}=py, \quad \sigma_{y\infty}=\tau_{z\infty}=0$$

For this case we can put,

TABLE 2.1. σ_s/f_1 in Case I ($f_2=0$)

a/b	$\xi\rightarrow$ $A/at\downarrow$	0°	15°	30°	45°	60°	75°	90°
0.15	0	2.547 (2.54)	2.315 (2.31)	1.689 (1.69)	0.853 (0.86)	0.043 (0.05)	−0.532 (−0.54)	−0.738 (−0.74)
	0.25	1.772	1.631	1.247	0.729	0.219	−0.150	−0.284
	0.5	1.429	1.322	1.032	0.638	0.249	−0.034	−0.136
	1.0	1.098	1.021	0.812	0.528	0.245	0.040	−0.035
	2.0	0.835	0.780	0.629	0.423	0.219	0 070	0.016
0.25	0	2.139 (2.16)	1.916 (1.92)	1.334 (1.33)	0.616 (0.62)	0.024 (0.02)	−0.299 (−0.30)	−0.038 (−0.39)
	0.25	1.533	1.401	1.052	0.601	0.190	−0.077	−0.167
	0.50	1.260	1.160	0.892	0.541	0.214	−0.006	−0.082
	1.0	0.988	0.915	0.718	0.457	0.209	0.039	−0.021
	2.0	0.763	0.710	0.565	0.371	0.185	0.056	−0.010

TABLE 2.2. σ_s/f_2 in Case I ($f_1=0$)

a/b	$\xi\rightarrow$ $A/at\downarrow$	0°	15°	30°	45°	60°	75°	90°
0.15	0	−0.780	−0.518	0.194	1.156	2.102	2.784	3.031
	0.25	−0.237	−0.084	0.334	0.900	1.462	1.870	2.019
	0.5	−0.072	0.040	0.347	0.764	1.178	1.479	1.589
	1.0	0.034	0.112	0.326	0.616	0.904	1.114	1.191
	2.0	0.079	0.134	0.283	0.486	0.687	0.834	0.887
0.25	0	−0.612 (−0.61)	−0.327 (−0.33)	0.444 (0.44)	1.459 (1.46)	2.407 (2.41)	3.032 (3.03)	3.241 (3.24)
	0.25	−0.094	0.064	0.491	1.060	1.603	1.978	2.110
	0.5	0.050	0.163	0.469	0.876	1.267	1.538	1.634
	1.0	0.132	0.209	0.415	0.690	0.954	1.138	1.202
	2.0	0.156	0.208	0.349	0.536	0.714	0.838	0.882

* Values in ( ) are reported by Howland

(i) $\alpha,\ \gamma=0$ and $A_n=C_n=0$ (n: odd)

(ii) $\alpha^*,\ \gamma^*,\ \alpha_0,\ \gamma_0$ and $A_n,\ C_n$ (n: even) are all imaginary.

and $\alpha^*=-\frac{p}{4}i,\ \gamma^*=\frac{p}{4}i$ from the conditions at infinity.

Arrived equations are;

$$\left.\begin{aligned} &C'_m=2(1-Q)A'_{m-2}+\sum_{n=2}^{\infty}A'_n(1-mQ)nF(n,\ m),\\ &\frac{pa}{8}\delta(m-2)+(m+1)A'_m-C'_{m+2}-\sum_{n=2}^{\infty}A'_n(m+1)nF(n,\ m+2)\\ &\qquad+\sum_{n=2}^{\infty}A'_n n(n+m)F(n,\ m)-\sum_{n=2}^{\infty}C'_n nF(n,\ m)=0, \end{aligned}\right\} \tag{14}$$

where $m,\ n$ are positive even integers.

$$\begin{aligned} &\varphi(\zeta)+\overline{\varphi(\zeta)}\\ &=-\frac{pa}{2}\cos\xi-2\sum_{n=2}^{\infty}nA'_n\Big[(-1)^{2/n}\cos(n+1)\xi-\sum_{n=2}^{\infty}mF(n,\ m)(-1)^{2/m}\cos(m-1)\xi\Big]. \end{aligned} \tag{15}$$

TABLE 3.1. σ_s/S in Case II

a/b	$\xi\rightarrow$ $A/at\downarrow$	0°	15°	30°	45°	60°	75°	90°
0.15	0	0.000 (0.00)	2.276 (2.28)	3.974 (3.97)	4.641 (4.64)	4.070 (4.07)	2.373 (2.37)	0.000 (0.00)
	0.25	0.000	1.303	2.266	2.631	2.292	1.329	0.000
	0.5	0.000	0.952	1.654	1.918	1.669	0.967	0.000
	1.0	0.000	0.662	1.150	1.332	1.158	0.670	0.000
	2.0	0.000	0.464	0.806	0.933	0.810	0.469	0.000
0.25	0	0.000 (0.00)	2.693 (2.70)	4.875 (4.88)	6.048 (6.05)	5.717 (5.71)	3.561 (3.56)	0.000 (0 00)
	0.25	0.000	1.494	2.649	3.172	2.868	1.718	0.000
	0.5	0.000	1.074	1.894	2.251	2.017	1.199	0.000
	1.0	0.000	0.736	1.294	1.529	1.362	0.805	0.000
	2.0	0.000	0.511	0.896	1.056	0.937	0.552	0.000

TABLE 3.2. τ/S in Case II

a/b	$\xi\rightarrow$ $A/at\downarrow$	0°	15°	30°	45°	60°	75°	90°
0.15	0	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.25	1.201	1.045	0.613	0.014	−0.599	−1.059	−1.229
	0.5	1.631	1.417	0.829	0.015	−0.814	−1.432	−1.660
	1.0	1.985	1.724	1.006	0.014	−0.991	−1.738	−2.013
	2.0	2.227	1.933	1.126	0.013	−1.112	−1.946	−2.253
0.25	0	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.25	1.369	1.215	0.777	0.116	−0.656	−1.331	−1.609
	0.5	1.830	1.616	1.012	0.120	−0.887	−1.736	−2.079
	1.0	2.197	1.933	1.193	0.117	−1.072	−2.050	−2.439
	2.0	2.441	2.143	1.311	0.112	−1.195	−2.255	−2.672

* Values in ( ) are reported by Atumi

7. Numerical Results for circular Holes with uniform Reinforcement

Some numerical results have been obtained, using the digital computer NEAC 2203 in our university. As seen immediately, it is difficult to obtain exact solutions from the equations described above. Consequently a finite number of coefficients in the infinite series were retained. But the convergence of the values of them was quite good in all cases, where retained number was six. In order to illustrate its degree, it must be noted that the final results agreed with the one obtained retaining only four coefficients by four significant decimals.

The stresses around holes were computed at intervals of and some of them are shown in Tables 2, 3, and Figs. 3∼5.

8. Conclusions

Plane stress problems have been solved for a infinite sheet containing reinforced circular holes arranged in a row under several types of loading at infinity.

i) Our results agreed exactly with the other authors', as might be expected, in the limiting case for the reinforcement vanished and for a single circular hole with various amount of reinforcement.

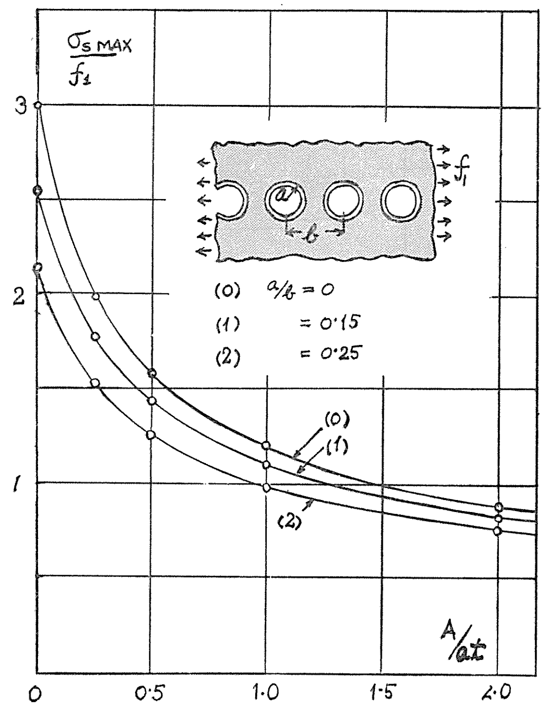
$\sigma_{S\,MAX}$ / f_1
3
2
1
(0) a/b = 0
(1) = 0·15
(2) = 0·25
(0)
(1)
(2)
A/at
0
0·5
1·0
1·5
2·0

FIG. 3

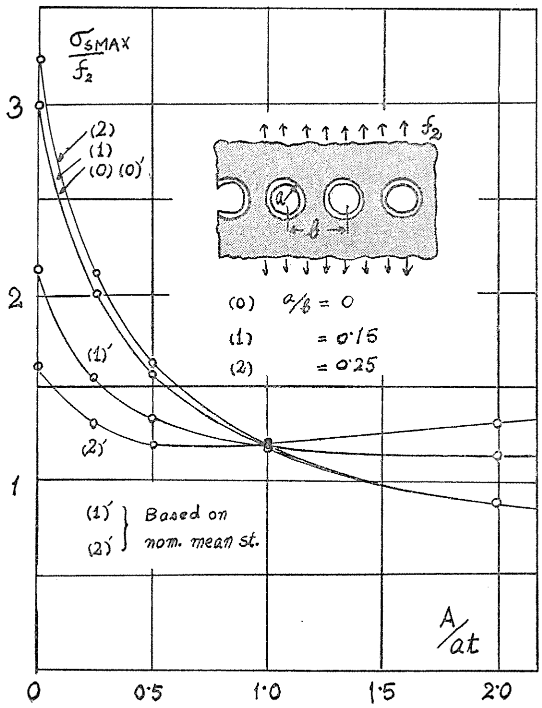
$\sigma_{S\,MAX}$ / f_2
3
2
1
(2)
(1)
(0) (0)'
(1)'
(2)'
(0) a/b = 0
(1) = 0·15
(2) = 0·25
(1)' (2)' Based on nom. mean st.
A/at
0
0·5
1·0
1·5
2·0

FIG. 4

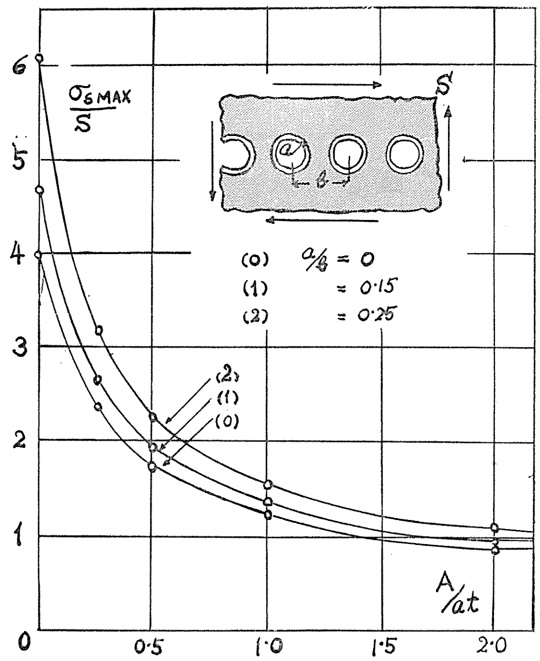

FIG. 5

ii) As the hole spacing decreases, the stress concentration factors around holes are changed substantially in the range of light reinforcement and somewhat in the range of heavier as shown in Figs.

iii) Neautral holes are also achieved in a loading case uniformly stretching in both directions with $A/at=1$.

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