SHEAR BUCKLING OF SIMPLY-SUPPORTED INFINITELY LONG PLATES REINFORCED BY OBLIQUE STIFFENERS

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§ 1. Introduction

In Ref. (1), Crook and Rockey presented a solution to the shear buckling of clamped and simply-supported infinitely long plates reinforced by transverse stiffeners spaced equally. This paper presents a solution to the shear buckling of simply-supported infinitely long plates reinforced by oblique stiffeners as shown in Fig. 1.

All stiffeners are assumed to have the same flexural and extensional rigidity and not to have the tortional one. The method used in this paper is the strain-energy method used in Ref. (1).

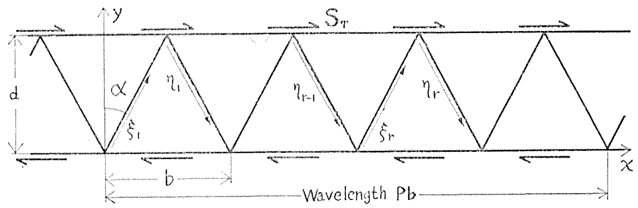

FIG. 1. Model considered.

§ 2. The deflection function $w(x, y)$

We use the same type of deflection function with Cook and Rockey, namely, the deflection function is assumed to be given by a double Fourier series as follows;

$$w = \sum_{m=1,2,\dots} \sum_{n=1,2,\dots} a_{mn} \sin\frac{2m\pi}{Pb}x \cdot \sin\frac{n\pi}{d}y + \sum_{m=0,1,2,\dots} \sum_{n=1,2,\dots} b_{mn} \cos\frac{2m\pi}{Pb}x \cdot \sin\frac{n\pi}{d}y. \tag{1}$$

It satisfies term by term the boundary conditions for the present case, *i.e.*,

(a) $w = 0$ when $y = 0, d$,

(b) $\dfrac{\partial^2 w}{\partial y^2} = 0$ when $y = 0, d$.

Note that it is only necessary to consider the case that the coefficient P of the wavelength Pb along the plate is an integer, in the same reasons as in Ref. (1).

§ 3. Calculations of elements of the matrix

Performing the variation on the total energy of the system per wavelength, we obtain the equations to get the buckling modes. The elements of the square matrix of coefficients being multiplied to the column vector $\begin{Bmatrix} \mathbf{X} \\ \mathbf{Y} \end{Bmatrix}$, where $\mathbf{X}=\{a_{mn}\}$ and $\mathbf{Y}=\{b_{mn}\}$, in the equation in matrix form are now calculated.

3.1. The energy of the plate: V_P

$$V_P=\frac{D}{2}\int_0^d\int_0^{Pb}\left[(\nabla^2 w)^2+2(1-\nu)\left\{\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right\}\right]dxdy \tag{2}$$

Substituting the assumed deflection function w in (2) and performing the integrations, we obtain

$$V_P=\frac{DPb\pi^4}{8d^3}\sum_m\sum_n\left[\left(\frac{2md}{Pb}\right)^2+n^2\right]^2(a_{mn}^2+b_{mn}^2)(1+\delta_{(m)}), \tag{3}$$

the summations being taken over the obvious values of m and n, *i.e.*, $n=1, 2, 3, \dots$, $m=1, 2, 3, \dots$ for a_{mn}, and $n=1, 2, 3, \dots$, $m=0, 1, 2, 3, \dots$ for b_{mn}. And

$$\delta(x)=\begin{cases}1 \text{ if } x=0,\\ 0 \text{ otherwise.}\end{cases}$$

From (3), we obtain

$$\left.\begin{aligned}
\frac{\partial^2 V_P}{\partial a_{mn}\partial a_{pq}}&=\frac{DPb\pi^4}{4d^3}\left[\left(\frac{2md}{Pb}\right)^2+n^2\right]^2\cdot\delta(m-p)\cdot\delta(n-q),\\
\frac{\partial^2 V_P}{\partial b_{mn}\partial b_{pq}}&=\frac{DPb\pi^4}{4d^3}\left[\left(\frac{2md}{Pb}\right)^2+n^2\right]^2\cdot[\delta(m-p)\delta(n-q)+\delta(m+p)\cdot\delta(n-q)],\\
\frac{\partial^2 V_P}{\partial a_{mn}\partial b_{pq}}&=0.
\end{aligned}\right\} \tag{4}$$

3.2. The work done, during buckling, by the shear force S acting in the plane of the plate: T_P

$$\begin{aligned}
T_P&=-S\int_0^d\int_0^{Pb}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}dxdy\\
&=-4\pi S\sum_m\sum_n\sum_{\substack{q\\(n+q)\text{odd}}}\frac{mnq}{n^2-q^2}a_{mn}b_{mq}.
\end{aligned} \tag{5}$$

From (5), we obtain

$$\left.\begin{aligned}
&\frac{\partial^2 T_P}{\partial a_{mn}\partial a_{pq}}=0,\qquad \frac{\partial^2 T_P}{\partial b_{mn}\partial b_{pq}}=0,\\
&\frac{\partial^2 T_P}{\partial a_{mn}\partial b_{pq}}=-4\pi S\frac{mnq}{n-q}\mu(n+q)\cdot\delta(m-p),
\end{aligned}\right\} \tag{6}$$

where

$$\mu(x) = \begin{cases} 1/x \text{ if } x = \text{odd}, \\ 0 \quad \text{otherwise}. \end{cases}$$

The function $\mu(x)$ is introduced in this paper for convenience' sake.

3.3. The energy of the oblique stlffeners: V_S

$$V_S = \frac{EI}{2}\Big[\sum_{r=1}^{P} \int_0^{d/\cos\alpha} \Big(\frac{\partial^2 w}{\partial \xi_r^2}\Big)^0 d\xi_r + \sum_{r=1}^{P} \int_0^{d/\cos\alpha} \Big(\frac{\partial^2 w}{\partial \eta_r^2}\Big)^2 d\eta_r \Big], \tag{7}$$

where ξ_r, η_r are axes along the r-th oblique stiffeners (see Fig. 1).
Using the results of Appendix I, we obtain

$$\left.\begin{aligned}
\frac{\partial^2 V_S}{\partial a_{mn}\partial a_{pq}} &= \frac{2\pi^4 EI\sin^3\alpha}{(Pb)^3}[m+nP)^2(p+qP)^2\delta(r_1) - (m+nP)^2(p-qP)^2\cdot\{\delta(r_3)+\delta(r_4)\} \\
&\quad - (m-nP)^2\cdot(p+qP)^2\cdot\{\delta(r_5)+\delta(r_6)\} + (m-nP)^2\cdot(p-qP)^2\cdot\{\delta(r_7)+\delta(r_8)\}], \\
\frac{\partial^2 V_S}{\partial b_{mn}\partial b_{pq}} &= \frac{2\pi^4 EI\sin^3\alpha}{(Pb)^3}[m+nP)^2\cdot(p+qP)^2\delta(r_1) - (m+nP)^2(p-qP)^2\cdot\{\delta(r_3)-\delta(r_4)\} \\
&\quad - \{m-nP\}^2\cdot(p+qP)^2\cdot\{\delta(r_5)-\delta(r_6)\} + (m-nP)^2\cdot(p-qP)^2\cdot\{\delta(r_7)-\delta(r_8)\}], \\
\frac{\partial^2 V_S}{\partial a_{mn}\partial b_{pq}} &= 0,
\end{aligned}\right\} \tag{8}$$

where

$$\left.\begin{aligned}
r_1 &= \frac{m-p}{P}+n-q,\ r_2 = \frac{m+p}{P}+n+q,\ r_3 = \frac{m-p}{P}+n+q,\ r_4 = \frac{m+p}{P}+n-q, \\
r_5 &= \frac{m-p}{P}-n-q,\ r_6 = \frac{m+p}{P}-n+q,\ r_7 = \frac{m-p}{P}-n+q,\ r_8 = \frac{m+p}{P}-n-q.
\end{aligned}\right\} \tag{9}$$

3.4. The work done, during buckling, by the direct forces ($\pm R$) acting in the oblique stiffeners: T_S

$$T_S = -\frac{R}{2}\Big[\sum_{r=1}^{P} \int_0^{d/\cos\alpha} \Big(\frac{\partial w}{\partial \xi_r}\Big)^2 d\xi_r - \sum_{r=1}^{P} \int_0^{d/\cos\alpha} \Big(\frac{\partial w}{\partial \eta_r}\Big)^2 d\eta_r \Big] \tag{10}$$

Using the results of Appendix II, we obtain

$$\left.\begin{aligned}
&\frac{\partial^2 T_S}{\partial a_{mn}\partial a_{pq}} = 0, \quad \frac{\partial^2 T_S}{\partial b_{mn}\partial b_{pq}} = 0, \\
&\frac{\partial^2 T_S}{\partial a_{mn}\partial b_{pq}} = -\pi R\frac{\sin\alpha}{Pb}[(m+nP)\cdot(p+qP)\cdot\{\mu(r_1)+\mu(r_2)\} \\
&\quad - (m+nP)\cdot(p-qP)\cdot\{\mu(r_3)+\mu(r_4)\} - (m-nP)\cdot(p+qP)\{\mu(r_5)+\mu(r_6)\} \\
&\quad + (m-nP)\cdot(p-qP)\cdot\{\mu(r_7)+\mu(r_8)\}]
\end{aligned}\right\} \tag{11}$$

§4. The division of the applied shear force

The shear force (S_T) applied along the edges is divided into the shear force (S) in the plate and the direct forces ($\pm R$) in the oblique stiffeners.

$$S_T = S + S_R, \tag{12}$$

where

$$S_R = 2R \sin \alpha / b. \tag{13}$$

The ratio of division is determined by their stiffnesses. It is as follows. Consider the deformation due to S. The strain energy due to the deformation is

$$U_1 = \frac{S^2 bd}{2Gt}.$$

The amount of deformation is thus

$$\delta_1 = \frac{\partial U_1}{\partial (Sb)} = \frac{Sd}{Gt} = \frac{Sb}{2Gt \tan \alpha}.$$

Next, consider the deformation due to R. The strain energy due to the deformation is

$$U_2 = 2\frac{R^2}{2EA}\frac{b}{2 \sin \alpha} = \frac{S_R^2 b^3}{8EA \sin^3 \alpha}.$$

The amount of deformation is thus

$$\delta_2 = \frac{\partial U_2}{\partial (S_R b)} = \frac{S_R b^2}{4EA \sin^3 \alpha} = \frac{Rb}{2EA \sin^2 \alpha}.$$

Equalization of δ_1 and δ_2 leads to the result

$$\frac{R}{Sb} = \frac{EA \sin \alpha \cdot \cos \alpha}{Gtb}. \tag{14}$$

§ 5. Computation of the buckling shear force

5.1. The buckling equation

Performing the variation on the total energy $(V_p + V_s - T_p - T_s)$, we obtain the equation

$$\left[\begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{B} \end{pmatrix} + \lambda \begin{pmatrix} \mathbf{O} & \mathbf{C} \\ \tilde{\mathbf{C}} & \mathbf{O} \end{pmatrix}\right]\begin{Bmatrix} \mathbf{X} \\ \mathbf{Y} \end{Bmatrix} = 0, \tag{15}$$

using the above results.

In (15),

$$\mathbf{X} = \{a_{mn}\},\ \mathbf{Y} = \{b_{mn}\}, \tag{16}$$

$$\begin{aligned} \text{any element of } \mathbf{A} &= (m^2 + n^2P^2 \tan^2 \alpha)^2 \cdot \delta(m-p) \cdot \delta(n-q) \\ &+ \frac{\gamma \sin^4 \alpha}{2}[(m+nP)^2 \cdot (p+qP)^2 \delta(r_1) - (m+nP)^2 \cdot (p-qP)^2 \cdot \{\delta(r_3) + \delta(r_4)\} \\ &- (m-nP)^2 \cdot (p+qP)^2 \cdot \{\delta(r_5) + \delta(r_6)\} + (m-nP)^2 \cdot (p-qP)^2 \{\delta(r_7) + \delta(r_8)\}, \end{aligned}$$

$$\left.\begin{aligned}
&\text{any element of } \mathbf{B} = (m^2+n^2P^2\tan^2\alpha)^2\cdot[\delta(m-p)\cdot\delta(n-q)+\delta(m+p)\cdot\delta(n-q)]\\
&+\frac{\gamma\sin^4\alpha}{2}[(m+nP)^2(p+qP)^2\delta(r_1)-(m+nP)^2(p-qP)^2\cdot\{\delta(r_3)-\delta(r_4)\}\\
&-(m-nP)^2\cdot(p+qP)^2\cdot\{\delta(r_5)-\delta(r_6)\}+(m-nP)^2\cdot(p-qP)^2\{\delta(r_7)-\delta(r_8)\}],\\
&\text{any element of } \mathbf{C} = \frac{8P^3\tan^3\alpha}{\pi}\cdot\frac{mnq}{n-q}\cdot\mu(n+q)\cdot\delta(m-p)\\
&+\frac{\beta P^2\sin^2\alpha\cdot\cos^2\alpha\cdot\tan^3\alpha}{\pi}[(m+nP)\cdot(p+qP)\cdot\{\mu(r_1)+\mu(r_2)\}\\
&-(m+nP)\cdot(p-qP)\cdot\{\mu(r_3)+\mu(r_4)\}-(m-nP)\cdot(p+qP)\cdot\{\mu(r_5)+\mu(r_6)\}\\
&+(m-nP)\cdot(p-qP)\cdot\{\mu(r_7)+\mu(r_8)\}].
\end{aligned}\right\} \tag{17}$$

$\tilde{\mathbf{C}}$ is the transpose of the matrix $\mathbf{C}$, and $\mathbf{0}$ a zero-matrix. Furthermore in (15), (17)

$$\left.\begin{aligned}
&\lambda = Sd^2/\pi^2D \quad \text{(non-dimensional buckling shear force parameter)}\\
&\left.\begin{aligned}\gamma &= EI/Dd\sin\alpha\\ \beta &= EA/Gtd\sin\alpha\end{aligned}\right\}\ \text{(non-dimensional stiffness parameter).}
\end{aligned}\right\} \tag{18}$$

From (15), we obtain

$$\mathbf{AX}+\lambda\mathbf{CY}=\mathbf{0},\quad \lambda\tilde{\mathbf{C}}\mathbf{X}+\mathbf{BY}=\mathbf{0}.$$
$$\therefore\quad [\mathbf{A}-\lambda^2\mathbf{CB}^{-1}\tilde{\mathbf{C}}]\mathbf{X}=\mathbf{0}.$$

The condition for the above equation to have a non-zero solution for $\mathbf{X}$ gives the equation

$$\det[\mathbf{A}-\lambda^2\mathbf{CB}^{-1}\tilde{\mathbf{C}}]=0. \tag{19}$$

The buckling equation (19) gives λ^2,*s*. But we need only to have the positive minimum values of λ for the present. The calculation is done by the iteration method.

5.2. Buckling modes

When $\begin{Bmatrix}\mathbf{X}\\ \mathbf{Y}\end{Bmatrix}$, in (15), is split into some sets of $\begin{Bmatrix}\mathbf{X}\\ \mathbf{Y}\end{Bmatrix}$, the submatrices being multiplied to them may be independent with each other. As a natural course of the event the buckling equation (19) is split into some distinct sets. Each set corresponds to a particular buckling mode. This procedure is necessary in view of capacity of the computer available to us.

(i) $P=1$. In a_{mn} and b_{pq}

(i-a) $m+n=$even, $p+q=$odd,

(i-b) $m+n=$odd, $p+q=$even.

Set (i-a) has the property that $w(x,y)=w\left(\frac{b}{2}-x,\ d-y\right)$ being symmetric about the point $\left(\frac{b}{4}, \frac{b}{2}\right)$. Set (i-b) has the property that $w(x, y) = -w\left(\frac{b}{2} - x, d - y\right)$ being antisymmetric about the point $\left(\frac{b}{4}, \frac{d}{2}\right)$.

(ii) $P=2$. In a_{mn} and b_{pq}

$$\text{(ii-a)} \quad m, p = \text{even},$$
$$\text{(ii-b)} \quad m, p = \text{odd}.$$

Set (ii-a) is clealy included in the case that $P=1$. We need only to consider Set (ii-b).

(iii) $P=3$. In a_{mn} and b_{pq}

$$\text{(iii-a)} \quad m, p = 3j,$$

$$\text{(iii-b)} \quad \begin{pmatrix} m = 6j \pm 1 \\ n = \text{odd} \end{pmatrix}, \begin{pmatrix} m = 6j \pm 2 \\ n = \text{even} \end{pmatrix}, \begin{pmatrix} p = 6j \pm 1 \\ q = \text{even} \end{pmatrix}, \begin{pmatrix} p = 6j \pm 2 \\ q = \text{odd} \end{pmatrix},$$

$$\text{(iii-c)} \quad \begin{pmatrix} m = 6j \pm 1 \\ n = \text{even} \end{pmatrix}, \begin{pmatrix} m = 6j \pm 2 \\ n = \text{odd} \end{pmatrix}, \begin{pmatrix} p = 6j \pm 1 \\ q = \text{odd} \end{pmatrix}, \begin{pmatrix} p = 6j \pm 2 \\ q = \text{even} \end{pmatrix}.$$

where $j = 0, 1, 2, \ldots.$

Set (iii-a) is clealy included in the case that $P=1$. Set (iii-b) has the property that $w(x, y) = w\left(-\frac{3b}{2} - x, d - y\right)$. Set (iii-c) has the property that $w(x, y) = -w\left(\frac{3b}{2} - x, d - y\right)$.

(iv) $P=4, 5, 6, \ldots.$ In each of these cases, too, it is split into some sets in the same manner.

5.3. Numerical results

Numerical results have been obtained using NEAC-2203, the electric digital computer in the University of Nagoya. The values of the parameteres used in the computation were as follows;

(i) $\alpha = \frac{\pi}{6}, \frac{\pi}{8}.$

(ii) $P = 1, 2, 3.$

(iii) $\beta = 0, 0, 1.$

(iv) $\gamma = 0 \sim 50.$

Both the number of a_{mn} and that of b_{mn} were about twenty. The data obtained are plotted in Figs. 2 and 3. But some unnecessary numerical results are not plotted.

Note; $P=1+$ corresponds to (i-a) in §5.2, $P=1-$ to (i-b), $P=2$ to (ii-b), and $P=3+, 3-$ to (iii-b), (iii-c) respectively.

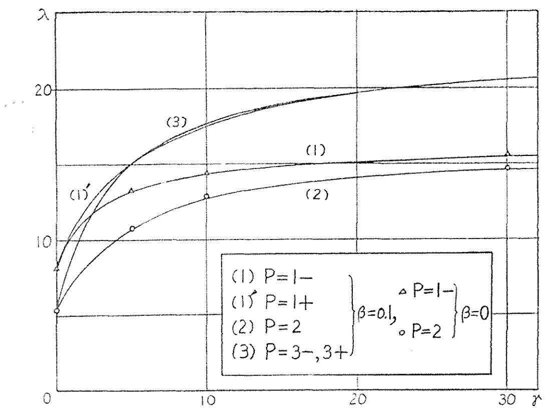

FIG. 2. $\alpha = \frac{\pi}{6}$.

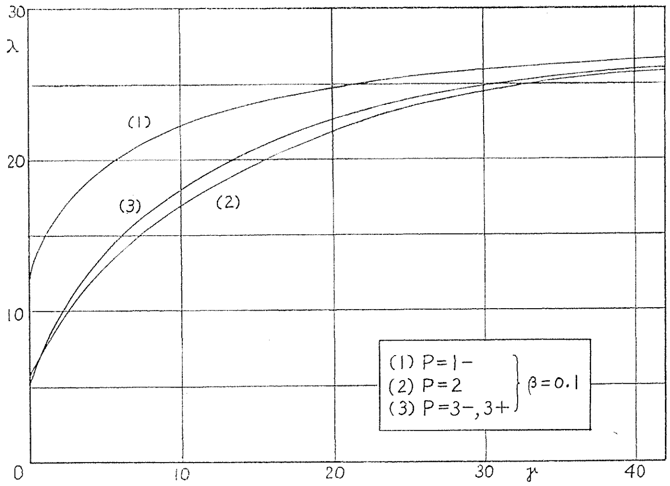

FIG. 3. $\alpha=\frac{\pi}{8}$

§ 6. Conclutions

(i) When $\alpha=\frac{\pi}{6}$, the critical buckling shear force parameter λ is determined only by the curve $P=2$, *i.e.*, the system buckles with the wavelength $2b$.

(ii) When $\alpha=\frac{\pi}{8}$, the curve $P=3$ comes into play at the range where γ is small.

(iii) The modes $P=3+$, $P=3-$ lead to the same buckling shear force parameter λ.

(iv) The buckling shear force parameter of the plate, λ, when $\beta=0.1$ is slightly less than that when $\beta=0$. But the buckling shear force parameter of the system, λ_T, when $\beta=0.1$ must be more than that when $\beta=0$. λ_T is determined by the following equation;—

$$\lambda_T=\frac{S_T d^2}{D\pi^2}=\lambda\left(1+\beta\frac{\sin^2 2\alpha}{4}\right).$$

§ 7. References

1) I. T. Cook, K. C. Rockey. Shear Buckling of Clamped and Simply-Supported Infinitely Long Plates Reinforced by Transverse Stiffeners. Aeronautical Quaterly, **13** (1962), 41.

[Appendix I]

In (7), using the following relations

$$\left(\frac{\partial^2 w}{\partial \xi_r^2}\right)^2 = \sum_{m,n,p,q} \left[\frac{\partial^2}{\partial \xi_r^2}\left\{a_{mn}\sin\frac{2m\pi}{Pb}x\cdot\sin\frac{n\pi}{d}y + b_{mn}\cos\frac{2m\pi}{Pb}x\cdot\sin\frac{n\pi}{d}y\right\}\right]$$
$$\times\left[\frac{\partial^2}{\partial \xi_r^2}\left\{a_{pq}\sin\frac{2p\pi}{Pb}x\cdot\sin\frac{q\pi}{d}y + b_{pq}\cos\frac{2p\pi}{Pb}x\cdot\sin\frac{q\pi}{d}y\right\}\right],$$

$$\left(\frac{\partial^2 w}{\partial \eta_r^2}\right)^2 = \sum\left[\frac{\partial^2}{\partial \eta_r^2}\{+\}\right]\times\left[\frac{\partial^2}{\partial \eta_r^2}\{+\}\right],$$

$$\frac{\partial^2}{\partial \xi_r^2} = \sin^2\alpha\frac{\partial^2}{\partial x^2} + \cos^2\alpha\frac{\partial^2}{\partial y^2} + 2\sin\alpha\cdot\cos\alpha\frac{\partial^2}{\partial x\partial y},$$

$$\frac{\partial^2}{\partial \eta_r^2} = \sin^2\alpha\frac{\partial^2}{\partial x^2} + \cos^2\alpha\frac{\partial^2}{\partial y^2} - 2\sin\alpha\cdot\cos\alpha\frac{\partial^2}{\partial x\partial y},$$

$$x = (r-1)b + \xi_r\sin\alpha = \left(r-\frac{1}{2}\right)b + \eta_r\sin\alpha,$$

$$y = \xi_r\cos\alpha = \alpha - \eta_r\cos\alpha,$$

$$\frac{b}{2\alpha} = \tan\alpha$$

and putting

$$\xi = x/b,$$

we obtain

$$\left.\begin{aligned}
\frac{\partial^2 V_S}{\partial a_{mn}\partial a_{pq}} &= \frac{16\pi^4\sin^3\alpha\cdot EI}{b^3p^4}\left[\int_0^P\left\{(m^2+n^2P^2)\sin\frac{2m\pi}{P}\xi\cdot\sin 2n\pi\xi - 2mnP\cos\frac{2m\pi}{P}\xi\cdot\cos 2n\pi\xi\right\}\right.\\
&\quad\left.\times\left\{(p^2+q^2P^2)\sin\frac{2p\pi}{P}\xi\cdot\sin 2q\pi\xi - 2pqP\cos\frac{2p\pi}{P}\xi\cdot\cos 2q\pi\xi\right\}d\xi\right],\\
\frac{\partial^2 V_S}{\partial b_{mn}\partial b_{pq}} &= \frac{16\pi^4\sin^3\alpha\cdot EI}{b^3P^4}\left[\int_0^P\left\{(m^2+n^2P^2)\cos\frac{2m\pi}{P}\xi\cdot\sin 2n\pi\xi + 2mnP\sin\frac{2m\pi}{P}\xi\cdot\cos 2n\pi\xi\right\}\right.\\
&\quad\left.\times\left\{(p^2+q^2P^2)\cos\frac{2p\pi}{P}\xi\cdot\cos 2q\pi\xi + 2pqP\sin\frac{2p\pi}{P}\xi\cdot\cos 2q\pi\xi\right\}d\xi\right],\\
\frac{\partial^2 V_S}{\partial a_{mn}\partial b_{pq}} &= \frac{16\pi^4\sin^3\alpha\cdot EI}{b^3P^4}\left[\int_0^P\left\{(m^2+n^2P^2)\sin\frac{2m\pi}{P}\xi\cdot\sin 2n\pi\xi - 2mnP\cos\frac{2m\pi}{P}\xi\cdot\cos 2n\pi\xi\right\}\right.\\
&\quad\left.\times\left\{(p^2+q^2P^2)\cos\frac{2p\pi}{P}\xi\cdot\cos 2q\pi\xi + 2pqP\sin\frac{2p\pi}{P}\xi\cdot\cos 2q\pi\xi\right\}d\xi\right].
\end{aligned}\right\}\tag{A.1}$$

In (A.1), using the trigonometric formulae and performing the integations, we obtain (8) in the main issue.

[Appendix II]

In the same mauner as in Appendix I, we obtain

$$\left.\begin{aligned}
\frac{\partial^2 T_S}{\partial a_{mn}\partial a_{pq}} &= -\frac{4\pi^2 R\sin\alpha}{bP^2}\Big[\sum_{r=1}^{P}\Big(\int_{r-1}^{r-1/2}-\int_{r-1/2}^{r}\Big)\Big(m\cos\frac{2mn}{P}\zeta\cdot\sin 2n\pi\zeta+nP\sin\frac{2m\pi}{P}\zeta\cdot\cos 2n\pi\zeta\Big)\\
&\quad\times\Big(p\cos\frac{2p\pi}{P}\zeta\cdot\sin 2q\pi\zeta+qP\sin\frac{2p\pi}{P}\zeta\cdot\cos 2q\pi\zeta\Big)d\zeta\Big],\\
\frac{\partial^2 T_S}{\partial b_{mn}\partial b_{pq}} &= -\frac{4\pi^2 R\sin\alpha}{bP^2}\Big[\sum_{r=1}^{P}\Big(\int_{r-1}^{r-1/2}-\int_{r-1/2}^{r}\Big)\Big(m\sin\frac{2m\pi}{P}\zeta\cdot\sin 2n\pi\zeta-nP\cos\frac{2m\pi}{P}\zeta\cdot\cos 2n\pi\zeta\Big)\\
&\quad\times\Big(p\sin\frac{2p\pi}{P}\zeta\cdot\sin 2q\pi\zeta-qP\cos\frac{2p\pi}{P}\zeta\cdot\cos 2q\pi\zeta\Big)d\zeta\Big],\\
\frac{\partial^2 T_S}{\partial a_{mn}\partial b_{pq}} &= -\frac{4\pi^2 R\sin\alpha}{bP^2}\Big[\sum_{r=1}^{P}\Big(\int_{r-1}^{r-1/2}-\int_{r-1/2}^{r}\Big)\Big(m\cos\frac{2m\pi}{P}\zeta\cdot\sin 2n\pi\zeta+nP\sin\frac{2m\pi}{P}\zeta\cdot\cos 2n\pi\zeta\Big)\\
&\quad\times\Big(-p\sin\frac{2p\pi}{P}\zeta\cdot\sin 2q\pi\zeta+qP\cos\frac{2p\pi}{P}\zeta\cdot\cos 2q\pi\zeta\Big)d\zeta\Big].
\end{aligned}\right\} \tag{A.2}$$

In (A. 2), using the following trigonometric formulae

$$\sum_{r=1}^{P}\Big(\int_{r-1}^{r-1/2}-\int_{r-1/2}^{r}\Big)\cos\frac{2k\pi}{P}\zeta d\zeta=0,$$

$$\sum_{r=1}^{P}\Big(\int_{r-1}^{r-1/2}-\int_{r-1/2}^{r}\Big)\sin\frac{2k\pi}{P}\zeta d\zeta=\begin{cases}\dfrac{2P^2}{\pi k} & \text{if } \dfrac{k}{P}=\text{odd},\\ 0 & \text{otherwise,}\end{cases}$$

where

$$\frac{k}{P}=\frac{m\pm p}{P}\pm n\pm q,$$

we obtain (11) in the main issue.