RESEARCH REPORTS

ON A RENEWAL PROBLEM

TAMIO ONO

Department of Engineering Mathematics

and IKUMA ISHII*

(Received Febrary 12, 1962)

§ 1. Introduction

In this note we prove a generalization of the fundamental theorem in renewal theory due to P. Erdös-W. Feller-H. Pollard.[1] Precisely speaking, suppose for instance some eggs are put in a ice-box. To fix our idea, we consider the total of them as "1". Assume they are distributed as $(\omega_0(t);\ -\infty < t < \infty)$, where $\omega_0(t)$ denotes the quantity of eggs that get unavailable by the first t-th day. The number t need not be an integer nor a positive number. Suppose we take off unavailable eggs and use the first worst quantity "λ" of remaining eggs during each day. Suppose then we put the quantity "$\omega_0(0)$" of eggs distributed as $(v(t);\ -\infty<t<\infty)$ and the quantity "λ" of eggs distributed as $(u(t);\ -\infty<t<\infty)$ into the ice-box, where $u(t)$ $(v(t))$ denotes the quantity of eggs that get unavailbe by the first $(t+1)$-th day and $\lim_{t\to\infty} u(t)=\lambda$ $(\lim_{t\to\infty} v(t)=1)$ respectively. Repeating this process, assume after n days we get the eggs in the ice-box distributed as $((B^n\omega_0)(t);\ -\infty<t<\infty)$, where $(B^n\omega_0)(t)$ denotes the quantity of eggs that get unavailable by the first $(n+t)$-th day starting from the day when we put eggs in the ice-box for the first time. We denote by Ω^B the set of the starting distributions ω_0, for which we can continue the process forever. The set Ω^B might be empty. We have the following two problems:

PROBLEM 1: Under what condition is Ω^B non-empty?

PROBLEM 2: Under what condition does $\lim_{n\to\infty}(B^n\omega_0)(t)$ exist for any t and for any $\omega_0\in\Omega^B$?

It is just the case of P. Erdös-W. Feller-H. Pollard[1] when $\lambda=0$ and $\omega_0(0)=1$.

In § 1, we define A-mappings (a kind of B-mappings without v) and determine the set of fixed functions of A (Prop. 1.1). In § 2, we define B-mappings and give an answer for Problem 1 (Prop. 2.1). In § 3, we state an answer for Problem 2 (Theorem 3.1) and related problems. We prove the main theorem (Theorem 3.1) in § 4.

§ 1. A-Mappings

We begin with the following

DEEINITION 1.1: A function ω defined on the real line is called a (α,β)-type

* Nagoya Technical College.

distribution function if it satisfies the following conditions:

(1.1) $$\omega(t_1) \leqq \omega(t_2) \text{ for } -\infty < t_1 \leqq t_2 < \infty,$$

(1.2) $$\lim_{t \to t_0+0} \omega(t) = \omega(t_0) \text{ for } -\infty < t < \infty, \quad \text{and}$$

(1.3) $$\lim_{t \to -\infty} \omega(t) = \alpha, \ \lim_{t \to \infty} \omega(t) = \beta.$$

Denote by Ω the set of $(0,1)$-type distribution functions and by $\Omega(\alpha, \beta)$ the set of (α, β)-type distribution functions. We introduce into Ω the weakest topology such that every projection: $\omega \to \omega(t)$ is continuous for all t; thus

(1.4) $$\omega_\iota \to \omega \text{ if and only if } \omega_\iota(t) \to \omega(t) \text{ for all } t,$$

where $\{\iota\}$ is an arbitrary directed set.

Given $0 \leqq \lambda \leqq 1$ and $u \in \Omega(0, \lambda)$, we construct the mapping A of Ω into Ω defined by

(1.5) $$(A\omega)(t) = \omega(t+1) + u(t) - \delta_\omega(t+1) \qquad (\omega \in \Omega),$$

where

(1.6) $$\delta_\omega(t) = \begin{cases} \omega(t) & \text{if } \omega(t) \leqq \lambda, \\ \lambda & \text{if } \lambda \leqq \omega(t). \end{cases}$$

We call A the A-mapping associated with (λ, u). Set

(1.7) $$t_\omega = \inf(t\,;\ \lambda \leqq \wedge(t+1)).$$

We call t_ω the *critical number* of ω with respect to A. Then

$$\begin{cases} t_\omega = -\infty & \text{if } \lambda = 0, \\ -\infty < t_\omega < \infty & \text{if } 0 < \lambda < 1, \\ t = & \text{if } \lambda = 1. \end{cases}$$

Lemma 1.1: A is continuous.

Proof: In view of (1.5), in order to prove Lemma 1.1, we need only to show that δ_ω is continuous. Suppose $\omega_\iota \to \omega$. It must be shown that, for a fixed t,

$$\lim \delta_{\omega\iota}(t) = \begin{cases} \omega(t) & \text{if } \omega(t) \leqq \lambda, \\ \lambda & \text{if } \lambda \leqq \omega(t). \end{cases}$$

If $\omega(t) < \lambda$, there exists an index ι_0 such that $\omega_\iota(t) < \lambda$ for $\iota_0 \leqq \iota$. Hence $\delta_{\omega\iota}(t) = \omega_\iota(t)$ for $\iota_0 \leqq \iota$. This implies that $\lim \delta_{\omega\iota}(t) = \omega(t)$. Similarly we can verify that $\lim \delta_{\omega\iota}(t) = \lambda$ if $\lambda < \omega(t)$. If $\omega(t) = \lambda$, for $\varepsilon > 0$, there exists an index ι_0 such that $|\omega_\iota(t) - \lambda| < \varepsilon$ for $\iota_0 \leqq \iota$. This implies that $|\delta_{\omega\iota}(t) - \lambda| < \varepsilon$ for $\iota_0 \leqq \iota$. Hence $\lim \delta_{\omega\iota}(t) = \lambda$. Thus the proof is completed.

Set

(1.8) $$w(t) = \sum_{n=0}^{\infty} (\lambda - u(t+n)).$$

Lemma 1.2: If $w(t) < \infty$ for some t, then $w(t) < \infty$ for all t.

Proof: Suppose $w(t') < \infty$ for some t'. Given t, we can select a natural number n' such that $t' \leqq t + n'$. Then

$$\begin{aligned} w(t) &= \sum_{n=0}^{\infty} (\lambda - u(t+n)) \\ &= \sum_{n=0}^{n'-1} (\lambda - u(t+n)) + \sum_{n=n'}^{\infty} (\lambda - u(t+n)) \\ &= \sum_{n=0}^{n'-1} (\lambda - u(t+n)) + \sum_{n=0}^{\infty} (\lambda - u(t'+n)) \\ &< \infty. \end{aligned}$$

This completes the proof.

Set

$$\wedge(t) = 1 - w(t). \tag{1.9}$$

Then $\wedge \in \Omega(-\infty, 1)$ if $w(t) < \infty$ for some t and $\wedge \in \Omega(-\infty, -\infty)$ if $w(t) = \infty$ for all t.

Definition 1.2: 1) An A-mapping associated with (λ, u) is said to be finite (infinite) if $w(t) < \infty$ for some t or $\lambda(1-\lambda) = 0$ ($w(t) = \infty$ for all t and $0 < \lambda < 1$).

2) An A-mapping associated with (λ, u) is said to be regular if A is finite and $0 < \lambda < 1$, semi-infinite if $\lambda = 1$, and singular if $\lambda = 0$, respectively.

Set

$$t(A) = \inf(t;\ \lambda \leqq \wedge(t+1)). \tag{1.10}$$

We call $t(A)$ the *critical number* of A. Then we have

$$\begin{cases} t(A) = -\infty & \text{if } A \text{ is singular,} \\ -\infty < t(A) < \infty & \text{if } A \text{ is regular,} \\ t(A) = \infty & \text{if } A \text{ is semi-infinite or infinite.} \end{cases}$$

If A is not singular, we construct the function $\omega(A)$ such that

$$\omega(A)(t) = \begin{cases} u(t) & \text{if } t < t(A), \\ \wedge(t) & \text{if } t(A) \leqq t, \end{cases} \tag{1.11}$$

and call it the *standard function* of A. If A is singular, every function in Ω is called the standard function of A. Given ω in Ω, we say that $\omega(A)$ is the *standard function* of *A at* ω if A is not singular and that ω itself is the *standard function* of *A at* ω if A is singular. For regular case we have

$$u(t) = \wedge(t) + \lambda - \wedge(t+1) \text{ for all } t. \tag{1.12}$$

Lemma 1.3: If A is regular or infinite, then $\omega(A)(t) = \mathrm{Max}(u(t), \wedge(t))$ for all t.

Proof: Since the others are obvious, we can assume that A is regular. Suppose $t < t(A)$. We have from (1.12) $u(t) \geqq \wedge(t)$, for $\wedge(t+1) \leqq \lambda$. Hence $\omega(A)(t) = u(t) = \mathrm{Max}(u(t), \wedge(t))$. Similarly we get $\omega(A)(t) = \wedge(t) = \mathrm{Max}(u(t), \wedge(t))$ if $t(A) \leqq t$. Thus the proof is completed.

LEMMA 1.4: $\omega(A)\in\Omega$ if A is finite and $\omega(A)\in\Omega(0,\lambda)$ if A is infinite.

Proof: The assertion is obvious if $\lambda=0$ or $t(A)=\infty$. Hence we need only to show Lemma 1.4 for regular case. Suppose A is regular. The condition (1.1) for $\omega(A)$ follows from Lemma 1.3. The conditions (1.2)-(1.3) for $\omega(A)$ follow from the definition of $\omega(A)$. Hence we have $\omega(A)\in\Omega$. This completes the proof.

We say that a function ω in Ω is a *fixed function* of A in Ω if $A\omega=\omega$. Denote by $\Omega(A)$ the set of fixed functions in Ω.

LEMMA 1.5: $\omega(A)\in\Omega(A)$ if A is finite.

Proof: Since the assertion is obvious for the other cases, we can assume that A is regular. We write ω for $\omega(A)$. If $t<t(A)$, then, by (1.11) and (1.10), $\omega(t)=u(t)$ and $\wedge(t+1)\lambda.<$ Hence, by (1.5) and (1.6), $(A\omega)(t)=u(t)$ if $\omega(t+1)=\wedge(t+1)$. If $t<t(A)$ and if $\omega(t+1)=u(t+1)$, then $\omega(t+1)\leqq\lambda$ and also $(A\omega)(t)=u(t)$. On the other hand, if $t(A)\leqq t$, then $\omega(t)=\wedge(t)$ and $\lambda\leqq\wedge(t+1)(=\omega(t+1))$. Hence, by (1.5), (1.6), and (1.12), $(A\omega)(t)=\wedge(t+1)+u(t)-\lambda=\wedge(t)$. Thus the proof is completed.

PROPOSITION 1.1: 1) $\Omega(A)=\Omega$ if A is singular, 2) $\Omega(A)$ is a singleton $\{\omega(A)\}$ if A is regular or semi-infinite, and 3) $\Omega(A)$ is empty if A is infinite and not semi-infinite.

Proof: The assertion 1) is obvious.

Proof of 2): The assertion is obvious for semi-infinite case. Suppose A is regular and suppose ω is a fixed function of A in Ω. If $t_\omega\leqq t$, we have from (1.5) and (1.6)

$$\omega(t)=\omega(t+n)-\sum_{j=0}^{n-1}(\lambda-u(t+j)). \tag{1.13}$$

By making $n\to\infty$, we get $\omega(t)=\wedge(t)$. Hence we have from (1.5) and (1.12) $\delta_\omega(t+1)=\omega(t+1)+u(t)-\omega(t)=\wedge(t+1)+u(t)-\wedge(t)=\lambda$, or $\lambda\leqq\wedge(t+1)$. This implies that $t(A)\leqq t_\omega$. If $t_\omega-1<t<t_\omega$, we have from (1.5) and (1.6) $\omega(t)=u(t)$ and $\wedge(t+1)=\omega(t+1)<\lambda$. This implies that $t_\omega\leqq t(A)$. Hence $t_\omega=t(A)$. If $t<t_\omega$, we have from (1.5) $\omega(t)=u(t)$. From these it follows that $\omega=\omega(A)$. By this combining with Lemma 1.5 we get 2).

Proof of 3): Suppose ω is a fixed function of A in Ω. If $t_\omega<\infty$, we have (1.13) for $t_\omega\leqq t$. This is impossible, for A is infinite. Hence $t_\omega=\infty$ and $\omega(t)=u(t)$ for all t. Since $\omega\in\Omega$, we have $\lambda=1$. This is also impossible, for A is not semi-infinite. Hence there is no fixed function of A in Ω and $\Omega(A)$ is empty.

This completes the proof.

The following lemma is merely a restatement of a fact stated in Proof of 2) in the proof of Prop. 1.1.

LEMMA 1.6: $t_{\omega(A)}=t(A)$ if A is finite.

§ 2. B-Mappings

Let Ω and $\Omega(\alpha,\beta)$ be sets defined in § 1. Given $0\leqq\lambda\leqq1$, $u\in\Omega(0,\lambda)$, and $v\in$

$\Omega(0,1)$, we construct the mapping B defined by

$$(2.1)\qquad (B\omega)(t)=\omega'(t+1)+u(t)+v(t)\omega(0)-\delta\omega'(t+1),$$

where

$$(2.2)\qquad \omega'(t)=\omega(t)-\omega(0)\ \text{for all } t, \text{ and}$$

$$(2.3)\qquad \delta\omega'(t)=\begin{cases}\omega'(t) & \text{if } \omega'(t)\leqq\lambda,\\ \lambda & \text{if } \lambda\leqq\omega'(t).\end{cases}$$

We say that B is definable at a function ω in Ω if $B\omega\in\Omega$. Denote by Ω_B the set of definable functions in Ω, then $\Omega_B=(\omega;\ \omega(0)\leqq 1-\lambda,\ \omega\in\Omega)$. We call the mapping B of Ω_B into Ω the B-mapping associated with (λ,u,v). It is easy to see that Ω_B is closed in Ω. For $\omega\in\Omega_B$, set

$$(2.4)\qquad t_{\omega'}=\inf\,(t;\ \lambda\leqq\omega'(t+1)).$$

We call $t_{\omega'}$ the ***critical number*** of ω with respect to B.

The proof of the following iemma is obtained similarly as in that of Lemma 1.1 and so it may be omitted.

Lemma 2.1: B is continuous.

Definition 2.1: A function ω in Ω is said to be stable with respect to B if $B^n\omega\in\Omega$ for all n.

Denote by Ω^B the set of stable functions in Ω_B with respect to B.

Lemma 2.2: Ω^B is closed in Ω.

Proof: Suppose $\omega_\iota\to\omega$ and $\omega_\iota\in\Omega^B$ for all ι, where $\{\iota\}$ is a directed set. Since $\omega_\iota\in\Omega_B$, we have $\omega\in\Omega_B$ and $B\omega_\iota\to B\omega$. Repeating this argument, we get $B^n\omega\in\Omega_B$. This implies that $\omega\in\Omega^B$. Thus the proof is completed.

Set

$$(2.5)\qquad w'(t)=\sum_{n=0}^{\infty}(1-v(t+n)).$$

The proof of the following lemma is quite similar as in that of Lemma 1.2 and so it may be omitted.

Lemma 2.3: If $w'(t)<\infty$ for some t, then $w(t)<\infty$ for all t.

Definition 2.2: A B-mapping B associated with (λ,u,v) is said to be 1) of Type I if $0<\lambda<1$, $n(0)^2+(1-v(0))^2\neq 0$, and if $w(0)\geqq 1$ in case that $u(0)=0$ and $w'(0)=\infty$, 2) of Type II if $\lambda=0$, 3) of type III if $0<\lambda<1$ and $u(0)^2+(1-v(0))^2=0$, 4) of Type IV if $\lambda=1$, and 5) of Type V if $0<\lambda<1$, $u(0)^2+(1-v(0))^2\neq 0$, $w(0)<1$, $u(0)=0$, and $w'(0)=\infty$.

Definition 2.3: A B-mapping B is said to be stable if Ω^B is non-empty.

Proposition 2.1: If a B-mapping B is of Type II or Type III, then B is stable. If B is of Type I or Type V, then it is stable when and only when $u(0)/(1-v(0))\leqq 1-\lambda$. If B is of Type IV, then it is stable when and only when $u(0)=0$.

Proof: Proof for Types I, V: Suppose Ω^B is non-empty and ω is in Ω^B. Then

$$\begin{aligned}(B^n\omega)(0) &= (B^{n-1}\omega)(1) + u(0) + v(0)(B^{n-1}\omega)(0) - \delta_B n - 1_{\omega'} \qquad (1)\\ &\geqq u(0) + v(0)(B^{n-1}\omega(0))\\ &\geqq \cdots\cdots\cdots\\ &\geqq u(0)(1 + v(0) + \cdots + v(0)^{n-1}) + v(0)^n\omega(0).\end{aligned}$$

By making $n \to \infty$, we get

(2.6) $$u(0)/(1 - v(0)) \leqq \underline{\lim}\ (B^n\omega)(0) \leqq 1 - \lambda.$$

Hence the condition is necessary. Suppose next $u(0)/(1-v(0)) \leqq 1-\lambda$. We show that, if $u(0)/(1-v(0)) \leqq \omega(0) \leqq 1-\lambda$, then $u(0)/(1-v(0)) \leqq (B\omega)(0) \leqq 1-\lambda$. Since $\omega(0) \leqq 1-\lambda$, B is definable at ω. If $\omega'(1) \leqq \lambda$, then $(B\omega)(0) = u(0) + v(0)\omega(0)$, and so $u(0)/(1-v(0)) \leqq (B\omega)(0) \leqq 1-\lambda$. On the other hand, if $\lambda < \omega'(1)$, then $(B\omega)(0) = u(0) + v(0)\omega(0) + \omega'(1) - \lambda \geqq u(0) + v(0)\omega(0) \geqq n(0)/(1-v(0))$, for $u(0)/(1-v(0)) \leqq \omega(0)$. And also $(B\omega)(0) = u(0) + v(0)\omega(0) + \omega'(1) - \lambda = \omega(1) - \lambda - (\omega(0) - (u(0) + v(0)\omega(0))) \leqq \omega(1) - \lambda \leqq 1-\lambda$. Hence we get the above assertion. Repeating this argument, we can conclude that ω is stable with respect to B if $u(0)/(1-v(0)) \leqq \omega(0) \leqq 1-\lambda$. Hence Ω^B is non-empty.

Proof for Type II: If it is the case, then $\Omega_B = \Omega$ and so $\Omega^B = \Omega$.

Proof forType III: Suppose $\omega \in \Omega_B$. Then $\omega(0) = 1-\lambda$. If $\omega'(1) \leqq \lambda$, then we have $(B\omega)(0) = \omega(0) \leqq 1-\lambda$. If $\lambda < \omega'(1)$, then we have also $(B\omega)(0) = \omega'(1) + \omega(0) - \lambda = \omega(1) - \lambda \leqq 1-\lambda$. Hence $B\omega \in \Omega_B$. By repeating this argument, we get $\omega \in \Omega^B$. Thus we have $\Omega^B = \Omega_B$ and Ω^B is non-empty.

Proof for Type IV: The condition is necessary, because $\Omega_B = (\omega;\ \omega(0) = 0, \omega \in \Omega)$ and $(B\omega)(0) = u(0)$ for ω in Ω_B. Suppose $u(0) = 0$, then $u \in \Omega_B$ and $Bu = u$. Hence $u \in \Omega^B$ and Ω^B is non-empty.

This completes the proof.

Let B be a B-mapping of Type III associated with (λ, u, v) and ω be a function in Ω^B. We have from (2.1) and (2.2)

$$\omega(0) \leqq (B\omega)(0) \leqq (B^2\omega)(0) \leqq \cdots = 1 - \lambda.$$

Hence there exists the limit of $\{(B^n\omega)(0)\}$. Denote it by $\beta_0(\omega)$.

Let B be a stable B-mapping associated with (λ, u, v). Set

(2.7) $$\mu' = 1 + w'(0),$$

(2.8) $$\beta' = n(0)/(1 - v(0)) \text{ for Type I, II, IV, V cases,}$$

(2.9) $$\beta'' = (1 - w(0))/\mu' \quad (\beta'' = 0 \text{ if } w(0) + \beta' w'(0) = \infty),$$

(2.10) $$\beta = \begin{cases} \left.\begin{array}{l} 1 - \lambda \text{ if } \beta' = 1 - \lambda \\ \text{Max } (\beta', \beta'') \text{ if } \beta' \neq 1 - \lambda \end{array}\right\} & \text{for Type I, V cases,} \\ 1/\mu' & \text{for Type II case,} \\ \beta_0 \ (\text{Min}(\text{Max}(0, \beta''), 1 - \lambda) \leqq \beta_0 \leqq 1 - \lambda) & \\ & \text{for Type III case,} \\ 0 & \text{for Type IV case.} \end{cases}$$

We call the standard function of the A-mapping associated with $(\lambda+\beta, u+\beta v)$

the *standard function* of B (except for Type V case) and denote it by $\omega(B)$. (We set $\omega(B)(t)=u(t)=0$ for $t<0$ and $\omega(B)(t)=w(0)-w(t)$ for $t\geqq 0$ if B is of Type V). Given ω in Ω^B, we call $\omega(B)$ the standard function of B at ω if B is not of Type III, and we call the standard function of the A-mapping associated with $(\lambda+\beta_0(\omega),\ u+\beta_0(\omega)v)$ the *standard function* of B *at* ω.

DEFINITION 2.4: A B-mapping B associated with (λ, u, v) is said to be finite (resp. semi-infinite, infinite) if the A-mapping associated with $(\lambda+\beta, u+\beta v)$ is finite (resp. semi-infinite, infinite) for all β.

The following lemma is an immediate consequence of Lemma 1.4.

LEMMA 2.5: $\omega(B)\in\Omega$ if B is finite and $\omega(B)\in\Omega(0,\lambda+\beta)$ if B is infinite.

LEMMA 2.6: If B is infinite and of Type I or II, then one of the following conditions occur: a) $w(0)=\infty$, and b) $w(0)<\infty$, $w'(0)=\infty$, and $\beta'\neq 0$.

Proof: Suppose $w(0)<\infty$. Since B is infinite, $w(0)+\beta w'(0)=\infty$. Hence $w'(0)=\infty$ and $\beta\neq 0$. Since $w'(0)=\infty$, $\beta''=0$. Hence $\beta'=\beta\neq 0$. Thus the proof is completed.

We say that a function ω in Ω^B is a fixed function of B if $B\omega=\omega$. Denote by $\Omega(B)$ the set of fixed functions of B in Ω^B.

PROPOSITION 2.2: Let B be a stable B-mapping associated with (λ, u, v), then we have 1) $\Omega(B)$ is the set of standards functions of B in Ω and 2) $\Omega(B)$ is empty if and only if one of the following conditions are satisfied: a) B is finite and of Type I or III with $1-\lambda<\beta''$, b) B is finite and of Type V, c) B is infinite and of Type I with $\lambda+\beta\neq 1$, and d) B is infinite and of Type II.

Proof: We prove the assertions for Type I, II, V cases, because the others are verified similarly. We first show that $\Omega(B)=$the singleton $\{\omega(B)\}$ if $\Omega(B)$ is non-empty. Suppose $\omega\in\Omega(B)$. If $\omega'(1)\leqq\lambda$, then $\omega(0)=u(0)+v(0)\omega(0)$ and so $\omega(0)=\beta'$. Hence, $\omega(t)=\omega(t+1)+n(t)+\beta'v(t)-\delta(t+1)$ for all t, where $\delta(t)=\omega(t)$ if $\omega(t)\leqq\lambda+\beta'$ and $\delta(t)=\lambda+\beta'$ if $\lambda+\beta'\leqq\omega(t)$. This shows that ω is a fixed function of the A-mapping associated with $(\lambda+\beta', u+\beta'v)$. If $\beta'=1-\lambda$, then B is semi-infinite and $\omega=\omega(B)$. Hence we can assume that $\beta'<1-\lambda$. Since $\omega(0)=u(0)+\beta'v(0)$, we have from Lemma 1.3 $\beta'\geqq\wedge(0)=1-(w(0)+\beta'w'(0))$, or $\beta'\geqq\beta''$. Hence $\omega=\omega(B)$. On the other hand, if $\lambda\leqq\omega'(1)$, then $\lambda\leqq\omega'(n)$ for all n. Hence $\omega(0)=\omega(n)-\sum_{j=0}^{\infty}(\lambda-u(j)+(1-v(j))\omega(0))$. By making $n\to\infty$, we get $\omega(0)=1-(w(0)+w'(0)\omega(0))$, or $\omega(0)=\beta''$. Hence, $\omega(t)=\omega(t+1)+u(t)+\beta''v(t)-\delta(t+1)$, where $\delta(t)=\omega(t)$ if $\omega(t)\leqq\lambda+\beta''$ and $\omega(t)=\lambda+\beta''$ if $\lambda+\beta''\leqq\omega(t)$. This shows that ω is a fixed function of the A-mapping associated with $(\lambda+\beta'', u+\beta''v)$. Since $\omega(0)=\wedge(0)$, we have from Lemma 1.3 $\beta''\geqq u(0)+\beta''v(0)$, or $\beta''\geqq\beta'$. Hence $\omega=\omega(B)$.

In the rest of the proof, we show that $\omega(B)\notin\Omega(B)$ if and only if all the conditions a)-d) are not satisfied. Assume first B is finite. It is obvious that $\omega(B)\notin\omega(B)$ if $1-\lambda<\beta''$. Suppose $\beta'<\beta''\leqq 1-\lambda$. Then $w(0)<\infty$ and $w'(0)<\infty$. Hence B is not of Type V. Since $u(0)+\beta''v(0)\leqq\beta''=1-(w(0)+\beta''w'(0))$, we have $\omega(B)(0)=\beta''$. From this combining with Lemma 2.5 we have $\omega(B)\in\Omega(B)$ in this case. Suppose next $\beta''\leqq\beta'$. If B is not of Type V, then $u(0)+\beta'v(0)=\beta'$

$\geqq 1-(w(0)+\beta' w'(0))$. Hence, $\omega(B)(0)=\beta'$ and so, by Lemma 2.5, $\omega(B)\in\Omega(B)$. If B is of Type V, then $\omega(B)(0)\neq\beta'=\beta''=0$ and $\omega(B)$ is not a fixed function of B. Assume next B is infinite, then we have $\omega(B)(t)=u(t)+\beta v(t)$ for all t. We also have $\beta''=0$. In fact, if $\beta''>0$, then $w(0)+\beta' w(0)<\infty$ and hence $w(0)<\infty$, $w'(0)<\infty$, or B would be finite. Hence $\beta=\beta'$. Since $\omega(B)(0)=u(0)+\beta' v(0)=\beta'$, $\omega(B)\in\Omega(B)$ if and only if $\lambda+\beta=\lambda+\beta'\neq 1$.

This completes the proof.

§ 3. Main Theorem

THEOREM 3.1: Let B be a stable B-mapping associated with (λ, u, v), then $\lim B^n\omega_0$ exists for all ω_0 in Ω^B if and only if

(*) the greatest common divisor of $(n; p(n)\neq 0,\ n\geqq 1)=1$

when B' is finite and of Type I or II and $\beta'<\beta''\leqq 1-\lambda$, where $p(1)=v(0)$ and $p(n)=v(n-1)-v(n-2)$ $(n\geqq 2)$, and if it is the case, the limit is equal to the standard function of B at ω_0. Moreover the limit is in Ω if and only if B is finite or semi-infinite.

COROLLARY 3.1: Let A be an A-mapping associated with (λ, u), then $\lim A^n\omega_0$ exists for all ω_0 in Ω, and the limit is equal to the standard functin of A at ω_0. Moreover the limit is in Ω if and only if A is finite or semi-infinite.

COROLLARY 3.2: (P. Erdös-W. Feller-H. Pollard[1]): Let $\{\eta_1\}$ be a sequence of real numbers such that

(3.1) $$\eta_i\geqq 0 \text{ for all } i, \text{ and}$$

(3.2) $$\sum_{i=1}^{\infty}\eta_i=1,$$

and let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of real numbers determined by

(3.3) $$\lambda_{n+1}=\eta_1\lambda_n+\cdots+\eta_{n+1}\lambda_0 \quad (n\geqq 0) \quad \text{and} \quad \lambda_0=1,$$

then $\lim \lambda_n$ exists if and only if

(**) the greatest common divisor of $(n;\ \eta_n\neq 0,\ n\geqq 1)$ is equal to 1.

And, if it is the case,

(3.4) $$\lim \lambda_n=\Big(\sum_{i=1}^{\infty} i\eta_i\Big)^{-1}.$$

In this section we prove the above two corollaries by making use of Theorem 3.1. Theorem 3.1 will be proved in the next section.

Proof of Corollary 3.1: The assertion is obvious if $\lambda=0$. Suppose $\lambda>0$, then there exists anatural number r such that $1<r\lambda$. We first assume that $u(t)=w_0(t)=0$ for $t<r$. Construct v in Ω such that $v(t)=1$ if $t\geqq 0$ and $v(t)=0$ if $t<0$. Denote by B the B-mapping associated with (λ, u, v). Since $u(0)=1-v(0)=0$, B is of Type III. We show that

(3.5) $$(B^n\omega_0)(s)\leqq \text{Max}\,(0,\ 1-(r-s)\lambda) \text{ for all } n\ (s\leqq r).$$

In fact, it is true for $s=r$. Suppose it is true for $k<s\leqq r$. We then have from

(2.1)

$$(B^{n+1}\omega_0)(k)=(B^n\omega_0)(k+1)-\delta B^n\omega_0'(k+1)$$
$$\leqq \mathrm{Max}(0,\ (B^n\omega_0)(k+1)-\lambda)$$
$$\leqq \mathrm{Max}(0,\ 1-(r-k)\lambda)\qquad (n\geqq 0),$$

and $(B^0\omega_0)(k)=0$. Thus we get (3.5). Since $1<r\lambda$, we must have $(B^n\omega_0)(0)=0$ for all n. Hence $\beta_0(\omega_0)=0$ and $\omega(A)=\omega(B)$, the standard function of B at ω_0. This implies that $A^n\omega_0=B^n\omega_0$ for all n and $\lim A^n\omega_0=\lim B^n\omega_0=\omega(B)=\omega(A)$. By a suitable translation of t, thus, we get $\lim A^n\omega_0=\omega(A)$ if $u(t)=\omega_0(t)=0$ for $t<r$, where r is an arbitrary number, Set

$$u'(t)=\begin{cases} u(t) \text{ if } r\leqq t,\\ 0 \text{ if } t<r,\end{cases}\qquad \tilde{\omega}_0(t)=\begin{cases}\omega_0(t) & \text{if } r\leqq t,\\ 0 & \text{if } t<r.\end{cases}$$

Denote by A' the A-mapping associated with (λ, u'), then $(A^n\omega_0)(t)=((A')^n\tilde{\omega}_0(t)$ for all n $(r\leqq t)$ and $\lim (A^n\omega_0)(t)=\lim (A')^n\tilde{\omega}_0)(t)=\omega(A)(t)$ for $r\leqq t$. Since r is arbitrary, we get $\lim A^n\omega_0=\omega(A)$. This completes the proof.

Proof of Corollary 3.2: Set $v(t)=\sum_{i\leqq t}\eta_{i+1}$ $(-\infty<t<\infty)$ and construct the B-mapping B associated with $(0,0,v)$. Since $\lambda=0$, B is of Type II. Set

$$\omega_0(t)=\begin{cases}1 \text{ for } 0\leqq t,\\ 0 \text{ for } t<0,\end{cases}\tag{3.6}$$

and $\gamma_n=(B^n\omega_0)(0)$ $(n\geqq 0)$. We have from (2.1) and (2.2)

$$\gamma_{n+1}=1-(1-\eta_1)\gamma_n-\cdots-(1-\eta_1-\cdots-\eta_{n+1})\gamma_0,\tag{3.7}$$

Set

$$\xi_i=\sum_{j=}^{\infty}\eta_j\ (i\geqq 1).\tag{3.8}$$

We have from (3.2) and (3.7)

$$\xi_1\gamma_n+\cdots+\xi_{n+1}\gamma_0=1 \text{ for all } n\geqq 0.\tag{3.9}$$

Set $\varphi(z)=\sum_{n=0}^{\infty}\gamma_n z^n$, $P(z)=\sum_{n=1}^{\infty}\eta_n z^n$, and $Q(z)=\sum_{n=0}^{\infty}\xi_{n+1}z^n$. These functions are regular in $|z|<1$. We have from (3.9) $\varphi(z)Q(z)=(1-z)^{-1}$ and from (3.8) $1-P(z)=(1-z)Q(z)$. Hence we get $\varphi(z)\ (1-P(z))=1$, or (3.3). On the other hand, we have

$$\mu'=1+w'(0)=1+\sum_{i=1}^{\infty}\xi_i=\sum_{i=1}^{\infty}i\eta_i.\tag{3.10}$$

Thus we get $\lim \gamma_n=\lim\ (B^n\omega_0)(0)=1/\mu'=(\sum_{i=1}^{\infty}i\eta_i)^{-1}$ if (*) (and so (**)) holds when $\mu'<\infty$. Denote by m the greatest common diviror of $(n;\ \eta_n\neq 0,\ n\geqq 1)$. If $m>1$, we have

$$\lambda_{m(n+1)}=\eta_m\lambda_{mn}+\cdots+\eta_{mn}\lambda_0\quad (n\geqq 0),\tag{3.11}$$

and the greatest common divisor of $(n;\ \eta_{mn} \neq 0,\ n \geqq 1)$ is equal ot 1 .Hence $\lim_{n\to\infty} \lambda_{mn} = (\sum_{i=1}^{\infty} i\eta_{mi})^{-1} = m/\mu'$. But we can prove that $\lim \lambda_n = 1/\mu'$ if $\lim \lambda_n$ exists. Therefore, $\lim \lambda_n$ does not exist if $m>1$ and $\mu'<\infty$. Thus the proof is completed.

§ 4. Proof of Theorem 3.1

Let B be a stable B-mapping associated with (λ, u, v) and ω_0 be a function in Ω^B. Set

$$\beta_n = (B^n\omega_0)(0) \text{ for } n \geqq 0. \tag{4.1}$$

Lemma 4.1: For each $n \geqq 0$ there exists $-\infty \leqq i_n \leqq \infty$ such that

$$1-(B^{n+1}\omega_0)(i) = 1-\omega_0(n+i+1) + \sum_{j=0}^{n}(\lambda - u(i+j) + (1-v(i+j))\gamma_{n-j}) \quad \text{for } i_n \leqq i, \text{ and} \tag{4.2}$$

$$(B^{n+1}\omega_0)(i) = u(i) + v(i)\beta_n \text{ for } i \leqq i_n - 1. \tag{4.3}$$

Proof: If $n=-1$ we have $(B^0\omega_0)(i) = \omega_0(i)$ for all i, and we can consider i_{-1} as $-\infty$. Hence we may prove Lemma 4.1 by induction starting from $n=-1$. Suppose Lemma 4.1 is true for $k=n-1$. Note that $i_{n-1} = t_{B^n\omega_0}' + 1$, where $t_{B^n\omega_0}'$ denotes the critical number of $B^n\omega_0$ with respect to B. Using this we can conclude that i_n exists and equals to $[t_{B^n\omega_0}']$, where $[\cdot]$ denotes Gauss' symbol. Thus the proof is completed.

Lemma 4.2: If B is infinite and of Type I or II and if $\beta' \leqq \gamma_n \leqq 1-\lambda$ $(n \geqq 0)$, then $i_n \to \infty$ $(n \to \infty)$.

Proof: Suppose $I = (n;\ i_n \leqq i)$ is an infinite set, then we have from (4.2)

$$1 \geqq \overline{\lim}\,(1-(B^{n+1}\omega_0)(i)) \geqq w(i) + \beta' w'(i).$$

This is impossible because of Lemma 2.6. Hence I is finite for all i. This shows that $\underline{\lim}\ i_n = \infty$, or $\lim i_n = \infty$. Thus the proof is completed.

Lemma 4.3: If B is of Type I or II or V, if $\beta' \leqq \gamma_r \leqq 1-\lambda$ for all r, and if $i_n = 0$ for some n, then, by setting

$$\gamma_{n+k+1} = \omega_0(n+k+1) - \sum_{j=0}^{n+k}(\lambda - u(j) + (1-v(j))\gamma_{n+k-j}) + a(k), \tag{4.4}$$

we have

$$|a(k)| \leqq \sum_{j=0}^{k}(\lambda - u(n+j) + (1-v(n+j))\beta'). \tag{4.5}$$

Proof: Set

$$(B^{n+k+1}\omega_0)(i) = \omega_0(n+k+i+1) - \sum_{j=0}^{n+k}(\lambda - u(i+j) + (1-v(i+j))\gamma_{n+k-j}) + a(k,i). \tag{4.6}$$

We have from (2.1), (2.2), and (4.6) $a(k,i) = a(k-1, i+1) + (\lambda - \delta)$, where $\delta = \delta'_\omega(i+1)$ and $\omega = B^{n+k}\omega_0$. Since $a(0,i) \geqq 0$, we have $a(k,0) \geqq a(0, k+i) \geqq 0$ for all k, i

$\geqq 0$. We prove

$$(4.7) \qquad a(k,i) \leqq \sum_{j=0}^{k} (\lambda - u(n+j) + (1-v(n+j))\beta').$$

Since $\beta' \leqq \gamma_r$ for all r, we have

$$\begin{aligned}(B^{n+k}\omega_0)^1(i+1) &= \omega_0(n+k+i+1) - \omega_0(n+k) + \sum_{j=0}^{i}(\lambda - u(j)) - \sum_{j=0}^{i}(\lambda - u(n+k+j)) \\ &\quad + \sum_{j=0}^{n+k}(v(i+j+1) - v(j))\gamma_{n+k-j-1} + a(k-1, i+1) - a(k-1,0) \\ &\geqq \lambda - u(0) - (\lambda - u(n+k)) + \sum_{j=0}^{n+k}(v(j+1) - v(j))\beta' - (a(k-1,0) - a(k-1,i+1)) \\ &\geqq \lambda - (\lambda - u(n+k) + (1-v(n+k))\beta' + a(k-1,0) - a(k-1,i+1)).\end{aligned}$$

If $(B^{n+k}\omega_0)'(i+1) \geqq \lambda$, then $a(k,i) = a(k-1, i+1)$. If $(B^{n+k}\omega_0)'(i+1) < \lambda$, then $a(k,i) - a(k-1,i+1) \leqq \lambda - u(n+k) + (1-v(n+k))\beta' + a(k-1,0) - a(k-1,i+1)$, or $a(k,i) \leqq \lambda - u(n+k) + (1-v(n+k))\beta' + a(k-1,0)$. Hence we can prove (4.7) by induction. If we put $i=0$ in (4.7), we get (4.5). Thus the proof is completed.

LEMMA 4.4: $\lim \gamma_n$ exists under the assumption (*) of Theorem 3.1.

Proof: Type I, II, V Cases. It is easy to see that

$$(4.8) \qquad \text{if } \gamma_n \leqq \beta', \text{ then } \gamma_n \leqq \gamma_{n+1}, \text{ and}$$

$$(4.9) \qquad \text{if } \beta' \leqq \gamma_n, \text{ then } \beta' \leqq \gamma_{n+1}.$$

Moreover, if $\gamma_0 \leqq \gamma_1 \leqq \cdots \leqq \beta'$, then $\lim \gamma_n$ exists. Hence we can assume without loss of generality that

$$(4.10) \qquad \beta' \leqq \gamma_n \leqq 1 - \lambda \text{ for all } n.$$

If $i_n = 1$ for almost all n, then we have $\gamma_{n+1} = u(0) + v(0)\gamma_n$ for almost all n, or $\lim \gamma_n = \beta'$. Hence $\lim \gamma_n$ exists for infinite case because of Lemma 4.2. Hence we can assume without loss of generality that B is finite and that $(n;\ i_n = 0)$ is an infinite set. Arrange $(n;\ i_n = 0)$ as $\{n_k\}$, $n_1 < n_2 < \cdots$.

In the rest of the proof of Type I, II, V case, we devide it into two cases. We first assume that $\beta'' \leqq \beta'$ and that B is not of Type V. By Lemma 4.3, given $\varepsilon > 0$, there exists a natural number n_0 such that

$$|1 - \gamma_{n+1} - \sum_{j=0}^{\infty}(-u(j) + (1-v(j))\gamma_{n-j})| \leqq \varepsilon$$

for $n = n_k \geqq n_0$, where we set $\gamma_r = \beta'$ if $r < 0$. Then,

$$\begin{aligned}1 &= \gamma_{n+1} + (1 - n + 1) \\ &\geqq \beta' + w(0) + w'(0)\beta' - \varepsilon \\ &\geqq \beta'' + w(0) + w'(0)\beta'' - \varepsilon \\ &\geqq 1 - \varepsilon.\end{aligned}$$

This implies that $\beta' = \beta'' = \beta$ and that $|\gamma_{nk+1} - \beta| \leqq 2\varepsilon$ for $n_k \geqq n_0$. For $n_k < n < n_{k+1}$ $(n_k \geqq n_0)$, we have $\gamma_{n+1} = u(0) + v(0)\gamma_n$ and $|\gamma_{n+1} - \beta| \leqq 2\varepsilon$. This shows that $\lim \gamma_n$ exists.

We next assume that $\beta' < \beta''$ or that B is of Type V. Set

(4.11) $$\gamma_{n+1}=1-w(0)-\sum_{j=0}^{n}(1-v(j))\gamma_{n-j}+b_n.$$

We have from (4.2)

(4.12) $$b_n\to 0 \quad (n\to\infty).$$

We have from (4.12) $w(0)\leqq 1$. If $w(0)=1$, then $0\leqq\gamma_{n+1}\leqq b_n$ and so $\lim\gamma_{n+1}=0$. Assume $w(0)<1$. Set $\gamma'_n=\gamma_n/(1-w(0))$ and $b'_n=b_n/(1-w(0))$, then

(4.13) $$\gamma'_{n+1}=1-\sum_{j=0}^{n}(1-v(j))\gamma'_{n-j}+b'_n.$$

Set $\xi_1=1$, $\xi_{i+2}=1-v(i)$ $(i\geqq 0)$, and $\eta_i=\xi_i-\xi_{i+1}$ $(i\geqq 1)$. Using φ, P in § 3, and $R(z)=\sum_{n=0}^{\infty}(1+b'_n)z^{n+1}$, we have from (4.13) $\varphi(1-P)=(1-z)R+Q$, where Q is a suitable polynomial of z, or

(4.14) $$\gamma'_{n+1}=\eta_1\gamma'_n+\cdots+\eta_{n+1}\gamma'_0+c'_n \text{ for almost all } n, \text{ where}$$

(4.15) $$c'_n=b'_n-b'_{n-1}\to 0 \ (n\to\infty).$$

By a slight modification of W. Feller [2, Chap. 12, Sec. 7], from (4.12)-(4.15) it follows that $\lim\gamma'_n$ exists (and equals to $1/\mu'$). Hence $\lim\gamma_n$ exists (and equals to β'').

Type III Case: If it is the case, $\lim\gamma_n=\beta_0(w_0)$ (cf. § 2).

Type IV Case: It is obvious that $\lim\gamma_n=0$ for this case.

This completes the proof.

LEMMA 4.5: $\sum_{j=0}^{n}(1-v(j))\gamma_{n-j}\to 0$ or $1-w(0)$ if B is finite and of Type I or II or V, $\beta'=0$, and $w'(0)=\infty$.

Proof: We have $\beta'=\beta''=0$. If $(n;\ i_n=0)$ is finite, then $\gamma_{n+1}=v(0)\gamma_n$ for almost all n $(0\leqq v(0)<1)$ and so $\sum_{j=0}^{n}(1-v(j))\gamma_{n-j}\to 0$. If $(n;\ i_n=0)$ is infinite, then we have from (4.11) $w(0)\leqq 1$. If $w(0)=1$, then $\gamma_n\to 0$ and $\sum_{j=0}^{n}(1-v(j))\gamma_{n-j}\to 0$. If $w(0)<1$, then $\gamma_n\to\beta''=0$ and $\sum_{j=0}^{n}(1-v(j))\gamma_{n-j}\to 1-w(0)$. Thus the proof is completed.

Proof of Theorem 3.1: Finite Case: We first assume that $\lim\gamma_n=1-\lambda$. Given $-\infty<t<\infty$ and a subsequence $\{n'\}$ of $\{n\}$ such that $\lim(B^{n+1}\omega_0)(t)$ exists, there exists a subsequence $\{n''\}$ of $\{n'\}$ such that $\lim(B^{n''+1}\omega_0)(t+i)$ and $\lim(B^{n''}\omega_0)(t+i)$ exist for all integers $i\geqq 0$. Since $\lim\gamma_n=1-\lambda$, we have from (2.1) $\lim(B^{n''+1}\omega_0)(t+i)=u(t+i)+v(t+i)\gamma$, where $\gamma=\lim\gamma_n$. This shows that $\lim(B^n\omega_0)(t)$ exists for all t, and equals to $u(t)+v(t)\gamma$.

We next assume that $\lim\gamma_n<1-\lambda$ and $\sum_{j=0}^{n}(1-v(j))\gamma_{n-j}\to 0$. If B is of Type IV, then $\sum_{j=0}^{n}(1-v(j))\gamma_{n-j}=0$ for all n. Notice that $\sum_{j=0}^{n}(1-v(t+j))\gamma_{n-j}\to 0$ for all t. There exists n_0 and t_0 such that

(4.16) $$1-(B^{n+1}\omega_0)(t)\leqq 1-\omega_0(n+t+1)+\sum_{j=0}^{n}(\lambda-u(t+j)+(1-v(t+j))\gamma_{n-j})$$
$$\leqq 1-\lambda-\gamma_{n+1} \text{ for } n_0\leqq n \text{ and } t_0\leqq t,$$

or $(B^{n+1}\omega_0)'(t) \geqq \lambda$. Hence we have from (2.1)

$$(B^{n-1}\omega_0)(t) = \omega_0(u+t+1-n_0) - \sum_{j=0}^{n}(\lambda - u(t+j) + (1-v(t+j))\gamma_{n-j}) \text{ for } n_0 < n \text{ and } t_0 \leqq t.$$

By making $n \to \infty$, we get $\lim(B^{n+1}\omega_0)(t) = \wedge(t) = 1-(w(t)+\gamma w'(t))$ for $t_0 \leqq t$, where $\gamma = \lim \gamma_n$. In view of (2.1), if $\lim(B^n\omega_0)(t+1)$ exists, then $\lim(B^n\omega_0)(t)$ exists. Hence $\lim(B^n\omega_0)(t)$ exists for all t. Denote it by ω. We show that $\omega \in \Omega$. It is easy to verify that ω satisfies the condition (1.1). The condition (1.2) for ω is an immediate consequence of the fact that $\omega(t) = \wedge(t)$ for $t_0 \leqq t$ and

$$\omega(t) = \omega'(t+1) + u(t) + v(t)\gamma - \delta'_\omega(t+1). \tag{4.17}$$

The first part of (1.3) follows from the fact that $\omega(t) = u(t)+v(t)\gamma$ for $t<0$. The second part of (1.3) follows from the fact that $\omega(t) = \wedge(t)$ for $t_0 \leqq t$. Thus we get $\omega \in \Omega$. From (4.17) it follows that ω is a fixed function of B in Ω. Hence $\omega = \omega(B)$, the standard function of B at ω_0. If B is of Type V, then it is finite and has no fixed function in Ω. Hence we also saw that $\sum_{j=0}^{n}(1-v(j))\gamma_{n-j} \to 1 - w(0)$ if B is of Type V.

We have the same result when $w'(0) < \infty$. By the proof of Lemma 4.5, $\sum_{j=0}^{n}(1-v(j))\gamma_{n-j} \to 0$ if B is finite, of Type I or II, $\beta' = 0$, and $w'(0) = \infty$. Hence, $\sum_{j=0}^{n}(1-v(j))\gamma_{n-j} \to 1 - w(0) \neq 0$ if and only if B is of Type V.

We finally assume that $\gamma < 1-\lambda$ and that $\sum_{j=0}^{n}(1-v(j))\gamma_{n-j} \to 1-w(0) \neq 0$ (or equivalently, B is of Type V). We have $\gamma = 0$ and $\sum_{j=0}^{n}(1-v(t+j))\gamma_{n-j} \to 1-w(0)$ for all $t \geqq 0$. Moreover, by the proof of Lemma 4.5, $(n;\ i_n = 0)$ is an infinite set. Set

$$(B^{n+1}\omega_0)(t) = \omega_0(n+t+1) - \sum_{j=0}^{n}(-u(t+j) + (1-v(t+j))\gamma_{n-j} + s(n,t). \tag{4.18}$$

By a similar argument as in the proof of Lemma 4.3, we can conclude that

$$s(n,t) \to 0 \qquad (n \to \infty). \tag{4.19}$$

By making $n \to \infty$ in (4.18), we thus get $\lim(B^{n+1}\omega_0) = w(0) - w(t)$ for $t \geqq 0$. On the other hand, we have $(B^{n+1}\omega_0)(t) = u(t) + v(t)\gamma_n$ for $t<0$ and $\lim(B^{n+1}\omega_0)(t) = u(t) = 0$ for $t<0$.

Infinite Case: We have from Lemma 4.2 that $i_n \to \infty$. Hence, given t, $(B^{n+1}\omega_0)(t) = u(t)+v(t)\gamma_n$ for almost all n, and $\lim(B^{n+1}\omega_0)(t) = u(t)+v(t)\gamma$, where $\gamma = \lim \gamma_n$. It is easy to see that γ is the β described in § 2.

This completes the proof.

References

1) P, Erdös, W. Feller, and H. Pollard, A Theorem on Power series, Bulletin of the American Mathematical Society. Vol. **55**, pp. 201-204 (1949).

2) W. Feller, An introduction to Probability Theory and its Applications, Volume I (1950).