ON THE VIBRATION OF AN AEROFOIL

II. PROPAGATION OF DISTURBANCE IN A LINEAR ELASTIC CHAIN

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§ 0. Introduction

The second paper on the vibration of aerofoil treats the propagation of disturbance in a linear elastic chain with coupled harmonic oscillators. In the first paper[1] emphasis was laid on the study of the discrete elastic chain, instead of a continuous aerofoil. Here we shall make clear the dynamical behavior of the linear elastic chain.

As for the statistical dynamical properties of the discrete chain, it is a well-known fact that if one assumes a canonical distribution at the initial instant of time, the evolution of the ensemble constitutes a stationary Gaußian process of ergodic character. The analytical properties of correlation functions are also investigated in various cases. Interesting problems concerning the localized vibration[1][2] in such a system have been also studied, treating the cases of locally reduced mass, of local hard springs, and also of applied external forces of high frequency.

In the present paper the authors made a study of a chain of *infinite* length under certain conditions, or rather, of a chain of *finite* length under certain periodic conditions. At the initial instant of time, it is assumed that half the system of the chain is perfectly at rest, while the other half has temperature T which measures the mean square of the momenta of particles. In other words, half the system is distributed canonically with temperature T and the other half is fixed at its dynamically equilibrium position.

Starting from this initial ensemble we shall pursue the timal behavior of the ensemble, which evolves in course of time according to the law of classical dynamics. By introducing the Schrödinger coordinates[3] and by means of the solutions expanded in Bessel functions[3][4], one can easily pursue the timal behavior of the correlation functions of the ensemble. After a sufficiently long time, we found that the whole system approaches a stationary state with temperature $T/2$. The result obtained in this paper is seen to be quite the same as given recently by means of Fourier series.[5] We also found that the average values of the potential energy of each particle approach the same stationary value after a sufficiently long time. Thus the equipartition of the kinetic and the potential energies is proved to establish. However, the mathematical method we used here is easier to understand and simpler by far than that by means of Fourier series; because one can take explicitly into account the initial conditions of the ensemble and also because the covariance matrix[6] in the distribution function (or, to be more precise, probability density function) of the canonical ensemble at the initial instant of time has been already diagonalized by means of the dynamical solutions expanded in Bessel functions.

§ 1. Periodic Linear Elastic Chain

We shall consider a linear elastic chain consisting of an infinite number of coupled harmonic oscillators. Let M be a mass of each particle and K the force constant of interaction between the nearest particles. Then the equations of motion of the system read in the Schrödinger coordinates[1)3)4)] as follows:

$$2\frac{dy_n(\tau)}{d\tau}=y_{n+1}(\tau)-y_{n-1}(\tau), \qquad \text{for } -\infty<n<+\infty \tag{1}$$

where

$$y_{2n}=\frac{\xi_n}{\sigma\omega},$$

$$y_{2n+1}=\frac{r_{n,n+1}-\bar{r}}{\sigma},$$

$$\tau=2\,\omega t,$$

$$\omega=\sqrt{K/M},$$

with $r_{n,n+1}=$ the distance between the n-th and the $(n+1)$-th particles, $\bar{r}=$ the distance of particles at equilibrium, $\xi_n=$ the velocity of the n-th particles, $\sigma=$ the root mean square of y_{2n+1}, and $t=$ time. Equations (1) have a system of solutions:

$$y_n(\tau)=\sum_{\nu=-\infty}^{+\infty}a_\nu J_{\nu-n}(\tau), \qquad \text{for } -\infty<n<+\infty \tag{2}$$

with integration constants a_ν and Bessel function $J_m(\tau)$ of order m.

The solutions (2) of equations of motion of an infinite chain can be also used in the case of a system of finite length under certain conditions. Now, let us take a finite chain of $2N$ particles with both ends free. The particles are numbered from $-N+1$ to N. Then we have a system of equations

$$\frac{d}{d\tau}y_{-2N+2}=\frac{1}{2}y_{-2N+3}, \qquad \frac{d}{d\tau}y_{2N}=-\frac{1}{2}y_{2N-1},$$

and

$$\frac{d}{d\tau}y_n=\frac{1}{2}(y_{n+1}-y_{n-1}), \quad \text{for } n=-2N+3,\ -2N+4,\ldots,-1,0,1,\ldots,2N-1.$$

Inserting (2) into the above equations, we obtain, for the integration constants:

$$a_{-2N+1}=0, \qquad a_{2N+1}=0,$$

and

$$a_{\nu-2N+1}=(-1)^{1+\nu}a_{-\nu-2N+1},\ a_{2N+1+\nu}=(-1)^{1+\nu}a_{2N+1-\nu}. \qquad \text{for any integers } \nu.$$

From these expressions, we can write all the integration constants a_ν in (2) in terms of $a_{-2N+2}, a_{-2N+3}, \ldots, a_{2N}$, which are respectively the initial values of $y_{-2N+2}, y_{-2N+3}, \ldots, y_{2N}$. In a compact form, the integration constants can be written as follows:

$$a_{\nu+k\cdot 4N}=\begin{cases} a_\nu, & \text{for } k: \text{ even integers} \\ (-1)^\nu a_{2-\nu}, & \text{for } k: \text{ odd integers} \end{cases} \tag{3}$$

with any integers ν.

Inserting (3) into (2), we obtain

$$y_n(\tau)=\sum_{\nu=-2N+2}^{2N} a_\nu\cdot\sum_{k=-\infty}^{+\infty}\{J_{\nu+2k\cdot 4N-n}+ + (-1)^\nu J_{2-\nu+(2k+1)\cdot 4N-n}\}. \qquad \text{for } -2N+2\leq n\leq 2N$$

The equivalent expression was also obtained by Klein and Prigogine[4)].

If we replace the periodic condition (3) by another periodic condition for a cyclic chain:

$$y_{n\pm 4N}(\tau)=y_n(\tau), \qquad \text{for } -\infty<n<+\infty$$

we obtain, for the integration constants,

$$a_n=a_{n+4sN}, \qquad \text{for } -\infty<n<+\infty \tag{3'}$$

with any integers s.

In this case, we have solutions of the system:

$$y_n(\tau)=\sum_{\nu=-2N+2}^{2N+1} a_\nu\cdot\sum_{k=-\infty}^{+\infty} J_{\nu+4kN-n}. \qquad \text{for } -2N+2\leq n\leq 2N+1$$

In both cases of:

(i) the chain with free ends (3),

and

(ii) the cyclic chain (3′),

we have similar solutions, if their length is sufficiently long. This can be explained as follows:

viz., if we take sufficiently short time as compared with the time needed for disturbance to propagate to the end of the system (i.e. $\tau\ll 2N$), the expression of the solution of equation for the finite chain reduces to*:

$$y_n(\tau)=\sum_{\nu=-2N+2}^{2N+1} a_\nu J_{\nu-n}(\tau). \qquad \text{for } -2N+2\leq n\leq 2N \tag{4}$$

§ 2. Initial Ensemble

Now, let us introduce an initial statistical ensemble in which particles Nos.

* cf. Appendix III.

0, -1, -2, . . . , and $(-N+1)$ are fixed at their equilibrium positions, *viz.*,

$$y_0(\tau=0)=y_{-1}(\tau=0)=y_{-2}(\tau=0)=\cdots=y_{-2N+3}(\tau=0)=y_{-2N+2}(\tau=0)=0. \qquad (5)$$

Then, from (4) and (5), we obtain at once

$$a_0=a_{-1}=a_{-2}=\cdots=a_{-2N+3}=a_{-2N+2}=0. \qquad (6)$$

From (6), the solutions (4) under periodic conditions (3′), are expressed as follows:

$$y_n(\tau)=\sum_{\nu=1}^{2N+1}a_\nu J_{\nu-n}(\tau). \qquad \text{for } -\infty<n<+\infty \qquad (7)$$

Further we assume that the random variables y_m at $\tau=0$, *i.e.* a_m $(m=1, 2, \ldots, 2N+1)$ are distributed with canonical distribution at temperature T. Accordingly, we take the distribution function (or, to be more precise, probability density function) $W(a_1, a_2, \ldots, a_{2N+1})$ for a_ν $(\nu=1, 2, \ldots, 2N+1)$ as follows:

$$\begin{aligned} W(a_1, a_2, \ldots, a_{2N+1}) &= \left(\frac{\sigma^2K}{2\pi kT}\right)^{N+1/2}\cdot\exp\left[-\frac{\sigma^2K}{2kT}\left\{\sum_{n=1}^{N}a_{2n}^2+\sum_{n=0}^{N}a_{2n+1}^2\right\}\right] \\ &= \left(\frac{\sigma^2K}{2\pi kT}\right)^{N+1/2}\cdot\exp\left[-\frac{\sigma^2K}{2kT}\cdot\sum_{s=1}^{2N+1}a_s^2\right], \end{aligned} \qquad (8)$$

where k is the Boltzmann constant. Here, the expression,

$$H=\frac{\sigma^2K}{2}\cdot\sum_{s=1}^{2N+1}a_s^2\left(=\frac{\sigma^2K}{2}\sum_{s=-2N+2}^{2N+1}a_s^2\right),$$

is the Hamiltonian of the system at $\tau=0$.

This ensemble with distribution function (8) has the averages

$$\left.\begin{aligned} \langle a_{2n}\cdot a_{2m}\rangle_{Av} &= \frac{kT}{\sigma^2K}\delta_{n,m}, \\ \langle a_{2n+1}\cdot a_{2m}\rangle_{Av} &= 0, \\ \text{and} \qquad \langle a_{2n+1}\cdot a_{2m+1}\rangle_{Av} &= \frac{kT}{\sigma^2K}\delta_{n,m}, \end{aligned}\right\} \qquad (9)$$

with Kronecker's delta $\delta_{n,m}$.

§ 3. Correlation Functions

Now, let us take the correlation functions of the n-th and $(n+m)$-th particles, which we shall define as follows:

$$\Theta_{n,m}(\tau)=\frac{\sigma^2K}{k}\langle|y_{2n}(\tau)y_{2n+2m}(\tau)|\rangle_{Av}, \qquad \text{for } -N+1\leq n,\ n+m\leq N \qquad (10)$$

and

$$U_{n,m}(\tau) = \frac{\sigma^2 K}{k} \langle |y_{2n+1}(\tau)\, y_{2n+2m+1}(\tau)| \rangle_{Av}. \qquad \text{for} \quad -N+1 \leq n,\ n+m \leq N \quad (11)$$

Now, we shall pursue the statistical behavior of the dynamical system (7) in course of time. By means of (7), we find the functional form of $\Theta_{n,m}(\tau)$ at any instant of time τ:

$$\begin{aligned}
\Theta_{n,m}(\tau) &= \frac{\sigma^2 K}{k} \langle | \sum_{\nu=1}^{2N+1} a_\nu J_{\nu-2n}(\tau) \cdot \sum_{\mu=1}^{2N+1} a_\mu J_{\mu-2n-2m}(\tau) | \rangle_{Av} \\
&= \frac{\sigma^2 K}{k} [\, | \sum_{k,i=1}^{N} \langle a_{2k} \cdot a_{2i} \rangle_{Av} J_{2k-2n}(\tau) J_{2i-2n-2m}(\tau) \\
&\quad + \sum_{k,i=0}^{N} \langle a_{2k+1} \cdot a_{2i+1} \rangle_{Av} J_{2k+1-2n}(\tau) J_{2i+1-2n-2m}(\tau) \\
&\quad + \sum_{k=1}^{N} \sum_{i=0}^{N} \langle a_{2k} \cdot a_{2i+1} \rangle_{Av} J_{2k-2n}(\tau) J_{2i+1-2n-2m}(\tau) \\
&\quad + \sum_{k=0}^{N} \sum_{i=1}^{N} \langle a_{2k+1} \cdot a_{2i} \rangle_{Av} J_{2k+1-2n}(\tau) J_{2i-2n-2m}(\tau) | \,].
\end{aligned} \qquad (12)$$

Inserting (9) into (12), we obtain

$$\Theta_{n,m}(\tau) = T \cdot | \sum_{k=-2n-m+1}^{2N-2n-m+1} J_{k+m}(\tau) J_{k-m}(\tau) |. \qquad (13)$$

The similar calculation as in (12) and (13) leads to:

$$U_{n,m}(\tau) = T \cdot | \sum_{k=-2n-m}^{2N-2n-m} J_{k+m}(\tau) J_{k-m}(\tau) |. \qquad (14)$$

For the special case: $m=0$ in (13) and (14), we have

$$\Theta_{n,0}(\tau) = T \cdot [\sum_{k=-2n+1}^{2N-2n+1} J_k^2(\tau)], \qquad (15)$$

and

$$U_{n,0}(\tau) = T \cdot [\sum_{k=-2n}^{2N-2n} J_k^2(\tau)]. \qquad (16)$$

The expressions (15) and (16) correspond to the kinetic and the potential energies* of the n-th particle, respectively. The expression (15) is somewhat different from Teramoto's[5], who employed Fourier series. Moreover, we find that the expression (16) is very similar to (15), but the details are somewhat different.

For the sake of reference, we shall take the correlation function of the n-th and the $(n+m)$-th particles, which we shall define as follows:

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$$\text{kinetic energy} = \frac{1}{2} k \cdot \Theta_{n,0}(\tau),$$

and

$$\text{potential energy} = \frac{1}{4} k \cdot [U_{n,0}(\tau) + U_{n-1,0}(\tau)].$$

$$V_{n,m}(\tau)=\frac{\sigma^2 K}{k}\langle|y_{2n}(\tau)y_{2n+2m+1}(\tau)|\rangle_{AV}. \qquad \text{for } -N+1\leq n,\ n+m\leq N \quad (17)$$

From (9) we obtain

$$V_{n,m}(\tau)=T\cdot|\sum_{k=-2n+1}^{2N-2n+1}J_k(\tau)J_{k-2m-1}(\tau)|,$$
$$=T\cdot|\sum_{k=-2n-m+1/2}^{2N-2n-m+1/2}J_{k+m+1/2}(\tau)J_{k-m-1/2}(\tau)|. \quad (18)$$

For the case: $m=0$, the expression (18) reduces to:

$$V_{n,0}(\tau)=T\cdot|\sum_{k=-2n+1/2}^{2N-2n+1/2}J_{k+1/2}(\tau)J_{k-1/2}(\tau)|, \quad (19)$$

which represents the correlation of velocities and positions of the n-th particle. The summation of k in (18) and (19) covers half odd integers.

§ 4. Approach to the Stationary State

If we consider the case: $N\to+\infty$, we have the following results** for the correlation functions (13) and (14):

$$\Theta_{n,m}(\tau)=T\cdot|\sum_{k=-2n-m+1}^{+\infty}J_{k+m}(\tau)J_{k-m}(\tau)|$$

$$=\begin{cases}\frac{T}{2}[|\delta_{m,0}+(-1)^{m+1}J_m^2(\tau)+2\sum_{k=-2n-m+1}^{0}J_{k+m}(\tau)J_{k-m}(\tau)|], & \text{for } 2n>-m \quad (20)\\ \frac{T}{2}[|\delta_{m,0}+(-1)^{m+1}J_m^2(\tau)-2\sum_{k=1}^{-2n-m}J_{k+m}(\tau)J_{k-m}(\tau)|], & \text{for } 2n<-m \quad (21)\\ \frac{T}{2}[|\delta_{m,0}+(-1)^{m+1}J_m^2(\tau)|], & \text{for } 2n=-m \quad (22)\end{cases}$$

and

$$U_{n,m}(\tau)=T\cdot|\sum_{k=-2n-m}^{+\infty}J_{k+m}(\tau)J_{k-m}(\tau)|$$

$$=\begin{cases}\frac{T}{2}[|\delta_{m,0}+(-1)^{m+1}J_m^2(\tau)+2\sum_{k=-2n-m}^{0}J_{k+m}(\tau)J_{k-m}(\tau)|], & \text{for } 2n>-m-1 \quad (23)\\ \frac{T}{2}[|\delta_{m,0}+(-1)^{m+1}J_m^2(\tau)-2\sum_{k=1}^{-1-2n-m}J_{k+m}(\tau)J_{k-m}(\tau)|], & \text{for } 2n<-m-1 \quad (24)\\ \frac{T}{2}[|\delta_{m,0}+(-1)^{m+1}J_m^2(\tau)]. & \text{for } 2n=-m-1 \quad (25)\end{cases}$$

In the case of $m\neq 0$, the correlation functions (20)~(25) mentioned above take the same value zero at $\tau=0$, and they finally vanish at $\tau=+\infty$. This means that the correlation between velocities of different particles and the correlation between positions of different particles vanish asymptotically.

** cf. Appendix I.

For $m=0$, however, the expressions (20) ~ (22) and (23) ~ (24) reduce respectively to:

$$\Theta_{n,0}(\tau)=\begin{cases}\frac{T}{2}\cdot[1+J_0^2(\tau)+2\sum_{k=1}^{2n-1}J_k^2(\tau)], & \text{for } n\geq 1 \quad (26)\\ \frac{T}{2}\cdot[1-J_0^2(\tau)-2\sum_{k=1}^{|2n|}J_k^2(\tau)], & \text{for } n\leq -1 \quad (27)\\ \frac{T}{2}\cdot[1-J_0^2(\tau)], & \text{for } n=0 \quad (28)\end{cases}$$

and

$$U_{n,0}(\tau)=\frac{T}{2}\cdot[1+J_0^2(\tau)+2\sum_{k=1}^{2n}J_k^2(\tau)], \qquad \text{for } n\geq 1 \tag{29}$$

$$=\frac{T}{2}\cdot[1-J_0^2(\tau)-2\sum_{k=1}^{|2n+1|}J_k^2(\tau)], \qquad \text{for } n\leq -1 \tag{30}$$

$$=\frac{T}{2}\cdot[1+J_0^2(\tau)]. \qquad \text{for } n=0 \tag{31}$$

The expressions (20) ~ (28) are very much the same as those obtained by means of Fourier series.[5]

The kinetic temperatures (26) ~ (28) have initial values, respectively, of

$$\Theta_{n,0}(\tau)=T, \qquad \text{for } n\geq 1$$

$$\Theta_{n,0}(\tau)=0, \qquad \text{for } n\leq -1$$

and

$$\Theta_{n,0}(\tau)=0, \qquad \text{for } n=0$$

while, when τ goes to infinity, they approach[5][8] the same stationary value $T/2$, *i.e.*

$$\lim_{\tau\to+\infty}\Theta_{n,0}(\tau)=\frac{T}{2}, \tag{32}$$

for $n>0$, $n=0$, and $n<0$.

At any instant of time τ, which lies in the interval: $0\leq\tau<+\infty$, it is easily seen that

(i) if $n\geq l\geq 1$, we obtain

$$\frac{T}{2}\leq\Theta_{l,0}(\tau)\leq\Theta_{n,0}(\tau)\leq T, \tag{33}$$

and

(ii) if $n\leq l\leq 0$, we obtain

$$0\leq\Theta_{n,0}(\tau)\leq\Theta_{l,0}(\tau)\leq\frac{T}{2}. \tag{34}$$

The expressions for the potential energies (29) ~ (31), have initial values of:

$$U_{n,0}(\tau)=\begin{cases}T, & \text{for } n\geq 0\\ 0. & \text{for } n\leq -1\end{cases}$$

We found also that, for $0\leq\tau<+\infty$,
(i) if $n\geq l\geq 0$, we obtain:

$$\frac{T}{2}\leq U_{l,0}(\tau)\leq U_{n,0}(\tau)\leq T, \tag{35}$$

and

(ii) if $n\leq l\leq -1$, we obtain:

$$0\leq U_{n,0}(\tau)\leq U_{l,0}(\tau)\leq\frac{T}{2}, \tag{36}$$

while, the expressions (29)~(31) approach the same stationary value $T/2$, when τ goes to infinity,
i.e.

$$\lim_{\tau\to+\infty}U_{n,0}(\tau)=\frac{T}{2}, \tag{37}$$

for $n>0$, $n=0$, and $n<0$.

Therefore, we can say that the expressions (32) and (37) establish the equipartition of the kinetic and the potential energies. The kinetic and the potential energies of each particle approach the same stationary value $kT/4$, when τ goes to infinity.

On the other hand, the correlation functions $V_{n,m}(\tau)$ behave somewhat differently. From (18) we obtain for the case: $N\to+\infty$,

$$V_{n,m}(\tau)=T\cdot[|\sum_{k=1/2}^{+\infty+1/2}J_{k+m+1/2}(\tau)J_{k-m-1/2}(\tau)+$$
$$+\sum_{k=-2n-m+1/2}^{-1/2}J_{k+m+1/2}(\tau)J_{k-m-1/2}(\tau)|], \qquad \text{for } 2n>-m \tag{38}$$
$$=T\cdot[|\sum_{k=1/2}^{+\infty+1/2}J_{k+m+1/2}(\tau)J_{k-m-1/2}(\tau)-$$
$$-\sum_{k=1/2}^{-2n-m-1/2}J_{k+m+1/2}(\tau)J_{k-m-1/2}(\tau)|], \qquad \text{for } 2n<-m. \tag{39}$$
$$=T\cdot|\sum_{k=1/2}^{+\infty+1/2}J_{k+m+1/2}(\tau)J_{k-m-1/2}(\tau), \qquad \text{for } 2n=-m \tag{40}$$

where the summation with regard to k covers half odd integers.

From the above expressions and by means of the formula***:

$$\lim_{\tau\to+\infty}\sum_{k=1/2}^{+\infty+1/2}J_{k+\alpha}(\tau)J_{k-\alpha}(\tau)=\frac{\sin(\pi\alpha)}{2\pi\alpha}, \tag{41}$$

we find:

*** cf. Appendix II.

$$\lim_{\tau\to+\infty} V_{n,m}(\tau) = T\cdot\left|\frac{(-1)^m}{\pi(2m+1)}\right|\cdot \qquad \text{for } -N+1\le n,\ n+m\le N \tag{42}$$

The expression (42) means that the correlation between velocities and positions of particles never vanish asymptotically. We also find that the correlation function $V_{n,m}(\tau)$ between distant particles is smaller than that between near particles, *viz.* if $0\le|l|\le|r|$, then we have, in the case: $\tau\to+\infty$,

$$0 < V_{n,r}(\tau) \le V_{n,l} \le \frac{T}{\pi}\cdot \tag{43}$$

For $m=0$ in the above expressions, we obtain:

$$\lim_{\tau\to+\infty}\lim_{N\to+\infty} V_{n,0}(\tau) = \frac{T}{\pi}, \tag{44}$$

which is the same result obtained directly from (19) in the case: $N\to+\infty$ and $\tau\to+\infty$. The expressions (42) and (44) represent the existence of the instantaneous flow of energy at every point in the system.

It is worth while remarking[5] that our results (26)~(28) and (29)~(31) have similar properties to the solution:

$$\Theta(x,\ t) = \frac{T}{2}\left\{1+\frac{1}{a\sqrt{\pi t}}\int_0^x \exp\left[-x^2/(4a^2t)\right]dx\right\}, \tag{45}$$

of the classical equation of heat flow in an infinite rod:

$$\frac{\partial\Theta}{\partial t} = a^2\frac{\partial^2\Theta}{\partial x^2}\,,$$

with the initial condition:

$$\Theta(x,\ t=0) = \begin{cases} T, & \text{for } x>0 \\ 0. & \text{for } x\le 0 \end{cases}$$

The expressions (26)~(28), (29)~(31) and (45), represent the process of thermal conduction to establish the uniform temperature in the system. The energies (26)~(28) and (29)~(31) oscillate with increasing time; while the solution (45) for $x<0$ increases monotonously and the solution (45) for $x>0$ decreases monotonously, with increasing time. In brief, all the expressions above mentioned approach the same stationary value $T/2$ after a sufficiently long time. It should be added, however, that the timal behavior of (26)~(28) and (29)~(31) is essentially different from the process of thermal diffusion in a rod given by (45).

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Appendix I

If we consider the case: $N\to+\infty$ in (13), we shall have:

$$\frac{\Theta_{n,m}(\tau)}{T}=|\sum_{k=-2n-m+1}^{+\infty}J_{k+m}(\tau)J_{k-m}(\tau)|$$

$$=\begin{cases}[|\sum_{k=1}^{+\infty}J_{k+m}(\tau)J_{k-m}(\tau)+\sum_{k=-2n-m+1}^{0}J_{k+m}(\tau)J_{k-m}(\tau)|], & \text{for } 2n>-m\\ [|\sum_{k=1}^{+\infty}J_{k+m}(\tau)J_{k-m}(\tau)-\sum_{k=1}^{-2n-m}J_{k+m}(\tau)J_{k-m}(\tau)|], & \text{for } 2n<-m\\ |\sum_{k=1}^{+\infty}J_{k+m}(\tau)J_{k-m}(\tau)|. & \text{for } 2n=-m\end{cases} \quad \text{(I-1)}$$

The expression involved in the above is easily calculated as follows:

$$\sum_{k=1}^{+\infty}J_{k+m}(\tau)J_{k-m}(\tau)=\frac{1}{2}\{\sum_{k=-\infty}^{+\infty}J_{k+m}(\tau)J_{k-m}(\tau)+(-1)^{m+1}J_m^2(\tau)\}$$

$$=\frac{1}{2}\{\delta_{m,-m}+(-1)^{m+1}J_m^2(\tau)\}$$

$$=\frac{1}{2}\{\delta_{m,0}+(-1)^{m+1}J_m^2(\tau)\}, \quad \text{(I-2)}$$

for m and k integers.

From (I-1) and (I-2), we obtain the results (20), (21), and (22).

Appendix II

We shall demonstrate the following formulae:

$$\sum_{k=(\nu+1)/2}^{+\infty+(\nu+1)/2}J_{k+\alpha}(z)J_{k-\alpha}(z)=\frac{1}{\pi}\int_0^{\pi/2}d\theta\cos(2\alpha\theta)\cdot\int_0^{2z\cos\theta}J_\nu(\xi)d\xi, \quad \text{(II-1)}$$

$$k\in\left\{\left(\frac{\nu+1}{2}+\text{non-negative integers}\right)\right\}$$

and

$$\lim_{z\to+\infty}\sum_{k=(\nu+1)/2}^{+\infty+(\nu+1)/2}J_{k+\alpha}(z)J_{k-\alpha}(z)=\frac{\sin(\pi\alpha)}{2\pi\alpha}, \quad \text{(II-2)}$$

for any real α, not necessarily an integer.

To prove these formulae, we take formula (*cf.* Watson: *Bessel Functions*[7], p. 150):

$$J_{\nu+\alpha}(z)J_{\nu-\alpha}(z)=\frac{2}{\pi}\int_0^{\pi/2}J_{2\nu}(2z\cos\theta)\cos(2\alpha\theta)d\theta, \quad \text{(II-3)}$$

for a complex number ν with $\Re_e(2\nu)>-1$.

For any real μ, we obtain:

$$\sum_{\mu>-1/2}^{+\infty}J_{\mu+\alpha}(z)J_{\mu-\alpha}(z)=\frac{2}{\pi}\sum_{\mu>-1/2}^{+\infty}\int_0^{\pi/2}J_{2\mu}(2z\cos\theta)\cos(2\alpha\theta)d\theta$$

$$=\frac{2}{\pi}\int_0^{\pi/2}d\theta\cos(2\alpha\theta)\cdot\sum_{\mu>-1/2}^{\infty}J_{2\mu}(2z\cos\theta)$$

$$= \frac{2}{\pi}\int_0^{\pi/2} d\theta \cos(2\alpha\theta) \cdot \sum_{\mu=r+(\nu+1)/2>-1/2}^{\infty} J_{2r+\nu+1}(2z\cos\theta). \quad \text{(II-4)}$$

with real $\nu > -2$, and non-negative integers r.
Accordingly, from (II-4), we obtain:

$$\sum_{\mu=(\nu+1)/2}^{+\infty+(\nu+1)/2} J_{\mu+\alpha}(z)J_{\mu-\alpha}(z) = \frac{2}{\pi}\int_0^{\pi/2} d\theta \cos(2\alpha\theta) \cdot \sum_{r=0}^{\infty} J_{2r+\nu+1}(2z\cos\theta)$$

$$\mu \in \left\{\left(\frac{\nu+1}{2} + \text{non-negative integers}\right)\right\}$$

$$= \frac{1}{\pi}\int_0^{\pi/2} d\theta \cos(2\alpha\theta)\int_0^{2z\cos\theta} J_\nu(\xi)d\xi, \quad \text{(II-5)}$$

for $\nu > -2$.

For the special cases: $\nu = 0$ and $\nu = 1$ in (II-5), we have:

$$\sum_{\mu=1/2}^{+\infty+1/2} J_{\mu+\alpha}(z)J_{\mu-\alpha}(z) = \frac{1}{\pi}\int_0^{\pi/2} d\theta \cos(2\alpha\theta)\int_0^{2z\cos\theta} J_0(\xi)d\xi, \quad \text{(II-6)}$$

$$\mu \in \{\text{positive half odd integers}\}$$

and

$$\sum_{\mu=1}^{+\infty} J_{\mu+\alpha}(z)J_{\mu-\alpha}(z) = \frac{1}{\pi}\int_0^{\pi/2} d\theta \cos(2\alpha\theta)\int_0^{2z\cos\theta} J_1(\xi)d\xi. \quad \text{(II-7)}$$

$$\mu \in \{\text{positive integers}\}$$

We take the limit: $z \to +\infty$ in (II-5) and obtain:

$$\lim_{z\to+\infty} \sum_{\mu=(\nu+1)/2}^{+\infty+(\nu+1)/2} J_{\mu+\alpha}(z)J_{\mu-\alpha}(z) = \frac{1}{\pi}\int_0^{\pi/2} d\theta \cos(2\alpha\theta)\int_0^{\infty} J_\nu(\xi)d\xi$$

$$\mu \in \left\{\left(\frac{\nu+1}{2} + \text{non-negative integers}\right)\right\}$$

$$= \frac{1}{\pi}\int_0^{\pi/2} \cos(2\alpha\theta)d\theta \times 1$$

$$= \frac{\sin(\pi\alpha)}{2\pi\alpha}, \quad \text{(II-8)}$$

for any finite ν with $\nu > -1$, and any real α.

For the special case: $\alpha = \text{integers} \neq 0$ in (II-8), we obtain:

$$\lim_{z\to+\infty} \sum_{\mu=(\nu+1)/2}^{+\infty+(\nu+1)/2} J_{\mu+\alpha}(z)J_{\mu-\alpha}(z) = 0. \quad \text{(II-9)}$$

$$\mu \in \left\{\left(\frac{\nu+1}{2} + \text{non-negative integers}\right)\right\}$$

In the case: $\alpha \to 0$, however, we obtain from (II-8):

$$\lim_{z\to+\infty} \sum_{\mu=(\nu+1)/2}^{+\infty+(\nu+1)/2} J_\mu^2(z) = \frac{1}{2}. \quad \text{(II-10)}$$

$$\mu \in \left\{\left(\frac{\nu+1}{2} + \text{non-negative integers}\right)\right\}$$

For the special case of $\nu=1$ in (II-9) and (II-10), we have:

$$\lim_{z\to+\infty}\sum_{k=1}^{+\infty} J_{k+\alpha}(z)J_{k-\alpha}(z)=0,$$

and

$$\lim_{\to+\infty}\sum_{k=1}^{+\infty} J_k^2(z)=\frac{1}{2},$$

which coincide completely with the results obtained from (I-2) for the case: $\tau\to+\infty$.

The expression (II-8) with $\nu=0$ proves (41) for any real α.

Appendix III

We take the identities:

$$\sum_{k=-\infty}^{+\infty} J_{2p+4kN}(\tau)=\frac{1}{2N}\sum_{r=-N}^{N-1}\cos\frac{2p\pi r}{2N}\cdot\cos\left(\tau\sin\frac{\pi r}{2N}\right).$$

for p: integers (III-1)

and

$$\sum_{k=-\infty}^{+\infty} J_{2p+1+4kN}(\tau)=\frac{1}{2N}\sum_{r=-N}^{N-1}\sin\frac{(2p+1)\pi r}{2N}\cdot\sin\left(\tau\sin\frac{\pi r}{2N}\right),$$

for p: integers (III-2)

and consider the case: $N\to+\infty$.

The summations in the right-hand sides of (III-1) and (III-2) are replaced by the integrations in the limit: $N\to+\infty$, and we obtain, respectively,

$$\frac{2}{\pi}\int_0^{\pi/2}\cos(2px)\cdot\cos(\tau\sin x)dx=J_{2p}(\tau),\qquad \text{for } p\text{: integers} \qquad \text{(III-3)}$$

and

$$\frac{2}{\pi}\int_0^{\pi/2}\sin\{(2p+1)x\}\cdot\sin(\tau\sin x)dx=J_{2p+1}(\tau).\quad \text{for } p\text{: integers} \quad \text{(III-4)}$$

From (III-3) and (III-4), we have:

$$\lim_{N\to+\infty}\sum_{k=-\infty}^{+\infty} J_{m+4kN}(\tau)=J_m(\tau).\qquad \text{for any integers } m \qquad \text{(III-5)}$$

The expression (III-5) proves the approximation (4).

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