COMPLEX TRANSFORMATION OF THE DIFFERENTIAL EQUATIONS OF THIN ELASTIC SHELLS BY THE CONSISTENT THEORY WHEN POISSON'S RATIO IS EQUAL TO ZERO

Osamu MATSUOKA

Department of Architectural Engineering

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Introduction

This paper is concerned with the complex transformation of the differential equations of equilibrium and of compatibility of homogeneous isotropic shells by the consistent theory[1] which is based on the first approximation of Love, when Poisson's ratio is equal to zero. The complex unknown functions can be expressed by the two different terms, that is, the first term is of resultant moments and of the strains of the middle surface of the shells, and the second term is of the displacement components of the middle surface and the stress functions. The complex transformation on the shells in curvilinear orthogonal coordinate was first appeared in the Novozhilov's book[2] in which the complex auxiliary functions are of the resultant moments and forces.

The Love's first approximation by which the consistent shell theory is derived falls into the following two assumptions.

(a) Each element on the normal to the undeformed middle surface of the shells remains on the same normal to the deformed middle surface after deformation. And distance between each element on the normal does not change after deformation. Thus, the normal stress in the normal direction is assumed to be neglected.

(b) t/R is neglected because of the small quantities against unity where t is the constant thickness of the shells and R is the minimum radius of the middle surface of the shells.

Owing to the above assumptions the resultant forces become symmetry and the strain vanishes in body motion.

Since the shells are treated in general coordinates, the tensor notation is used. For example, any term in which the same index appears twice stands for the sum of all such terms obtained by giving this index its complete range of values.

The symbols used herein are as follows,

- x^j : coordinates of any point on the middle surface of the shells
- x^3 : coordinates of the normal direction of the middle surface of the shells
- g^{jk}, g_{jk} : the first fundamental contravariant and covariant tensor of the middle surface of the shells, respectively

$g = \det(g_{jk})$

$H_{jk},\ H^{j}_{k}$: the second fundamental covariant and mixed tensors of the middle surface of the shells, respectively

$\left\{ {j \atop kl} \right\}$: Christoffel's three index symbols

$\epsilon^{jk},\ \epsilon_{jk}$: Eddington's symbols, $\epsilon^{12} = -\epsilon^{21} = \dfrac{1}{\sqrt{g}},\quad \epsilon^{11} = \epsilon^{22} = 0,\quad \epsilon_{12} = -\epsilon_{21} = \sqrt{g},\quad \epsilon_{11} = \epsilon_{22} = 0$

E : Young's modulus

ν : Poisson's ratio

t : constant thickness of the shells

$u_j,\ w$: covariant tangential component and normal component of the displacement of the middle surface of the shells, respectively

e_{jk} : covariant strain tensor at any point on the shells

ε_{jk} : covariant strain tensor of the middle surface of the shells

κ_{jk} : covariant curvature change tensor of the middle surface of the shells

$U_p,\ W$: stress functions of covariant and scalar, respectively

N^{jk} : covariant resultant force

M^{j}_{k} : mixed resultant moment

∇_j : covariant differential with respect to x^j, for example,

$$\nabla_j B_p = B_{p,j} - \left\{ {a \atop jp} \right\} B_a$$

$$\nabla_j B^p = B^p{}_{,j} + \left\{ {p \atop ja} \right\} B^a$$

$$\nabla_j B_{pq} = B_{pq,j} - \left\{ {a \atop jp} \right\} B_{aq} - \left\{ {b \atop jq} \right\} B_{pb}$$

$$\nabla_j B^p_q = B^p_{q,j} + \left\{ {p \atop ja} \right\} B^a_q - \left\{ {b \atop jq} \right\} B^p_b$$

$$\nabla_j B^{pq} = B^{pq}{}_{,j} + \left\{ {p \atop ja} \right\} B^{aq} + \left\{ {q \atop jb} \right\} B^{pb}$$

$$\nabla_j (u_l H^l_k) = (u_l H^l_k)_{,j} - \left\{ {a \atop jk} \right\} u_l H^l_a$$

$$\nabla_j w_{,p} = w_{,p,j} - \left\{ {a \atop jp} \right\} w_{,a}$$

$$\nabla_p \nabla_q B_{ab} = (\nabla_q B_{ab})_{,p} - \left\{ {j \atop pq} \right\} \nabla_j B_{ab} - \left\{ {k \atop pa} \right\} \nabla_q B_{kb} - \left\{ {l \atop pb} \right\} \nabla_q B_{al}$$

$$\nabla_p \nabla_q B^a_b = (\nabla_q B^a_b)_{,p} + \left\{ {a \atop pj} \right\} \nabla_q B^j{}_b - \left\{ {k \atop pq} \right\} \nabla_k B^a{}_b - \left\{ {l \atop pb} \right\} \nabla_q B^a_l$$

$$\nabla_p (B^c{}_d C^d_c) = (B^c_d C^d_c)_{,p} + \left\{ {c \atop pj} \right\} B^j_d C^d_c + \left\{ {d \atop pk} \right\} B^c{}_d C^k_c - \left\{ {l \atop pd} \right\} B^c_l C^d_c - \left\{ {m \atop pc} \right\} B^c{}_d C^d_m$$

Lower index k following commer means the partial differential with respect to

x^k such as $A_{p,k}=\dfrac{\partial A_p}{\partial x^k}$

1. Fundamental Relations

Following fundamental relations are shown in the author's paper.[1] u_j, N^{jk}, M^i_k, etc. appeared in this paper are not the physical quantities but the tensorial. The positive directions of these and the relations between these tensorial and physical quantities are shown in reference[3].

Strain

$$e_{jk}=\varepsilon_{jk}-x^3\kappa_{jk} \tag{1-1}$$

$$\varepsilon_{jk}=\nabla_{(j}u_{k)}-H_{jk}w \tag{1-2}$$

$$\kappa_{jk}=\nabla_{(k}(u_lH^l_{k)}+w,_{j)})-\frac{1}{2}\epsilon_{p(j}H^p_{k)}\epsilon^{ab}\nabla_a u_b \tag{1-3}$$

where $_{(\ )}$ indicates the symmetric quantity, for example,

$$B_{a(b}C_{c)d}=\frac{1}{2}(B_{ab}C_{cd}+B_{ac}C_{bd})$$

Resultant moment and force-strain relations

$$\left.\begin{aligned}N^{jk}&=\frac{Et}{2(1+\nu)}\left(g^{ja}g^{kb}+g^{jb}g^{ka}+\frac{2\nu}{1-\nu}g^{jk}g^{ab}\right)\varepsilon_{ab}\\ M^k_j&=\frac{Et^3}{24(1+\nu)}\epsilon_{jp}\left((g^{ka}g^{pb}+g^{kb}g^{pa}+\frac{2\nu}{1-\nu}g^{kp}g^{ab}\right)\kappa_{ab}\end{aligned}\right\} \tag{2-1}$$

$$\left.\begin{aligned}\varepsilon_{kj}&=\frac{1+\nu}{2Et}\left(g_{ka}g_{jb}+g_{kb}g_{ja}-\frac{2\nu}{1+\nu}g_{kj}g_{ab}\right)N^{ab}\\ \kappa_{kj}&=\frac{6(1+\nu)}{Et^3}\epsilon^{pb}\left(g_{ka}g_{jb}+g_{kb}g_{ja}-\frac{2\nu}{1+\nu}g_{kj}g_{ab}\right)M^a_p\end{aligned}\right\} \tag{2-2}$$

Equilibrium Equation

The third term of the equilibrium equation of the tangential direction of the middle surface of the shells is the modified term due to the approximate expression of the curvature change tensor (1-3), where the equilibrium equations are obtained from the principle of the virtual work of the potential energy of strain, excluding body force.

Normal direction

$$N^{kj}H_{jk}+\epsilon^{ab}\nabla_p\nabla_a M^p_b=0 \tag{3-1}$$

Tangential direction ($j=1,2$)

$$-\nabla_k N^{kj}+\epsilon^{kq}H^j_p\nabla_k M^p_b+\frac{1}{2}\epsilon^{pj}\nabla_p(H_b{}^aM_a{}^b)=0 \tag{3-2}$$

Compatibility Equation

The quantity with the sign-means the quantity after deformation, and the first and second fundamental tensors of the middle surface of the shells after deformation can then be written as follows,

$$\bar{g}_{jk} = g_{jk} + 2\,\varepsilon_{jk}$$
$$\bar{H}_{jk} = H_{jk} + \kappa_{jk} + H^{p}_{(j}\varepsilon_{pk)}$$

The compatibility equations obtained from Gauss's and Mainardi-Codazzi's equations become as follows,

$$\epsilon^{hl}\epsilon^{jk}H_{hj}\kappa_{kl} + \epsilon^{hl}\epsilon^{jk}\nabla_h\nabla_j\varepsilon_{kl} = 0 \tag{4-1}$$
$$-\epsilon^{hl}\epsilon^{jk}\nabla_h\kappa_{kl} + H^{j}_{p}\epsilon^{ac}\epsilon^{pb}\nabla_a\varepsilon_{bc} + \frac{1}{2}\epsilon^{pj}\epsilon^{qf}\nabla_p(H^{c}_{q}\varepsilon_{fc}) = 0 \tag{4-2}$$

Stress Functions

Owing to the similarity of the forms of Eqs. (3-1) and (4-1), and (3-2) and (4-2), the stress function U_p and W which satisfies Eqs. (3-1) and (3-2) identically can be introduced as follows,

$$N^{kj} = \epsilon^{km}\epsilon^{jl}\{\nabla_{(m}(U_pH^{p}_{l)} + W_{,l)}) - \frac{1}{2}\epsilon_{q(l}H^{q}_{m)}\epsilon^{ab}\nabla_aU_b\} \tag{5-1}$$
$$M^{k}_{j} = \epsilon^{kq}(\nabla_{(j}U_{q)} - H_{jq}W) \tag{5-2}$$

The boundary conditions corresponding to these relations are shown in the previous paper[1].

2. Complex Transformation

There exist two methods for solving shell problems. In the first method, three components of the displacement of the middle surface of the shells, or three stress functions may be chosen as unknown functions. In the second method, since there exist six equations, three in equilibrium and the other three in compatibility in which six quantities, ε_{jk} and κ_{jk}, or ε_{jk} and M^{j}_{k}, or N^{jk} and κ_{jk}, or N^{jk} and M^{j}_{k} are contained, one of above four couples can be expressed by the unknown six functions.

Considering the similarity of the forms between Eqs. (3-1) and (4-1), and Eqs. (3-2) and (4-2), the complex auxiliary functions can be introduced, when Poisson's ratio is equal to zero.

Resultant moment and force-strain relations can be written as follows in the case of $\nu=0$.

$$\left.\begin{aligned} N^{kj} &= Etg^{ka}g^{jb}\varepsilon_{ab} \\ M^{k}_{j} &= -\frac{Et^3}{12}\epsilon_{pj}g^{ka}g^{pb}\kappa_{ab} \\ \varepsilon_{kj} &= \frac{1}{Et}g_{ka}g_{jb}N^{ab} \\ \kappa_{kj} &= -\frac{12}{Et^3}\epsilon^{bp}g_{ka}g_{jb}M^{a}_{b} \end{aligned}\right\} \tag{6}$$

Setting the mixed tensor A^{b}_{a} as follows,

$$A^{b}_{a} = M^{b}_{a} + i\frac{Et^2}{\sqrt{12}}\epsilon^{bf}\varepsilon_{af}, \qquad i=\sqrt{-1} \tag{7}$$

and making and calculating the following equations with the use of Eq. (6),

$$(3-1)+i\frac{Et^2}{\sqrt{12}}(4-1),\ (3-2)+i\frac{Et^2}{\sqrt{12}}(4-2)$$

above equations can be shown by M^k_j and ε_{kj}, as follows,

$$\left.\begin{aligned}&\epsilon^{jb}\nabla_k\nabla_j A^k_b+i\frac{\sqrt{12}}{t}\epsilon^{jm}g^{kp}g_{ma}H_{kj}A^a_p=0\\&-i\frac{\sqrt{12}}{t}\epsilon^{jm}g^{kp}g_{ma}\nabla_k A^a_p+\epsilon^{kq}H^j_p\nabla_k A^p_q+\frac{1}{2}\epsilon^{pj}\nabla_p(H^a_b A^b_a)=0\end{aligned}\right\}\quad(8)$$

where $A^j_j=0$

A^b_a denoted in Eq. (7) can be rewritten with u_j, w and U_j, W by using Eqs. (1-2) and (4-2),

where

$$\left.\begin{aligned}&A^b_a=\epsilon^{bf}(\nabla_{(a}\alpha_{f)}-H_{af}\Phi)\\&\alpha_q=U_q+i\frac{Et^2}{\sqrt{12}}u_q,\quad \Phi=W+i\frac{Et^2}{\sqrt{12}}w\end{aligned}\right\}\quad(9)$$

α_q and Φ play as covariant and scalar, respectively.

From Eqs. (7) and (8) the resultant moments and the strains of the shells are expressed as the unknown functions, and from Eqs. (8) and (9) stress functions and displacement components are expressed as the unknown quantities.

It is convenient to use the contravariant tensor $A^{pq}=g^{sq}A^p_s$ for the special coordinate. Equations (8) become as follows,

$$\begin{aligned}&\epsilon^{jb}g_{pb}\nabla_k\nabla_j A^{kp}+i\frac{\sqrt{12}}{t}\epsilon^{jm}g_{ma}H_{kj}A^{ak}=0\\&-i\frac{\sqrt{12}}{t}\epsilon^{jm}g_{ma}\nabla_k A^{ak}+\epsilon^{kq}H^j_p g_{qs}\nabla_k A^{ps}+\frac{1}{2}\epsilon^{pj}\nabla_p(H_{sb}A^{bs})=0\end{aligned}\quad(10)$$

where A^{pq} is not symmetry, and $g_{jk}A^{kj}=0$

Conclusions

In the previous study the author derived the approximate bending theory of shells in which a complex auxiliary function made of the stress function of Airry's type and the normal displacement is taken as unknown quantities. However, this theory is not applicable to solve the all shell problems. For example it can not be used to the torque problem of shells of revolution. In this present work, the author extended his theory to solve the general shell problems.

Equations (7), (8) and (9) are introduced from the consistent shell theory based on the Kirichhoff-Love assumption, in the case of Poisson's ratio being equal to zero. These equations are in the simple forms, and it is obvious to reduce the difficulties lying in solving general static shell problems.

As to which is convenient (7) or (9) to solve shell problems will be dicided from the conditions of the shape and the boundary of shells.

References

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3) Y. Yokowo and O. Matsuoka, "Approximation of the Bending Theory of Shells". Proceedings of the Third Japan National Congress for Applied Mechanics, 1953.