WAKE FLOW WITH AXIAL PRESSURE GRADIENT FAR BEHIND FROM THE BODY

MICHIRU YASUHARA and MASAKAZU GOTŌ

Department of Aeronautical Engineering

(Received May 31, 1963)

Summary

Laminar and turbulent flow solutions in the wake with axial pressure gradient are treated for a two-dimensional, imcompressible flow. A similar solution is obtained within the wake approximation, which is applicable to the flow far downstream from the body with any pressure gradient. In the laminar wake if the volocity $U(x)$ outside the wake is proportional to x^m, then the width of the wake is proportional to $x^{(1-m)/2}$ and the velocity defect on the wake axis to $x^{(-3m+1)/2}$, and in the turbulent wake the width to $x^{(-3m+1)/2}$ and the velocity defect to $x^{(-m+1)/2}$.

Introduction

In the wake behind a flat plate at zero incidence, as shown in Fig. 1, or at some distances behind a body, as shown in Fig. 2, the boundary layer equation can be applied. As is well known, at a large distance from the body, the velocity defect u_1 in the wake becomes small compared with the velocity $U(x)$ outside the wake, and the solutions were already given in the case without axial pressure gradient.[1) 2)] In the present paper we extend these solutions to cases with axial pressure gradient.

In the case without axial pressure gradient, it is known that the flow in the wake at a large distance from the body becomes similar and the crosswise velocity can be assumed to be negligible. In the wake with axial pressure gradient, however, as will be shown in the following sections, the crosswise velocity v can not be neglected.

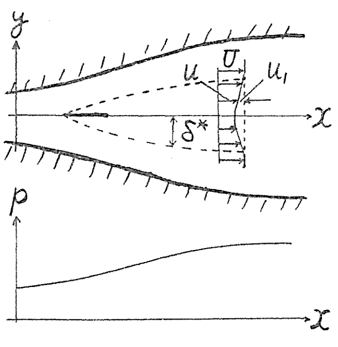

FIG. 1. Schematic pressure and velocity distribution in the wake behind a flat plate.

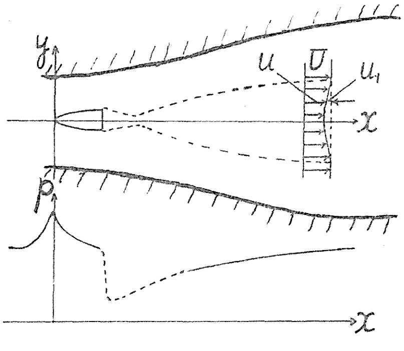

FIG. 2. Schematic pressure and velocity distribution in the wake behind a blunt based body.

§ 1. Laminar Wake

We first consider a laminar wake with axial pressure gradient. The boundary layer equations for a two-dimensional, imcompressible, steady flow are given by:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2}$$

where $U(x)$ is the x-wise velocity component just outside the wake. Now we introduce the velocity defect $u_1(x, y)$ as follows:

$$u(x, y)=U(x)-u_1(x, y), \tag{3}$$

and apply the appoximation that $U(x)\gg u_1(x, y)$, which is applicable in the wake far behind from the body.

Then, the boundary layer equations (1), (2) combined with Eq. (3) give:

$$\frac{\partial(Uu_1)}{\partial x}+v\frac{\partial u_1}{\partial y}=\nu\frac{\partial^2 u_1}{\partial y^2}, \tag{4}$$

$$\frac{dU}{dx}+\frac{\partial v}{\partial y}=0, \tag{5}$$

where quadratic terms in u_1 are neglected. The continity equation (5) can be integrated immediately to give:

$$v=-\frac{dU}{dx}y, \tag{6}$$

because $(v)_{y=0}=0$. Although in the wake without axial pressure gradient the velocity v can be neglected, yet in the present case as shown in Eq. (6) v must be taken into acount. Introducing Eq. (6) into Eq. (4) the differential equation for the velocity defect becomes:

$$\frac{\partial(Uu_1)}{\partial x}-\frac{dU}{dx}y\frac{\partial u_1}{\partial y}=\nu\frac{\partial^2 u_1}{\partial y^2}. \tag{7}$$

We introduce a new valiable η, and assume the similar solution for the velocity defect u_1 as follows:

$$u_1(x, y)=h(x)f(\eta), \qquad \eta=\frac{y}{g(x)}, \tag{8}$$

where $g(x)$ can be considered as a measure of the wake width, and $h(x)$ gives the velocity defect along the wake axis. Introducing Eq. (8) into Eq. (7), we have:

$$f''+\frac{g(gU)'}{\nu}\eta f'-\frac{g^2(Uh)'}{\nu h}f=0. \tag{9}$$

As we assume that the velocity defect u_1 has a similar solution, so that f

may depend on η alone, coefficients of the second and third terms of the left-hand side of Eq. (9) must be constants. Without loss of generality, these coefficients can be taken to be constants, separately, since there are two free factors $g(x)$ and $h(x)$ to be chosen suitably. Therefore we can set as follows:

$$\left.\begin{aligned} \frac{g(gU)'}{\nu} &= 1, \\ \frac{g^2(hU)'}{\nu h} &= -1. \end{aligned}\right\} \tag{10}$$

These are the differential equations determining $g(x)$ and $h(x)$. From Eqs. (10) $g(x)$ and $h(x)$ can be integrated to give:

$$\left.\begin{aligned} g(x) &= \frac{U(x_0)}{U(x)}\left[2\nu\int_{x_0}^{x}\frac{U(x)}{U^2(x_0)}dx+g_0^2\right]^{1/2}, \\ h(x) &= h_0\frac{U(x_0)}{U(x)}\exp\left[-\int_{x_0}^{x}\frac{\nu}{\frac{U^2(x_0)}{U(x)}\left\{2\nu\int_{x_0}^{x}\frac{U(x)}{U^2(x_0)}dx+g_0^2\right\}}dx\right], \end{aligned}\right\} \tag{11}$$

with the conditions that:

$$U(x)=U(x_0),\ g(x)=g_0,\ h(x)=h_0 \text{ at } x=x_0.$$

When $g_0=0$, Eqs. (11) become:

$$\left.\begin{aligned} g(x) &= \frac{\sqrt{2\nu}}{U(x)}\left[\int_{x_0}^{x}U(x)dx\right]^{1/2}, \\ h(x) &= \frac{C}{U(x)}\exp\left[-\frac{1}{2}\int_{x_0}^{x}\frac{U(x)}{\int_{x_0}^{x}U(x)dx}dx\right], \end{aligned}\right\} \tag{12}$$

where $C\{=h_0U(x_0)\}$ is a constant to be determined physically. Such a method of derivation is very similar to Görtler's one.[3]

With $g(x)$ and $h(x)$ given by Eqs. (11) or (12), Eq. (9) is reduced to:

$$f''+\eta f'+f=0, \tag{13}$$

which is the same as that for the case without axial pressure gradient. The boundary conditions are as follows:

$$\frac{\partial u_1}{\partial y}=0,\ f=1 \text{ at } y=0, \text{ and } u_1=0 \text{ as } y\to\infty. \tag{14}$$

Integrating Eq. (13) with these boundary conditions, we have:

$$f(\eta)=e^{-1/2\,\eta^2}. \tag{15}$$

The above derivation shows that the profile of the velocity defect in the wake at a large distance from the body becomes similar, even in the case with axial pressure gradient. It is important to note that the profile of the velocity defect $f(\eta)$ is independent of $U(x)$.

As a particular example, we consider the case when $U(x)$ is of the form:

$$U(x) = U_0\left(\frac{x}{L}\right)^m, \quad x_0 = 0, \quad g_0 = 0: \tag{16}$$

were L is a characteristic length, and U_0 is a constant. Introducing Eq. (16) into Eqs. (12), we obtain:

$$\left.\begin{aligned} g(x) &= \left\{\frac{2\nu L}{U_0(1+m)}\right\}^{1/2}\left(\frac{x}{L}\right)^{(1-m)/2}, \\ h(x) &= C' U_0\left(\frac{x}{L}\right)^{-(3m+1)/2}, \end{aligned}\right\} \tag{17}$$

and the velocity defect $u_1(x, y)$ in the wake is:

$$u_1(x, y) = C' U_0\left(\frac{x}{L}\right)^{-(3m+1)/2} \exp\left[-\frac{1}{2}\frac{y^2}{\dfrac{2\nu L}{U_0(1+m)}\left(\dfrac{x}{L}\right)^{1-m}}\right], \tag{18}$$

where C' is a nondimensional constant. This means that the width of the wake develops proportionally to $x^{(1-m)/2}$ and the velocity defect to $x^{-(3m+1)/2}$, when the velocity $U(x)$ outside the wake is proportional to x^m.

§ 2. Turbulent Wake

In the case of turbulent wake, the boundary layer equations are:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = U\frac{dU}{dx} + \frac{1}{\rho}\frac{\partial \bar{\tau}}{\partial y}, \tag{19}$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0, \tag{20}$$

where-denotes averaged value. The shearing stress $\bar{\tau}$ includes the viscous and the Reynolds stress. In the wake the viscous stress can be assumed small compared with the Reynolds stress. If we assume that the Prandtl's mixing length theory can be applicable also in the case with axial pressure gradient for the Reynolds stress, then:

$$\bar{\tau} = \rho l^2\left(\frac{\partial \bar{u}}{\partial y}\right)^2, \tag{21}$$

where l is the mixing length, assumed to be proportional to the wake depth and independent of y.

As in the laminar case, we introduce the velocity defect $\bar{u}_1(x, y)$ and apply the wake approximation on Eqs. (19), (20), then we have:

$$\frac{\partial(U\bar{u}_1)}{\partial x} - \frac{dU}{dx}y\frac{\partial \bar{u}_1}{\partial y} = -2\,l^2\frac{\partial \bar{u}_1}{\partial y}\frac{\partial^2 \bar{u}_1}{\partial y^2}. \tag{22}$$

Introducing a new variable η, and assuming that the similar solution exists in the next form:

$$\bar{u}_1(x, y) = H(x)f(\eta), \qquad \eta = \frac{y}{G(x)}, \tag{23}$$

where $G(x)$ and $H(x)$ are functions of x to be determined later, we get the next equation:

$$f'f'' - \frac{G^2(GU)'}{2\,l^2H}\eta f' + \frac{G^3(HU)'}{2\,l^2H^2}f = 0. \tag{24}$$

All coefficients in Eq. (24) must be independent of x, and we can choose $G(x)$ and $H(x)$ so as to satisfy:

$$\left.\begin{aligned} \frac{G^2(GU)'}{2\,l^2H} &= 1, \\ \frac{G^3(HU)'}{2\,l^2H^2} &= -1. \end{aligned}\right\} \tag{25}$$

These determine $G(x)$ and $H(x)$ as functions of $U(x)$. If we assume that the mixing length $l(x)$ is proportional to the mixing depth $G(x)$, then:

$$l(x) = \kappa G(x), \tag{26}$$

where κ is a proportionarity constant to be determined experimentaly. From Eqs. (25) and (26), we have:

$$\left.\begin{aligned} G(x) &= \frac{U(x_0)}{U(x)}\left[4\,\kappa^2 G_0H_0\int_{x_0}^{x}\frac{1}{U(x)}\,dx + G_0^2\right]^{1/2}, \\ H(x) &= \frac{G_0H_0U(x_0)}{U(x)}\left[4\,\kappa^2 G_0H_0\int_{x_0}^{x}\frac{1}{U(x)}\,dx + G_0^2\right]^{-1/2}, \end{aligned}\right\} \tag{27}$$

with the conditions that:

$$U(x) = U(x_0),\ G(x) = G_0,\ H(x) = H_0 \text{ at } x = x_0.$$

When $G_0 = 0$, Eqs. (27) become:

$$\left.\begin{aligned} G(x) &= \frac{2\,\kappa\sqrt{C_1}}{U(x)}\left[\int_{x_0}^{x}\frac{1}{U(x)}\,dx\right]^{1/2}, \\ H(x) &= \frac{\sqrt{C_1}}{2\,\kappa U(x)}\left[\int_{x_0}^{x}\frac{1}{U(x)}\,dx\right]^{-1/2}, \end{aligned}\right\} \tag{28}$$

where $C_1\{=G_0H_0U^2(x_0)\}$ is a constant to be determined later.

If we choose $G(x)$ and $H(x)$ so as to satisfy Eqs. (27), Eq. (24) is reduced to:

$$f'f'' - \eta f' - f = 0, \tag{29}$$

and the boundary conditions are:

$$\frac{\partial \bar{u}_1}{\partial y} = 0 \text{ at } y = 0, \qquad \bar{u}_1 = 0 \text{ at } y = G(x). \tag{30}$$

From Eqs. (29) and (30), we have:

$$f(\eta) = \frac{2}{9}(1 - \eta^{3/2})^2, \tag{31}$$

and the velocity defect $\bar{u}_1(x, y)$ in the wake becomes,

$$\bar{u}_1 = \frac{2\sqrt{C_1}}{18\,\kappa U(x)}\left[\int_{x_0}^{x} \frac{1}{U(x)}\,dx\right]^{-1/2}\left[1 - \left\{\frac{y}{G(x)}\right\}^{3/2}\right]^2. \tag{32}$$

As well as the laminar case, we can conclude that even in the wake with axial pressure gradient the profile of the velocity defect in the turbulent wake also becomes similar at a large distance behind from the body, and further, the similarity profile $f(\eta)$ is independent of $U(x)$.

When $U(x)$ is of the form:

$$U(x) = U_0\left(\frac{x}{L}\right)^m, \qquad x_0 = 0,\ G_0 = 0: \tag{33}$$

then from Eqs. (28) and (32), we have:

$$\left.\begin{aligned} G(x) &= \sqrt{10}\,\kappa\left[\frac{C_1' L^2}{(1-m)}\right]^{1/2}\left(\frac{x}{L}\right)^{(1-3m)/2}, \\ H(x) &= \frac{\sqrt{10}\,U_0}{4\,\kappa}[C_1'(1-m)]^{1/2}\left(\frac{x}{L}\right)^{(m-1)/2}. \end{aligned}\right\} \tag{34}$$

$$\bar{u}_1 = \frac{\sqrt{10}}{18\,\kappa}U_0[C_1'(1-m)]^{1/2}\left(\frac{x}{L}\right)^{-(m+1)/2}\left[1 - \left\{\frac{y}{G(x)}\right\}^{3/2}\right]^2, \tag{35}$$

where

$$C_1 = \frac{10}{4}C_1' U_0^3 L.$$

From the above, we can conclude that, in the turbulent case, if the axial velocity $U(x)$ outside the wake is proportional to x^m, the width of the wake develops proportionally to $x^{(1-3m)/2}$, and the velocity defect to $x^{(m-1)/2}$. These results become the same as the usual wake solution when $m = 0$.

§ 3. Determination of $C(C')$ and $C_1(C_1')$

The previous similar solutions include unknown constants $C(C')$ or $C_1(C_1')$ to be determined physically, which will be considered in this section. The momentum integral equation can be applied not only to the boundary layer, but also to the wake, and we have:

$$\frac{\tau_0}{\rho} = \frac{d}{dx}(U^2\theta) + \delta^* U\frac{dU}{dx}. \tag{36}$$

where τ_0 is the shearing stress on the wall, and on the wake axis $\tau_0 = 0$, δ^* the displacement thickness, and θ the momentum thickness.

Integrating Eq. (36) from the leading edge of the flat plate to the wake at a large distance from the plate, we have:

$$D = \rho U_w^2 \theta_w - p_w \delta_w^* + \int_0^{\delta_w^*} p d\delta^*, \tag{37}$$

where $2D$ is the friction drag on both sides of the plate, and the subscript w denotes quantities in the wake. When $U(x) = U_0\left(\frac{x}{L}\right)^m$, θ_w and δ_w^* follow for laminar and turbulent wake respectively:

$$\left.\begin{aligned} \text{laminar}: \ \theta_w \doteqdot \delta_w^* &= \int_0^{\infty}\left(1-\frac{u}{U}\right)dy = C'\left[\frac{\nu L\pi}{U_0(1+m)}\right]^{1/2}\left(\frac{x}{L}\right)^{-3m}. \\ \text{turbulent}: \ \theta_w \doteqdot \delta_w^* &= \int_0^{G(x)}\left(1-\frac{\bar{u}}{U}\right)dy = \frac{1}{4}C_1'L\left(\frac{x}{L}\right)^{-3m}. \end{aligned}\right\} \tag{38}$$

The first term of the right hand-side of Eq. (37) is the momentum defect, the second term the pressure force acting on the flow from $y=0$ to $y=\delta_w^*$ from the back, and the third term the axial component of the total pressure force acting on the imaginary surface of the outer edge of the effectively displaced region from the leading edge of the body to the wake.

Without axial pressure gradient, the second and third terms are canceled out, and only the first term remains finite. So that unknown constant can be combined immediately with the drag. However if the flow has the axial pressure gradient they can not be combined immediately, and the displacement thickness distribution $\delta^*(x)$, and the pressure distribuion $p(x)$ have influence upon flow quantities. Thus, if the distribution $\delta^*(x)$ is known, unkown constants $C(C')$ or $C_1(C_1')$ can be determined by employing Eq. (37), or from Eqs. (38) they can be determined by measuring experimentally δ_w^* at a given station in the wake.

Conclusions

(1) It is concluded that the velocity distribution in the wake at a large distance from the body and the growth of the wake width are influenced by axial pressure gradient.

(2) Within the wake approximation, the flow is represented by similar solutions in both laminar and tubulent wakes, even in the case with axial pressure gradient, so that the velocity defect in the wake (at a large distance from the body) is always kept similar.

(3) When the velocity outside the wake is proportional to x^m, in the laminar case the wake width develops proportionally to $x^{(1-m)/2}$ and the velocity defect to $x^{-(3m+1)/2}$, and in the turbulent case the width to $x^{(1-3m)/2}$ and the velocity defect to $x^{-(m+1)/2}$.

References

1) H. Schlichting: Boundary Layer Theory, Pergamon Press, 1955, p. 138.
2) H. Schlichting: Boundary Layer Theory, Pergamon Press, 1955, p. 492.
3) H. Görtler: Eine Neue Reihenentwicklung Für Laminare Grenzschichten, ZAMM, **32**, 1952, p. 270-271.