ON THE LONGITUDINAL OSCILLATION OF AN AEROPLANE IN TURBULENT AIR

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Résumé

In the present paper the dynamical response of the longitudinal motion of an aeroplane in turbulent air is treated under the assumption of small deviation from steady flight and the gust of wind acting on the aeroplane is assumed to have a spectrum with a Gaussian white noise. The dynamical response of the motion of the plane was found to be quite similar to the response in a linear system of coupled oscillators excited by random inputs. Under the assumption of a stationary simple Markoffian process, the transition probability which governs the random motion of the aeroplane was obtained by the Fokker-Planck equation. The average values of the motion of the plane were given as functions of time and their long-time behavior was also investigated with the eigenfunction expansion of solution of the stochastic differential equation.

§ 0. Preliminaries

In treating the dynamical random process of an aeroplane, we can avail of two kinds of main approaches in physics: that of Brownian motion and that of kinetic theory. The first of these is embodied in Onsager's theory of irreversible process, and also in diffusion type equations such as the Fokker-Planck equation.[1][2]

An equivalent mathematical formulation is Langevin's stochastic differential equation. The second approach, *viz.*, that of kinetic theory, is crystalized in the Boltzmann equation. Perhaps the approach of kinetic theory may be put aside here, because of the nature of the phenomenological motion of the aeroplane.

In both approaches, however, the most important point is the concept of a friction coefficient. Moreover, these various equations are based upon assumptions, such as the "Stoßzahlansatz" and the Markoffian character of the motion.

To date, attempts have been made to replace these assumptions by something more satisfactory and to start from a purely dynamical equation in order to deduce its stochastic offspring. This would be perhaps a satisfactory formulation of the statistical mechanics of irreversible processes. In such a deduction, one should introduce certain *well defined averaging procedures*. Some studies in this direction have been made by Kirkwood,[3] and Prigogine and his collaborators,[4] by "coarse-graining in time" to get statistical dynamical equations from purely dynamical ones, and along the same line of study, fundamental processes in the dynamical model of an aerofoil were treated under the influence of random inputs,[5] and the nature of the energy transfer in the system was also made clear.[6]

Apart from the irreversible process, Rice[7] has given a practical method for treatment of random-in-time phenomena and shown the importance of frequency spectra. His method has been adopted by Miles[8] and others,[9] especially in the field of random noise of aeronautical problems.

In the present paper, the authors take the purely dynamical equation of motion of an aeroplane in a gust of wind and try to obtain Fokker-Planck's stochastic differential equation, where the friction coefficient comes from the stable character of the aeroplane at a steady flight. The statistical dynamical response of the aeroplane and the joint and the conditional probabilities of the responce were given under the assumption of the Markoffian charactar of the motion. The eigenfunction-expansion of the solutions of the stochastic equations and the average values of the disturbed velocities of the plane were also given.

§ 1. Equations of Longitudinal Motion of the Aeroplane in Turbulent Air

After the gust of wind $V(t)$, an aeroplane at a steady symmetric horizontal flight, suffers small disturbances u, w, and θ. u and w are the disturbed velocities of the plane in the horizontal and the downward directions, respectively. θ represents the disturbed angle of rotation of the plane about its center of gravity. Then the equations of longitudinal motion[10] of the plane in the perturbed state read as follows:

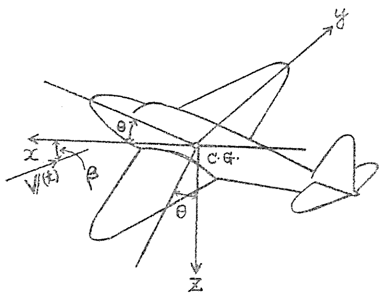

FIG. 1. Coordinate system

$$\left.\begin{aligned}
&\tau\frac{du}{dt}+x_u\cdot u+x_w\cdot w+x_w\cdot u_0\cdot\theta=(x_u\cdot\cos\beta+x_w\cdot\sin\beta)V(t),\\
&\tau\frac{dw}{dt}-\tau u_0\cdot\frac{d\theta}{dt}+z_u\cdot u+z_w\cdot w=(z_u\cdot\cos\beta+z_w\cdot\sin\beta)V(t),\\
\text{and}\quad&\\
&\tau^2 u_0\frac{d^2\theta}{dt^2}+m_q\cdot\tau u_0\cdot\frac{d\theta}{dt}+m_w\cdot\mu w=m_w\cdot\mu\sin\beta\cdot V(t),
\end{aligned}\right\}\qquad(1)$$

where β is the angle between the direction of the gust and the x-axis. The symbols x_u, x_w, z_u, z_w, m_q and m_w show non-dimensional quantities, usually called stability derivatives:

$$\left.\begin{aligned}
&(x_u,\ x_w,\ z_u,\ z_w) = \left(2\,C_D,\ \frac{\partial C_D}{\partial \alpha} - C_L,\ 2\,C_L,\ \frac{\partial C_L}{\partial \alpha} + C_D\right),\\
&(K_B m_w,\ K_B m_q) = \left(\frac{\partial C_m}{\partial \alpha} + \frac{\partial C'_L}{\partial \alpha'}\cdot\frac{S'l}{Sc},\ \frac{\partial C'_L}{\partial \alpha'}\cdot\frac{S'l^2}{Sc^2}\right),\\
&\text{and}\\
&(C_D,\ C_L,\ C_m) = \left(X\Big/\left(\frac{1}{2}\ \rho u_0^2 S\right),\ Z\Big/\left(\frac{1}{2}\rho u_0^2 S\right),\ M\Big/\left(\frac{1}{2}\rho u_0^2 cS\right)\right).
\end{aligned}\right\}\quad(2)$$

with

$$X = \text{lift}, \qquad Z = \text{drag},$$

M = moment of force about the y-axis,
and

$$\alpha = \text{angle of incidence.}$$

The symbols S, c and l are, respectively, the wing area, the semichord-length of the aerofoil, and the distance between the center of gravity and the tail's aerodynamic center. The prime indicates the values with respect to the tail. The symbols τ and μ are the non-dimensional parameters defined by:

$$\tau = \frac{m}{\rho u_0 S}, \quad \text{and} \quad \mu = \frac{m}{\rho u_0 S}, \tag{3}$$

where m signifies the mass of the aeroplane, and ρ the density of air. For the sake of simplicity, we shall write:

$$\tau\frac{d}{dt} = \frac{d}{dt'}, \quad \text{and} \quad u_0\theta = \theta'. \tag{4}$$

Let us take four independent random variables, y_i $(i=1,\ 2,\ 3,\ 4)$:

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \equiv \begin{pmatrix} u - V_{0x}\cdot\tau \\ w - V_{0z}\cdot\tau \\ u_0\cdot\theta \\ u_0\cdot\dot{\theta} \end{pmatrix}, \tag{5}$$

where

$$V_{0x} = x\text{-component of the average value of the gust,}$$

and

$$V_{0z} = z\text{-component of the average value of the gust.}$$

Then the equation (1) becomes:

$$\left.\begin{aligned}
&\sum_{j=1}^{4}\left(\frac{dy_i}{dt'} + G_{ij}\cdot y_j\right) = E_i\cdot V_1(t'\tau),\\
&\text{or in vector notation:}\\
&\dot{\mathbf{y}} + \mathbf{G}\cdot\mathbf{y} = \mathbf{E}\cdot V_l(t'\tau),
\end{aligned}\right\}\quad(6)$$

with velocity deviation $V_l(t'\tau)$ of the gust from $\mathbf{V}_0$, where $\mathbf{G}$ and $\mathbf{E}$ are the square matrix and the column vector, respectively defined as follows:

$$\mathrm{G}=\begin{pmatrix} x_u, & x_w, & x_w, & 0 \\ z_u, & z_w, & 0, & -1 \\ 0, & 0, & 0, & -1 \\ 0, & \mu m_w, & 0, & m_q \end{pmatrix}, \text{ and } \mathrm{E}=\begin{pmatrix} x_u\cos\beta+x_w\sin\beta \\ z_u\cos\beta+z_w\sin\beta \\ 0 \\ \mu m_w\sin\beta \end{pmatrix}. \tag{7}$$

The dot indicates the differentiation with respect to time t'.

§ 2. Statistical Properties of the Gust

We have some amount of experimental information[12] about the frequency spectrum of gusts in the atmosphere. Unfortunately, however, the results obtained are sometimes different from each other, and at present we can not determine the precise functional form of the spectrum of the gusts. We know also that the average response of the linear system does not depend very mcuh upon the precise functional form of the spectrum of the inputs. Accordingly, for the sake of simplicity, we shall assume, for the gust $\mathbf{V}(t)$ which acts as external inputs to our linear system (6), the following properties:

$$\left.\begin{array}{l} \text{i) } \mathbf{V}(t)=\mathbf{V}_0(t)+\mathbf{V}_l(t), \\ \text{ii) } \langle\mathbf{V}(t)\rangle_{Av}=\mathbf{V}_0(t)+\langle\mathbf{V}_l(t)\rangle_{Av}=\mathbf{V}_0(t), \\ \text{iii) } \langle\mathbf{V}_l(t_1)\cdot\mathbf{V}_l(t)\rangle_{Av}=2D\cdot\delta(t_1-t_2), \end{array}\right\} \tag{8}$$

and

iv) The probability distribution of $\mathbf{V}_l(t)$ is Gaussion,

where the angular bracket $\langle\mathbf{V}(t)\rangle_{Av}$ indicates the average value of $\mathbf{V}(t)$. The third equation of (8) indicates that the gust $\mathbf{V}_l(t)$ has a white spectrum of hight D.

§ 3. The Fokker-Planck Equation

We shall assume that the random process governing the dynamical behaviour of the aerorplane is of a stationary, simple Markoffian process. We can determine, from the following relation (9), the joint probability $W(\mathbf{y}_1, \mathbf{y}_2; t')$, which is the probability of finding $\mathbf{y}$ in the range $(\mathbf{y}_1, \mathbf{y}_1+d\mathbf{y}_1)$ at any time t_0' and in the range $(\mathbf{y}_2, \mathbf{y}_2+d\mathbf{y}_2)$ at a later time $t_0'+t'$, *viz.*,

$$W(\mathbf{y}_1, \mathbf{y}_2; t')=W(\mathbf{y}_1)\cdot P(\mathbf{y}_1|\mathbf{y}_2; t'), \tag{9}$$

where $P(\mathbf{y}_1|\mathbf{y}_2; t')$ is the conditional probability function which is the probability that when $\mathbf{y}$ takes a value $\mathbf{y}_1$ at any time t_0', one finds $\mathbf{y}$ in the range $(\mathbf{y}_2, \mathbf{y}_2+d\mathbf{y}_2)$ at a later time $t_0'+t'$.

The probability density function $P(\mathbf{y}_1|\mathbf{y}_2; t')$ satisfies the following conditions:

$$\left.\begin{array}{ll} \text{i)} & P(\mathbf{y}_1|\mathbf{y}_2;\ t')\geq 0, \\ \text{ii)} & \int_{-\infty}^{+\infty} P(\mathbf{y}_1|\mathbf{y}_2;\ t')d\mathbf{y}_2=1, \\ \text{iii)} & \int_{-\infty}^{+\infty} W(\mathbf{y}_1)\cdot P(\mathbf{y}_1|\mathbf{y}_2;\ t')\,d\mathbf{y}_1=W(\mathbf{y}_2), \end{array}\right\} \qquad (10)$$

and
the Smoluchowski, equation:

$$\text{iv)}\ P(\mathbf{y}_1|\mathbf{y}_2;\ t')=\int_{-\infty}^{+\infty} d\mathbf{z}\cdot P(\mathbf{y}_1|\mathbf{z};\ t_0')\cdot P(\mathbf{z}|\mathbf{y}_2;\ t'-t_0'). \qquad (11)$$

The n-th moment of $\mathbf{y}$ about the coordinate $\mathbf{z}$ after a short interval of time $\Delta t'$ is defined as follows:

$$a_n(\mathbf{z},\ \Delta t')=\int_{-\infty}^{+\infty} d\mathbf{y}(\mathbf{y}-\mathbf{z})^n P(\mathbf{z}|\mathbf{y};\ \Delta t'). \qquad (12)$$

Now, we shall assume that only the first and the second moments take finite values while higher moments all vanish in the limit: $\Delta t'\to 0$. Then the expression (11) is replaced by the well-known Fokker-Planck equation:[2)]

$$\frac{\partial P(\mathbf{y}_1|\mathbf{y}_2;\ t')}{\partial t'}=\sum_{i=1}^{4}\frac{\partial}{\partial y_i}[A_i\cdot P(\mathbf{y}_1|\mathbf{y}_2;\ t')]+\sum_{i,j=1}^{4}B_{ij}\frac{\partial^2 P(\mathbf{y}_1|\mathbf{y}_2;\ t')}{\partial y_i\partial y_j}, \qquad (13)$$

where

$$\begin{aligned} A_i &= \lim_{\Delta t'\to 0}\frac{1}{\Delta t'}\int_{-\infty}^{+\infty}\Delta y_i\cdot P(\mathbf{y}_1|\mathbf{y}_2;\ t')d(\Delta\mathbf{y}) \\ &= \lim_{\Delta t'\to 0}\frac{\langle\Delta y_i\rangle_{Av}}{\Delta t'}, \qquad (i=1,2,3,4) \end{aligned} \qquad (14)$$

and

$$\begin{aligned} B_{ij} &= \frac{1}{2}\lim_{\Delta t'\to 0}\frac{1}{\Delta t'}\int_{-\infty}^{+\infty}(\Delta y_i)(\Delta y_j)\cdot P(\mathbf{y}_1|\mathbf{y}_2;\ t')\,d(\Delta\mathbf{y}) \\ &= \frac{1}{2}\lim_{\Delta t'\to 0}\frac{\langle\Delta y_i\cdot\Delta y_j\rangle_{Av}}{\Delta t'}. \end{aligned} \qquad (15)$$

§ 4. Transition Probability

The equations of motion of the aeroplane in the gust are given by the linear equation (6), while the transition probability function $P(\mathbf{y}_1|\mathbf{y}_2;\ t')$ is determined by the solution of the partial differential equation (13). The first moment (*i.e.* average value) defined by equation (14), can be computed as follows:

Integrating equation (6) over a short interval of time $\Delta t'$, we shall obtain:

$$\Delta\mathbf{y}=(-\mathbf{G}\cdot\mathbf{y})\Delta t'+\int_{t'}^{t'+\Delta t'}\mathbf{E}\cdot V_l(t'\tau)dt', \qquad (16)$$

and

$$\mathbf{A} = \lim_{\Delta t' \to 0} \frac{\langle \Delta \mathbf{y} \rangle_{Av}}{\Delta t'} = -\mathbf{G} \cdot \mathbf{y},$$

i.e.

$$\mathbf{A} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} & & & \\ & -\mathbf{G} & & \\ & & & \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}. \tag{17}$$

In a similar manner, the second moment $\mathbf{B}$ with components B_{ij} $(ij = 1, 2, 3, 4)$ becomes:

$$\begin{aligned} B_{ij} &= \frac{1}{2} \lim_{\Delta t' \to 0} \frac{\langle \Delta y_i \cdot \Delta y_j \rangle_{Av}}{\Delta t'} \\ &= \frac{1}{2} \lim_{\Delta t' \to 0} \frac{A_i A_j \cdot (\Delta t')^2 + (E_i E_j / \tau^2) \int_{t'}^{t'+\Delta t'} \int \langle V_I(\xi) V_I(\eta) \rangle_{Av} d\xi \, d\eta}{\Delta t'} \\ &= \frac{D}{\tau^2} E_i E_j, \end{aligned}$$

i.e

$$\mathbf{B} = \frac{D}{\tau^2} \begin{pmatrix} E_1^2, & E_1 E_2, & 0, & E_1 E_4 \\ E_2 E_1, & E_2^2, & 0, & E_2 E_4 \\ 0, & 0, & 0, & 0 \\ E_4 E_1, & E_4 E_2, & 0, & E_4^2 \end{pmatrix}. \tag{18}$$

Here, we used the equation (8).

Now, let us transform $\mathbf{y}$ into $\mathbf{z}$,

$$\mathbf{y} = \mathbf{C} \cdot \mathbf{z}, \quad y_i = \sum_{j=1}^{4} c_{ij} y_j, \qquad (i = 1, 2, 3, 4) \tag{19}$$

with the matrix $\mathbf{C}$, which satisfies the relation:

$$\mathbf{C} \cdot (-\mathbf{G}) = \mathbf{R} \cdot \mathbf{C}, \quad \sum_{j=1}^{4} c_{ij} (-G_{jl}) = \lambda_i c_{il}, \qquad (i, j = 1, 2, 3, 4) \tag{20}$$

where $\mathbf{R}$ is a diagonal matrix:

$$\mathbf{R} = \| \lambda_i \delta_{ij} \|.$$

The eigenvalues of the matrix λ_i $(i = 1, 2, 3, 4)$ are given by the characteristic equation:

$$\det |G_{ij} + \lambda \delta_{ij}| = 0. \tag{21}$$

If the aeroplane is dynamically stable, we have $\Re_e(\lambda_i) < 0$ for $i = 1, 2, 3, 4$.

By means of the linear transformation (19), the equation (13) becomes:

$$\frac{\partial P(\mathbf{z}, t')}{\partial t'} = -\sum_{i=1}^{4} \lambda_i \frac{\partial}{\partial z_i} [z_i \cdot P(\mathbf{z}, i')] + \sum_{i,j=1}^{4} \sigma_{ij} \frac{\partial^2 P(\mathbf{z}, t')}{\partial z_i \partial z_j}. \tag{22}$$

where $\|\sigma_{ij}\| \equiv \mathbf{S} = \mathbf{C}\cdot\mathbf{B}\cdot\tilde{\mathbf{C}}$,
with $\tilde{\mathbf{C}}$, the transposed matrix of $\mathbf{C}$. For the sake of simplicity, we write $P(\mathbf{z}, t')$ instead of $P(\mathbf{z}_1|\mathbf{z}_2; t')$.

We introduce the Fourier tranasform of $P(\mathbf{z}, t')$:

$$f(\mathbf{s}, t') = \iiiint_{-\infty}^{+\infty} d\mathbf{z} P(\mathbf{z}, t') \exp[-i\tilde{\mathbf{s}}\mathbf{z}]$$

$$= \iiiint_{-\infty}^{+\infty} dz_1 dz_2 dz_3 dz_4 P(\mathbf{z}, t') \exp[-i\sum_{j=1}^{4} s_j z_j]. \tag{23}$$

From the equation (22), we shall see that $f(\mathbf{s}, t')$ satisfies the following differential equation:

$$\frac{\partial f}{\partial t'} - \sum_{i=1}^{4} \lambda_i s_i \frac{\partial f}{\partial s_i} = -f \sum_{i,j=1}^{4} \sigma_{ij} s_i s_j. \tag{24}$$

By means of the linear transformation:

$$\mathbf{s} = \mathbf{D}\cdot\mathbf{v}, \qquad s_i = \sum_{j=1}^{4} D_{ij} v_j, \tag{25}$$

the equation (24) can be transformed into the equation of diagonal form:

$$\frac{\partial f}{\partial t'} - \sum_{i=1}^{4} q_i v_i \frac{\partial f}{\partial v_i} = -f \sum_{i=1}^{4} a_i^2 v_i^2, \tag{26}$$

with

$$(\mathbf{D}^{-1}\mathbf{R}\mathbf{D})_{ij} = q_i \cdot \delta_{ij},$$

and

$$(\tilde{\mathbf{D}}\mathbf{S}\mathbf{D})_{ij} = a_i^2 \cdot \delta_{ij},$$

where $\mathbf{D}^{-1}$ and $\tilde{\mathbf{D}}$ are the inverse and the transposed matrices of $\mathbf{D}$.

The characteristic equations of (26) are:

$$\frac{dt'}{1} = -\frac{dv_1}{q_1 v_1} = -\frac{dv_2}{q_2 v_2} = -\frac{dv_3}{q_3 v_3} = -\frac{dv_4}{q_4 v_4} = \frac{-df}{f \sum_{i=1}^{4} a_i^2 v_i^2}. \tag{27}$$

Accordingly, the general solution of the equation (26) is:

$$f(\mathbf{v}, t') = \phi\{v_1 \exp(q_1 t'), v_2 \exp(q_2 t'), v_3 \exp(q_3 t'), v_4 \exp(q_4 t')\} \exp\left[\frac{1}{2} \sum_{i=1}^{4} \frac{a_i^2}{q_i} v_i^2\right], \tag{28}$$

with an arbitrary function ϕ.

Next, let us take as an initial condition,

$$P(\mathbf{z}; t' = 0) = \delta(\mathbf{z} - \mathbf{z}_0), \tag{29}$$

with $\mathbf{z}_0$, the initial value of $\mathbf{z}$.

The Fourier transform of (29) becomes:

$$f(\mathbf{s};\ t'=0)=\exp[-i\tilde{\mathbf{z}}_0\mathbf{s}]. \tag{30}$$

Thus the function ϕ at the initial time $t'=0$, takes the form:

$$\phi(\mathbf{v};\ 0)=\exp\Big[-i\tilde{\mathbf{z}}_0\mathbf{D}\mathbf{v}-\frac{1}{2}\tilde{\mathbf{v}}\mathbf{H}^{-1}\mathbf{v}\Big], \tag{31}$$

with

$$(\mathbf{H})_{ij}=\frac{q_i}{a_i^2}\delta_{ij},\qquad (\mathbf{H}^{-1})_{ij}=\frac{a_i^2}{q_i}\delta_{ij}.$$

From (28) and (31) we shall obtain

$$f(\mathbf{v};\ t')=\exp\Big[-i\sum_{i,j=1}^{4}z_{i0}D_{ij}v_j\cdot\exp(q_jt')+ + \frac{1}{2}\sum_{i=1}^{4}\frac{1}{H_i^2}v_i^2\{1-\exp(2q_it')\}\Big], \tag{32}$$

i.e.

$$f(\tilde{\mathbf{s}};\ t')=\exp\Big[-i\tilde{\mathbf{z}}_0\tilde{\mathbf{L}}\mathbf{D}^{-1}\mathbf{s}+\frac{1}{2}\mathbf{s}\tilde{\mathbf{D}}^{-1}\mathbf{K}\mathbf{D}^{-1}\mathbf{s}\Big]. \tag{33}$$

where

$$(\mathbf{L})_{ij}=D_{ji}\cdot\exp(q_it'),$$

$$(\mathbf{K})_{ij}=\frac{1}{H_i^2}\cdot\delta_{ij}\cdot\{1-\exp(2q_it')\}.\qquad (i,j=1,2,3,4)$$

The expression (30) is the characteristic function of the 4 dimensional Gaussian distribution with the average value,

$$\bar{\mathbf{z}}\equiv\langle\mathbf{z}\rangle_{Av}=\tilde{\mathbf{D}}^{-1}\mathbf{L}\cdot\mathbf{z}_0, \tag{34}$$

and the covariance,

$$\langle(z_i-\bar{z}_i)(z_j-\bar{z}_j)\rangle_{Av}=(\tilde{\mathbf{D}}^{-1}\mathbf{K}\mathbf{D}^{-1})_{ij}. \tag{35}$$

The average value and the covariance satisfy the following equations:

$$\frac{d\langle\mathbf{z}\rangle_{Av}}{dt'}+\mathbf{R}\langle\mathbf{z}\rangle_{Av}=0, \tag{36}$$

and

$$\frac{\partial(\overline{z_iz_j})}{\partial t'}=(\lambda_i+\lambda_j)\cdot(\overline{z_iz_j})+\sigma_{ij}.\qquad (i,j=1,2,3,4) \tag{37}$$

Since $\Re_e(q_i)<0$ for $i=1,2,3,4$, the expression (34) vanishes for $t'\to+\infty$, *i.e.*

$$\lim_{t'\to+\infty}\langle\mathbf{z}\rangle_{Av}=0.$$

For the covariance we have:

$$\lim_{t'\to\infty} \mathbf{K}^{-1} = \mathbf{H}.$$

The transition probability $P(z, t')$ becomes:

$$P(\mathbf{z}_0 | \mathbf{z}\,;\ t') = \frac{\det \mathbf{D}}{(2\pi)^2\sqrt{\det \mathbf{K}}} \exp\Big[-\frac{1}{2} \widetilde{\{\tilde{\mathbf{z}} - \tilde{\mathbf{D}}^{-1}\mathbf{L}\mathbf{z}_0\}} \times \times \mathbf{D}\mathbf{K}^{-1}\tilde{\mathbf{D}}\{\mathbf{z} - \tilde{\mathbf{D}}^{-1}\mathbf{L}\mathbf{z}_0\} \Big]. \quad (38)$$

Accordingly, we shall find the transition probability function in the limit: $t'\to+\infty$, as follows:

$$P(\mathbf{z}\,;\ \infty) \frac{\prod_{k=1}^{4} H_k \cdot \det \mathbf{D}}{(2\pi)^2} \cdot \exp\Big[-\frac{1}{2} \tilde{\mathbf{z}}\mathbf{D}\mathbf{H}\tilde{\mathbf{D}}\mathbf{z} \Big]. \quad (39)$$

Transforming back to the original variable $\mathbf{y}$, we can rewrite the average value of $\mathbf{y}$ as follows:

$$\langle \mathbf{y} \rangle_{Av} = \exp[-\mathbf{G}t'] \cdot \mathbf{y}_0, \quad (40)$$

and the covariance matrix as follows:

$$\mathbf{M} = \mathbf{C}^{-1} \cdot \mathbf{K} \cdot \tilde{\mathbf{C}}^{-1}, \quad (41)$$

$$M_{ij} = \langle (y_i - \bar{y}_i)(y_j - \bar{y}_j) \rangle_{Av},$$

where $\mathbf{y}_0$ is the initial value of $\mathbf{y}$; $\mathbf{C}^{-1}$ and $\tilde{\mathbf{C}}$ are the inverse and the transposed matrices of $\mathbf{C}$, respectively.

Here we shall find that $\langle \mathbf{y} \rangle_{Av}$ also vanishes, and that $\mathbf{M}$, the covariance matrix of $\mathbf{y}$, is of diagonal form.

For $t'\to+\infty$, we shall see that the transition probability function $P(\mathbf{y}, t')$ of the disturbances of the aeroplane approaches the probability density function of the 4-dimensional Gaussian distribution:

$$\lim_{t'\to\infty} P(\mathbf{y}\,;\ t') = \frac{1}{(2\pi)^2\sqrt{\det \mathbf{M}}} \exp\Big[-\frac{1}{2} \tilde{\mathbf{y}}\mathbf{M}^{-1}\mathbf{y} \Big]$$

$$= \prod_{i=1}^{4} \frac{1}{\sqrt{2\pi M_i}} \exp\Big[-\frac{y_i^2}{2M_i} \Big]. \quad (42)$$

From (42), we shall see that the transition probability for $t'\to+\infty$ becomes:

$$\lim_{t'\to+\infty} P(\mathbf{y}_1 | \mathbf{y}_2\,;\ t') = P(\mathbf{y}_2). \quad (43)$$

The quantity $\langle \mathbf{y} \rangle_{Av}$ satisfies the following equation:

$$\frac{d\langle \mathbf{y} \rangle_{Av}}{dt'} + \mathbf{G} \cdot \langle \mathbf{y} \rangle_{Av} = 0, \quad (44)$$

which shows that $\langle y\rangle_{Av}$ behaves like the velocity of a particle in a damping medium.

For $t'\to+\infty$, the joint probability function $W(y_1, y_2; t')$ approaches the product of $W(y_1)$ and $W(y_2)$, *viz.*

$$\lim_{t'\to+\infty} W(y_1, y_2; t') = W(y_1)\cdot W(y_2), \tag{45}$$

with

$$\lim_{t'\to+\infty} P(y_1|y_2; t') = W(y_2). \tag{46}$$

§ 5. Eigenfunction Expansion of the Solution of the Stochastic Equation

After diagonalizing the matrix $\|\sigma_{ij}\|$ in the equation (22), we shall obtain the following form of the Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = \sum_{i=1}^{4} \frac{1}{\tau_i}\left[\frac{\partial}{\partial z_i}(z_i P) + \frac{\partial^2}{\partial z_i^2} P\right]. \tag{47}$$

We can easily see that, with an arbitrary initial probability density function $W(z_i, t=0)$, the probability density function $W(z_i, t)$ at any instant of time t, satisfies the same equation:

$$\frac{\partial W(z, i)}{\partial t} = \sum_{j=1}^{4} \frac{1}{\tau_j}\left[\frac{\partial}{\partial z_j}\left(z_j W(z, t) + \frac{\partial^2}{\partial z_j^2} W(z, t)\right)\right]. \tag{48}$$

Separating variables t and z, we shall obtain the eigen-function[11] of the partial differential equation (48) as follows:

$$W_{k_1 k_2 k_3 k_4}(z_i, t) = \exp\left[-\frac{t}{\tau} - \frac{1}{2}\sum_{i=1}^{4} z_i^2\right]\prod_{i=1}^{4} He_{k_i}(z_i), \tag{49}$$

where $He_{k_i}(z_i)$ is an Hermite polynomial of degree k_i, and the time constant τ is given by the relation:

$$\frac{1}{\tau} = \sum_{i=1}^{4} \frac{k_i}{\tau_i}. \tag{50}$$

The expression (50), where k_i can take all the non-negative integers, gives the spectrum of all the relaxation times τ_i $(i=1, 2, 3, 4)$.

The average values of the powers of z_i, *i.e.* the moments of z_i, satisfy simpler differential equations, *e.g.* equations (36) and (37).

From the equation (39), we can also easily see that, if the initial distribution is Gaussian, then we shall obtain a Gaussian distribution at $t\to+\infty$.

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