NOTE ON SHEATH FORMATION

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Abstract

We have discussed a plasma-sheath transition as a function of the electric current delivered to the sheath, using an one-dimensional, collisionless model and a set of macroscopic equations.

When the conditions $\gamma^2=\sum_{e,i} m_s v_{s_0}^2/\sum_{e,i} kT_s=1$ and $E_0=0$, (where the subscript s stands for e (electron) and i (ion), m_s is the mass of a particle, v_{s_0} the velocity at the plasma-sheath boundary, T_s the temperature and E_0 the electric field at the boundary, (i) for $m_i v_{i_0}^2/kT_i>1$ (or $m_e v_{e_0}^2/kT_e<1$) the pure ion sheath is formed without any local excess of electron space charge, (ii) for $m_i v_{i_0}^2/kT_i=1$ (or $m_e v_{e_0}^2/kT_e=1$) any sheath is not established and (iii) for $m_i v_{i_0}^2/kT_i<1$ (or $m_e v_{e_0}^2/kT_e>1$) the pure electron sheath is developed. When $\gamma^2\neq 1$ and $E_0=0$ or $\neq 0$, then the formation of sheath is much complicated. Thus the conditions $\gamma=1$ and $E_0=0$ lead to the establishment of much simple sheathes among the various sheathes and are including the familiar Bohm criterion when $v_{e_0}\to 0$.

Furthermore, one may find that the ideal Langmuir probe curve, in which the conditions $\gamma^2=1$ and $E_0=0$ are imposed, deviates from the classical probe theory.

§ 1. Introduction

The problem of a plama-sheath transition has been discussed by D. Bohm[1] and found that a stable ion sheath is formed when ions enter into the plasma-sheath boundary with velocities of the order $(kT_e/m_i)^{1/2}$ or greater, where m_i is the mass of ion and T_e the electron temperature. He has also studied[3] that the maximum ion current which we can draw out from the plasma is of the order of the magnitude $j_{ib}\simeq en_0(kT_e/m_i)^{1/2}$, where e is the charge of ion and n_0 the number density of ions at the plasma sheath boundary.

These statements are based on the assumptions that the strength of electric field at the boundary, E_0, is nearly equal to zero and the ion temperature is also. These assumptions seem to be wrong[2], especially when E_0 is finite then a stable ion sheath is well established for the ions with velocities lower than $(kT_e/m_i)^{1/2}$. Without these assumptions, J. E. Allen *et al.*[4] have re-examined and improved the above theory.

Recently, P. L. Auer[5] has done research on the role of ion current in the formation of sheath in a low pressure positive column and found that the familiar Bohm criterion is not essential but a minimum ion current is required to be delivered to the sheath.

That statement suggests to have to study the formation of sheath as a function of electric current. The reader will certainly know that the Langmuir probe immersed in a plasma can drag out a positive or negative electric current from the plasma by applying electric potential to the probe. Thus we must study the problem of a plasma-sheath transition as a function of electric current, which is the purpose of this note and which has been suggested by H. K. Wimmel[6].

§ 2. Fundamental Equations and Integrations

Consider an one-dimensional steady state system composed with ions and electrons. Their number densities, n_i and n_e, ars equal each other at the plane $(0, y, z)$ of the rectangular coordinates and charges move towards a plane collector electrode placed on the plane (d, y, z). We shall assume electron-ion collisions infrequent in the system and, furthermore, assume that such asystem is described by a set of macroscopic equations.

The set of equations is written as

$$\frac{\partial}{\partial x} n_s v_s = 0, \qquad (s = e \text{ or } i) \tag{1}$$

$$n_s m_s v_s \frac{\partial}{\partial x} v_s = e_s n_s E - \frac{\partial}{\partial x} p_s, \tag{2}$$

$$p_s \simeq n_s k T_s, \tag{3}$$

$$\varepsilon_0 \frac{\partial E}{\partial x} = \sum e_s n_s, \tag{4}$$

$$E = -\frac{\partial V}{\partial x}, \tag{5}$$

where n_s in the number density, v_s the velocity, m_s the mass of a pareticle, e_s the charge, E the strength of electric field, p_s the partial pressure, k the Boltzmann constant, T_s the temperature, ε_0 the dielectric constant in a vacuum and V the electric potential.

When $v_s = 0$ and T_s constant all over the system, then Eqs. (2) and (3) lead

$$n_s \propto exp(-e_s V/kT_s), \tag{6}$$

which is the density distribution in an electric potential being in the thermal eqilibrium. When v_s is finite, then T_s is no more constant, but it, perhaps, changes adiabatically as it is usual in hydrodynamics. Thus whether the temperature T_s is constant or adiabatic is depend on the velocity v_s. We shall here assume T_s constant, because this note is only a qualitative discussion and most of experiments, regarding the Langmuir probe measurement, supports the exponential form of Eq. (6).

The equations (1)∼(5) can be integrated as follows,

$$n_i v_i = \text{constant}, \tag{7}$$

$$n_e v_e = \text{constant}, \tag{8}$$

$$m_i n_i v_i^2 + m_e n_e v_e^2 - \frac{\varepsilon_0 E^2}{2} + (n_i k T_i n_e k T_e) = \text{constant}, \tag{9}$$

$$\frac{m_i v_i^2}{2} + eV + kT_i \, ln \cdot n_i = \text{constant}, \tag{10}$$

$$\frac{m_e v_e^2}{2} - eV + kT_e \, ln \cdot n_e = \text{constant}. \tag{11}$$

The equation (9) expresses the conservation of stress and Eqs. (10) and (11) mean the conservation of energy.

We shall now impose the following boundary conditions, namely

$$n_i = n_e = n_0,$$

$$v_i = v_{i_0}, \quad v_e = v_{e_0},$$

$$E = E_0 \text{ and } V = 0,$$

at $x=0$.

§ 3. The N_i-N_e Diagram

For the simplification of calculations, we wish to use various dimensionless quantities, namely

$$C_s^2 = m_s v_s^2/kT_s, \quad N_s = n_s/n_0, \quad U_s = v_s/v_{s_0},$$

$$\Psi = \varepsilon_0(E^2 - E_0^2)/2\,n_0 kT_e \text{ and } \varphi = eV/kT_e.$$

With these notations and the boundary conditions given in § 2, Eq. (9) becomes

$$\Psi = \Psi_i + \Psi_e, \tag{12}$$

where

$$\Psi_i = \frac{T_i}{T_e}\left\{ C_i^2\left(\frac{1}{N_i} - 1\right) - (1 - N_i)\right\}. \tag{13}$$

and

$$\Psi_e = C_e^2\left(\frac{1}{N_e} - 1\right) - (1 - N_e). \tag{14}$$

The equations (10) and (11) are rewritten as

$$-\varphi = \frac{T_i}{T_e}\left\{\frac{C_i^2}{2}\left(\frac{1}{N_i^2} - 1\right) + ln \cdot N_i\right\}, \tag{15}$$

$$\varphi = \left\{\frac{C_e^2}{2}\left(\frac{1}{N_e^2} - 1\right) + ln \cdot N_e\right\}. \tag{16}$$

In Fig. 1, the curve $T_e\varphi/T_i$ was plotted as a function of N_i for the various values of C_i. From the figure, one may see that $T_e\varphi/T_i$ has a maximum value at $N_i = C_i$. Furthermore, we can plot φ as a function of N_e as shown in Fig. 2 from which we know that φ takes a minimum value at $N_e = C_e$.

From Eqs. (15) and (16), we can easily find the relation between N_i and N_e eliminating φ which is determined when C_i and C_e are given. Some examples of the relation between N_i and N_e were shown in Figs. 3 and 4. The relation N_i vs. N_e will be called, "The N_i-N_e diagram". In Figs. 3 and 4, the point ($N_i = 1$, $N_e = 1$) means the plasma-sheath boundary and the broken straight line expresses $N_i = N_e$. In the region upper than the straight line $N_i = N_e$, the ion density is larger than the electron density, whereas the lower region gives $N_i < N_e$.

From Eqs. (15) and (16) or Figs. 3 and 4, one may conclude the following terms:

(1) For $C_i^2 = 1$ and $C_e^2 = 1$, the solution is only the point ($N_i = 1$, $N_e = 1$), which

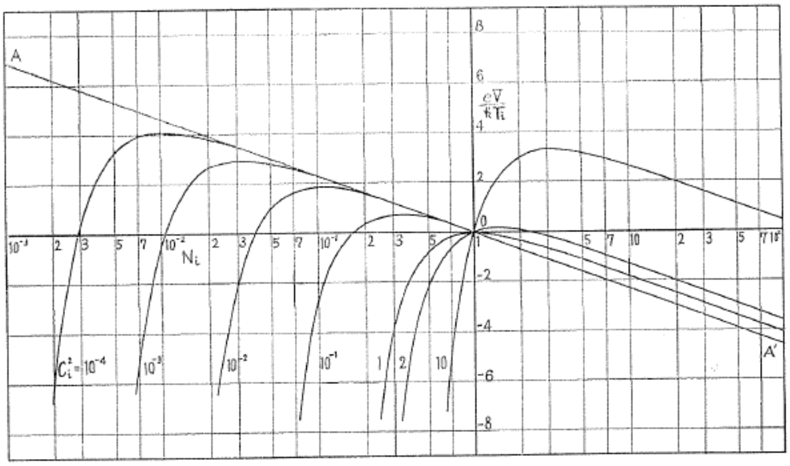

FIG. 1. The curves of $T_e\varphi/T_i(=eV/kT_i)$ vs. $N_i(=n_i/n_0)$. The straight line A–A' shows $n_i/n_0=exp(-eV/kT_i)$.

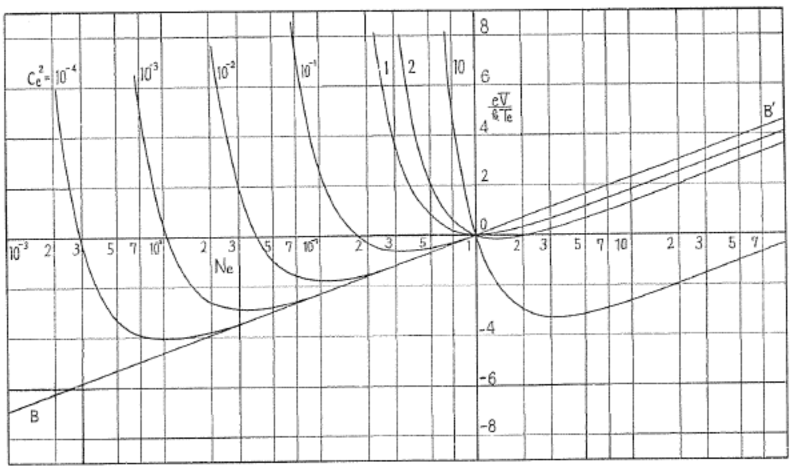

FIG. 2. The curves of φ vs. $N_e(=n_e/n_0)$. The straight line B–B' shows $n_e/n=exp(eV/kT_e)$.

implies that any sheath does not develope and the electric potential of the collector drawing out the electric current from plasma source is identical to that of plasma source.

(2) The N_i-N_e diagram generally is a closed curve or a loop. The loop intersects with the straight line $N_i=N_e$ at two points which are determined by the following equation,

$$\frac{\gamma^2}{2}\left(\frac{1}{N_i^2}-1\right)+ln\cdot N_i=0, \tag{17}$$

where

$$\gamma^2=\frac{T_iC_i^2+T_eC_e^2}{T_i+T_e}.$$

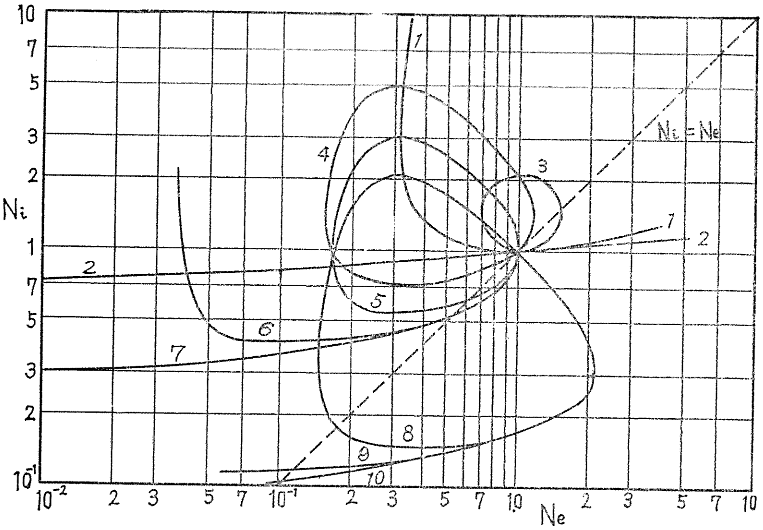

FIG. 3. The N_i-N_e diagram, which is obtained by eliminating φ in Figs. 1 and 2. We have assumed $T_i=T_e$. The curve 1 is for ($C_i^2=10$, $C_e^2=1$), 2 ($C_i^2=10$, $C_i^2=10$, $C_e^2=10^{-4}$), 3 ($C_i^2=2$, $C_e^2=1$), 4 ($C_i^2=2$, $C_e^2=10^{-1}$), 5 ($C_i^2=1$, $C_e^2=10^{-2}$), 6 ($C_i^2=1$, $C_e^2=10^{-2}$), 7 ($C_i^2=1$, $C_e^2=10^{-3}$), 8 ($C_i^2=10^{-1}$, $C_i^2=10^{-1}$), 9 ($C_i^2=10^{-1}$, $C_e^2=10^{-2}$), 10 ($C_i^2=10^{-1}$. $C_e^2=10^{-3}$).

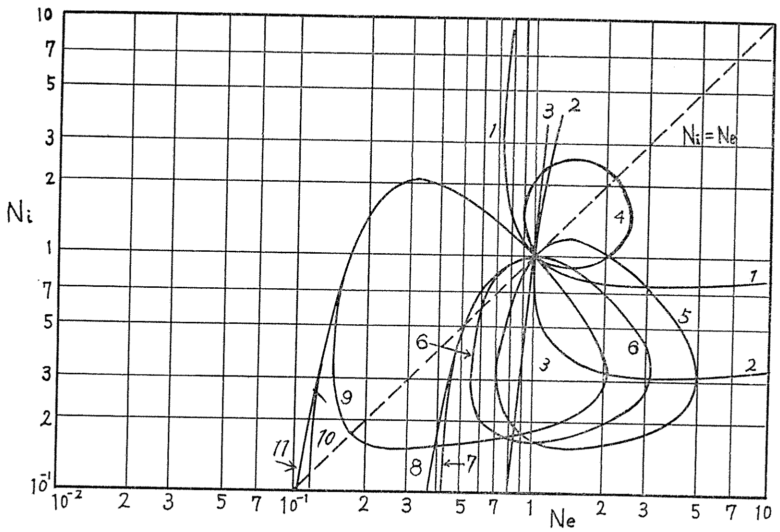

FIG. 4. The N_i-N_e diagram, where $T_e=T_i$. The curve 1 is for ($C_e^2=10$, $C_i^2=10$), 2 ($C_e^2=10$, $C_i^2=1$), 3 ($C_e^2=10$, $C_i^2=10^{-4}$), 4 ($C_e^2=2$, $C_i^2=2$), 5 ($C_e^2=2$, $C_i^2=10^{-1}$), 6 ($C_e^2=1$, $C_i^2=10^{-1}$), 7 ($C_e^2=1$, $C_i^2=10^{-2}$), 8 ($C_e^2=1$, $C_i^2=10^{-3}$, 9 ($C_e^2=10^{-1}$, $C_i^2=10^{-1}$), 10 ($C_e^2=10^{-1}$, $C_i^2=10^{-2}$) and 10 ($C_e^2=10^{-1}$, $C_i^2=10^{-3}$).

One of two intersecting points is $N_i=N_e=1$ and independent of γ^2. The other point, denoting it by $(N_i)_a$, is depend on γ^2 as it was tabulated in Table 1. From Table 1, one may see that $(N_i)_a>1$ for $\gamma^2>1$ and $(N_i)_a<1$ for $\gamma^2<1$. For $\gamma^2=1$, two intersecting points are identical each other and, only when $\gamma^2=1$, the loop is in contact with the straight line $N_i=N_e$ at the point $(N_i=1,\ N_e=1)$. Thus if the condition $\gamma^2=1$ is satisfied then the loop will be located in the region either $N_i\geq N_e$ or $N_i\leq N_e$, whereas if $\gamma^2\neq 1$ then a part of loop will be in the region $N_i\leq N_e$ and the residual part in the region $N_i\geq N_e$. In the other words, when $\gamma^2=1$ one may find either a pure ion excess sheath or a pure electron excess sheath, whereas when $\gamma^2\neq 1$ a complicated sheath is formed just as space charge is partly positive and partly negative.

TABLE 1. The values of $(N_i)_a$ and $T_e\phi((N_i)_a)/(T_e+T_i)$ for the various values of γ^2

γ^2	$(N_i)_a$	$\frac{T_e}{T_e+T_i}\phi((N_i)_a)$
10	147	136
1	1	0
10^{-1}	0.165	-0.3
10^{-2}	0.0295	-0.7
10^{-3}	0.00106	-0.9
0	0	-1.0

(3) We can easily calculate (dN_e/dN_i) at the boundary using Eqs. (15) and (16), namely

$$\left(\frac{dN_e}{dN_i}\right)_{x=0}=\alpha=-\frac{T_i}{T_e}\left(\frac{1-C_i^2}{1-C_e^2}\right). \tag{18}$$

from which $\alpha=0$ when $C_i^2=1$, α infinite when $C_e^2=1$ and $\alpha=1$ when $\gamma^2=1$. Thus only when $\gamma^2=1$, (dN_i/dx) at the boundary is equal to (dN_e/dx) at the point.

(4) Excepting two intersecting points, there are four noticeable points on the loop, i.e. N_i is maximum or minimum at $N_e=C_e$ and N_e is also at $N_i=C_i$. We shall denote these extreame values as $(N_i)_{max}$, $(N_i)_{min}$, $(N_e)_{max}$ and $(N_e)_{min}$.

Thus far only we have discussed the relation of N_i to N_e which generally is a loop in the N_i-N_e diagram. However, we can not say that all of the points on the loop is existent in a real physical system.

§ 4. The Electric Field

We shall now discuss the dimensionless electric field Ψ. The functions Ψ_e and Ψ_i are identical to each other in their algebraic forms, so that only Ψ_e was plotted as a function of N_e in Fig. 5 which shows that Ψ_e has a minimum value at $N_e=C_e$. Although Ψ is a function of both N_i and N_e, we can rewrite Ψ as a function of only N_i using the N_i-N_e diagram, i.e. $\Psi=\Psi(N_i)$.

The function $\Psi(N_i)$ takes various extreme values at the following values of N_i.

$$N_i=N_e,$$

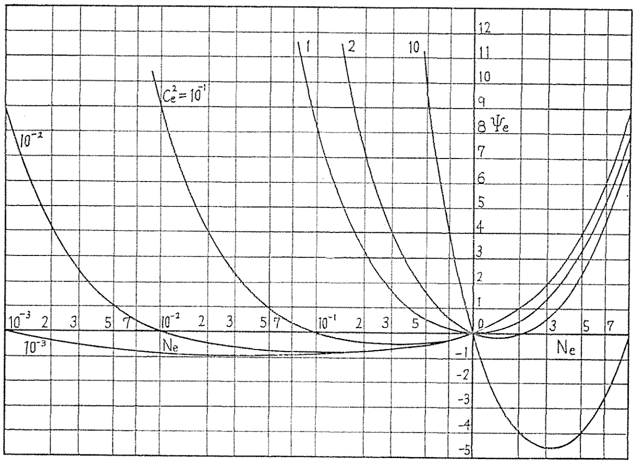

FIG. 5. The curves of Ψ_e as a function of N_e.

$$N_i = N_e = 1 \tag{19 a}$$

and

$$N_i = C_i. \tag{19 b}$$

The function $\Psi(N_i)$ is minimum or maximum at $N_i = C_i$ depending on whether $N_i > N_e$ or $N_i < N_e$ because $dN\Psi/dN_i^2$ is written as

$$\frac{d^2\Psi}{dN_i^2} = \frac{T_i}{T_e}\left\{2(N_i - N_e) + \frac{N_e(1 + T_i/T_e)(N_i^2 - C_i^2)(N_e^2 - \gamma^2)}{N_i^2(N_e^2 - C_e^2)}\right\}, \tag{20}$$

For the values of N_i given by Eq. (19 a), the function $\Psi(N_i)$ is also maximum and the value of $\Psi(N_i)$ at $N_i = N_e = (N_i)_a$, $\Psi((N_i)_a)$, can be calculated by the following equation,

$$\frac{T_e}{T_i + T_e}\Psi(N_i)_a) = \gamma^2\left(\frac{1}{(N_i)_a} - 1\right) - (1 - (N_i)_a),$$

and was tabulated in Table 1. From the table one may see $\Psi(N_i)_a) < 0$ for $\gamma^2 < 1$, $\Psi((N_i)_a) > 0$ for $\gamma^2 > 1$ and $\Psi((N_i)_a) = 0$ for $\gamma^2 = 1$.

The function $\Psi(N_i)$ was sketched in Figs. 6, 7 and 8 for $\gamma^2 = 1$, $\gamma^2 < 1$ and $\gamma^2 > 1$ respectively. In these figures, the right-hand side gives the relation $\Psi(N_i)$ vs. N_i and the left-hand side is the N_i-N_e diagram. Furthermore, the notations *max.* and *min.* mean the maximum and minimum values of $\Psi(N_i)$ and the symboles (*max*) and (*min*) express the extreme values of $\Psi(N_e)$.

In any real physical system, the square of the electric field E must be positive or zero, i.e. $E^2 \geq 0$ or $2n_0kT_e\Psi/\varepsilon_0 + E_0^2 \geq 0$. Furthermore, for the problem of a plasma-sheath transition being in a steady state, the electric field should be continuous and both N_i and N_e have to change continuously starting $N_i = N_e = 1$. When $\gamma^2 = 1$ and $E_0 = 0$, the above reqirements lead, seeing Fig. 6, that the allowable range of N_i is $(1, (N_i)_{min})$, where $(N_i)_{min}$ is the minimum value of N_i or the value of N_i at $N_e = \dot{C}_e$. When $\gamma^2 = 1$ and $E_0 \neq 0$, the allowable ranges of N_i are $(1, (N_i)_{min})$ and $(1, N_i > 1)$, where in the later range the upper limit of N_i

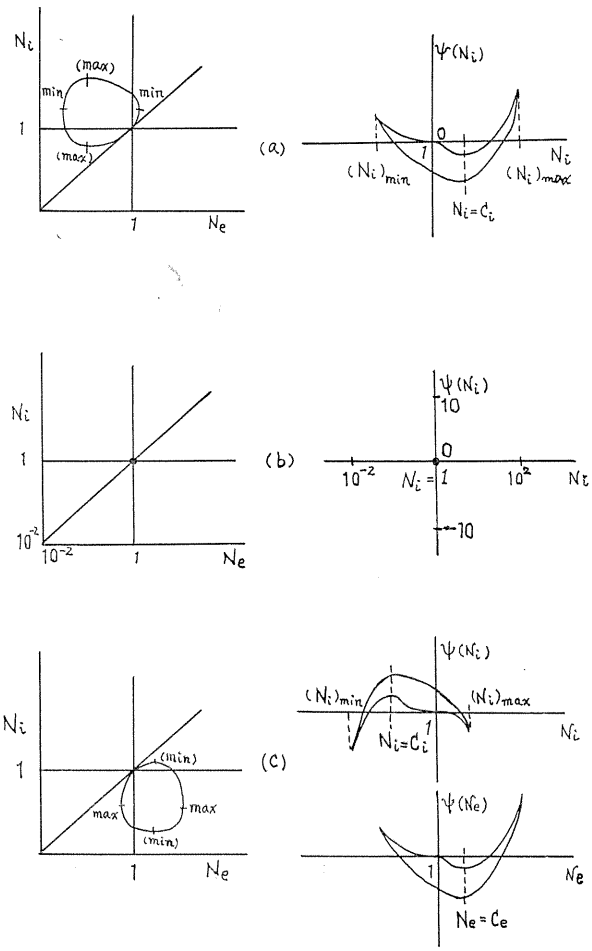

FIG. 6. The left-hand side is the N_i-N_e diagram and the right-hand side is the function $\Psi\ (=\varepsilon_0(E^2-E_0^2)/2\,n_0kT_e)$, where $\gamma^2=1$. (a) $C_i^2>1$ (or $C_e^2<1$), (b) $C_i^2=1$ (or $C_e^2=1$) and (c) $C_i^2<1$ (or $C_e^2>1$). The notations *max* and *min* are the maximum and minimum values of $\Psi(N_i)$ and (*max*) and (*min*) are those of $\Psi(N_e)$.

may be determined by $E=0$. It is not yet clear whether such two ranges of N_i is connected with an instability or not. When $\gamma^2\neq1$ and $E_0=0$, seeing Figs. 7 and 8, situations are much complicated. Only we can say that for Figs. 7 (a), 7 (e) and 8 (c) the solution is $N_i=N_e=1$.

§ 5. The Ideal Langmuir Probe Curve

Thus far, there has been appeared three parameters C_i, C_e and E_0. The last parameter would be determined by an inherent character of plasma source and the electric current being delivered to the sheath. Let j be the density of

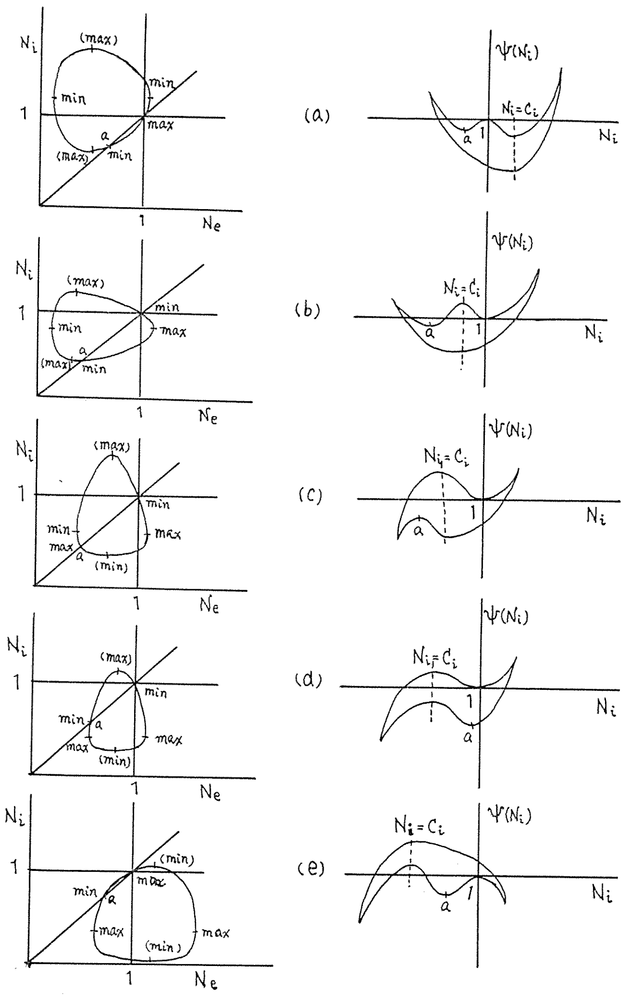

FIG. 7. See the caption of Fig. 6, where $\gamma^2<1$.

electriccurrent and let σ be the conductivity of plasma. Then E_0 is given by the following equation,

$$E_0 = j/\sigma.$$

The density of electric current j, which is measurable, can be written as

$$j = en_0(v_{i_0} - v_{e_0}),$$

so that either C_i or C_e is able to replace by the measurable quantity j. However, either C_i or C_e is left undetermined. That difficulty may be settled by a measurement of momentum.

We shall now calculate the Langmuir probe curve[7] imposing the following conditions,

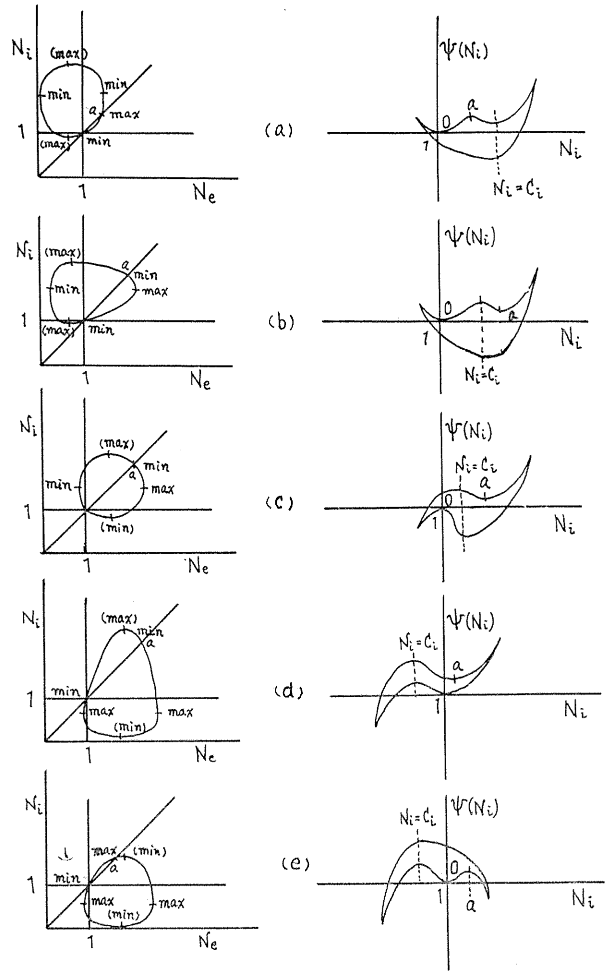

FIG. 8. See the caption of Fig. 6, where $\gamma^2 > 1$.

$$\gamma^2 = 1 \text{ and } E_0 = 0.$$

These conditions seem to be of more general than the familiar Bohm criterion leading to the formation of a stable ion sheath, because:

(i) When $C_i^2 > 1$ or $(C_e^2 < 1$ since $\gamma^2 = 1)$, then the pure ion excess sheath is formed, i.e. the number density of ions is equal or greater than that of electrons all over the system. As $v_{e_0} \to 0$, the velocity of ions at the plasma-sheath boundary becomes

$$v_{i_0} = \{(kT_e + kT_i)/m_i\}^{1/2},$$

from which the density of electric current delivered to the sheath, j_{is}, is written as

$$j_{is} = en_0\{(kT_i + kT_e)/m_i\}^{1/2}.$$

(ii) When $C_i^2=1$ (or $C_e^2=1$), then any sheath is not formed in the system and therefore the electric potential of collector is equal to that of plasma source. In this case, the electric current flowing into the collector, j_p, is calculated as follows,

$$j_p = -en_0\{(kT_e/m_e)^{1/2}-(kT_i/m_i)^{1/2}\} \simeq -en_0(kT_e/m_e)^{1/2}.$$

(iii) When $C_i^2<1$ (or $C_e^2>1$), then the pure electron excess sheath is established. As $v_{i_0}\to 0$, the velocity of electrons at the boundary and the electric current dragged out from plasma are written as

$$v_{e_0}=\{(kT_i+kT_e)/m_e\}^{1/2},$$
$$j_{es}=-en_0\{(kT_i+kT_e)/m_e\}^{1/2},$$

respectively.

We can clculate the Langmuir probe curve by the following procedure: (1) For $C_i^2>1$, the range of N_i is $(1, (N_i)_{\min})$ and only the range is significant physically. In the other words, the range of N_e is $(1, C_e)$. With the definition

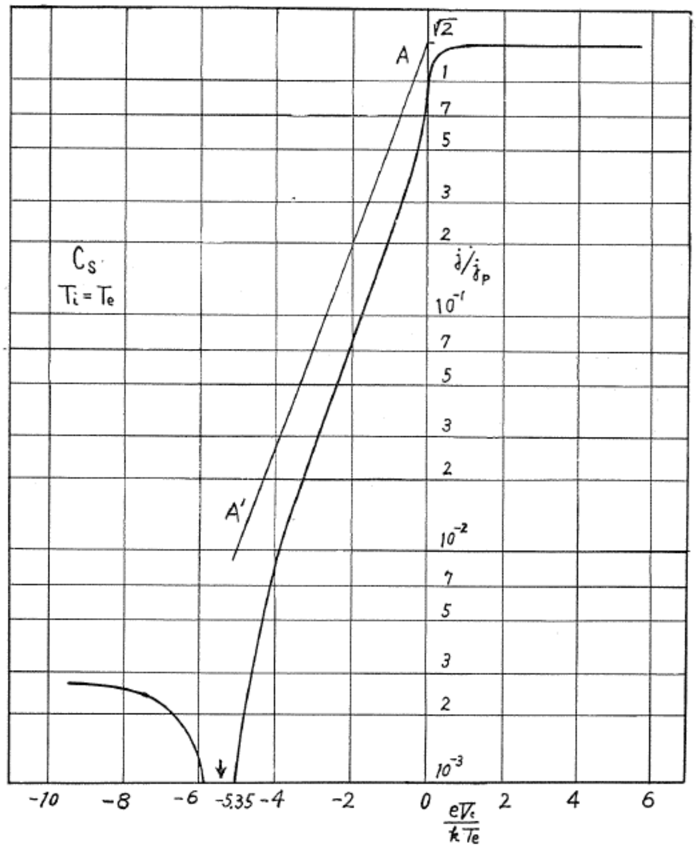

FIG. 9. The ideal Langmuir probe curve, in which the conditions $\gamma^2=1$ and $E_0=0$ are imposed, for C_s assuming $T_i=T_e$. In the figure, the straigh tline A-A' shows the relation $j/j_{es}=\exp(eV_c/kT_e)$.

of j, $\gamma^2=1$ and Eq. (16) substituting $N_e=C_e$, we can obtain the $j-V_c$ curve. Here V_c is the electric potential of collector relative to that of plasma source. (2) For $C_i^2<1$, the significant range of N_e is $(1, (N_e)_{min})$ which corresponds to the range $(1, C_i)$. In the same way as for $C_i^2>1$, we can get the relation of j to V_c. Thoretical probe curve obtained by the above procedure will be called, "The ideal Langmuir probe curve".

In Fig. 9 we showed the ideal Langmuir probe curve for C_s plasmg assuming $T_e=T_i$. The straight line A–A' in the figure gives the relation $j=j_{es}exp(eV_c/kT_e)$ proposed by I. Langmuir. We must note that dj/dV_c is infinite at the plasma potential. Furthermore, a linear part, that $\ln \cdot j$ is proportinal to eV_c/kT_e, is in a narrow range of eV_c/kT_e. The lighter the mass of ion, the narrower the range of eV_c/kT_e.

§6. Conclusion

We have discussed the problem of a plasma-sheath transition as a function of the electric current drawn out from plasma, using an one-dimensional collisionless model and a set of macroscopic equations.

The equations give the relation between the ion density and the electron density which is a loop in the $n_i/n^0-n_e/n_0$ diagram. The loop is determined when two parameters $C_i^2=m_i v_{i0}^2/kT_i$ and $C_e^2=m_e v_{e0}^2/kT_e$. These loops are classified into two groups depending on whether $\gamma^2=1$ or $\gamma^2\neq1$. When $\gamma^2=1$, then the loop is in the region $n_i\geq n_e$ or $n_i\leq n_e$, whereas when $\gamma^2\neq1$ then the loop is partly in the region $n_i\geq n_e$ and partly $n_i\leq n_e$.

The $n_i/n_0-n_e/n_0$ diagram and the equation of stress lead to the following conclusions: When $\gamma^2=1$ and $E_0=0$ are satisfied, then (i) for $C_i^2>1$ (or $C_e^2<1$) the pure ion excess sheath is formed, viz. the ion space charge is equal or greater than the electron density all over the system. (ii) For $C_i^2=1$ (or $C_e^2=1$), any sheath is not developed and the system is electrical neutral, viz. the electric potential of the collector drawing out the electric current is identical to the plasma potential. (iii) For $C_i^2<1$ (or $C_e^2>1$) the pure electron excess sheath is established. When $\gamma^2\neq1$ and $E_0=0$ or $E_0\neq0$, then sheathes are much complicated.

In §5 we have discussed the Langmuir probe curve, and calculated the ideal Langmuir probe curve in which the conditions $\gamma^2=1$ and $E_0=0$ are imposed. The ideal Langmuir probe curve deviats from the classical theory of probe proposed by I. Langmuir and shows that resistance in the system is equal to zero at the plasma potential.

(Note: A set of equations Eqs. (1)-(5) are solved by making use of the following integration,

$$x=\int\frac{dV/dn_i}{E(n_i)}dn_i,$$

$$\text{or}\qquad =\int\frac{dV/dn_e}{E(n_e)}en_e.)$$

I wish to add to this note the following sentence: when plasma source is uniform and semi-infinite in the region $x\leq0$, then the conditions $\gamma^2=1$ and $E_0=$

0 are necessary and sufficient for the establishment of sheath in the region $x>0$.

From Poisson's equation, it follows

$$\varepsilon_0\left(\frac{dE}{dx}\right)_{x=0} = e(n_i - n_e)_{x=0} = 0,$$

and the second-order derivative of E is written as

$$\varepsilon_0\left(\frac{d^2E}{dx^2}\right) = e\left\{\left(\frac{dn_i}{dx}\right)_{x=0} - \left(\frac{dn_e}{dx}\right)_{x=0}\right\}.$$

These two equations imply that when $(d^2E/dx^2)_{x=0} \neq 0$ then the electric field E is minimum or maximum. Since our plasma source is uniform and semi-infinite, so the electric field can not take any extreme value at $x=0$ and inflect at there; i.e. $(d^2E/dx^2)_{x=0}$ must be equal to zero. By making use of Eqs. (15) and (16), the condition $(d^2E/dx^2)_{x=0}=0$ is rewritten as

$$\gamma^2 = 1,$$

which is independent of whether $E_0=0$ or $\neq 0$.

Furthermore, it is required that

$$\left(\frac{d^nE}{dx^n}\right)_{x=0} = 0, \qquad (n \geq 3)$$

This requirement is satisfied only when $E_0=0$, because when $E_0=0$, in general, $(d^n n_i/dx^n)_{x=0}=0$ and $(d^n n_e/dx^n)_{x=0}=0$ from Eqs. (15) and (16), where $n \geq 1$. When $E_0 \neq 0$, however, they are different from zero Therefore, when $E_0 \neq 0$, the electric field has an inflection point at $x=0$.

Thus the opening sentence has been proved. If plasma source is finite and if a sheath is allowable to form in the region $x \leq 0$, which means that plasma is something like a plane, then conditions leading to the formation of sheath in the region $x>0$ will no more be only those of $\gamma^2=1$ and $E_0=0$.

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