ON THE SCATTERING OF SOUND WAVE IN A NON-ISOTROPIC TURBULENCE

Éi Iti TAKIZAWA, Akihiko KUMAZAWA, and Keiko KOBAYASI

Institute of Aeronautical and Space Science

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Résumé—With probability density functions of vortices of different inclinations, radii and circulations, the problem of sound scattering in a turbulent medium is treated. The anisotropy as well as the isotropy of the turbulence is defined by means of the probability density function. Scattering intensity and power are calculated in the anisotropic and the isotropic cases.

§ 1. Preliminaries and Notations

The problem of sound scattering in a homogeneous, isotropic turbulence has been treated by some authors.[1] Especially, Obuchow[2] took account of velocity correlation in the turbulence. Tatarskiy[3] investigated the fluctuation of scattered sound by means of Obuchow's theory. Ellison[4] considered wave propagation in a medium, in which refractive index varies randomly in co-ordinate space. It is sometimes difficult to have a relationship between the fluctuation of the refractive index and the turbulent velocities. Lighthill[5] did not assume the isotropy of the turbulence. The velocity correlation in an *isotropic* turbulence is rather widely investigated. Little is known, however, about the velocity correlation in a *non-isotropic* turbulence. Our present knowledge[6] about the mechanism of the *decay of anisotropic turbulence* is also very little.

On the other hand, Müller[7] treated the turbulence as a statistical superposition of vortices, whose axes were orientated spacially at random. In such a model of vortex, one can easily take into account the *anisotropy* (non-isotropy) of a turbulence, because a sort of measure of anisotropy can be expressed by some parameters involved in the probability density function, whose random variable is to be taken as angle of inclination of vortices with regard to the direction of incident sound wave. Accordingly, we shall consider in the present paper phenomena of sound scattering by means of a single vortex model presented by Müller. Then, we describe the anisotropy of turbulence by a superposition of statistically orientated vortices of different radii and circulations.

Notations

r, φ, z	:	cylindrical co-ordinates,
t	:	time,
θ	:	angle of vortex axis with regard to the wave vector of incident sound,
λ	:	wave-length of incident wave,
c	:	velocity of sound,

Γ : circulation of vortex filament,
$\mu=\Gamma/(2\pi\lambda c)$, $R=r/\lambda$, $Z=z/\lambda$, $\tau=ct/\lambda$,
Φ_0 : velocity potential of incident sound wave,
$\mu\Phi_1$: velocity potential of scattered wave,
$\Phi=\Phi_0+\mu\Phi_1$,
r_a and r_i : outer and inner radius of scattering region,
$R_a=r_a/\lambda$, $R_i=r_i/\lambda$,
L_{sc} : scattering power emitted from scattering region of unit length,
l : scattering power emitted from scattering region of unit volume,
L : total scattering power emitted from turbulent region of volume V,
$M=\frac{1}{\pi r_a^2 c}\int_0^{2\pi}\int_0^{r_a}\frac{\Gamma}{2\pi r}rdrd\varphi=\frac{\Gamma}{\pi r_a c}$: mean Mach number,
$W(R_a, \theta, \mu; \mathbf{r})$: probability density function at co-ordinate point $\mathbf{r}$,
σ and σ_j : standard deviations.

§ 2. Scattering of Sound Wave by a Single Vortex-Filament

Let a steady vortex of circulation Γ occupy the region:

$$r_i \leq r \leq r_a, \qquad -\infty < z < +\infty,$$

in cylindrical co-ordinates (r, φ, z). The fluid is at rest in the region out of the vortex. As shown in Fig. 1, an incident plane sound wave of wave length λ goes through the vortex.

The velocity potential of the incident sound is taken to be

$$\Phi_0 = A\cdot\exp[2\pi i(\tau - R\sin\theta\cos\varphi - Z\cos\theta)]. \tag{1}$$

Let Φ be the velocity potential of the wave and be written as

$$\Phi = \Phi_0 + \mu\Phi_1, \tag{2}$$

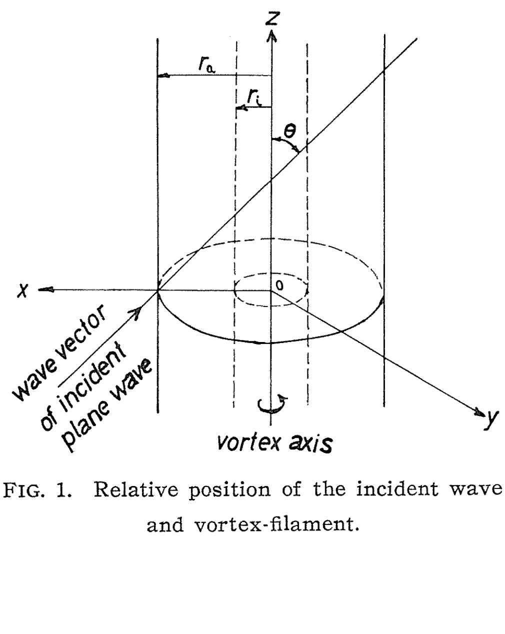

FIG. 1. Relative position of the incident wave and vortex-filament.

where $\mu\Phi_1$ is the velocity potential of the scattered sound by the vortex. Then, Φ satisfies the following equations:[7) 2)]

$$\Delta\Phi - \frac{\partial^2\Phi}{\partial\tau^2} = \frac{2\mu}{R^2}\frac{\partial^2\Phi}{\partial\varphi\partial\tau}, \quad \text{for } R_i \leq R \leq R_a, \tag{3}$$

and

$$\Delta\Phi - \frac{\partial^2\Phi}{\partial\tau^2} = 0, \quad \text{for } R < R_i \text{ and } R > R_a. \tag{4}$$

Assuming that $|\mu| \ll 1$, and neglecting the quantities of $O(\mu^2)$, we obtain from (3) and (4)

$$\Delta\Phi_1 - \frac{\partial^2\Phi_1}{\partial\tau^2} = \frac{2}{R^2}\frac{\partial^2\Phi_0}{\partial\varphi\partial\tau}, \quad \text{for } R_i \leq R \leq R_a, \tag{5}$$

and

$$\Delta\Phi_1 - \frac{\partial^2\Phi_1}{\partial\tau^2} = 0, \quad \text{for } R < R_i \text{ and } R > R_a, \tag{6}$$

as usual in the Born approximation.

Boundary conditions[8) 9)] at $R = R_a$ are:

$$[\Phi_1]_{R_a+0} - [\Phi_1]_{R_a-0} = \frac{1}{2\pi i R_a^2}\left[\frac{\partial\Phi_0}{\partial\varphi}\right]_{R_a}, \tag{7}$$

and

$$\left[\frac{\partial\Phi_1}{\partial R}\right]_{R_a+0} - \left[\frac{\partial\Phi_1}{\partial R}\right]_{R_a-0} = \frac{-1}{2\pi i R_a^2}\left[\frac{\partial^2\Phi_0}{\partial R\partial\varphi}\right]_{R_a}. \tag{8}$$

Boundary conditions at $R = R_i$ are:

$$[\Phi_1]_{R_i-0} - [\Phi_1]_{R_i+0} = \frac{1}{2\pi i R_i^2}\left[\frac{\partial\Phi_0}{\partial\varphi}\right]_{R_i}, \tag{9}$$

and

$$\left[\frac{\partial\Phi_1}{\partial R}\right]_{R_i-0} - \left[\frac{\partial\Phi_1}{\partial R}\right]_{R_i+0} = \frac{-1}{2\pi i R_i^2}\left[\frac{\partial\Phi_0}{\partial R\partial\varphi}\right]_{R_i}. \tag{10}$$

At $R = +\infty$, it is required that

$$\lim_{R\to+\infty} \Phi_1 = 0. \tag{11}$$

Under the conditions (7)~(11), assuming that the "Aufpunkt" (r, φ, z) is far from the vortex (*i. e.* $R \sin\theta \gg R_a$), we obtain the following solution from (5)~(6) for the case $R_i \to 0$:

$$\begin{aligned} &\mu\Phi_1(R, \varphi, Z, \theta) \\ &= \frac{\mu\pi A}{\sqrt{R\sin\theta}} \cdot \exp\left[\frac{i\pi}{4}\right] \cdot \exp\left[2\pi i(\tau - R\sin\theta - Z\cos\theta)\right] \\ &\times \left[J_0\left(4\pi R_a \sin\theta \sin\frac{\varphi}{2}\right) - 1\right] \cdot \left[-1 + 2\sin^2\theta \sin\frac{\varphi}{2}\right] \cdot \cot\frac{\varphi}{2}, \end{aligned} \tag{12}$$

with $J_0(\xi)$ Bessel function of order zero.

Scattering intensity I_{sc} and scattering power L_{sc} emitted per unit length of the vortex are

$$I_{sc} = \frac{\mu^2 \pi^2}{R \sin \theta} \cdot F(R_a, \varphi, \theta) \cdot I_0, \tag{13}$$

and

$$\frac{L_{sc}}{L_0} = \frac{1}{2} \frac{\mu^2 \pi^2}{R_a \sin \theta} \int_0^{2\pi} F(R_a, \varphi, \theta)\, d\varphi, \tag{14}$$

where $L_0 = 2 r_a \sin \theta \cdot I_0$ is the sound power falling on the vortex per unit length, I_0 is the sound intensity of the incident wave (1), and

$$F(R_a, \varphi, \theta) = \left[J_0\left(4\pi R_a \sin \theta \sin \frac{\varphi}{2}\right) - 1 \right]^2 \times \left[1 - 2 \sin^2 \theta \sin^2 \frac{\varphi}{2} \right]^2 \cdot \cot^2 \frac{\varphi}{2}. \tag{15}$$

Scattering power l per unit volume is

$$l = \frac{L_{sc}}{\pi r_a^2}, \tag{16}$$

which has the following expressions in the limitting cases:

$$l = \mu^2 \pi^6 R_a^2 \sin^4 \theta \,(\sin^4\theta + 4 \cos^4\theta) \frac{I_0}{\lambda}, \quad \text{for } R_a \ll 1, \tag{17}$$

and

$$l = 32\, \mu^2 \pi^2 \left(1 - \frac{2}{\pi}\right) \sin \theta \cdot \frac{I_0}{r_a}. \quad \text{for } R_a \gg 1. \tag{18}$$

§ 3. Scattering of Sound Wave by Turbulence

In §2, we have discussed the scattering of sound by a single vortex. In order to apply the results obtained in §2 to the turbulent medium, we shall assume that the turbulence can be described by the statistical superposition of vortices of different radii r_a, different circulations Γ and different inclinations θ with respect to the propagation vector of the incident sound wave. As different vortices are statistically mutually independent, the intensities of the scattered waves produced by them can be simply added. An assumption is made, also, that the time required for the incident sound to pass through the vortex is small as compared with the time in which the vortex undergoes any sensible change. This assumption is equivalent to the requirment that $R_a \gg \mu$.

Let $W(R_a, \theta, \mu; \mathbf{r})\, dR_a\, d\theta\, d\mu$ be the probability such that the volume element $d\mathbf{r}$ at a co-ordinate point $\mathbf{r}$ belongs to a vortex, whose radius falls in the region between R_a and $R_a + dR_a$, dimensionless criculation $\Gamma/(2\pi\lambda c)$ between μ and $\mu + d\mu$, and angle of inclination with regard to the wave vector of the incident wave between θ and $\theta + d\theta$. The probability density funcion $W(R_a, \theta, \mu; \mathbf{r})$ has to satisfy the following condition:

$$\int_{\mu=-\infty}^{+\infty} \int_{\theta=0}^{\pi} \int_{R_a=0}^{+\infty} W(R_a, \theta, \mu; \mathbf{r})\, dR_a d\theta d\mu = 1. \tag{19}$$

Statistical independence of R_a, θ and μ, implies the factorization of probability density functions, *i.e.*

$$W(R_a, \theta, \mu; \mathbf{r}) = f(R_a; \mathbf{r}) \cdot g(\theta; \mathbf{r}) \cdot h(\mu; \mathbf{r}), \tag{20}$$

which we shall consider hereafter.

With this probability density function W in (19), we can express the character of the turbulence (*e.g.* the homogeneity, the non-isotropy and the isotropy) in the following manner:

(*1*) *Homogeneity*

The turbulence is homogeneous, if $W(R_a, \theta, \mu; \mathbf{r})$ is independent of $\mathbf{r}$. In symbolical expression, the homogeneity of turbulence means

$$\frac{dW(R_a, \theta, \mu; \mathbf{r})}{d\mathbf{r}} = 0. \tag{21}$$

(*2*) *Isotropy*

If the probability density function W is independent of the inclination of the vortex, *i.e.*

$$\frac{\partial W(R_a, \theta, \mu; \mathbf{r})}{\partial \theta} = 0,$$

or

$$W(R_a, \theta, \mu; \mathbf{r}) = \frac{1}{\pi} \cdot W_1(R_a, \mu; \mathbf{r}), \tag{22}$$

or

$$g(\theta; \mathbf{r}) = \frac{1}{\pi},$$

we shall call this isotropy "θ-isotropy".

(*3*) *Extreme Non-Isotropy*

When all the vortex has parallel axes, making an angle to the direction of the wave vector of the incident wave, we call this non-isotropy "extreme non-isotropy". In this case the probability density function W is expressed as

$$W(R_a, \theta, \mu; \mathbf{r}) = \delta(\theta - \theta_0) \cdot W_2(R_a, \mu; \mathbf{r}). \tag{23}$$

(*4*) *Anisotropy*

We shall call the turbulence "θ-anisotropic", if $W(R_a, \theta, \mu; \mathbf{r}; \alpha_n)$ is expressed by a probability density function, which, in extreme cases, is reduced to (22) and (23) with limitting values of parameters α_n involved in W. In another word, the "θ-anisotropy" (non-isotropy) means that W has an intermediate character between (22) and (23). For example, we can take W to be Gaußian with regard to θ:

$$W(R_a, \theta, \mu; \mathbf{r}) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(\theta - \theta_m)^2}{2\sigma^2}\right] \cdot W_3(R_a, \mu; \mathbf{r}), \tag{24}$$

where θ_m and σ^2 are parameters. The expression (24) is reduced to (23) when

$\sigma \to 0$. If we take (24) as a probability density function which describes the anisotropy of turbulence, the parameter σ^2 serves as a measure of the anisotropy of turbulence.

Analogous considerations hold also for R_a and μ, if the radius and circulation of the vortex vary statistically in the turbulent medium. The probability density function $f(R_a)$ for $R_a \geq 0$ may be taken to be

(i) Gaußian:

$$f(R_a) = \frac{1}{\sqrt{2\pi}\,\sigma_1} \cdot \exp\left[-\frac{(R_a - R_m)^2}{2\sigma_1^2}\right]. \tag{25}$$

with two positive parameters R_m and σ_1,

(ii) χ^2-distribution of degrees of freedom $\nu \geq 2$:

$$f(R_a) = \frac{\beta}{2^{\nu/2}\,\Gamma(\nu/2)} (\beta R_a)^{(\nu-2)/2} \cdot \exp\left[-\frac{\beta R_a}{2}\right], \tag{26}$$

with a positive parameter β,

(iii) with a parameter $\alpha = \frac{\beta}{2} > 0$ and $\nu = 2$ in (26):

$$f(R_a) = \alpha \cdot \exp\left[-\alpha R_a\right], \tag{27}$$

or

(iv) Rayleigh distribution:

$$f(R_a) = \frac{R_a}{\sigma_2^2} \cdot \exp\left[-\frac{R_a^2}{2\sigma_2^2}\right], \tag{28}$$

with a parameter σ_2,
etc.

As for the distribution for μ, we may take also the probability density function $h(\mu)$ as

(i) Gaußian;

$$h(\mu) = \frac{1}{\sqrt{2\pi\,\sigma_3^2}} \cdot \exp\left[-\frac{(\mu - \mu_m)^2}{2\sigma_3^2}\right], \qquad \text{for } -\infty < \mu < +\infty \tag{29}$$

(ii) χ^2-distribution of degrees of freedom $s \geq 2$:

$$h(\mu) = \frac{\gamma}{2^{s/2}\,\Gamma(s/2)} (\gamma\mu)^{(s-2)/2} \cdot \exp\left[-\frac{\gamma\mu}{2}\right]. \tag{30}$$

with a positive parameter γ, and for $\mu \geq 0$,
or

(iii) Rayleigh distrbution:

$$h(\mu) = \frac{\mu}{\sigma_4^2} \cdot \exp\left[-\frac{\mu^2}{2\sigma_4^2}\right], \tag{31}$$

with a parameter σ_4,
etc.

If all vortices have the same radius R_0 or the same value of circulation μ_0, we can put

$$f(R_a) = \delta(R_a - R_0), \tag{32}$$

or

$$h(\mu) = \delta(\mu - \mu_0). \tag{33}$$

Assuming (21), (33) and Gaußian distribution both for θ and R_a just as in (24) and (25), we write the joint probability density function $W(R_a, \theta, \mu)$ as follows:

(i) anisotropic case:

$$W(R_a, \theta, \mu) = \frac{1}{2\pi\sigma\sigma_1} \cdot \exp\left[-\frac{(\theta-\theta_m)^2}{2\sigma^2}\right] \cdot \exp\left[-\frac{(R_a-R_m)^2}{2\sigma_1^2}\right] \cdot \delta(\mu-\mu_0).$$
$$\text{for } 0 \leq \theta \leq \pi \text{ and } R_a \geq 0 \tag{34}$$

As is well known, the expression (34) leads to the Rayleigh distribution if we put

$$\xi^2 = (\theta-\theta_m)^2 + (R_a-R_m)^2$$

and

$$\tan\eta = (R_a-R_m)/(\theta-\theta_m).$$

(ii) isotropic case:

$$W(R_a, \theta, \mu) = \frac{1}{\sqrt{2\pi}\,\pi\,\sigma_1} \cdot \exp\left[-\frac{(R_a-R_m)^2}{2\sigma_1^2}\right] \cdot \delta(\mu-\mu_m). \qquad \text{for } R_a \geq 0 \tag{35}$$

In the above expressions (34) and (35), an assumption should be made that

$$2\sigma < \theta_m < \pi - 2\sigma \text{ and } 2\sigma_1 < R_m, \tag{36}$$

in order to keep the relative error, which may occur in calculating the value of the probability, smaller than 5%.

If we take other probability density functions (for example χ^2-distribution for $R_a \geq 0$), there does not occur such a trouble as to assume the range of parameters.

§ 4. Scattering Power in a Turbulence

Total scattering power L emitted from a turbulent volume V is calculated by means of the probability density function W in (19) and (16).

$$L = \iiint_V \int_{\mu=-\infty}^{+\infty} \int_{\theta=0}^{\pi} \int_{R_a=0}^{\infty} W \cdot I \cdot dR_a\, d\theta\, d\mu\, d\mathbf{r}. \tag{37}$$

Calculation is made for the θ-anisotropic and the θ-isotropic cases. As the first example, we take (32), (33), and $g(\theta)$ to be Gaußian as shown in (34), and obtain:

(A) $g(\theta)$ and $f(R_a)$ are Gaußian

(i) θ-anisotropic case

$$L = V \cdot \frac{\mu^2 \pi^6}{2^7} \cdot \frac{I_0}{\lambda} R_0^2 \cdot (47-56 \cos 2\theta_0 \cdot \Xi^2 + 12 \cos 4\theta_0 \cdot \Xi^8 - 8 \cos 6\theta_0 \cdot \Xi^{18} + 5 \cos 8\theta_0 \cdot \Xi^{32}), \tag{38}$$

for the case $R_0 \ll 1$, with $\Xi = \exp[-\sigma^2]$ and $\sigma \ll 1$, $\sigma_1 \ll 1$.

$$L = 32 \cdot V \cdot \mu^2 \pi^2 \cdot \left(1 - \frac{2}{\pi}\right) \cdot \frac{I_0}{r_0} \cdot \sin \theta_0 \cdot \exp\left[-\frac{\sigma^2}{2}\right]. \tag{39}$$

for the case $R_0 \gg 1$.

(ii) θ-isotropic case

$$L = \frac{47}{128} V \cdot \mu^2 \pi^6 R_0^2 \cdot \frac{I_0}{\lambda}. \qquad \text{for } R_0 \ll 1 \tag{40}$$

and

$$L = 64\, V \cdot \mu^2 \pi \cdot \left(1 - \frac{2}{\pi}\right) \cdot \frac{I_0}{r_0}. \qquad \text{for } R_0 \gg 1 \tag{41}$$

The asymptotes of (38)$\sim$(41) are plotted against R_0 in Fig. 2., for $\mu = \frac{1}{2} M R_0$ with the mean Mach number M. If we fix the radius of vortex r_a as constant, R_a is proportional to the frequency ω of the incident wave. Accordingly, the scattering power is proportional to the fifth power of frequency in the region of small ω, and is proportional to the second power of frequency in the region of large ω. The scattering power becomes maximum for $\theta_0 = 90°$ and $\sigma = 0$, and decreases with $\theta \to 0$ or with $\theta \to \pi$.

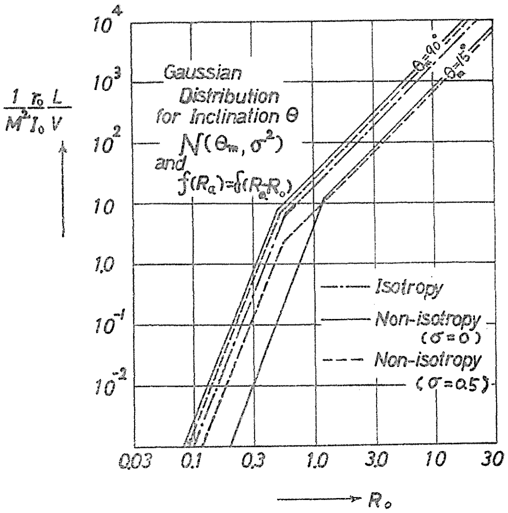

FIG. 2. Total scattering power versus radius of vortex R_0.

As another example, we calculate the scattering power by means of (27).

(B) $g(\theta)$ is Gaußian and $f(R_a)$ is χ^2-distribution with $\nu=2$.

(i) θ-anisotropic case

$$L = V \cdot \Big[\frac{\mu^2\pi^6}{2^7} \cdot \frac{I_0}{\lambda} \cdot (47 - 56\cos 2\theta_0 \cdot \Xi^2 + 12\cos 4\theta_0 \cdot \Xi^8 - 8\cos 6\theta_0 \cdot \Xi^{18} + 5\cos 8\theta_0 \cdot \Xi^{32}) \times \Big\{ -e^{-\alpha}\Big(1 + \frac{2}{\alpha} + \frac{2}{\alpha^2}\Big) + \frac{2}{\alpha^2}\Big\} - 32\mu^2\pi^2\Big(1 - \frac{2}{\pi}\Big) \cdot \frac{I_0}{\lambda} \cdot \sin\theta_0 \cdot \Xi^{1/2} \cdot \alpha \cdot Ei(-\alpha)\Big], \quad (42)$$

and

(ii) θ-isotropic case

$$L = \frac{47}{128} \cdot V \cdot \mu^2\pi^6 \cdot \frac{I_0}{\lambda} \cdot \Big\{ -e^{-\alpha}\Big(1 + \frac{2}{\alpha} + \frac{2}{\alpha^2}\Big) + \frac{2}{\alpha^2}\Big\} - V \cdot 64\mu^2\pi \cdot \Big(1 - \frac{2}{\pi}\Big) \cdot \frac{I_0}{\lambda} \cdot \alpha \cdot Ei(-\alpha), \quad (43)$$

with exponential integral $Ei(\xi)$. In this case, the scattering power is proportional to the third power of frequency of the incident wave, *i.e.* propotional to the third power of $(1/\lambda)$ if other parameters are kept constant. The graphs of (42) and (43) are plotted against $1/\lambda$ in Fig. 3, with $N=\Gamma/(\pi c)$.

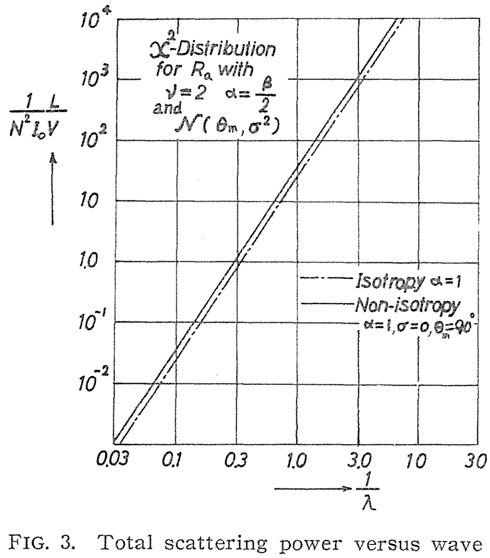

FIG. 3. Total scattering power versus wave number $1/\lambda$.

§ 5. Discussions

1) In quantum mechanical analogy, we shall call the operator $(2\mu/R^2) \cdot \partial^2/\partial\varphi\partial\tau$ appeared in the right hand side of the equation (3) "scattering potential". In equation (3) this potential comes essentially from the induced velocity of the

fluid caused by the vortex-filament lying at the z-axis. If Φ is proportional to $\exp[2\pi i(\tau - R\sin\theta\cos\varphi)]$, the scatttering potential becomes to be proportional to $1/R$. As the type of the scattering potential is Coulombian, its integral diverges at $R=+\infty$, and we have assumed that the potential is only effective within a range $R_i \leq R \leq R_a$. The divergence at $R=+\infty$ can be avoided if we take the potential as a form:

$$\exp[-bR]/R, \tag{44}$$

with positive b. In this case, the parameter b measures the screening effect of the fluctuating velocity field of the scattering medium.

2) The error of probability caused by cutting off the skirts of the probability density functions of Gaußian type (34) and (35), should be kept to be small (less than 5%). Accordingly, the assumption (36) for θ_m and R_m is to be taken into account. If θ_m or R_m does not satisfy the conditions (36), we should take other probability density function which is defined in the interval $0 \leq \theta \leq \pi$ for θ, or in the interval $R_a \geq 0$ for R_a.

3) The upper limit R_u of the radial extension of the scattering region should be necessarily taken into account, if we take the joint probability density function similar to $W(R_u, \theta, \mu; \mathbf{r})$, which is defined as the probability such that the volume element $d\mathbf{r}$ at a co-ordinate point $\mathbf{r}$ belongs to a vortex, whose radius falls in the region between R_u and $R_u + dR_u$. Accordingly, if we take the scattering potential of the form (44), we may take R_u such that

$$\exp[-bR_u] = 1/2, \ i.e.\ R_u = (\log 2)/b.$$

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