COLLISIONLESS EXPANSION OF GASES OF FINITE AMOUNT INTO VACUUM

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(Received October 30, 1963)

Summary

The expansion of gas molecules into vacuum ejected in a finite period is investigated on the basis of the collisionless Boltzmann equation. It is shown that the cloud of molecules diffuses into space advancing with mass velocity and that the distribution of density is fairly deformed by individual mass velocity.

1. Introduction

The ejection of gas into space is one of the fundamental problems in astronautical sciences. Some ambient properties in space can be measured by the emission of sample gas molecules and momentum produced by jet is generally used to propel space vehicles and to control their orientations. As a fundamental aspect of this problem the expansion of finite amount of gas molecules ejected into vacuum with a mass velocity in one direction is studied in the present paper, being based on the collisionless Boltzmann equation.

There has been some investigations on free expansion of gas molecules. Keller[1] has studied one dimensional expansion from the boundary of semi-infinite mass of gas particles. Molmud[2] has calculated the concentration of molecules in free expansion of symmetric gas clouds. Narasimha[3] has derived a general formulation of collisionless expansion and has given results on expansion of point cloud, symmetric and asymmetric clouds of gases and on continuous ejection through point sources.

In the present paper the general formulation introduced by Narasimha is extended to apply to solve the expansion of gas ejected for a finite period, which has much reality in space sciences comparing with continuous emission. The initial molecular velocity distribution is assumed to be Maxwellian.

2. The general expression of collisionless free molecular flow

The basic formulation is to find the expression of molecular velocity distribution function, which gives the number density of molecules of velocity $\mathbf{V}(u, v, w)$ at position $\mathbf{X}(x, y, z)$ and time t per unit volume in physical and velocity space. This function denoted by $f=f(\mathbf{X}, t; \mathbf{V})$ is governed by Boltzmann's equation. For a monoatomic gas with no external forces it is given

$$\frac{\partial f}{\partial t}+\mathbf{V}\frac{\partial f}{\partial \mathbf{X}}=[G(f)-fL(f)]+Q(\mathbf{X}, t;\ \mathbf{V}),$$

where $G(f)-fL(f)$ stands for the collision integrals and Q is the number of molecules which emits at X, t per unit volume of X, V space and per unit time. When the effect of collision is neglected, Boltzmann's equation becomes simply

$$\frac{\partial f}{\partial t}+V\frac{\partial f}{\partial X}=Q(X,\ t;\ V), \tag{1}$$

where V takes all possible values.

This kind of problem should satisfies the initial condition which takes the following form:

$$f(X,\ t;\ V)=f_0(X;\ V)\quad \text{at}\quad t=0, \tag{2}$$

where f_0 is an given arbitrary function of X and V. Eq. (1) is a linear first order partial differential equation and can be solved by the method of characteristics. Introducing a prameter s the characteristic differential equations are expressed by

$$\frac{dt}{ds}=1,\quad \frac{dX}{ds}=V,\quad \frac{df}{ds}=Q \tag{3}$$

The integration of Eq. (3) gives

$$t=s,\quad X_s=Vs+X_0,\quad f=\int_0^s Qds+f_0 \tag{4}$$

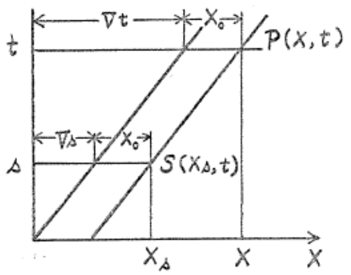


FIG. 1. Characteristic variables.

where X_0 can be eliminated by the relation $X_s=X$ at $s=t$, therefore, by $X_0=X-Vt$. It is seen that s represents the historical time of particle with velocity V and that $X_s=X-Vt+Vs$ is its position as shown in Fig. 1. The particle passed the point **S** at time s with velocity t can reach point **P** at time t.

Function f at time t can, therefore, be written by

$$f(X,\ t;\ V)=f_0(X-Vt;\ V)+\int_0^t Q(X-Vt+Vs,\ s;\ V)ds \tag{5}$$

This is the general solution of distribution function derived by Narasimha and the density, velocity and other quantities can be calculated by using Eq. (5).

3. Jet flow emitted for a finite period

The collisionless free molecular flow ejected from a point nozzle continuously for a certain finite period is considered. The gas molecules are ejected with a given mean velocity U at a certain number rate $\dot{N}(t)$ per unit time. To express the jet from a point source Dirac delta function $\delta(X)$ is introduced, which has value 1 when integrated including origin. Assuming Maxwellian velocity distribution the number rate of gas flow ejected from the point source can be expressed by

$$Q=\delta(\mathbf{X}_s)\dot{N}(s)\left(\frac{\beta}{\pi}\right)^{3/2}\exp\{-\beta(\mathbf{V}-\mathbf{U})^2\} \tag{6}$$

where β is the parameter of velocity distribution and is expressed by using the most probable velocity V_m, mass of the molecule m, gas constant for one molecule k, gas constant for one mole R, Avogadro number A and absolute temperature T as follows:

$$\beta=\frac{1}{V_m^2}=\frac{m}{2\,kT}=\frac{mA}{2\,RT} \tag{7}$$

The whole space is assumed to be vacuum before ejection, therefore, the initial condition

$$f_0=0 \tag{8}$$

is applied.

Substituting Eqs. (6) and (8) into Eq. (5), function f can be expressed by

$$f(\mathbf{X},\,t\,;\,\mathbf{V})=\left(\frac{\beta}{\pi}\right)^{3/2}\exp\{-\beta(\mathbf{V}-\mathbf{U})^2\}\int_0^t\delta(\mathbf{X}-\mathbf{V}t+\mathbf{V}s)\dot{N}(s)ds \tag{9}$$

In order to see the velocity dependence of delta function by changing variable from $\mathbf{X}-\mathbf{V}t+\mathbf{V}s$ to $\mathbf{V}-\mathbf{X}/(t-s)$, the following transformation introduced by Narasimha is used:

$$\sigma=\frac{1}{t-s}\quad\text{or}\quad s=t-\frac{1}{\sigma}$$

and therefore

$$\delta(\mathbf{X}-\mathbf{V}t+\mathbf{V}s)=\sigma^3[-\delta(\mathbf{V}-\mathbf{X}\sigma)]$$

Then Eq. (9) is transformed to

$$f(\mathbf{X},\,t\,;\,\mathbf{V})=\left(\frac{\beta}{\pi}\right)^{3/2}\exp\{-\beta(\mathbf{V}-\mathbf{U})^2\}\int_{1/t}^{\infty}[-\delta(\mathbf{V}-\mathbf{X}\sigma)]\dot{N}\left(t-\frac{1}{\sigma}\right)\sigma d\sigma \tag{10}$$

Flow quantities can be calculated by this equation with proper functional expression for $\dot{N}$, which governs the source strength with time.

Density and velocity distribution of gas cloud can be derived by integrating mf and $m\mathbf{V}f$ in whole velocity space respectively.

$$\begin{aligned}\rho(\mathbf{X},\,t)&=m\left(\frac{\beta}{\pi}\right)^{3/2}\int_{1/t}^{\infty}\sigma\dot{N}d\sigma\int_{-\infty}^{\infty}\exp\{-\beta(\mathbf{V}-\mathbf{U})^2\}[-\delta(\mathbf{V}-\mathbf{X}\sigma)]D\mathbf{V}\\&=m\left(\frac{\beta}{\pi}\right)^{3/2}\int_{1/t}^{\infty}\sigma\dot{N}\left(t-\frac{1}{\sigma}\right)\exp\{-\beta(\mathbf{X}\sigma-\mathbf{U})^2\}d\sigma\end{aligned} \tag{11}$$

$$\begin{aligned}\mathbf{u}(\mathbf{X},\,t)&=\frac{m}{\rho}\left(\frac{\beta}{\pi}\right)^{3/2}\int_{1/t}^{\infty}\sigma\dot{N}d\sigma\int_{-\infty}^{\infty}\mathbf{V}\exp\{-\beta(\mathbf{V}-\mathbf{U})^2\}[-\delta(\mathbf{V}-\mathbf{X}\sigma)]D\mathbf{V}\\&=\frac{m}{\rho}\mathbf{X}\left(\frac{\beta}{\pi}\right)^{3/2}\int_{1/t}^{\infty}\sigma^2\dot{N}\left(t-\frac{1}{\sigma}\right)\exp\{-\beta(\mathbf{X}\sigma-\mathbf{U})^2\}d\sigma\end{aligned} \tag{12}$$

To integrate these equations it is more convenient to take x axis in the direction of vector $\mathbf{U}$ and to deform

$$-\beta(\mathbf{X}\sigma-\mathbf{U})^2=-\beta[(X\sigma-U\cos\theta)^2+U^2\sin^2\theta]$$

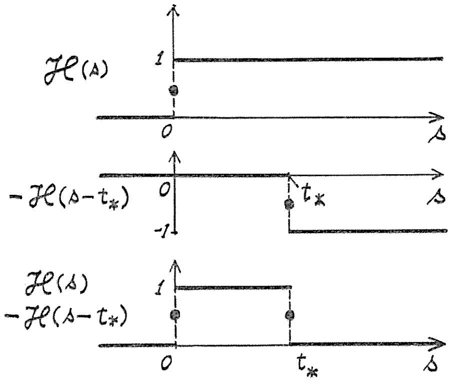


FIG. 2. Heaviside step function.

where $X=|\mathbf{X}|=x/\cos\theta$ is magnitude of position vector and θ is its direction cosine to x axis.

Now flow quantities of jet emitted with constant rate $(\dot{N})$ from time 0 to time t_* will be calculated. The variations of source strength can be expressed by using Heaviside step function $\mathscr{H}(s)$ shown in Fig. 2.

$$\dot{N}\left(t-\frac{1}{\sigma}\right)=(\dot{N})[\mathscr{H}(s)-\mathscr{H}(s-t_*)]=(\dot{N})\left[\mathscr{H}\left(t-\frac{1}{\sigma}\right)-\mathscr{H}\left(t-t_*-\frac{1}{\sigma}\right)\right] \tag{13}$$

For this expression of $\dot{N}$ Eqs. (11) and (12) can be explicitly integrated by applying variable transformation $\omega=\sqrt{\beta}\,(X\sigma-U\cos\theta)$. The solutions are given in the following form:

$$\rho(\mathbf{X},t)=I_\rho(\mathbf{X},t)-I_\rho(\mathbf{X},t-t_*),\quad \mathbf{u}(\mathbf{X},t)=[I_u(\mathbf{X},t)-I_u(\mathbf{X},t-t_*)]/\rho(\mathbf{X},t) \tag{14}$$

where

$$\begin{aligned}I_\rho(\mathbf{X},t)&=m(\dot{N})\left(\frac{\beta}{\pi}\right)^{3/2}\exp\{-\beta U^2\sin^2\theta\}\int_{1/t}^{\infty}\exp(-\omega^2)\mathscr{H}\left(t-\frac{1}{\sigma}\right)\sigma d\sigma\\&=\frac{m(\dot{N})\sqrt{\beta}}{2\pi^{3/2}X^2}\exp\{-\beta U^2\sin^2\theta\}[\exp(-\kappa^2)+\sqrt{\pi}\sqrt{\beta}\,U\cos\theta\,\mathrm{erfc}(\kappa)]\end{aligned} \tag{15}$$

$$\begin{aligned}\mathbf{I}_u(\mathbf{X},t)&=\mathbf{X}m(\dot{N})\left(\frac{\beta}{\pi}\right)^{3/2}\exp\{-\beta U^2\sin^2\theta\}\int_{1/t}^{\infty}\exp(-\omega^2)\mathscr{H}\left(t-\frac{1}{\sigma}\right)\sigma^2 d\sigma\\&=\frac{\mathbf{X}}{X}\frac{m(\dot{N})}{2\pi^{3/2}X^2}\exp\{-\beta U^2\sin^2\theta\}\left[\sqrt{\beta}\left(\frac{X}{t}+U\cos\theta\right)\exp(-\kappa^2)+\frac{\sqrt{\pi}}{2}\right.\\&\qquad\left.(1+2\beta U^2\cos^2\theta)\,\mathrm{erfc}(\kappa)\right]\end{aligned} \tag{16}$$

and

$$\kappa=\sqrt{\beta}\left(\frac{X}{t}-U\cos\theta\right)=\sqrt{\beta}\left(\frac{x}{\cos\theta}\frac{1}{t}-U\cos\theta\right) \tag{17}$$

Non-dimensional exprssion will be preferable to show general form and numerical result. As the unit of quantities the most probable velocity V_m, ejection time t_*, length L, density ρ_* and the number rate of ejection of molecules $(\dot{N})$ are introduced by defining:

$$V_m=\frac{1}{\sqrt{\beta}},\quad L=V_m t_*,\quad \rho_*=\frac{mN}{(V_m t_*)^3},\quad (\dot{N})=\frac{N}{t_*}$$

where N is the whole number of molecules. Non-dimensional quantities are denoted by

$$\frac{X}{V_m t_*} = \lambda, \quad \frac{x}{V_m t_*} = \xi, \quad \frac{\sqrt{y^2+z^2}}{V_m t_*} = \eta, \quad \frac{t}{t_*} = \tau$$

and non-dimensionalized κ is denoted by

$$\Lambda_\tau = \frac{\lambda}{\tau} - \frac{U}{V_m}\frac{\xi}{\lambda} = \kappa \tag{18}$$

Introducing similar parameter

$$\Omega_\tau = \frac{\lambda}{\tau} + \frac{U}{V_m}\frac{\xi}{\lambda} \tag{19}$$

and other parametric expressions

$$\left.\begin{aligned} G_\tau &= \exp(-\Lambda_\tau^2) + \sqrt{\pi}\frac{U}{V_m}\frac{\xi}{\lambda}(1-\mathrm{erf}\,\Lambda_\tau) \\ H_\tau &= \Omega_\tau \exp(-\Lambda_\tau^2) + \frac{\sqrt{\pi}}{2}\left(1+2\frac{U^2}{V_m^2}\frac{\xi^2}{\lambda^2}\right)(1-\mathrm{erf}\,\Lambda_\tau) \end{aligned}\right\} \tag{20}$$

The final form of density and velocity distribution is given by

$$\frac{\rho}{\rho_*} = \frac{1}{2\pi^{3/2}}\frac{1}{\lambda^2}\exp\left\{-\left(\frac{U}{V_m}\right)^2\left(\frac{\eta}{\lambda}\right)^2\right\}[G_\tau - G_{\tau-1}] \tag{21}$$

$$\frac{\mathbf{u}}{V_m} = \frac{\mathbf{X}}{X}\frac{H_\tau - H_{\tau-1}}{G_\tau - G_{\tau-1}} \tag{22}$$

The magnitude of velocity is

$$\frac{|\mathbf{u}|}{V_m} = \frac{H_\tau - H_{\tau-1}}{G_\tau - G_{\tau-1}} \tag{23}$$

Eq. (22) shows that velocity vectors are all radial.

4. Jet flow with continuous emission

When the jet is started to emit at time 0 and continues ejection with constant rate $(\dot{N})$, the function of source strength should be put

$$\dot{N}\left(t-\frac{1}{\sigma}\right) = (\dot{N})\mathscr{H}(s) = (\dot{N})\mathscr{H}\left(t-\frac{1}{\sigma}\right) \tag{24}$$

This case is already calculated by Narasimha[3] and the results are shown by the present simbols as follows

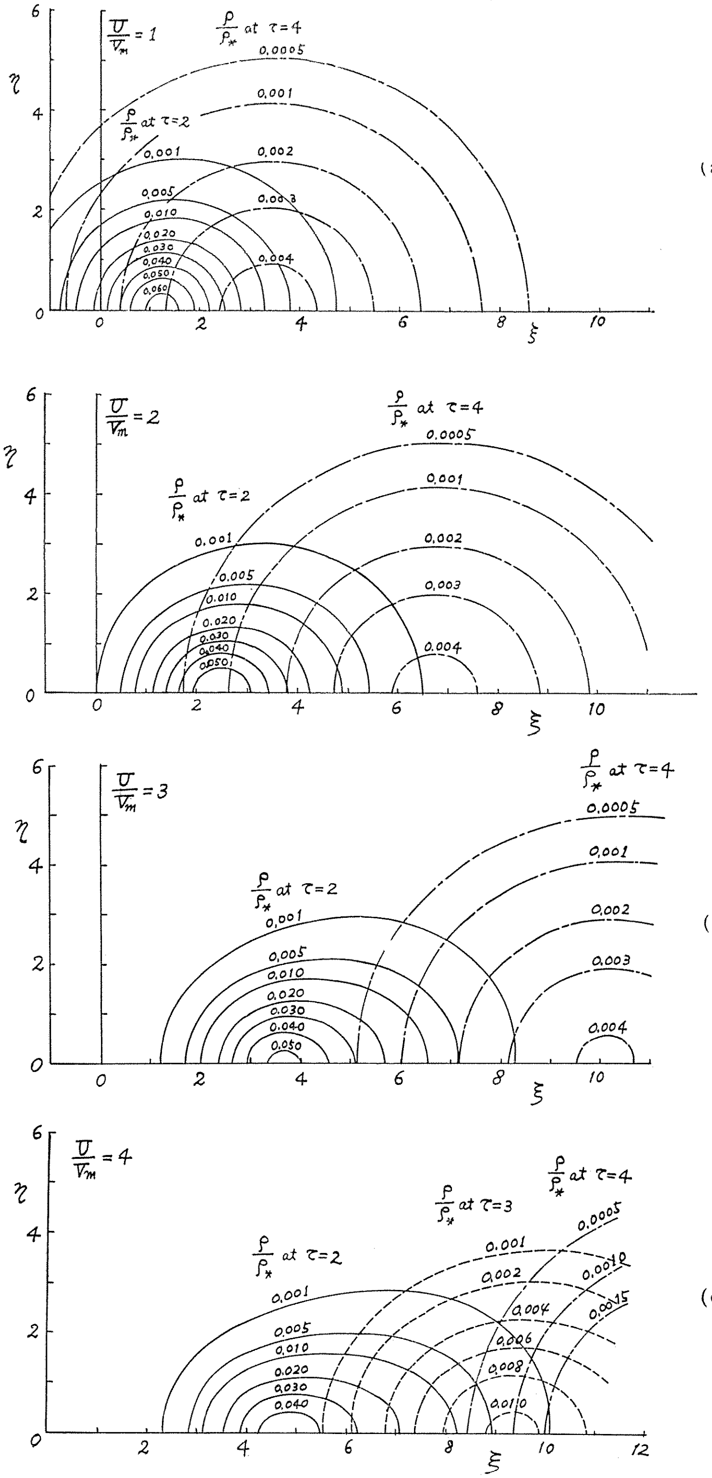
$$\frac{\rho}{\rho_*} = \frac{1}{2\pi^{3/2}}\frac{1}{\lambda^2}\exp\left\{-\left(\frac{U}{V_m}\right)^2\left(\frac{\eta}{\lambda}\right)^2\right\}G_\tau \tag{25}$$

$$\frac{\mathbf{u}}{V_m} = \frac{\mathbf{X}}{X}\frac{H_\tau}{G_\tau} \tag{26}$$

where unit of time t_* is used for only a reference of time.

5. Numerical examples of free molecular jet

Several examples of numerical calculation on time variations of equi-density



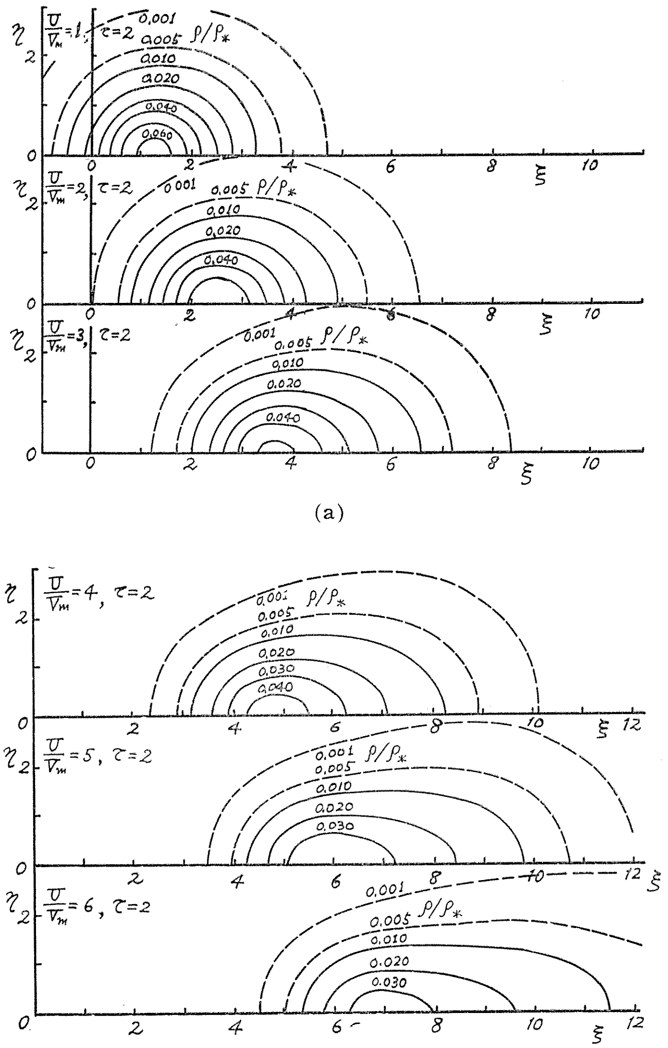
(a) $U/V_m=1$

(b) $U/V_m=2$

(c) $U/V_m=3$

(d) $U/V_m=4$

FIG. 3. Variations of equi-density contours.



(a)

(b)

FIG. 4 (a), (b). Deformation of concentration by different mean velocity.

lines in the meridian plane are shown in Fig. 3.

The aspects of expansion of gas clouds translating with mean motion can be seen in clear form. The peak point of maximum density is moving by a lower velocity than the mean velocity itself, which may be caused by the fact that the gas molecules are more concentrated close to the origin.

The pattern of equi-density lines is highly affected by the mean velocity U/V_m as shown in Fig. 4. When there is no mean motion the concentration should be co-spherical. The higher the velocity of the gas ejection, the flatter becomes the pattern and the more it is elongated in the direction of ξ axis. This nature suggests the fundamental concept of an idea to measure the velocity

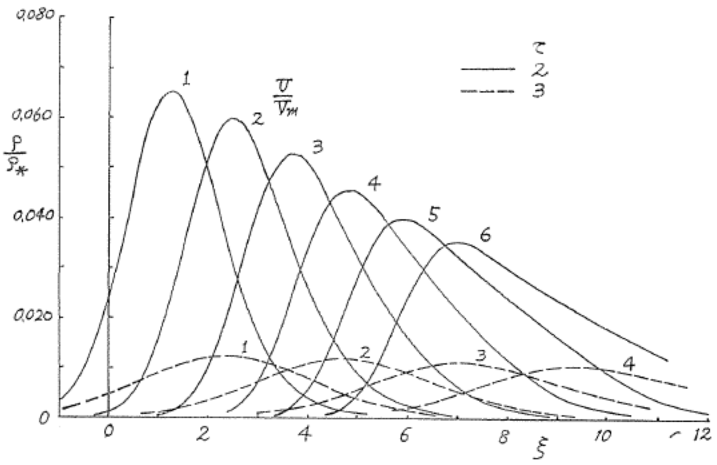


FIG. 5. Density distribution along axis.

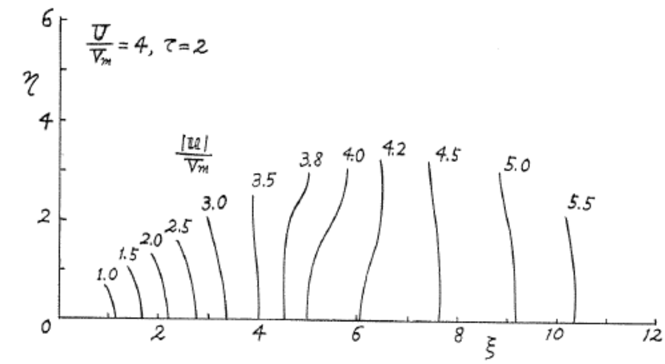


FIG. 6. An example of velocity distribution.

of body in vacuum. Density distributions along ξ axis are shown in Fig. 5. It is found that particles are relatively accumulated in the front part of density pattern with increasing mean velocity.

In Fig. 6 an example of equi-velocity lines is presented.

Acknowledgement

The authors are grateful to the financial aid by the Scientific Research Fund of the Ministry of Education.

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